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ON THE PERIPHERAL SPECTRUM OF UNIFORMLY ERGODIC POSITIVE OPERATORS ON C^*-ALGEBRAS

ULRICH GROH

1. Let $\mathfrak{A}$ be a unital C^*-algebra and let $T \in \mathscr{L}(\mathfrak{A})$. T is called *uniformly ergodic*, if the averages $T_n := n^{-1} \sum_{k=0}^{n-1} T^k$ converge in the uniform operator topology to an operator $P \in \mathscr{L}(\mathfrak{A})$. P is a projection onto the fixed space $F(T) := \{x \in \mathfrak{A} : Tx = x\}$ and $TP = PT = P$. We call P the ergodic projection associated with T. If $T \in \mathscr{L}(\mathfrak{A})$ is positive, i.e. $T(\mathfrak{A}_+) \subseteq \mathfrak{A}_+$ where $\mathfrak{A}_+$ is the positive cone of $\mathfrak{A}$, then the following holds (see [6, Theorem 5]):

1.1. PROPOSITION. *If T is a positive operator on a C^*-algebra the following assertions are equivalent:*

(a) *T is uniformly ergodic.*

(b) *$r(T) \leqslant 1$ and 1 is a pole of the resolvent $\mathrm{R}(\lambda, T)$ of order $\leqslant 1$.*

(c) *$r(T) \leqslant 1$ and $\lim_{\lambda \searrow 1} (\lambda - 1)\mathrm{R}(\lambda, T)$ exists in the uniform operator topology.*

If a positive operator is uniformly ergodic then the associated ergodic projection, the residue of the resolvent at 1, and the limit in (c) are equal. For the notations concerning the *spectrum* $\sigma(T)$, *spectral radius* $r(T)$, *resolvent* $\mathrm{R}(\lambda, T)$, *pole of the resolvent*, etc., we refer to the book of Dunford-Schwartz [3].

Suppose $r = r(T)$ is a pole of the resolvent for a positive operator on a C^*-algebra. Then r is a pole of maximal order on the peripheral spectrum $\sigma(T) \cap r\Gamma$ of T, where $\Gamma = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ ([10, App. 2.2]). Moreover, if T is *irreducible*, i.e. leaves no non trivial closed face of $\mathfrak{A}_+$ invariant (see, e.g., [4, Definition 2.2]), then r has order one App. ([10, App. 3.2 (i)]). In our main theorem we will show that for an uniformly ergodic, irreducible and completely positive operator the peripheral spectrum consists entirely of first order poles of the resolvent. For the definition and properties of completely positive maps we refer to [13, IV.3]. Since for an irreducible positive operator $r(T) > 0$ ([4, 2.3]) we may assume without loss of generality $r(T) = 1$ in the theorem below.

1.2. MAIN THEOREM. *Let T be an irreducible and completely positive operator with spectral radius $r(T) = 1$ on a C^*-algebra. If T is uniformly ergodic the following assertions are true:*

(a) *The peripheral spectrum of T is the group Γ_h of all h-th roots of unity for some $h \geq 1$ and every $\alpha \in \Gamma_h$ is a simple eigenvalue of T and a pole of first order of the resolvent $R(\lambda, T)$.*

(b) *T has a representation*

$$T = \sum_{k=0}^{h-1} \varepsilon^k P_k + R$$

where ε is a primitive h-th root of unity, P_k are projections of rank 1 commuting with T, $P_{k_1} \cdot P_{k_2} = \delta_{k_1,k_2} \cdot P_{k_1}$ $(0 \leq k_1, k_2 \leq h-1)$, and R is an operator commuting with T and with spectral radius $r(R) < 1$.

Resulting from this spectral theoretical theorem, for which a proof will be given in the next section, we obtain a connection between various operator theoretical properties. Recall that an operator T is called *quasi-compact* if there exists a compact operator K and a positive integer n such that $\|T^n - K\| < 1$ (see, e.g., [3, VIII.8], [14]). We call an operator S on a Banach space E *partially periodic* if there exists $m_0 \in \mathbf{N}$ such that $S(I - S^{m_0}) = 0$. Then $S^{m_0}(I - S^{m_0}) = S(I - S^{m_0}) = 0$, hence S^{m_0} is a projection. Let $E_1 := S^{m_0}(E)$ and $E_2 := (I - S^{m_0})(E)$; then $E = E_1 \oplus E_2$, E_1 and E_2 are S-invariant, $(S|E_1)^{m_0} = I_{E_1}$ and $S|E_2 = 0$.

1.3. THEOREM. *Let T be an irreducible, completely positive operator on a C^*-algebra $\mathfrak{A}$. Then the following assertions are equivalent:*

(a) *T is uniformly ergodic.*

(b) *T is quasi-compact and the set $\{(\lambda - 1)R(\lambda, T) : \lambda > 1\}$ is uniformly bounded in $\mathscr{L}(\mathfrak{A})$.*

(c) *There exists a partially periodic operator $S \in \mathscr{L}(\mathfrak{A})$ such that* $\lim_n \|T^n - S^n\| = 0$.

(d) *$r(T) \leq 1$ and 1 is a pole of the resolvent of order ≤ 1.*

Proof. (a) $\Rightarrow$ (b) Under the assumptions we obtain $r(T) \leq 1$ and $\{(\lambda - 1)R(\lambda, T) : \lambda > 1\}$ is bounded in $\mathscr{L}(\mathfrak{A})$ (Proposition 1.1(b)). If $r(T) < 1$ then T is quasi-compact, so we assume $r(T) = 1$. In that case T has a representation $T = Q + R$ with $r(R) < 1$ and Q an operator of finite rank (Theorem 1.2). Since

$$\lim_n \|(T - Q)^n\| = \lim_n \|R^n\| = 0$$

there exists $n \geq 1$ and a compact operator K on $\mathfrak{A}$ such that $\|T^n - K\| < 1$.

(b) $\Rightarrow$ (c) The boundedness condition implies $r(T) \leq 1$ (use [4, 2.1]). If $r(T) < 1$ then the assertion (c) is fulfiled. If $r(T) = 1$ then $\sigma(T) \cap \Gamma$ is the group

Γ_h of all h-th roots of unity for some $h \geqslant 1$ ([4, 3.1(2)], [3, VIII.8.4]). Then T^h is quasi-compact, hence uniformly ergodic ([3, VIII.8.4]), $\sigma(T^h) \cap \Gamma = \{1\}$ and the fixed space of T^h is equal to the linear subspace generated by the eigenvectors of T pertaining to the eigenvalues in Γ_h. If P is the ergodic projection associated with T^h define $S := TP$ and note that $TP = PT$. Then S is partially periodic, since $S^h = PT^h = P$ and $T^m - S^m = T^m(I - P) = (T(I - P))^m$ norm converges to zero since $r(T(I - P)) < 1$.

(c) $\Rightarrow$ (a) Let $n_0 \in \mathbf{N}$ such that S^{n_0} is a projection. By assumptions the powers of T^{n_0} converge to S^{n_0}, hence T^{n_0} is uniformly ergodic. Since

$$T_n - T_m = \left(\sum_{k=0}^{n_0-1} T^k\right)((T^{n_0})_n - (T^{n_0})_m),$$

the averages T_n converge in $\mathscr{L}(\mathfrak{A})$.

(a) $\Leftrightarrow$ (d) See Proposition 1.1. ▨

2. In this section we give a proof of our main theorem. First we need some preparations which we state separately.

2.1. PROPOSITION. *Let T be a contraction on a Banach space E with adjoint $T^* \in \mathscr{L}(E^*)$. Then for every peripheral eigenvalue α of T,* $\ker(\alpha - T^*)$ *separates the points of* $\ker(\alpha - T)$. *In particular,* $\dim\ker(\alpha - T) \leqslant \dim\ker(\alpha - T^*)$.

Proof. Since for every $\alpha \in \mathbf{C}$ of modulus one $\ker(\alpha - T) = F(\alpha^* \cdot T)$ it suffices to prove that $F(T^*)$ separates the points of $F(T)$. Let $\mathfrak{B}$ be an ultrafilter on $[1, \infty)$ which converges to 1. Since the unit ball U^0 of E^* is $\sigma(E^*, E)$-compact and invariant under T^*, there exist for each $\psi \in U^0$ the linear form $\psi_0 := \lim_{\mathfrak{A}} (\lambda - 1)\mathrm{R}(\lambda, T^*)\psi$. Since T^* is $\sigma(E^*, E)$-continuous and $T^*\mathrm{R}(\lambda, T^*) = \mathrm{R}(\lambda, T^*) - I$ we conclude $\psi_0 \in F(T^*)$. Now take $0 \neq x_0 \in F(T)$ and choose $\psi \in U^0$ such that $\psi(x_0) \neq 0$. From the considerations above it follows that $\psi_0(x_0) = \psi(x_0) \neq 0$, hence $0 \neq \psi_0 \in F(T^*)$ and $F(T^*)$ separates the points of $F(T)$. ▨

In the proof of our main theorem we use the so called ultrapower $\hat{E}$ of a Banach space E with respect to a free ultrafilter $\mathfrak{U}$ on $\mathbf{N}$. For the construction of $\hat{E}$ we refer to [11, V.1.4]. Recall in particular that for $T \in \mathscr{L}(E)$ we have $\sigma(T) = \sigma(\hat{T})$ and $\mathrm{A}\sigma(T) = \mathrm{A}\sigma(\hat{T}) = \mathrm{P}\sigma(\hat{T})$ where $\hat{T}$ is the canonical extension of T to $\hat{E}$ and $\mathrm{A}\sigma(T)$ (resp. $\mathrm{P}\sigma(T)$) denotes the approximate point spectrum of T (resp. point spectrum of T).

2.2. PROPOSITION. *Let E be a Banach space and $T \in \mathscr{L}(E)$. If $\alpha \in \mathrm{P}\sigma(\hat{T})$ and* $\dim\ker(\alpha - \hat{T}) < \infty$, *then the following hold:*

(a) $\alpha \in \mathrm{P}\sigma(T)$ *and* $\dim\ker(\alpha - T) = \dim\ker(\alpha - \hat{T})$.

(b) *If T is a contraction and $\alpha \in \Gamma$ then α is a pole of the resolvent* $R(\lambda, T)$.

Proof. (a) Let $\alpha \in P\sigma(\hat{T})$ and suppose $\dim \ker(\alpha - \hat{T}) < \infty$. We prove first that $\alpha \in P\sigma(T)$. Let (x_n) be a normalized sequence in E such that $\lim_n \|(\alpha - T)x_n\| = 0$. Since $\alpha \in P\sigma(T)$ as soon as (x_n) has a convergent subsequence, we assume to the contrary that there is no such subsequence. Thus we may assume that there exists $\delta > 0$ such that $\|x_{n_1} - x_{n_2}\| \geqslant \delta$ for all positive integers $n_1 \neq n_2$. For $i \in \mathbf{N}$ let $\hat{x}_i$ be the image of (x_{n+i}) in $\hat{E}$ and note that $\hat{x}_i \in \ker(\alpha - \hat{T})$ with $\|\hat{x}_i\| = 1$. Since $\ker(\alpha - \hat{T})$ is finite-dimensional the sequence $(\hat{x}_i)$ has an accumulation point in $\ker(\alpha - \hat{T})$. Thus there exist positive integers $j_1 < j_2$ such that $\|\hat{x}_{j_1} - \hat{x}_{j_2}\| \leqslant \delta/2$ which leads to a contradiction. Hence we proved $\alpha \in P\sigma(T)$.

Obviously $\dim \ker(\alpha - T) \leqslant \dim \ker(\alpha - \hat{T})$. If $\dim \ker(\alpha - T) < \dim \ker(\alpha - \hat{T})$ then there exist $\gamma > 0$ and a normalized $\hat{x} \in \ker(\alpha - \hat{T})$ with the property $\gamma \leqslant \|\hat{x} - \hat{y}\|$ for all $y \in \ker(\alpha - T)$. Since every $(x_n) \in \hat{x}$ has a convergent subsequence by the result of the last paragraph, there exists $0 \neq z \in \ker(\alpha - T)$ such that $\gamma \leqslant \|\hat{z} - \hat{y}\|$ for all $y \in \ker(\alpha - T)$, a contradiction. Thus $\dim \ker(\alpha - T) = \dim \ker(\alpha - \hat{T})$ as desired.

(b) Let $\dim \ker(\alpha - T) = n$ and choose $x_1, \ldots, x_n$ linearly independent in $\ker(\alpha - T)$. By Proposition 2.1 there exist $\varphi_1, \ldots, \varphi_n$ in $\ker(\alpha - T^*)$ such that $\varphi_i(x_j) = \delta_{i,j}$ $(i, j = 1, \ldots, n)$. Let $M := \bigcap_{i=1}^{n} \ker \varphi_i$, then $E = \ker(\alpha - T) \oplus M$ and $T(M) \subseteq M$. We claim $\alpha \notin \sigma(T|M)$; for if not, there exists an approximative eigenvector sequence (y_n) in M pertaining to α. Since by the proof of (a) this sequence has a convergent subsequence, $M \cap \ker(\alpha - T) \neq \{0\}$, a contradiction. But α is a pole of the resolvent $R(\lambda, T|\ker(\alpha - T))$, thus a pole of $R(\lambda, T)$. ∎

Let $0 \leqslant T \in \mathscr{L}(\mathfrak{A})$ be irreducible and suppose that the spectral radius $r := r(T)$ is a pole of the resolvent $R(\lambda, T)$. By the irreducibility of T there exists an invertible element $z \in \mathfrak{A}_+$ with $Tz = rz$ ([10, App. 3.2 (ii)]). Then the operator

$$Sx := r^{-1} z^{-1/2} T(z^{1/2} x z^{1/2}) z^{-1/2}$$

satisfies $S\mathbf{1} = \mathbf{1}$, S is similar to $r^{-1}T$ via $S = r^{-1} U^{-1} \cdot T \cdot U$ where U is the completely positive mapping $x \mapsto z^{1/2} x z^{1/2}$ and S is completely positive iff T is completely positive.

We are now prepared to prove our main theorem which we state here in a slightly different form. But [4, Remark 3, p. 313] and the consideration above shows that the new formulation is equivalent to the one given in the first section.

2.3. THEOREM. *Let T be an identity preserving, completely positive operator on a C^*-algebra $\mathfrak{A}$, let $\varphi \in \mathfrak{A}^*$ be a faithful, T^*-invariant state and let T be uniformly ergodic with associated ergodic projection $P = \varphi \otimes \mathbf{1}$. Then the peripheral spectrum of T consists entirely of first order poles of the resolvent* $R(\lambda, T)$.

Proof. We select a free ultrafilter $\mathfrak{U}$ on $\mathbf{N}$ and embed $\mathfrak{A}$ into the $\mathfrak{U}$-product $\hat{\mathfrak{A}}$. If we define $\|\hat{x}\| := \lim_{\mathfrak{U}} \|x_n\|$, $\hat{x} \in \hat{\mathfrak{A}}$, $(x_n) \in \hat{x}$, then $\hat{\mathfrak{A}}$ is a C^*-algebra with unit. By [11, V.1.2] $\hat{T}$ is uniformly ergodic with ergodic projection $\hat{P} = \hat{\varphi} \otimes \mathbf{1}$ where $\hat{\varphi}$ is defined by $\hat{\varphi}(\hat{x}) = \lim_{\mathfrak{U}} \varphi(x_n)$ whenever $(x_n) \in \hat{x}$. Moreover, using [13, IV.3.4] it is easy to see that $\hat{T}$ is completely positive. Thus the second adjoint $\hat{T}^{**}$ of $\hat{T}$ is completely positive and identity preserving on the W^*-algebra $\hat{\mathfrak{A}}^{**}$. Since $\hat{T}$ is a contraction the assertion follows from Proposition 2.1, Proposition 2.2 and the following lemma.

LEMMA. *Let $\mathfrak{A}$ be a W^*-algebra, let $T \in \mathscr{L}(\mathfrak{A})$ be identity preserving and completely positive with preadjoint $T_* \in \mathscr{L}(\mathfrak{A}_*)$. If T is uniformly ergodic with ergodic projection $P = \varphi \otimes \mathbf{1}$, then every peripheral eigenvalue of T_* is simple.*

Proof. By assumption the ergodic projection P, and therefore the linear form φ, is positive, identity preserving and $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-continuous. Let $s(\varphi)$ be the support projection of φ in $\mathfrak{A}$ and let $\mathfrak{A}_\varphi$ be the W^*-algebra $s(\varphi)\mathfrak{A}s(\varphi)$. Then the operator T_φ on $\mathfrak{A}_\varphi$ given by $T_\varphi(x) := s(\varphi)(Tx)s(\varphi)$ $(x \in \mathfrak{A}_\varphi)$ is well defined, identity preserving, completely positive and uniformly ergodic. Moreover the ergodic projection associated with T_φ is given by $P_\varphi := \varphi \otimes s(\varphi)$, thus the fixed space of T_φ is one-dimensional.

Let $\alpha \in P\sigma(T_*) \cap \Gamma$ and let $\psi_\alpha \in \mathfrak{A}_*$ be a normalized eigenvector pertaining to α with polar decomposition $\psi_\alpha = R_{v_\alpha} |\psi_\alpha|$ ([12, 5.16]). Using Schwarz inequality for completely positive operators on C^*-algebras ([1], [13, IV.3.8]) we obtain for every $x \in \mathfrak{A}$:

$$|\psi_\alpha(x)|^2 = |\psi_\alpha(Tx)|^2 = |\,|\psi_\alpha|((Tx)v_\alpha)|^2 \leqslant$$
$$\leqslant |\psi_\alpha|\,(T(x)T(x)^*) \leqslant (T_*|\psi_\alpha|)\,(xx^*).$$

Since

$$\|\psi_\alpha\| = \|\,|\psi_\alpha|\,\| = |\psi_\alpha|\,(T\mathbf{1}) = \|T_*|\psi_\alpha|\,\|$$

it follows from [13, III.4.6] that $|\psi_\alpha| \in F(T_*)$. Because of $T_*(\psi_\alpha^*) = \alpha^*\psi_\alpha^*$ we obtain by the same arguments $|\psi_\alpha^*| \in F(T)$. Hence $|\psi_\alpha| = |\psi_\alpha^*| = \varphi$ since the fixed of T_* is spanned by φ. But $R_{v_\alpha^*} |\psi_\alpha^*|$ is the polar decomposition of ψ_α^* ([12, E.5.11, p. 131]), thus by [12, 5.16] $v_\alpha v_\alpha^* = v_\alpha^* v_\alpha = s(\varphi)$. In particular, v_α is unitary in $\mathfrak{A}_\varphi$. Since

$$\varphi((T_\varphi v_\alpha^* - \alpha v_\alpha^*)^*(T_\varphi v_\alpha^* - \alpha v_\alpha^*)) \leqslant$$
$$\leqslant \varphi(v_\alpha v_\alpha^*) - \alpha\varphi((T_\varphi v_\alpha)v_\alpha^*) - \alpha^*\varphi(v_\alpha(T_\varphi v_\alpha^*)) + \varphi(v_\alpha v_\alpha^*) =$$
$$= 2\varphi(\mathbf{1}) - \alpha\psi_\alpha^*(Tv_\alpha) - (\alpha\psi_\alpha^*(Tv_\alpha))^* =$$
$$= 2\varphi(\mathbf{1}) - \psi_\alpha^*(v_\alpha) - \psi_\alpha^*(v_\alpha)^* = 0$$

and since φ is faithful on $\mathfrak{A}_\varphi$ it follows $T_\varphi v_\alpha^* = \alpha v_\alpha^*$. Therefore, using [4, p. 314 (3)], $\alpha T_\varphi = M^{-1} \circ T_\varphi \circ M$ where M is the isometry $x \mapsto v_\alpha^* x$ on $\mathfrak{A}_\varphi$. Since the fixed space of T_φ is one-dimensional, α is a simple eigenvalue of T_φ. If ψ_1 and ψ_2 are normalized elements in $\ker(\alpha - T_*)$ with polar decomposition $\psi_1 = R_{v_1}\varphi$ and $\psi_2 = R_{v_2}\varphi$, then $v_1 = \mu v_2$ for some $\mu \in \mathbf{C}$ by the considerations above. Thus $\psi_1 = \mu\psi_2$ and $\dim \ker(\alpha - T_*) = 1$ as desired. ▨

2.4. Corollary. *Let $T \in \mathscr{L}(\mathfrak{A})$ satisfy the assumption of Theorem 2.3. Then the following assertion hold:*

(a) *There exists a primitive h-th root of unity ε such that $\sigma(T) \cap \Gamma = \{\varepsilon^k : 0 \leqslant k \leqslant h - 1\}$.*

(b) *T has a representation $T = \sum_{k=0}^{h-1} \varepsilon^k P_k + R$ where P_k are projections of rank 1 commuting with T, $P_{k_1} \circ P_{k_2} = \delta_{k_1,k_2} \cdot P_{k_1}$ $(0 \leqslant k_1, k_2 \leqslant h - 1)$ and R is an operator commuting with T and with spectral radius $r(R) < 1$.*

(c) *There exists a unitary eigenvector u of T pertaining to ε such that $P_k = (L_{u^k}\varphi) \otimes u^{-k}$ $(0 \leqslant k \leqslant h - 1)$.*

Proof. (a), (b) Since 1 is isolated in $\sigma(T)$ and $\sigma(T) \cap \Gamma \subseteq \mathrm{P}\sigma(T)$ the assertions follow from [4, 3.1(2)] and [3, VII.3].

(c) It follows from the proof of [4, 3.1(2)] that there exists a unitary $u \in \mathfrak{A}$ such that $Tu = \varepsilon u$. Since P_k is a projection onto the eigenvector space spanned by u^k we obtain $P_k = M^{-k} \circ P \circ M^k$ $(0 \leqslant k \leqslant h - 1)$ where M is the operator $x \mapsto ux$ and $P = \varphi \otimes \mathbf{1}$. A simple computation yields the assertion. ▨

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ULRICH GROH
Mathematisches Institut,
Universität Tübingen,
Auf der Morgenstelle 10, 74 Tübingen,
West Germany.

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