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QUASITRIANGULAR ALGEBRAS ARE "MAXIMAL"

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Quasitriangular operators were introduced by Halmos [5] as those operators T for which there was an increasing sequence of finite rank projections P_n tending to I such that $\lim \|P_n^{\perp} T P_n\| = 0$. In [2], Arveson considered the quasitriangular algebra $\mathrm{QT}(\mathscr{P})$ as the set of operators T which are quasitriangular with respect to $\{P_n\}$. He showed that $\mathrm{QT}(P) = \mathrm{alg}\,\mathscr{P} + \mathscr{K}(\mathscr{H})$, where $\mathscr{K}(\mathscr{H})$ is the ideal of compact operators. Since that time, there has been a lot of interest in these algebras. If $\mathscr{M}$ is an arbitrary subspace lattice, let $\mathrm{Q\,alg}\,\mathscr{M}$ denote the norm closure of $\mathrm{alg}\,\mathscr{M} + \mathscr{K}(\mathscr{H})$. In this paper, we give an affirmative answer to a conjecture in [4] by showing that if $\mathrm{Q\,alg}\,\mathscr{M}$ contains a quasitriangular algebra, then $\mathrm{Q\,alg}\,\mathscr{M}$ is also quasitriangular or is all of $\mathscr{B}(\mathscr{H})$. In this sense, quasitriangular algebras are maximal among compact perturbations of reflexive algebras.

When this problem was raised in [4], an important special case was resolved. In this paper, a careful analysis of the lattice structure of $\mathscr{M}$ will be used to reduce the problem to the special case. In [4], $\mathrm{Q\,alg}\,\mathscr{M}$ was defined as $\mathrm{alg}\,\mathscr{M} + \mathscr{K}(\mathscr{H})$ without norm closure. However, we have adopted the closure as being more natural. This has proved to be the case for commutative lattices in [1]. In [4], $\mathrm{Q\,alg}\,\mathscr{M}$ was automatically closed if $\mathscr{L}$ was a commutative AF lattice. All of the results in [4] go through if $\mathrm{Q\,alg}\,\mathscr{M}$ is replaced by its closure. The only place where nontrivial differences must be made is in Theorem 5.5. We will outline the changes we require later in this paper.

All Hilbert spaces in this paper are separable. Subspace lattices are assumed to be complete and closed in the strong operator topology. As in [4], a lattice is called AF if every element is the sup of finite rank elements of the lattice. For commutative lattices, it suffices that the identity is the up of finite rank elements. The symbols $\vee$ and $\wedge$ will denote the lattice operations of sup(closed linear span) and inf(intersection).

THEOREM. *Suppose* $\mathrm{Q\,alg}\,\mathscr{M}$ *is a proper subalgebra of* $\mathscr{B}(\mathscr{H})$ *containing a quasitriangular algebra. Then* $\mathrm{Q\,alg}\,\mathscr{M}$ *is quasitriangular.*

LEMMA 1. *Suppose* $\mathrm{Q\,alg}\,\mathscr{M}$ *is a proper subalgebra of* $\mathscr{B}(\mathscr{H})$ *containing a quasitriangular algebra* $\mathrm{QT}(\mathscr{P})$. *Then* $\mathscr{M}$ *contains only finite and cofinite elements. The* *sup*, M_∞, *of all finite projections and the inf*, N_∞, *of all cofinite projections are both cofinite. The lattice* $\mathcal{M} \vee M_\infty^\perp$ *generated by* $\{0, M \vee M_\infty^\perp : M \in \mathcal{M}\}$ *is* AF.

Proof. If A belongs to alg $\mathcal{M}$ and K is compact, then $M^\perp(A+K)M = M^\perp KM$ is compact for all M in $\mathcal{M}$. By continuity, this property extends to Q alg $\mathcal{M}$, and a fortiori to QT($\mathcal{P}$). Thus each M in $\mathcal{M}$ is essentially invariant for QT($\mathcal{P}$). By [3], M belongs to $\pi\mathcal{P} = \{\dot{0}, \dot{I}\}$. That is, M is either finite or cofinite.

By Lemma 5.2 of [4], both N_∞ and M_∞ are cofinite. Thus every projection in $\mathcal{M} \vee M_\infty^\perp$ is finite or cofinite. To show that it is AF, consider a cofinite projection L. As M_∞ is the sup of all finite elements of $\mathcal{M}$, there is an increasing sequence M_k with $\vee M_k = M_\infty$. Consider $L_k = (M_k \vee M_\infty^\perp) \wedge L$. Clearly the dimension dim L_k tends to infinity, so $L' = \vee L_k$ is infinite and hence $N_\infty \vee M_\infty^\perp \leqslant L' \leqslant L$. In particular, $N_\infty \vee M_\infty^\perp = \vee N_k$ where $N_k = (M_k \vee M_\infty^\perp) \wedge (N_\infty \wedge M_\infty^\perp)$.

Let v be any vector in $L\mathcal{H}$. Since $N_\infty^\perp$ is finite rank, for k sufficiently large $N_\infty^\perp(M_k \vee M_\infty^\perp)\mathcal{H} = N_\infty^\perp\mathcal{H}$. So for k sufficiently large, $M_k \vee M_\infty^\perp$ contains $v + n$ for some n in $N_\infty\mathcal{H}$. Since n can be approximated by elements of N_k, and hence of L_k for large k, it follows that v belongs to L'. So $\vee L_k = L$ and $\mathcal{M} \vee M_\infty^\perp$ i s AF. ∎

LEMMA 2. *If* Q alg $\mathcal{M}$ *contains a quasitriangular algebra* QT($\mathcal{P}$) *and* M_∞ *is as in Lemma* 1, *then there is a finite rank element* M_0 *in* $\mathcal{M}$ *such that* $\mathcal{M} \vee M_0 \vee M_\infty^\perp$ *is an* AF *chain, and thus* Q alg ($\mathcal{M} \vee M_0 \vee M_\infty^\perp$) *is quasitriangular.*

Proof. Theorem 4.2 and Lemma 5.4 of [4]. ∎

LEMMA 3. *Suppose* $\mathcal{P}$ *and* $\mathcal{Q}$ *are* AF *chains with* QT($\mathcal{P}$) *contained in* QT($\mathcal{Q}$). *Then* $\mathcal{Q}$ *is asymptotic to a subchain of* $\mathcal{P}$. *Furthermore there is a chain* $\mathcal{P}'$ *containing* $\mathcal{Q}$ *such that* QT($\mathcal{P}'$) = QT($\mathcal{P}$).

Proof. The first statement follows from Theorem 2.7 of [4]. Because of the finite dimensional character of these chains, it is easy to extend $\mathcal{Q}$ to a chain $\mathcal{P}'$ which is asymptotic to $\mathcal{P}$. (That is, $\lim\|P_n' - P_n\| = 0$.) Thus the characterization of QT($\mathcal{P}$) mentioned in the introduction [2] shows that QT($\mathcal{P}'$) = QT($\mathcal{P}$). ∎

Now we will investigate carefully the structure of $\mathcal{M}$. By Lemmas 2 and 3, we may suppose that $\mathcal{M} \vee M_0 \vee M_\infty^\perp$ is a chain $\mathcal{Q} = \{Q_k, k \geqslant 0\}$ containing a chain $\mathcal{P} = \{P_n, n \geqslant 0\}$ such that QT($\mathcal{P}$) is contained in Q alg $\mathcal{M}$. Furthermore, we may (and do) stipulate that $P_0 = M_0 \vee M_\infty^\perp$ and $\dim(P_{n+1} - P_n) = 1$ for $n \geqslant 0$ by enlarging $\mathcal{P}$ if required. Let n_k be the integers with $Q_k = P_{n_k}$.

Let $\mathcal{M}_k$ be the set of projections L in $\mathcal{M}$ such that $L \vee M_0 \vee M_\infty^\perp = Q_k$. For each k, let M_k be the sup of all L in $\mathcal{M}_k$. Since M_0 belongs to $\mathcal{M}$ and all finite L are dominated by M_∞, it must be that $M_0 \leqslant M_k \leqslant M_\infty$ and $M_k \vee M_\infty^\perp = Q_k$. Furthermore, $M_k \leqslant M_{k+1}$ and $\vee M_k = M_\infty$. It follows for L in $\mathcal{M}_k$ that $L \vee M_0 = M_k$. The modular law for subspaces of *finite dimensional* Hilbert spaces shows that for L in $\mathcal{M}_k$ and $j \leqslant k$,

$$(L \wedge M_j) \vee M_0 = M_j.$$

Hence $L \wedge M_j$ belongs to $\mathcal{M}_j$.

For $k \geqslant 1$, define $n_k = \min\{\dim(M_0 \wedge L) : L \in \mathcal{M}_k\}$. The remarks of the last paragraph show that n_k is an increasing sequence of integers bounded above by $\dim M_0$. So there is an integer k_0 so that $n_k = n_\infty$ is constant for $k \geqslant k_0$. If L_1 and L_2 belong to $\mathcal{M}_k$, then $\dim (L_1 \wedge L_2) \vee M_0 \geqslant \dim M_k - \dim M_0$. Thus $L_1 \wedge L_2$ belongs to $\mathcal{M}_l$ for some $l \geqslant k - m_0$. Here $m_k = \dim M_k$. Now for $k \geqslant k_0 + m_0$, take L in $\mathcal{M}_k$ with $\dim(M_0 \wedge L) = n_\infty$. If L_1 is any other projection in $\mathcal{M}_k$, then $N = L \wedge L_1 \wedge M_{k-m_0}$ belongs to $\mathcal{M}_{k-m_0}$. Plus $N \wedge M_0 \leqslant L \wedge M_0$ and both have dimension n_∞, hence we have equality. So $N \leqslant L \wedge M_{k-m_0}$, both belong to $\mathcal{M}_k$ and have the same intersection with M_0 and consequently $N = L \wedge M_{k-m_0}$. Thus every L_1 in $\mathcal{M}_k$ contains N as does every L' in $\mathcal{M}_{k'}$ for $k' > k$. This identifies a unique projection N_k in each $\mathcal{M}_k$, $k \geqslant k_0$ with $\dim(N_k \wedge M_0) = n_\infty$ and $N_k \leqslant L$ for every L in $\mathcal{M}_{k'}$ for $k' \geqslant k + m_0$.

Now

$$N_k = N_{k+m_0+1} \wedge M_k = (N_{k+m_0+1} \wedge M_{k+1}) \wedge M_k = N_{k+1} \wedge M_k.$$

So the N_k form a chain, and we extend this definition to $N_k = N_{k_0} \wedge M_k$ for $k < k_0$. Note that $\dim N_0 = n_\infty$. If L belongs to $\mathcal{M}_k$ for $k \geqslant k_1 = k_0 + m_0$, then $L \geqslant N_{k_0} \geqslant N_0$. So every L in $\mathcal{M}_k$, $k \geqslant k_i$, contains N_0. Thus if $L_0 = M_0 \ominus N_0$, then $L \vee L_0 = M_k$ for all L in $\mathcal{M}_k$ and $k \geqslant k_1$. The sup of the N_k must be N_∞ because $\vee N_k$ is infinite and thus at least N_∞. Yet if L_j is a sequence with sup N_∞, the L_j dominate the increasing sequence of N_k's and thus $\vee N_\infty \leqslant N_\infty$.

Let $\mathcal{N}_0 = \{M \in \mathcal{M} : M \leqslant M_{k_1}\}$ and $\mathcal{N}_\infty = \{M \in \mathcal{M} : M \geqslant N_\infty\}$, and let $\mathcal{M}_0$ be the sublattice of $\mathcal{M}$ generated by $\{\mathcal{N}_0, \mathcal{N}_\infty, N_k, k \geqslant k_1\}$. This lattice contains the chain M_k with $\sup M_\infty$, every L in M_k which is greater than N_k (since $L = (L \wedge M_0) \vee N_k$), every cofinite projection in $\mathcal{M}$ and all projection $L \leqslant M_{k_1}$. The only projections in $\mathcal{M} \setminus \mathcal{M}_0$ are those L in $\mathcal{M}_k$, $k > k_1$ which are not greater than N_k. Clearly Q alg$\mathcal{M}_0$ contains QT($\mathcal{P}$) also.

The lattice $\mathcal{M}_0$ almost fits into the mold required for Theorem 5.5 of [4]. We will show how that theorem can be modified to apply here. Choose unit vectors e_i in $(P_i - P_{i-1})\mathcal{H}$. Let $l_k = m_k - m_0 = \dim N_k - n_\infty$. It is easily seen that $M_k = \text{span}\{M_0, e_i, i = 1, \ldots, l_k\}$. Since $N_k \vee L_0 = M_k$ and $N_k \wedge L_0 = \{0\}$, there are unique constants $a_i \geqslant 0$ and unit vectors f_i in $L_0\mathcal{H}$ such that $N_k = \text{span}\{N_0, e_i + a_i f_i, i = 1, \ldots, l_k\}$.

LEMMA 4. $\sum_{i=1}^{\infty} a_i^2 < \infty$.

Proof. The proof is essentially the same as the proof of Theorem 5.5 of [4]. We will sketch the ideas here and indicate where any differences lie.

Denote by $T_{f\otimes e}$ the rank one operator which takes a vector x to $(x,e)f$. Define the Hilbert-Schmidt operators

$$H_p = -M_0 + \sum_{i=1}^{p} a_i T_{f_i \otimes e_i}.$$

An easy computation shows that $\|H_p\|_2^2 = m_{k_1} + \sum_{i=1}^{p} a_i^2$. The lemma fails only if $\lim_{p\to\infty} \|H_p\|_2 = \infty$. We shall suppose that this is the case and reach a contradiction.

The sequence $h_p = \|H_p\|_2^{-1} H_p$ is bounded in norm, and

$$\lim_{p\to\infty} h_p M_0 = \lim_{p\to\infty} - \|H_p\|_2^{-1} M_0 = 0$$

and

$$\lim_{p\to\infty} h_p e_n = \lim_{p\to\infty} \|H_p\|_2^{-1} a_n f_n = 0.$$

Thus h_p tends to zero in the strong operator topology. By Lemma 5.6 of [4], if C is any compact operator, $\lim_{p\to\infty} \|h_p C\|_2 = 0$.

Let Δ be the subset of $\mathbf{N}$ constructed in [4]. Let D be the orthogonal projection onto $\operatorname{span}\{e_k : k \in \Delta\}$. Since D belongs to alg $\mathscr{P}$, there is an operator $K = C + S$ so that $D + K$ belongs to alg $\mathscr{M}$ (and a fortiori to alg $\mathscr{M}_0$) with C compact and $\|S\| < 1/8$. Since M_k are invariant under D, they are also left invariant by K. In particular, K maps $M_0\mathscr{H}$ into itself. The proof of Theorem 5.5 in [4] now shows that there is a sequence n_i tending to ∞ so that $\|H_{n_i}K\|_2^2 \geqslant 32^{-1}\|H_{n_i}\|_2^2$. Hence

$$\|h_{n_i}C\|_2 \geqslant \|h_{n_i}K\|_2 - \|h_{n_i}S\|_2 \geqslant (4\sqrt{2})^{-1} - \|S\| > 0.05.$$

This contradicts the previous paragraph and justifies the lemma. ∎

LEMMA 5. *$\mathrm{Q}\,\mathrm{alg}\,\mathscr{M}_0$ is quasitriangular.*

Proof. First, notice that $\mathrm{Q}\,\mathrm{alg}\{M_k\} = \mathrm{QT}(\mathscr{Q})$. For if T belongs to alg $\mathscr{Q}$ or to alg$\{M_k\}$, then $M_\infty T M_\infty$ belongs to both, and $T - M_\infty T M_\infty$ is compact.

Let $E = M_\infty - M_{k_1} + \sum_{k>k_1} a_k T_{f_k \otimes e_k}$. The Hilbert-Schmidt norm of $M_{k_1}E$ is $(\sum_{k>k_1} a_k^2)^{1/2}$ which is finite. So $E = M_\infty - M_{k_1} + M_{k_1}E$ is bounded and in fact $I - E$ is compact. One readily verifies that $E = E^2 = E(M_\infty - M_{k_1})$. Suppose T belongs to alg$\{M_k\}$ and $n \leqslant l_p$.

$$ET(e_n + a_n f_n) = (M_\infty - M_{k_1})Te_n + \sum_{k>k_1} a_k(T(e_n + a_n f_n), e_k)f_k =$$

$$= \sum_{k=k_1+1}^{l_p} (Te_n, e_k)e_k + \sum_{k=k_1+1}^{l_p} a_k(Te_n, e_k)f_k = \sum_{k=k_1+1}^{l_p} = (Te_n, e_k)(e_k + a_k f_k).$$

Also $ETM_{k_1} = EM_{k_1}TM_{k_1} = 0$. Hence N_p is invariant for ET, as is every subspace of $M_{k_1}\mathscr{H}$. Also $ETM_k\mathscr{H} = ETN_k\mathscr{H}$ is contained in $N_k\mathscr{H}$ and thus in $N_\infty\mathscr{H}$ for all k. Hence $ETM_\infty\mathscr{H}$ is contained in $N_\infty\mathscr{H}$. Thus ETM_∞ leaves invariant all projections greater than N_∞. So ETM_∞ belongs to alg $\mathscr{M}_0$. Since $T - ETM_\infty$ is compact, QT($\mathscr{Q}$) is contained in Q alg $\mathscr{M}_0$. The reverse inclusion is trivial.

So Q alg $\mathscr{M}_0$ = QT($\mathscr{Q}$) and $T \to ETM_\infty$ takes alg $\mathscr{Q}$ into alg $\mathscr{M}_0$. This map is a homomorphism which induces the identity map on QT($\mathscr{Q}$)/$\mathscr{K}(\mathscr{H})$. To see this, note that $ETQ_{k_1} = EQ_{k_1}TQ_{k_1} = 0$. So

$$(ETM_\infty)(ESM_\infty) = ET(Q_{k_1}^\perp M_\infty E)SM_\infty = ETQ_{k_1}^\perp SM_\infty = ETSM_\infty.$$

In the other direction, it is readily verified that $T \to TM_\infty$ is a homomorphism of alg $\mathscr{M}_0$ into alg $\mathscr{Q}$. ▨

LEMMA 6. *There is an integer k_2 so that if L is a projection in $\mathscr{M}\setminus\mathscr{M}_0$, then $L \vee M_0 \leqslant M_{k_2}$.*

Proof. Suppose L belongs to $\mathscr{M}_k$ yet is not greater than N_k. Let l be the largest integer with $L \geqslant N_l$, so $l \geqslant k - m_0$. Let $L' = L \wedge M_{l+1}$. Then $L' \geqslant N_l$ but it is not greater than N_{l+1}. So L' must be the linear span of N_l, $L' \wedge M_0$, and a vector $e_{l+1} + g$ for some g in M_0.

Suppose A is any operator in alg $\mathscr{M}$. So both L' and N_{l+1} are invariant for A. Hence

$$A(e_{l+1} + a_{l+1}f_{l+1}) = \sum_{i=1}^{l+1} \alpha_i(e_i + a_if_i) + n, \quad n \in N_0.$$

$$A(e_{l+1} + g) = \sum_{i=1}^{l} \beta_i(e_i + a_if_i) + \beta_{l+1}(e_{l+1} + g) + m, \quad m \in L' \wedge M_0.$$

Since M_0 is invariant for A, $\alpha_i = (Ae_{l+1}, e_i) = \beta_i$, $i = 1, \ldots, l+1$. By subtracting, one obtains

$$A(a_{l+1}f_{l+1} - g) = (Ae_{l+1}, e_{l+1})(a_{l+1}f_{l+1} - g) + (n - m).$$

The vector $a_{l+1}f_{l+1} - g$ cannot belong to $L' \wedge M_0$, for then $e_{l+1} + a_{l+1}f_{l+1} = (e_{l+1} + g) + (a_{l+1}f_{l+1} - g)$ would belong to L' and hence $L' \geqslant N_{l+1}$. Let $v = (L' \wedge M_0)^\perp(a_{l+1}f_{l+1} - g)$. Since $L' \wedge M_0$ is invariant for A, $(L' \wedge M_0)^\perp A = (L' \wedge M_0)^\perp A(L' \wedge M_0)^\perp$, and hence

$$(Av, v) = (Ae_{l+1}, e_{l+1})(v, v).$$

Normalize v so that $\|v\| = 1$.

Now suppose that no integer k_2 as described in the statement of lemma exists. Then there are L_{k_i} in $\mathscr{M}_{k_i}$ such that L_{k_i} are not greater than N_{k_i} and $k_{i+1} > k_i + m_0$.

The previous paragraphs show that there are unit vectors v_i in M_0 and integers l_i (with $k_i - m_0 < l_i \leqslant k_i$) such that $(Av_i, v_i) = (Ae_{l_i}, e_{l_i})$ for all A in alg $\mathcal{M}$. Drop to a subsequence $\Lambda = \{i_j, j \geqslant 1\}$ so that $\lim_\Lambda v_i = v$ exists. Then $(Av, v) = \lim_\Lambda (Av_i, v_i) = \lim_\Lambda (Ae_{l_i}, e_{l_i})$.

Take the orthogonal projection D onto the span$\{e_{l_i} : i \in \Lambda_2 = \{i_{2j}, j \geqslant 1\}\}$. Choose K so that $D + K$ belongs to alg $\mathcal{M}$ and the essential norm $\|K\|_e < 1/3$. Then

$$\limsup |(Ke_i, e_i)| \leqslant \|K\|_e < 1/3.$$

Hence $\liminf_{\Lambda_2} |(D + Ke_{l_i}, e_{l_i})| \geqslant 2/3$ and $\limsup_{\Lambda \setminus \Lambda_2} |(D + Ke_{l_i}, e_{l_i})| \leqslant 1/3$. So $\lim(D + Ke_{l_i}, e_{l_i})$ does not exist. This contradiction establishes the lemma. ▨

Proof of Theorem. By replacing the role of M_{k_1} by M_{k_2}, the lattice $\mathcal{M}_0$ is in fact all of $\mathcal{M}$. By Lemma 5, we conclude that Q alg $\mathcal{M}$ is quasitriangular. ▨

REMARKS. 1) It follows from the proof of Lemma 5 that alg $\mathcal{M} + \mathcal{K}(\mathcal{H})$ is already closed.

2) The complete force of Q alg $\mathcal{M}$ containing QT($\mathcal{P}$) was not used. What was needed was the existence of a finite element M_0 and a cofinite element M_∞ in $\mathcal{M}$ such that $\mathcal{M} \vee M_0 \vee M_\infty^\perp$ was a chain, and that Q alg $\mathcal{M}$ contains an atomic masa.

3) There are homomorphisms of alg $\mathcal{Q}$ onto alg $\mathcal{M}$ and of alg $\mathcal{M}$ onto alg $\mathcal{Q}$ which induce the identity map on QT($\mathcal{Q}$)/$\mathcal{K}(\mathcal{H})$, as noted in Lemma 5. Thus if Q alg $\mathcal{M}$ and Q alg $\mathcal{N}$ are similar quasitriangular algebras, then there are homomorphisms between alg $\mathcal{M}$ and alg $\mathcal{N}$ which are inverses modulo the compact operators. It seems highly improbable that such a result holds in much greater generality.

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