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ON THE SPECTRAL SEMI-DISTANCE AND QUASI-NILPOTENT EQUIVALENCE FOR SYSTEMS OF OPERATORS

ŞTEFAN FRUNZĂ

The main result of this paper (Theorem 2) is that the "joint" spectral semi-distance is uniformly equivalent to the "cartesian" spectral semi-distance on any spectrally bounded set of commuting m-tuples of operators. As a corollary we get that the "joint" quasi-nilpotent equivalence coincides with the "cartesian" (or separate) quasi-nilpotent equivalence. We also discuss other possible spectral semi-distances and quasi-nilpotent equivalences for systems of operators which may be also reduced to "cartesian" notions.

1. Let X be a complex Banach space and $T, S \in L(X)$ be two continuous linear operators on X. For any positive integer $n \in \mathbf{N}$, denote

$$(T \diagdown S)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k T^{n-k} S^k.$$

Then T is said to be quasi-nilpotent equivalent to S, denoted $T \overset{q}{\sim} S$, if

$$\lim_n \|(T \diagdown S)^{[n]}\|^{1/n} = 0 = \lim_n \|(S \diagdown T)^{[n]}\|^{1/n}$$

(see [3]).

The spectral semi-distance between T and S is defined by

$$d(T, S) = \max(d_{sp}(T, S), d_{sp}(S, T))$$

where

$$d_{sp}(T, S) = \limsup_n \| (T \diagdown S)^{[n]} \|^{1/n}$$

(see [6] and [2]). It is obvious that $T \overset{q}{\sim} S$ if and only if $d(T, S) = 0$.

In [5] we generalized the notion of quasi-nilpotent equivalence to commuting systems of operators. For technical reasons we choose the following definition.

Denote by $L(X)^m_c$ the set of all commuting m-tuples in $L(X)^m$. For any system $T = (T_1, \ldots, T_m) \in L(X)^m_c$ and any multi-index $p = (p_1, \ldots, p_m)$ of positive integers, denote

$$T^p = T_1^{p_1} \dots T_m^{p_m} \quad \text{and} \quad |p| = p_1 + \dots + p_m .$$

For any two systems $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ in $L(X)_c^m$ and any multi-index $n = (n_1, \dots, n_m)$ denote

$$(T \setminus S)^{[n]} = \sum_{k_1=0}^{n_1} \dots \sum_{k_m=0}^{n_m} \binom{n_1}{k_1} \dots \binom{n_m}{k_m} (-1)^{|k|} T^{n-k} S^k$$

where $k = (k_1, \dots, k_m)$ and $n - k = (n_1 - k_1, \dots, n_m - k_m)$. T is said to be quasi--nilpotent equivalent to S if

$$\lim_k (\max_{|n|=k} \|(T \setminus S)^{[n]}\|)^{1/k} = 0 = \lim_k (\max_{|n|=k} \|(S \setminus T)^{[n]}\|)^{1/k}.$$

With that notion at hand we were able to prove the invariance of Taylor spectrum and other joint spectra with respect to quasi-nilpotent equivalence ([5], Section 4).

The spectral semi-distance between the systems T and S may be defined by

$$\mathrm{d}(T, S) = \max(\mathrm{d}_{\mathrm{sp}}(T, S), \mathrm{d}_{\mathrm{sp}}(S, T))$$

where

$$\mathrm{d}_{\mathrm{sp}}(T, S) = \limsup_k (\max_{|n|=k} \| (T \setminus S)^{[n]} \|)^{1/k}.$$

For what follows we need to express the spectral semi-distance between T and S by the commutator system $\mathrm{C}(T, S)$. Recall that if $T, S \in L(X)$ then the commutator operator of T and S is the operator $\mathrm{C}(T, S)$ defined on $L(X)$ by

$$\mathrm{C}(T, S)(A) = TA - AS, \quad A \in L(X).$$

One can easily see that $\mathrm{C}(T, S)$ may be also expressed as the difference of two commuting operators, $\mathrm{C}(T, S) = \mathrm{l}(T) - \mathrm{r}(S)$, where $\mathrm{l}(T)$ is the left multiplication by T and $\mathrm{r}(S)$ is the right multiplication by S on $L(X)$ ([4]). By using that we have

$$\mathrm{d}_{\mathrm{sp}}(T, S) = \limsup_n \|\mathrm{C}(T, S)^n(I)\|^{1/n}.$$

More generally, if $T = (T_1, \dots, T_m) \in L(X)_c^m$ and $S = (S_1, \dots, S_m) \in L(X)_c^m$, we define the commutator system by

$$\mathrm{C}(T, S) = (\mathrm{C}(T_1, S_1), \dots, \mathrm{C}(T_m, S_m))$$

which is a commuting m-tuple of operators on $L(X)$. Remark that

$$\mathrm{d}_{\mathrm{sp}}(T, S) = \limsup_k (\max_{|n|=k} \|\mathrm{C}(T, S)^m (I)\|)^{1/k}.$$

By the "cartesian" spectral semi-distance between T and S we mean

$$\mathrm{d}_0(T, S) = \max_i \mathrm{d}(T_i, S_i).$$

2. The proof of the main result will be reduced to the following auxiliary result.

THEOREM 1. *Let $C = (C_1, \ldots, C_m)$ be a commuting m-tuple of continuous linear operators on a Banach space L. Then for any $x \in L$ we have*

$$\limsup_k (\max_{|n|=k} \|C^n x\|)^{1/k} \leqslant (\max_j |C_j|_{sp})^{1-1/m} (\max_j \limsup_k \|C_j^k x\|^{1/k})^{1/m}.$$

Proof. If we denote

$$a = \max_j |C_j|_{sp}, \quad b = \max_j \limsup_k \|C_j^k x\|^{1/k}$$

then for any $\varepsilon > 0$, we can find $M = M(\varepsilon, x) \geqslant 1$ such that

(1) $\|C^n\| \leqslant M(a+\varepsilon)^{|n|}, \quad \|C_j^k x\| \leqslant M(b+\varepsilon)^k, \quad n = (n_1, \ldots, n_m), 1 \leqslant j \leqslant m, k \geqslant 1.$

Since $C_1, \ldots, C_m$ are mutually commuting we can write

$$C^n x = C^{n-n_l} C_l^{n_l} x, \tag{2}$$

where $n - n_l$ is the m-tuple having the same components as n except the l-component which is equal to zero. From (2) and (1) we get

$$\|C^n x\| \leqslant \|C^{n-n_l}\| \, \|C_l^{n_l} x\| \leqslant M^2 (a+\varepsilon)^{|n|-n_l} (b+\varepsilon)^{n_l}, \quad 1 \leqslant l \leqslant m. \tag{3}$$

If $n = (n_1, \ldots, n_m)$ fulfils the condition $|n| = k \geqslant 1$, we can find an index l, $1 \leqslant l \leqslant m$, such that $n_l \geqslant k/m$; applying (3) and using the obvious relation $(b+\varepsilon)/(a+\varepsilon) \leqslant 1$, we derive

$$\|C^n x\| \leqslant M^2 (a+\varepsilon)^{k-n_l} (b+\varepsilon)^{n_l} = M^2 (a+\varepsilon)^k [(b+\varepsilon)/(a+\varepsilon)]^{n_l} \leqslant$$
$$\leqslant M^2 (a+\varepsilon)^k [(b+\varepsilon)/(a+\varepsilon)]^{k/m}.$$

But this implies

$$\limsup_k (\max_{|n|=k} \|C^n x\|)^{1/k} \leqslant (a+\varepsilon)^{1-1/m} (b+\varepsilon)^{1/m}$$

and because ε may be chosen arbitrarily small, the proof is concluded.

A subset of m-tuples $T = (T_1, \ldots, T_m)$ in $L(X)_c^m$ will be called spectrally bounded if, for any $j = \overline{1, m}$, the set of spectral radii $|T_j|_{sp}$ is bounded.

THEOREM 2. *The "joint" spectral semi-distance is uniformly equivalent to the "cartesian" spectral semi-distance on any spectrally bounded subset of $L(X)_c^m$.*

Proof. It will be enough to show that, for any spectrally bounded subset of $L(X)_c^m$ and for any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$d_{sp}(T, S) < \varepsilon$$

whenever T and S belong to that subset and $d_{sp}(T_j, S_j) < \delta$, $j = \overline{1, m}$. For this purpose we apply Theorem 1 taking $L = L(X)$, $C = (C_1, \ldots, C_m)$ where $C_j = C(T_j, S_j)$, $j = \overline{1, m}$ and the identity operator as x. By using a Banach algebra argument we deduce that the spectral radii of C_j remain bounded for any $j = \overline{1, m}$ and consequently the number δ may be chosen as desired.

COROLLARY. *The "joint" quasi-nilpotent equivalence in $L(X)_c^m$ coincides with the "cartesian" quasi-nilpotent equivalence.*

3. Another natural possibility to define the "joint" spectral semi-distance and the "joint" quasi-nilpotent equivalence for m-tuples of operators could be to use

$$(T_1 \setminus S_1)^{[n_1]} \ldots (T_m \setminus S_m)^{[n_m]}$$

instead of $(T \setminus S)^{[n]}$. The fact that the corresponding "joint" notions can be reduced to the "cartesian" notions follows from Theorem 1 but it may be easily proved directly too. Between those extreme cases there are some other intermediate possibilities obtained by associating the terms of those two m-tuples in the same way and keeping the order. We can also apply Theorem 1 to deduce that all those "joint" notions may be reduced to the "cartesian" notions. Consequently all possible "joint" spectral semi-distances are uniformly equivalent to the "cartesian" spectral semi-distance on spectrally bounded sets of commuting m-tuples and all possible "joint" quasi-nilpotent equivalences coincide with the separate quasi-nilpotent equivalence.

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ŞTEFAN FRUNZĂ
Faculty of Mathematics,
University "Al. I. Cuza",
Calea 23 August 11, Jassy,
Romania.

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