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A RADON-NIKODYM THEOREM FOR POSITIVE LINEAR FUNCTIONALS ON $*$-ALGEBRAS

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1. INTRODUCTION

Noncommutative Radon-Nikodym theorem for von Neumann algebras has been investigated in detail, [7, 15, 19], but the study for algebras of unbounded operators seems to be hardly done except to [9].

In this paper we shall develop a Radon-Nikodym theorem in the context of unbounded operator algebras obtained by positive linear functionals on $*$-algebras.

For a positive linear functional, or more generally, for a positive invariant sesquilinear form φ on a $*$-algebra $\mathscr{A}$, the well-known GNS-construction yields a quartet $(\pi_\varphi, \lambda_\varphi, \mathscr{D}_\varphi, \mathscr{H}_\varphi)$ where $\mathscr{D}_\varphi$ is a dense subspace in a Hilbert space $\mathscr{H}_\varphi$, $\pi_\varphi(\mathscr{A})$ is a closed O_p^*-algebra on $\mathscr{D}_\varphi$ and λ_φ is a linear map of $\mathscr{A}$ into $\mathscr{D}_\varphi$ satisfying $\lambda_\varphi(xy) = \pi_\varphi(x)\lambda_\varphi(y)$ for each $x, y \in \mathscr{A}$. The Gudder's Radon-Nikodym theorem [9] asserts that if a positive linear functional ψ on a $*$-algebra $\mathscr{A}$ with identity e is strongly absolutely continuous with respect to a positive linear functional φ on $\mathscr{A}$ (that is, the map $\lambda_\varphi(x) \to \lambda_\psi(x)$ is closable), then there exists a positive self-adjoint operator H on $\mathscr{H}_\varphi$ such that $\psi(x) = (H\lambda_\varphi(x)|H\lambda_\varphi(e))$ for each $x \in \mathscr{A}$. However, the relation between the Radon-Nikodym derivative H and the O_p^*-algebra $\pi_\varphi(\mathscr{A})$ seems to be vague. With this view, we look again at a Radon-Nikodym theorem for $*$-algebras, and obtain the result that a positive invariant sesquilinear form ψ on a $*$-algebra $\mathscr{A}$ is strongly absolutely continuous with respect to a positive invariant sesquilinear form φ if and only if there exists a sequence $\{H_n\}$ of positive operators in the Powers' commutant $\pi_\varphi(\mathscr{A})'$ of the O_p^*-algebra $\pi_\varphi(\mathscr{A})$ such that

(a) $\{(H_n\lambda_\varphi(x)|\lambda_\varphi(y))\}$ converges for each $x, y \in \mathscr{A}$;

(b) $\{H_n^{1/2}\lambda_\varphi(x)\}$ converges in $\mathscr{H}_\varphi$ for each $x \in \mathscr{A}$;

(c) $\psi(x, y) = \lim_{n\to\infty} \varphi_{H_n}(x, y) \equiv \lim_{n\to\infty}(H_n\lambda_\varphi(x)|\lambda_\varphi(y))$ for each $x, y \in \mathscr{A}$.

Furthermore, we shall apply this result to the spatial theory for unbounded operator algebras.

2. POSITIVE INVARIANT SESQUILINEAR FORMS

Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathscr{H}$. By $\mathscr{L}(\mathscr{D})$ we denote the set of all linear operators of $\mathscr{D}$ into $\mathscr{D}$. Then $\mathscr{L}(\mathscr{D})$ is an algebra under the usual operations. By $\mathscr{L}^+(\mathscr{D})$ we denote the set of all linear operators A in $\mathscr{L}(\mathscr{D})$ such that $\mathscr{D}(A^*) \supset \mathscr{D}$ and the restriction A^+ of A^* to $\mathscr{D}$ is contained in $\mathscr{L}(\mathscr{D})$. Then $\mathscr{L}^+(\mathscr{D})$ becomes a $*$-algebra under the usual operations and the involution $A \to A^+$. An O_p^*-algebra $\mathfrak{A}$ on $\mathscr{D}$ is a $*$-subalgebra of $\mathscr{L}^+(\mathscr{D})$. Let $\mathfrak{A}$ be an O_p^*-algebra on $\mathscr{D}$. We define a seminorm $\|\cdot\|_A$ on $\mathscr{D}$ for $A \in \mathfrak{A}_I$ by $\|\xi\|_A = \|A\xi\|$, where $\mathfrak{A}_I$ is the O_p^*-algebra obtained by adjoining an identity operator to $\mathfrak{A}$. The induced topology $t_{\mathfrak{A}}$ on $\mathscr{D}$ is the locally convex topology generated by the collection of seminorms $\{\|\cdot\|_A;\ A \in \mathfrak{A}_I\}$. An O_p^*-algebra $\mathfrak{A}$ on $\mathscr{D}$ is called closed if $\mathscr{D}$ is complete in the induced topology $t_{\mathfrak{A}}$. The closedness of $\mathfrak{A}$ is equivalent to $\mathscr{D} = \bigcap_{A \in \mathfrak{A}} \mathscr{D}(\bar{A})$, where $\mathscr{D}(\bar{X})$ denotes the domain of the closure $\bar{X}$ of a closable operator X.

Let $\mathscr{A}$ be a $*$-algebra. A map φ of $\mathscr{A} \times \mathscr{A}$ into the complex number field $\mathbf{C}$ is said to be a sesquilinear form on $\mathscr{A}$ if $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z)$ and $\varphi(z, \alpha x + \beta y) = \bar{\alpha}\varphi(z, x) + \bar{\beta}\varphi(z, y)$ for each $x, y, z \in \mathscr{A}$ and $\alpha, \beta \in \mathbf{C}$. A sesquilinear form φ on $\mathscr{A}$ is called invariant (resp. positive) if $\varphi(xy, z) = \varphi(y, x^*z)$ for each $x, y, z \in \mathscr{A}$ (resp. $\varphi(x, x) \geqslant 0$ for each $x \in \mathscr{A}$). Let φ be a positive invariant sesquilinear form on $\mathscr{A}$. Then $N_\varphi = \{x \in \mathscr{A};\ \varphi(x, x) = 0\}$ is a left ideal in $\mathscr{A}$. For each $x \in \mathscr{A}$ we denote by $\lambda_\varphi(x)$ the coset of $\mathscr{A}/N_\varphi$ which contains x and define an inner product $(\,|\,)$ on $\lambda_\varphi(\mathscr{A})$ by $(\lambda_\varphi(x)|\lambda_\varphi(y)) = \varphi(x, y)$ for $x, y \in \mathscr{A}$. Let $\mathscr{H}_\varphi$ be the Hilbert space which is the completion of the pre-Hilbert space $\lambda_\varphi(\mathscr{A})$. We define a linear operator $\pi_\varphi^0(x)$ on $\lambda_\varphi(\mathscr{A})$ by $\pi_\varphi^0(x)\lambda_\varphi(y) = \lambda_\varphi(xy)$. We put

$$\mathscr{D}_\varphi = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\overline{\pi_\varphi^0(x)}) \quad \text{and} \quad \pi_\varphi(x) = \overline{\pi_\varphi^0(x)}/\mathscr{D}_\varphi .$$

Then π_φ is a $*$-homomorphism of $\mathscr{A}$ onto the closed O_p^*-algebra $\pi_\varphi(\mathscr{A})$ on $\mathscr{D}_\varphi$ which is said to be the operator-representation of $\mathscr{A}$ for φ, and λ_φ is a linear map of $\mathscr{A}$ into $\mathscr{D}_\varphi$ satisfying that $\lambda_\varphi(\mathscr{A})$ is dense in $\mathscr{D}_\varphi$ with respect to the induced topology $t_{\pi_\varphi(\mathscr{A})}$ and $\lambda_\varphi(xy) = \pi_\varphi(x)\lambda_\varphi(y)$ for each $x, y \in \mathscr{A}$, which is said to be the vector-representation of $\mathscr{A}$ for φ. The quartet $(\pi_\varphi, \lambda_\varphi, \mathscr{D}_\varphi, \mathscr{H}_\varphi)$ is said to be the GNS-construction for φ. Put

$$\mathscr{D}_\varphi^* = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\pi_\varphi(x)^*), \quad \pi_\varphi^*(x) = \pi_\varphi(x^*)^*/\mathscr{D}_\varphi^*$$

$$\mathscr{D}_\varphi^{**} = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\pi_\varphi^*(x)^*), \quad \pi_\varphi^{**}(x) = \pi_\varphi^*(x^*)^*/\mathscr{D}_\varphi^{**} .$$

Then π_φ^* is a homomorphism of $\mathscr{A}$ into $\mathscr{L}(\mathscr{D}_\varphi^*)$ and π_φ^{**} is a $*$-homomorphism of $\mathscr{A}$ into $\mathscr{L}^+(\mathscr{D}_\varphi^{**})$ satisfying that $\mathscr{D}_\varphi \subset \mathscr{D}_\varphi^{**} \subset \mathscr{D}_\varphi^*$, $\pi_\varphi(x)\xi = \pi_\varphi^{**}(x)\xi$ for each $x \in \mathscr{A}$ and $\xi \in \mathscr{D}_\varphi$, and $\pi_\varphi^{**}(x)\,\zeta = \pi_\varphi^*(x)\zeta$ for each $x \in \mathscr{A}$ and $\zeta \in \mathscr{D}_\varphi^{**}$.

Let f be a positive linear functional on $\mathscr{A}$. Then, a positive invariant sesquilinear form f^0 on $\mathscr{A}$ is defined by

$$f^0(x, y) = f(y^* x) \quad \text{for } x, y \in \mathscr{A}.$$

We simply denote by $(\pi_f, \lambda_f, \mathscr{D}_f, \mathscr{H}_f)$ the GNS-construction of f^0.

DEFINITION 2.1. Let φ be a positive invariant sesquilinear form on a $*$-algebra $\mathscr{A}$ and f be a positive linear functional on $\mathscr{A}$. If $\pi_\varphi(x) \in \mathscr{B}(\mathscr{H}_\varphi)$ for each $x \in \mathscr{A}$, where $\mathscr{B}(\mathscr{H}_\varphi)$ denotes the set of all bounded linear operators on $\mathscr{H}_\varphi$, then φ is called *admissible*. If π_φ is self-adjoint (that is, $\mathscr{D}_\varphi^* = \mathscr{D}_\varphi$), then φ is said to be a *Riesz form* on $\mathscr{A}$. If f^0 is admissible (resp. a Riesz form on $\mathscr{A}$), then f is called *admissible* (resp. a *Riesz functional* on $\mathscr{A}$).

Let φ be a positive invariant sesquilinear form on $\mathscr{A}$. We define the commutant $\pi_\varphi(\mathscr{A})'$ of the O_p^*-algebra $\pi_\varphi(\mathscr{A})$ as follows:

$$\pi_\varphi(\mathscr{A})' = \{C \in \mathscr{B}(\mathscr{H}_\varphi); (C\pi_\varphi(x)\lambda_\varphi(y)|\lambda_\varphi(z)) = (C\lambda_\varphi(y)|\pi_\varphi(x^*)\lambda_\varphi(z)) \text{ for each } x, y, z \in \mathscr{A}\}.$$

Then $\pi_\varphi(\mathscr{A})'$ is a weakly closed subspace of $\mathscr{B}(\mathscr{H}_\varphi)$ satisfying that $C^* \in \pi_\varphi(\mathscr{A})'$ for each $C \in \pi_\varphi(\mathscr{A})'$ and $C\pi_\varphi(x)\xi = \pi_\varphi^*(x)C\xi$ for each $x \in \mathscr{A}$, $\xi \in \mathscr{D}_\varphi$ and $C \in \pi_\varphi(\mathscr{A})'$. However, $\pi_\varphi(\mathscr{A})'$ is not necessarily an algebra. We see that if φ is a Riesz form then $\pi_\varphi(\mathscr{A})' \mathscr{D}_\varphi \subset \mathscr{D}_\varphi$ and $\pi_\varphi(\mathscr{A})'$ is a von Neumann algebra on $\mathscr{H}_\varphi$ [17].

3. RADON-NIKODYM THEOREM

In this section we develop a Radon-Nikodym theorem for positive invariant sesquilinear forms on $*$-algebras. We first generalize the classical concept of absolue continuity.

DEFINITION 3.1. Let $\mathscr{A}$ be a $*$-algebra and (φ, ψ) be a pair of positive invariant sesquilinear forms on $\mathscr{A}$.

(1) ψ is called *φ-absolutely continuous* if $\lambda_\varphi(x) \to \lambda_\psi(x)$ is a map.

(2) ψ is called *strongly φ-absolutely continuous* if $\lambda_\varphi(x) \to \lambda_\psi(x)$ is a closable map of $\mathscr{H}_\varphi$ into $\mathscr{H}_\psi$.

(3) ψ is called *φ-dominated* if $\lambda_\varphi(x) \to \lambda_\psi(x)$ is continuous.

(4) ψ is called *φ-singular* if for each $x \in \mathscr{A}$ there exists a sequence $\{x_n\}$ in $\mathscr{A}$ such that $\lim_{n\to\infty} \lambda_\varphi(x_n) = 0$ and $\lim_{n\to\infty} \lambda_\psi(x_n) = \lambda_\psi(x)$.

We now give our main results.

THEOREM 3.2. (Radon-Nikodym theorem). *Let $\mathscr{A}$ be a $*$-algebra and (φ, ψ) be a pair of positive invariant sesquilinear forms on $\mathscr{A}$. Then the following statements hold.*

(1) ψ *is* φ*-dominated if and only if there exists a bounded positive self-adjoint operator* H *such that* $\psi(x, y) = \varphi_H(x, y) \equiv (H\lambda_\varphi(x) | \lambda_\varphi(y))$ *for each* $x, y \in \mathscr{A}$. *Then* $H \in \pi_\varphi(\mathscr{A})'$.

(2) *The following statements are equivalent.*

(2.1) ψ *is strongly* φ*-absolutely continuous.*

(2.2) *There exists a positive self-adjoint operator* H *on* $\mathscr{H}_\varphi$ *whose domain contains* $\lambda_\varphi(\mathscr{A})$ *such that* $\psi(x, y) = \varphi_{H,H}(x, y) \equiv (H\lambda_\varphi(x)|H\lambda_\varphi(y))$ *for each* $x, y \in \mathscr{A}$.

Suppose that $\tau = \varphi + \psi$ *is a Riesz form on* $\mathscr{A}$. *Then the following statement* (2.3) *is equivalent to the above statements* (2.1) *and* (2.2).

(2.3) *There exists a sequence* $\{H_n\}$ *of positive operators in* $\pi_\varphi(\mathscr{A})'$ *such that*

(a) $\{(H_n\lambda_\varphi(x)|\lambda_\varphi(y))\}$ *converges for each* $x, y \in \mathscr{A}$;

(b) $\{H_n^{1/2}\lambda_\varphi(x)\}$ *converges in* $\mathscr{H}_\varphi$ *for each* $x \in \mathscr{A}$;

(c) $\psi(x, y) = \lim_{n\to\infty} \varphi_{H_n}(x, y)$ *for each* $x, y \in \mathscr{A}$.

THEOREM 3.3. (Lebesgue-decomposition theorem). *Let* $\mathscr{A}$ *be a* $*$*-algebra and* φ *be a positive invariant sesquilinear form on* $\mathscr{A}$. *Then every positive invariant sesquilinear form* ψ *on* $\mathscr{A}$ *such that* $\tau = \varphi + \psi$ *is a Riesz form is decomposed into the sum:*

$$\psi = \psi_a + \psi_s$$

of a strongly φ*-absolutely continuous positive invariant sesquilinear form* ψ_a *on* $\mathscr{A}$ *and a* φ*-singular positive invariant sesquilinear form* ψ_s *on* $\mathscr{A}$.

Proof of Theorem 3.2. (1) Put

$$T\lambda_\varphi(x) = \lambda_\psi(x) \quad \text{for } x \in \mathscr{A}.$$

Since ψ is φ-dominated, the map T can be extended to a continuous linear transform of $\mathscr{H}_\varphi$ into $\mathscr{H}_\psi$, which is also denoted by T. We easily see that $H = T^*T \in \pi_\varphi(\mathscr{A})'$ and $\psi(x, y) = (H\lambda_\varphi(x)|\lambda_\varphi(y))$ for each $x, y \in \mathscr{A}$.

(2) (2.1) $\Rightarrow$ (2.2) Suppose that ψ is strongly φ-absolutely continuous. Let T be the closure of the closable operator $\lambda_\varphi(x) \to \lambda_\psi(x)$ and let $T = UH$ be the polar decomposition of T. It is easily seen that H implies our assertion.

(2.2) $\Rightarrow$ (2.1) This is trivial.

Suppose that $\tau = \varphi + \psi$ is a Riesz form on $\mathscr{A}$. By (1) there exist elements $R, K \in \pi_\tau(\mathscr{A})'$ such that $R \geqslant 0$, $K \geqslant 0$, $R + K = I$, $\varphi(x, y) = (R\lambda_\tau(x)|\lambda_\tau(y))$ and $\psi(x, y) = (K\lambda_\tau(x)|\lambda_\tau(y))$ for each $x, y \in \mathscr{A}$. We now put

$$U_0\, \lambda_\varphi(x) = R^{1/2}\lambda_\tau(x) \quad \text{for } x \in \mathscr{A}.$$

Then U_0 can be extended to an isometry U of $\mathscr{H}_\varphi$ into $\mathscr{H}_\tau$. Let $R = \int_0^1 \lambda\, dE(\lambda)$ be the spectral resolution of R and $E(0)$ be the projection of $\mathscr{H}_\tau$ onto Ker R. Put

$$R_n = \int_{1/n}^1 \lambda\, dE(\lambda) \quad \text{for } n = 1, 2, \ldots .$$

Since R is contained in the von Neumann algebra $\pi_\tau(\mathscr{A})'$, and R and K commute, it follows that R_n, R_n^{-1}, $E(0)$ and K belong to $\pi_\tau(\mathscr{A})'$ and mutually commute. We now define a sequence $\{H_n\}$ in $\mathscr{B}(\mathscr{H}_\varphi)$ as follows:

$$H_n = U^* R_n^{-1} K U \quad \text{for } n = 1, 2, \ldots .$$

We see that H_n is a positive operator in $\pi_\varphi(\mathscr{A})'$. In fact, it is trivial that $H_n \geqslant 0$ and $H_n \in \mathscr{B}(\mathscr{H}_\varphi)$. The statement $H_n \in \pi_\varphi(\mathscr{A})'$ follows from the equality:

$$(H_n \pi_\varphi(x)\lambda_\varphi(y)|\lambda_\varphi(z)) = (U^* R_n^{-1} K U \lambda_\varphi(xy)|\lambda_\varphi(z)) =$$

$$= (R_n^{-1} K R^{1/2} \pi_\tau(x)\lambda_\tau(y)|R^{1/2}\lambda_\tau(z)) = (R_n^{-1} R K \pi_\tau(x)\lambda_\tau(y)|\lambda_\tau(z)) =$$

$$= (R_n^{-1} R K \lambda_\tau(y)|\pi_\tau(x^*)\lambda_\tau(z)) = (R_n^{-1} K U \lambda_\varphi(y)|U\pi_\varphi(x^*)\lambda_\varphi(z)) =$$

$$= (H_n \lambda_\varphi(y)|\pi_\varphi(x^*)\lambda_\varphi(z))$$

for each $x, y, z \in \mathscr{A}$. For each $x, y \in \mathscr{A}$ we have

$$\lim_{n\to\infty} \varphi_{H_n}(x, y) = \lim_{n\to\infty} (U^* R_n^{-1} K U \lambda_\varphi(x)|\lambda_\varphi(y)) =$$

$$= \lim_{n\to\infty} (R_n^{-1} R K \lambda_\tau(x)|\lambda_\tau(y)) =$$

(*)

$$= (K\lambda_\tau(x)|\lambda_\tau(y)) - (E(0)K\lambda_\tau(x)|\lambda_\tau(y)) =$$

$$= \psi(x, y) - (E(0)K\lambda_\tau(x)|\lambda_\tau(y)).$$

Since $\{H_n\}$ is a mutually commuting sequence and $H_n \geqslant H_m$ for $n > m$, it follows that $\{H_n^{1/2}\}$ is a mutually commuting sequence and $H_n^{1/2} \geqslant H_m^{1/2}$ for $n > m$. Hence, for $n > m$ and $x \in \mathscr{A}$ we have

$$\|H_n^{1/2}\lambda_\varphi(x) - H_m^{1/2}\lambda_\varphi(x)\|^2 \leqslant 2\{(H_n\lambda_\varphi(x)|\lambda_\varphi(x)) - (H_m\lambda_\varphi(x)|\lambda_\varphi(x))\},$$

so that we have by (*) that $\{H_n^{1/2}\lambda_\varphi(x)\}$ converges in $\mathscr{H}_\varphi$ for each $x \in \mathscr{A}$.

We show the implication (2.1) $\Rightarrow$ (2.3). Suppose that ψ is strongly φ-absolutely continuous. By the equality (*) it is sufficient to show that $(E(0)K\lambda_\tau(x)|\lambda_\tau(y)) = 0$ for each $x, y \in \mathscr{A}$. For each $x \in \mathscr{A}$ there is a sequence $\{x_n\}$ in $\mathscr{A}$ such that $\lim_{n\to\infty} \lambda_\tau(x_n) = E(0)\lambda_\tau(x)$. Then we have

$$\lim_{n\to\infty} U\lambda_\varphi(x_n) = \lim_{n\to\infty} R^{1/2}\lambda_\tau(x_n) = R^{1/2}E(0)\lambda_\tau(x) = 0$$

and

$$\lim_{n,m\to\infty} \|\lambda_\psi(x_n) - \lambda_\psi(x_m)\|^2 = \lim_{n,m\to\infty} (K(\lambda_\tau(x_n) - \lambda_\tau(x_m))|\lambda_\tau(x_n) - \lambda_\tau(x_m)) = 0.$$

Since $\lambda_\varphi(x) \to \lambda_\psi(x)$ is closable, it follows that $\lim_{n\to\infty} \lambda_\psi(x_n) = 0$. Hence, we have

$$(KE(0)\lambda_\tau(x)|\lambda_\tau(y)) = \lim_{n\to\infty} \psi(x_n, y) = 0.$$

We show now the implication (2.3) $\Rightarrow$ (2.1). Suppose that a sequence $\{H_n\}$ of positive operators in $\pi_\varphi(\mathscr{A})'$ satisfies the statements (a), (b) and (c) of (2.3). We define a linear operator H_0 on $\mathscr{H}_\varphi$ with domain $\lambda_\varphi(\mathscr{A})$ by

$$H_0\lambda_\varphi(x) = \lim_{n\to\infty} H_n^{1/2}\lambda_\varphi(x) \quad \text{for } x \in \mathscr{A}.$$

It is trivial that H_0 is a symmetric operator and so H_0 is a closable operator whose closure is denoted by H. We then see that

$$\psi(x, y) = \lim_{n\to\infty} (H_n\lambda_\varphi(x)|\lambda_\varphi(y)) =$$

$$= \lim_{n\to\infty} (H_n^{1/2}\lambda_\varphi(x)|H_n^{1/2}\lambda_\varphi(y)) = (H\lambda_\varphi(x)|H\lambda_\varphi(y))$$

for each $x, y \in \mathscr{A}$. Suppose that $\lim_{n\to\infty} \lambda_\varphi(x_n) = 0$ and $\lim_{n\to\infty} \lambda_\psi(x_n) = \xi$. Then, since $\{H\lambda_\varphi(x_n)\}$ is a Cauchy sequence and H is closed, we have $\lim_{n\to\infty} H\lambda_\varphi(x_n) = 0$. For each $y \in \mathscr{A}$ we have

$$(\xi|\lambda_\psi(y)) = \lim_{n\to\infty} (\lambda_\psi(x_n)|\lambda_\psi(y)) = \lim_{n\to\infty} (H\lambda_\varphi(x_n)|H\lambda_\varphi(y)) = 0,$$

and hence $\xi = 0$, which implies that $\lambda_\varphi(x) \to \lambda_\psi(x)$ is closable; that is, ψ is strongly φ-absolutely continuous. This completes the proof.

Proof of Theorem 3.3. By (*) in Theorem 3.2 we have the equality:

$$\psi(x, y) = \lim_{n\to\infty} \varphi_{H_n}(x, y) + (E(0)K\lambda_\tau(x)|\lambda_\tau(y))$$

for each $x, y \in \mathscr{A}$. Put

$$\psi_a(x, y) = \lim_{n\to\infty} \varphi_{H_n}(x, y) \quad \text{and} \quad \psi_s(x, y) = (E(0)K\lambda_\tau(x)|\lambda_\tau(y))$$

for $x, y \in \mathscr{A}$. By Theorem 3.2 ψ_a is a strongly φ-absolutely continuous positive invariant sesquilinear form on $\mathscr{A}$. We show that ψ_s is φ-singular. For each $x \in \mathscr{A}$ there is a sequence $\{x_n\}$ in $\mathscr{A}$ such that $\lim_{n\to\infty} \lambda_\tau(x_n) = E(0)\lambda_\tau(x)$. Then we have

$$\lim_{n\to\infty} U\lambda_\varphi(x_n) = R^{1/2}E(0)\lambda_\tau(x) = 0$$

and

$$\lim_{n\to\infty} \|\lambda_{\psi_s}(x_n) - \lambda_{\psi_s}(x)\|^2 =$$

$$= \lim_{n\to\infty} (KE(0)(\lambda_\tau(x_n) - \lambda_\tau(x))|\lambda_\tau(x_n) - \lambda_\tau(x)) = 0.$$

This completes the proof.

We give some examples of pair (φ, ψ) satisfying the condition that $\tau = \varphi + \psi$ is a Riesz form. We see that for pairs (φ, ψ) in the examples the same results as of Theorems 3.2, 3.3 hold.

EXAMPLE 3.4. (1) Let $\mathscr{A}$ be a symmetric $*$-algebra (that is, x^*x is quasi-regular for each $x \in \mathscr{A}$). Then, for each pair (φ, ψ), $\tau = \varphi + \psi$ is a Riesz form [12].

(2) Let $\mathscr{A}$ be a pseudo-complete hermitian locally convex $*$-algebra [3]; in particular, a locally convex GB^*-algebra [1,6]. Then, for each pair (φ, ψ), $\tau = \varphi + \psi$ is a Riesz form.

(3) We see that an admissible positive invariant sesquilinear form on a $*$-algebra is a Riesz form. Hence, if φ and ψ are admissible, then $\tau = \varphi + \psi$ is a Riesz form.

Let $\mathscr{A}$ be a locally convex $*$-algebra. An element x of $\mathscr{A}$ is said to be bounded if for some non-zero $\lambda \in \mathbf{C}$, the set $\{(\lambda x)^n ; n = 1,2, \ldots\}$ is bounded. The set of all bounded elements of $\mathscr{A}$ is denoted by $\mathscr{A}_0$.

(3.1) Let $\mathscr{A}$ be a locally convex $*$-algebra with $\mathscr{A} = \mathscr{A}_0$ (for example: (a) a normed $*$-algebra with continuous involution; (b) a locally convex $*$-algebra with continuous quasi-inverse, for example, the function algebra $\mathscr{D}(\mathbf{R})$ of infinitely differentiable functions on the real number field $\mathbf{R}$ with compact support, and the function algebra $\mathscr{S}(\mathbf{R})$ of infinitely differentiable functions which are rapidly decreasing as $|x| \to \infty$ together with their derivatives of all orders).

Then every continuous positive invariant sesquilinear form φ on $\mathscr{A}$ is admissible. In fact, for $x \in \mathscr{A}$ we define a positive invariant sesquilinear form φ_x on $\mathscr{A}$ by $\varphi_x(y, z) = \varphi(yx, zx)$ for $y, z \in \mathscr{A}$. We see [1] that $x \in \mathscr{A}_0$ if and only if

$$\beta(x) \equiv \sup_{p \in \mathscr{P}} \overline{\lim_{n\to\infty}} |P(x^n)|^{1/n} < \infty ,$$

where $\mathscr{P}$ is some basic set of seminorms which defines the topology of $\mathscr{A}$. Suppose $x \in \mathscr{A}$ and $h^* = h \in \mathscr{A}$. Then we have

$$|\varphi_x(h, h)|^{2^n} \leqslant \varphi(x, x)^{2^n-1}\varphi_x(h^{2^n}, h^{2^n})$$

for $n = 1, 2, \ldots$. By the continuity of φ we have

$$\|\pi_\varphi(h)\lambda_\varphi(x)\|^2 = |\varphi_x(h, h)| \leqslant \varphi(x, x)\beta(h)^2 = \|\lambda_\varphi(x)\|^2\beta(h)^2$$

for all $x \in \mathscr{A}$ and $h^* = h \in \mathscr{A}$, which implies that φ is admissible. This proof is analogous to the proof of ([18], Theorem 4.5.2).

We can also prove in the same way as in [17] that every positive invariant sesquilinear form on $\mathscr{A}$ (which is not necessarily continuous) is admissible if $\mathscr{A}$ is pseudo-complete.

(3.2) We give examples of locally convex $*$-algebras $\mathscr{A}$ which are not equal to $\mathscr{A}_0$, but every continuous positive invariant sesquilinear form on $\mathscr{A}$ is admissible:

(a) locally m-convex $*$-algebra [4]; (b) the Schwartz group algebra $\mathscr{D}(\mathbf{R})$ of infinitely differentiable functions on $\mathbf{R}$ with compact support, or more general, the Schwartz group algebra $\mathscr{D}(G)$ of a separable Lie group G [21].

REMARK. Let $\mathscr{A}$ be a locally convex $*$-algebra. Suppose that $\mathscr{A}_0$ is dense in $\mathscr{A}$. Then $\pi_\varphi(\mathscr{A})'$ is a von Neumann algebra on $\mathscr{H}_\varphi$ for every continuous positive invariant sesquilinear form φ on $\mathscr{A}$, so that the same results as of Theorems 3.2, 3.3 hold for each pair (φ, ψ) of continuous positive invariant sesquilinear forms on $\mathscr{A}$. In fact, it follows from the proof of (3.1) that $\overline{\pi_\varphi(\mathscr{A}_0)} \subset \mathscr{B}(\mathscr{H}_\varphi)$, and it is easily shown that $\pi_\varphi(\mathscr{A})' = \pi_\varphi(\mathscr{A}_0)'$ since φ is continuous and $\mathscr{A}_0$ is dense in $\mathscr{A}$.

We next show that the results of Theorems 3.2, 3.3 hold under the weaker condition than that of Example 3.4 (3).

THEOREM 3.5. *Let $\mathscr{A}$ be a $*$-algebra and (φ, ψ) be a pair of positive invariant sesquilinear forms on $\mathscr{A}$. If either φ or ψ is admissible, then the same results as of Theorems* 3.2, 3.3 *hold.*

Proof. Put $\tau = \varphi + \psi$. Let R, R_n^{-1}, $K, E(0)$, U and H_n be as in Theorem 3.2. Suppose that ψ is admissible. Then, for each $a \in \mathscr{A}$ there is a positive number γ_a such that

$$\|K^{1/2}\pi_\tau(a)\lambda_\tau(x)\| \leqslant \gamma_a\|K^{1/2}\lambda_\tau(x)\| \leqslant \gamma_a\|\lambda_\tau(x)\|$$

for each $x \in \mathscr{A}$, so that $\overline{K^{1/2}\pi_\tau(a)}$ is a bounded operator on $\mathscr{H}_\tau$ for every $a \in \mathscr{A}$. Furthermore, for each $x, y \in \mathscr{A}$ we have

$$R(K\pi_\tau(x))\lambda_\tau(y) = KR\pi_\tau(x)\lambda_\tau(y) =$$
$$= K\pi_\tau(x^*)^*R\lambda_\tau(y) = \overline{K\pi_\tau(x)}R\lambda_\tau(y)$$

since $K\pi_\tau(x) \subset \overline{K\pi_\tau(x^*)^*}$ and $K\pi_\tau(x)$ is bounded. Hence, $\overline{K\pi_\tau(x)}$ and R commute. We show $H_n = U^*R_n^{-1}KU \in \pi_\varphi(\mathscr{A})'$ for $n = 1,2, \ldots$. This follows from the equalities:

$$(H_n\pi_\varphi(x)\lambda_\varphi(y)|\lambda_\varphi(z)) = (R_n^{-1}RK\pi_\tau(x)\lambda_\tau(y)|\lambda_\tau(z)) =$$
$$= (R_n^{-1}R\lambda_\tau(y)|(K\pi_\tau(x))^*\lambda_\tau(z)) = (R_n^{-1}R\lambda_\tau(y)|\pi_\tau(x)^*K\lambda_\tau(z)) =$$
$$= (R_n^{-1}R\lambda_\tau(y)|K\pi_\tau(x^*)\lambda_\tau(z)) = (R_n^{-1}KU\lambda_\varphi(y)|U\pi_\varphi(x^*)\lambda_\varphi(z)) =$$
$$= (H_n\lambda_\varphi(y)|\pi_\varphi(x^*)\lambda_\varphi(z))$$

for each $x, y, z \in \mathscr{A}$. We can show the rest in the same way as in Theorems 3.2, 3.3. We can similarly show our arguments in case that φ is admissible.

4. APPLICATIONS

In this section we apply the Radon-Nikodym theorem (Theorems 3.2, 3.5) to the spatial theory for unbounded operator algebras. The spatial theory for unbounded operator algebras has been investigatied in [13, 20]. We now generalize Theorems 4.3, 4.4 in [13].

Let $\mathscr{A}$ and $\mathscr{B}$ be $*$-algebras, and φ and ψ be positive invariant sesquilinear forms on $\mathscr{A}$ and $\mathscr{B}$, respectively. Let σ be a $*$-isomorphism of $\mathscr{A}$ onto $\mathscr{B}$ satisfying $\sigma(\operatorname{Ker}\pi_\varphi) = \operatorname{Ker}\pi_\psi$. Putting

$$\tilde{\sigma}(\pi_\varphi(x)) = \pi_\psi(\sigma(x)) \quad \text{for } x \in \mathscr{A},$$

$\tilde{\sigma}$ is a $*$-isomorphism of the O_p^*-algebra $\pi_\varphi(\mathscr{A})$ onto the O_p^*-algebra $\pi_\psi(\mathscr{B})$. We denote by $I_{\text{sac}}((\mathscr{A}, \varphi), (\mathscr{B}, \psi))$ the set of all $*$-isomorphisms σ of $\mathscr{A}$ onto $\mathscr{B}$ satisfying that $\sigma(\operatorname{Ker}\pi_\varphi) = \operatorname{Ker}\pi_\psi$ and $\psi \circ \sigma$ is strongly φ-absolutely continuous.

THEOREM 4.1. *Let φ and ψ be positive invariant sesquilinear forms on $*$-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Suppose that either φ is admissible or φ is a Riesz form and ψ is admissible. If $\sigma \in I_{\text{sac}}((\mathscr{A}, \varphi), (\mathscr{B}, \psi))$, then both φ and ψ are admissible and there exists an isometry U of $\mathscr{H}_\psi$ into $\mathscr{H}_\varphi$ such that $\tilde{\sigma}(\pi_\varphi(x)) = U^*\pi_\varphi(x)U$ for each $x \in \mathscr{A}$.*

Proof. Suppose that φ is admissible and $\sigma \in I_{\text{sac}}((\mathscr{A}, \varphi), (\mathscr{B}, \psi))$. Then we may apply Theorem 3.5 to the positive invariant sesquilinear forms φ and $\psi \circ \sigma$, so that there exists a sequence $\{H_n\}$ of positive operators in the von Neumann algebra $\pi_\varphi(\mathscr{A})'$ such that $\{H_n^{1/2}\lambda_\varphi(x)\}$ converges in $\mathscr{H}_\varphi$ for each $x \in \mathscr{A}$ and $\psi(\sigma(x), \sigma(y)) = \lim_{n\to\infty} (H_n\lambda_\varphi(x)|\lambda_\varphi(y))$ for each $x, y \in \mathscr{A}$. We now put

$$\mu(x) = \lim_{n\to\infty} H_n^{1/2}\lambda_\varphi(x) \quad \text{for } x \in \mathscr{A}.$$

Then it follows since $H_n^{1/2} \in \pi_\varphi(\mathscr{A})'$ that μ is a linear map of $\mathscr{A}$ into $\mathscr{H}_\varphi$ satisfying that

$$\mu(xy) = \pi_\varphi(x)\lambda_\varphi(y)$$

and

$$(\lambda_\psi(\sigma(x))|\lambda_\psi(\sigma(y))) = (\mu(x)|\mu(y))$$

for each $x, y \in \mathscr{A}$. Hence, our assertion follows from ([13], Theorem 4.1).

In case that φ is a Riesz form and ψ is admissible, we can similarly prove the theorem since $\psi \circ \sigma$ is admissible and $\pi_\varphi(A)'$ is a von Neumann algebra. This completes the proof.

THEOREM 4.2. *Let φ and ψ be positive invariant sesquilinear forms on $*$-algebra $\mathscr{A}$ and $\mathscr{B}$, respectively, and let $\sigma \in I_{\text{sac}}((\mathscr{A}, \varphi), (\mathscr{B}, \psi))$. Suppose that both φ and $\varphi + \psi \circ \sigma$ are Riesz forms. Then there exists an isometry U of $\mathscr{H}_\psi$ into $\mathscr{H}_\varphi$ such that $U\mathscr{D}_\psi \subset \mathscr{D}_\varphi$ and $\tilde{\sigma}(\pi_\varphi(x))\xi = U^*\pi_\varphi(x)U\xi$ for each $x \in \mathscr{A}$ and $\xi \in \mathscr{D}_\psi$.*

Proof. We may apply Theorem 3.2 to the positive invariant sesquilinear forms φ and $\psi \circ \sigma$, so that we can prove the theorem in a similar way to Theorem 4.1.

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