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THE SPECTRUM OF HILBERT SPACE SEMIGROUPS

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1. INTRODUCTION

Suppose $\{P(t): t \geqslant 0\}$ is a strongly continuous semigroup of operators on a Hilbert space $\mathscr{H}$. We introduce the generator, A, of the semigroup by writing $P(t) = \exp(-tA)$ and consider the problem of determining the spectrum, $\sigma(P(t))$, given some knowledge of the operator A.

The inclusion

$$\sigma(\exp(-tA)) \supset \overline{\exp(-\sigma(tA))} \tag{1.1}$$

is known to hold for all such semigroups [5] but there are cases where the reverse inclusion fails [5]. (Here $\overline{\exp(-\sigma(tA))}$ is the closure of the set $\{e^{-z} : z \in \sigma(tA)\}$.) In fact, as is demonstrated in [5], it is possible to have $\sigma(A) = \emptyset$ while $\exp(-tA)$ has circles in its spectrum. There are of course examples where $\sigma(A) = \emptyset$ while $\sigma(\exp(-tA)) = \{0\}$ for $t > 0$ so that in general, $\sigma(\exp(-tA))$ is not determined by $\sigma(A)$ alone.

It follows from the assumed strong continuity of $P(t)$ that the bound

$$\|P(t)\| \leqslant K\exp(t\omega); \quad t \geqslant 0 \tag{1.2}$$

is valid for some $K \geqslant 1$ and $\omega \in \mathbf{R}$ [5]. If K can be set equal to one then $P(t)\exp(-t\omega) = \exp(-t(A+\omega))$ is a strongly continuous semigroup of contractions. In Hilbert space such semigroups are well studied and structure theorems comparable to the spectral theorem for normal operators are known [1].

Using this additional structure, L. Gearhart [2] showed that if $K = 1$ in (1.2) and $\mathscr{H}$ is separable then $\sigma(P(t))$ can be determined from a knowledge of $\sigma(A)$ and the behavior of $\|(z - A)^{-1}\|$ for z near infinity. Specifically, Gearhart proved the following result [2]. (In stating the result we use the notation $\rho(B)$ for the resolvent set of an operator B.)

THEOREM 1.1. ([2]). *Suppose* $\{\exp(-tA): t \geqslant 0\}$ *is a strongly continuous semigroup of operators on a separable Hilbert space with* $\|\exp(-tA)\| \leqslant \exp(t\omega)$ *for some* $\omega \in \mathbf{R}$. *Then*

(1) $\exp(-z_0) \in \rho(\exp(-A))$ *if and only if* $z_0 + 2\pi i n \in \rho(A)$ *for all integers* n *and*

$$\sup_{n \in \mathbf{Z}} \|(z_0 + 2\pi i n - A)^{-1}\| < \infty;$$

(2) $0 \in \rho(\exp(-A))$ *if and only if there are numbers* c_0 *and* $\omega_0 > 0$ *so that* $\{z : \operatorname{Re} z > \omega_0\} \subset \rho(A)$ *and for* $\operatorname{Re} z > \omega_0$

$$\|(z - A)^{-1}\| \leqslant c_0(\operatorname{Re} z)^{-1}.$$

Gearhart's proof of this theorem is not elementary. It is the purpose of this note to give a generalization of Theorem 1.1 with an elementary proof. This we do in Section 2.

In [4] Gearhart's theorem was applied to determine the essential spectrum of certain non-self adjoint partial differential operators related to the Stark-effect. In [3] Gearhart's theorem was used to generalize a theorem of Ichinose on the spectrum of $A_1 \otimes I + I \otimes A_2$. Given Theorem 2.1 of this paper, many of the results of [3] can be generalized further in an obvious way.

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2. A GENERALIZATION OF GEARHART'S THEOREM

In this section we will prove the following two results.

THEOREM 2.1. *Suppose* $\{\exp(-tA) : t \geqslant 0\}$ *is a strongly continuous semigroup of operators on a Hilbert space. Then conclusions* (1) *and* (2) *of Theorem* 1.1 *are valid.*

THEOREM 2.2. *Under the same hypotheses as in Theorem* 2.1, *the resolvent* $(z - \exp(-A))^{-1}$ *has at most a pole at* $z = 0$ *if and only if there are numbers* $c_j > 0$ *such that for all* z *with* $\operatorname{Re} z$ *sufficiently large*

$$\|(z - A)^{-1}\| \leqslant c_1 \exp(c_2 \operatorname{Re} z). \tag{2.1}$$

If given any $\varepsilon > 0$ *there is a* c_ε *such that*

$$\|(z - A)^{-1}\| \leqslant c_\varepsilon \exp(\varepsilon \operatorname{Re} z) \tag{2.2}$$

for $\operatorname{Re} z$ *large, then* $0 \in \rho(\exp(-A))$.

The main ingredient in our proof of these results is the Parseval relation for Fourier series:

LEMMA 2.3. *Suppose $\mathscr{H}$ is a Hilbert space and f and g are continuous functions on* [0, 1] *with values in $\mathscr{H}$. Define the Fourier coefficients of f as*

$$f_n = \int_0^1 e^{2\pi int} f(t)\,\mathrm{d}t$$

and similarly for g. Then

$$\int_0^1 (f(t), g(t))\mathrm{d}t = \sum_{n=-\infty}^{\infty} (f_n, g_n) \tag{2.3}$$

where the series on the right converges absolutely.

Proof of Lemma 2.3. Let $T \subset [0,1]$ and $\Lambda \subset \mathbf{C}$ be countable sets such that T is dense in [0,1] and Λ is dense in $\mathbf{C}$. Let $\mathscr{V}$ be the closure of the set of all vectors of the form

$$\sum_{i=1}^{N} \lambda_i f(t_i) + \sum_{j=1}^{N} \lambda_j' g(t_j)$$

where $N < \infty$, $\lambda_i, \lambda_i' \in \Lambda$, and $t_i, t_i' \in T$. It is clear that $\mathscr{V}$ is a separable Hilbert space containing $\operatorname{Ran} f \cup \operatorname{Ran} g$.

This reduces the problem to the case where $\mathscr{H}$ is separable and here the result is well known. ▨

We now begin the proof of Theorems 2.1 and 2.2. By adding a constant to A we can assume that $\|\exp(-tA)\| \leqslant K$ for all $t \geqslant 0$. Given $z \in \mathbf{C}$ let

$$a_n(z) = \int_0^1 \exp(-t(A-z))\exp(2\pi int)\mathrm{d}t \tag{2.4}$$

$$B(z) = 1 - \exp(-(A-z)). \tag{2.5}$$

An elementary integration shows that

$$B(z) = (A - z - 2\pi in)a_n(z). \tag{2.6}$$

Suppose $\exp(-z_0) \in \rho(\exp(-A))$. Then $B(z_0)$ is invertible so that (2.6) implies $z_0 + 2\pi in \in \rho(A)$ for all n and writing

$$R_n(z_0) = (A - z_0 - 2\pi in)^{-1}$$

we have

$$\|R_n(z_0)\| = \|B(z_0)^{-1}\| K \left(\int_0^1 \exp(t \operatorname{Re} z_0)\mathrm{d}t\right) \cdot$$

Hence $\sup_n \|(A - z_0 - 2\pi i n)^{-1}\| < \infty$. (This elementary argument is Gearhart's [2]. We have included it to keep the proof self-contained.)

To prove the converse consider the function

$$(2.7)\qquad q_\alpha(x) = \alpha^2 \int_0^1 \|\exp(-t(A - z_0))x\|^2 dt + \|x\|^2 - \|\exp(-(A - z_0))x\|^2.$$

We will first bound q_α from below. Since for $t \in [0,1]$

$$\|\exp(-(A - z_0))x\| = \|\exp(-(1 - t)(A - z_0)) \exp(-t(A - z_0))x\| \leqslant$$

$$\leqslant K\exp((1 - t) \operatorname{Re} z_0)\| \exp(-t(A - z_0))x\|$$

we have

$$\int_0^1 \|\exp(- t(A - z_0))x\|^2 dt \geqslant$$

$$(2.8)\qquad \geqslant K^{-2} \int_0^1 \exp(- 2(1 - t) \operatorname{Re} z_0)dt \|\exp(-(A - z_0))x\|^2 =$$

$$= K^{-2}[(1 - \exp(- 2\operatorname{Re} z_0))/2\operatorname{Re} z_0] \|\exp(- (A - z_0))x\|^2.$$

Here and in the following we assume $\operatorname{Re} z_0 \geqslant 0$ because since the spectral radius of $\exp(-A)$ is at most 1, $\exp(-z_0) \in \rho(\exp(-A))$ if $\operatorname{Re} z_0 < 0$. From (2.8) it follows that the choice

$$(2.9)\qquad \alpha = \begin{cases} K(2\operatorname{Re} z_0(1 - \exp(- 2\operatorname{Re} z_0))^{-1})^{1/2}; & \operatorname{Re} z_0 > 0 \\ K; & \operatorname{Re} z_0 = 0 \end{cases}$$

gives the lower bound

$$q_\alpha(x) \geqslant \|x\|^2.$$

We now bound q_α from above assuming that $\sup_n \|R_n(z_0)\| = M(z_0) < \infty$. Let $f(t) = g(t) = \exp(-t(A - z_0))x$ in (2.3). From (2.4) and (2.6) we have

$$(2.10)\qquad \int_0^1 \|\exp(- t(A - z_0))x\|^2 dt = \sum_{n=-\infty}^{\infty} \|R_n(z_0)B(z_0)x\|^2.$$

Let $w_0 = -\lambda + \mathrm{i}\,\mathrm{Im}\, z_0$ where $\lambda > 0$. Then by the resolvent equation

$$R_n(z_0) = C_n R_n(w_0)$$

with $C_n = 1 + (z_0 - w_0)R_n(z_0)$. Note that $\|C_n\| \leqslant 1 + (\mathrm{Re}\, z_0 + \lambda)M(z_0)$ so that

$$\|R_n(z_0)B(z_0)x\| \leqslant (1 + (\mathrm{Re}\, z_0 + \lambda)M(z_0))\| R_n(w_0)B(z_0)x\| \leqslant$$

$$\leqslant (1 + (\mathrm{Re}\, z_0 + \lambda)M(z_0))\|B(w_0)^{-1}\|\, \|R_n(w_0)B(w_0)B(z_0)x\|.$$

Since $\mathrm{Re}\, w_0 < 0$, $B(w_0)$ is certainly invertible. Define

$$\gamma = (1 + (\mathrm{Re}\, z_0 + \lambda)M(z_0))\|B(w_0)^{-1}\|.$$

Inserting the last estimate in (2.10) we have

$$\int_0^1 \|\exp(-t(A - z_0))x\|^2\, \mathrm{d}t \leqslant \gamma^2 \sum_{n=-\infty}^{\infty} \|R_n(w_0)B(w_0)B(z_0)x\|^2.$$

If we replace z_0 and x by w_0 and $B(z_0)x$ in (2.10) we have the identity

$$\int_0^1 \|\exp(-t(A - w_0))B(z_0)x\|^2\, \mathrm{d}t = \sum_{n=-\infty}^{\infty} \|R_n(w_0)B(w_0)B(z_0)x\|^2$$

so that

$$\int_0^1 \|\exp(-t(A - z_0))x\|^2 \mathrm{d}t \leqslant \gamma^2 \int_0^1 \|\exp(-t(A - w_0))B(z_0)x\|^2 \mathrm{d}t \leqslant$$

$$\leqslant (\gamma^2 K^2 \int_0^1 \exp(-2t\lambda)\mathrm{d}t)\, \|B(z_0)x\|^2 = \tag{2.11}$$

$$= \gamma^2 K^2 (1 - \exp(-2\lambda))\, (2\lambda)^{-1}\, \|B(z_0)x\|^2.$$

Now consider the remaining two terms in (2.7). We have

$$\|x\|^2 - \|\exp(-(A - z_0))x\|^2 = \|x\|^2 - \|(B(z_0) - 1)x\|^2 =$$

$$= 2\mathrm{Re}(x, B(z_0)x) - \|B(z_0)x\|^2 \leqslant \tag{2.12}$$

$$\leqslant 2\|x\|\, \|B(z_0)x\| - \|B(z_0)x\|^2.$$

Setting $\beta = \alpha\gamma K[(1 - \exp(-2\lambda))/2\lambda]^{1/2}$ and combining (2.11) and (2.12) we find the upper bound

$$q_\alpha(x) \leqslant (\beta^2 - 1)\|B(z_0)x\|^2 + 2\|x\|\,\|B(z_0)x\|. \tag{2.13}$$

Set $\|x\| = 1$ and let $u = \|B(z_0)x\|$. If α is chosen as in (2.9) then (2.13) combined with the lower bound $q_\alpha(x) \geqslant 1$ gives

$$0 \leqslant (\beta^2 - 1)u^2 + 2u - 1 = (\beta^2 - 1)(u - (1 - \beta)^{-1})(u - (1 + \beta)^{-1}) \tag{2.14}$$

where of course the equality in (2.14) holds only if $\beta \neq 1$. In any case (2.14) implies that $u \geqslant (1 + \beta)^{-1}$ which means

$$\|B(z_0)x\| \geqslant (1 + \beta)^{-1}\|x\| \tag{2.15}$$

for all $x \in \mathscr{H}$. Using the fact that $(\exp(-tA))^* = \exp(-tA^*)$, a similar argument implies

$$\|B(z_0)^* x\| \geqslant (1 + \beta)^{-1}\|x\|. \tag{2.16}$$

Thus $B(z_0)$ is invertible and

$$\|B(z_0)^{-1}\| \leqslant 1 + \beta. \tag{2.17}$$

This proves the first part of Theorem 2.1.

Now suppose that $0 \in \rho(\exp(-A))$. Then there is an $\omega_0 > 0$ so that if $\mathrm{Re}\, z > \omega_0$, $\exp(-z) \in \rho(\exp(-A))$ and

$$\|(\exp(-z) - \exp(-A))^{-1}\| \leqslant c.$$

Since $B(z)^{-1} = \exp(-z)(\exp(-z) - \exp(-A))^{-1}$ we thus have

$$\|B(z)^{-1}\| \leqslant c \exp(-\mathrm{Re}\, z).$$

From (2.6) it thus follows that for $\mathrm{Re}\, z > \omega_0$

$$\|(A - z)^{-1}\| \leqslant c\, K \exp(-\mathrm{Re}\, z) \int_0^1 \exp(t\, \mathrm{Re}\, z) dt \leqslant c_0 (\mathrm{Re}\, z)^{-1}.$$

The remainder of Theorem 2.1 is a consequence of Theorem 2.2. We thus assume that (2.1) holds for all z with $\mathrm{Re}\, z$ sufficiently large. We set $\lambda = \mathrm{Re}\, z$ in the explicit bound (2.17) and find that for $\mathrm{Re}\, z$ sufficiently large

$$\|B(z)^{-1}\| \leqslant 3c_1 K^2 (\mathrm{Re}\, z) \exp(c_2 \mathrm{Re}\, z).$$

Thus

$$\|(\exp(-z) - \exp(-A))^{-1}\| \leqslant 3c_1 K^2 (\mathrm{Re}\, z) \exp((1 + c_2)\mathrm{Re}\, z) \leqslant$$
$$\leqslant c \exp(N\, \mathrm{Re}\, z)$$

for some integer $N > 0$, or writing $w = \exp(-z)$

$$\|(w - \exp(-A))^{-1}\| \leqslant c\,|w|^{-N} \tag{2.18}$$

for sufficiently small $|w|$. We conclude from (2.18) that the function $f(w) = (w - \exp(-A))^{-1}$ has at most a pole of order N at $w = 0$.

Conversely, if $z = 0$ is at most a pole of the resolvent $(z - \exp(-A))^{-1}$, then (2.18) holds for some $N \geqslant 0$. Thus from (2.6)

$$\|(A - z)^{-1}\| \leqslant \|B(z)^{-1}\|\, K \exp(\operatorname{Re} z) \leqslant c\, K \exp(N \operatorname{Re} z).$$

Suppose now that given $\varepsilon > 0$, (2.2) holds for all large $\operatorname{Re} z$. Then from what we have just shown, $(z - \exp(-A))^{-1}$ has at most a pole at $z = 0$. If this pole is of order $N \geqslant 1$, then $\exp(-A)$ has a non-zero kernel. Suppose $\exp(-A)x = 0$. We will show that $x = 0$ and thus complete the proof.

From (2.11) we have (since $B(z)x = x$)

$$\int_0^1 \|\exp(-t(A - z))x\|^2\, dt \leqslant \|x\|^2 \gamma^2 K^2 (1 - \exp(-2\lambda))/2\lambda$$

for $\operatorname{Re} z$ large, where $\gamma = (1 + (\operatorname{Re} z + \lambda) M(z))\|B(w_0)^{-1}\|$. We choose $\lambda = \operatorname{Re} z$ and ε in $(0,1/2)$ so that (2.2) holds for all z with $\operatorname{Re} z$ large. Then

$$\left(\int_0^1 \|\exp(-tA)x\|^2 \exp(2t \operatorname{Re} z)\, dt\right)^{1/2} \leqslant d_\varepsilon (\operatorname{Re} z)^{1/2} \exp(\varepsilon \operatorname{Re} z)\|x\|.$$

Since

$$\int_0^1 \|\exp(-tA)x\|^2 \exp(2t \operatorname{Re} z) dt \geqslant \exp(4\varepsilon \operatorname{Re} z) \int_{2\varepsilon}^1 \|\exp(-tA)x\|^2\, dt$$

we have

$$\left(\int_{2\varepsilon}^1 \|\exp(-tA)x\|^2\, dt\right)^{1/2} \leqslant d_\varepsilon (\operatorname{Re} z)^{1/2} \exp(-\varepsilon \operatorname{Re} z)\|x\|$$

so that taking the limit $\operatorname{Re} z \to \infty$ we find $\exp(-tA)x = 0$ for $t \geqslant 2\varepsilon$. Thus $\exp(-tA)x = 0$ for all $t > 0$ and by continuity $x = 0$. ▨

Note: In a recent preprint [6], J. Howland has given an alternative proof of the first part of Theorem 2.1 using very different methods.

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