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NEST-SUBALGEBRAS OF VON NEUMANN ALGEBRAS: COMMUTANTS MODULO THE JACOBSON RADICAL

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This paper is a sequel to [10] and [11] in which properties of the class of nest subalgebras of von Neumann algebras have been systematically explored. A nest subalgebra of a von Neumann algebra (nsva) is an operator algebra of the form $\mathscr{B} \cap \operatorname{alg} \mathscr{N}$ where $\mathscr{B}$ is a von Neumann algebra and $\mathscr{N}$ is a nest of projections in $\mathscr{B}$. These algebras, which are natural generalizations of Ringrose's nest algebras [25], are closely related to other classes of nonselfadjoint algebras which have been extensively studied: most notably the triangular algebras [16], the subdiagonal algebras [1], and the algebras of analytic operators [20]. In [10] some basic properties of an nsva were given. The main structure results given there include a central decomposition into a direct integral of nest subalgebras of factor von Neumann algebras and a characterization of the Jacobson radical of an nsva which is the natural generalization of the characterization of the radical of a nest algebra given by Ringrose [25]. Subsequently in [11] the essential commutant (commutant modulo the compact operators) of an nsva was determined extending the result for von Neumann algebras due to Johnson and Parrot [15] and a result for nest algebras due to Christensen and Peligrad [4]. In addition some results concerning the Arveson distance estimate for a nsva (and other algebras) were obtained in [11]. The structure of the invariant subspace lattice of a nsva was studied in [10].

In this paper we consider structural properties related to the radical. We use the term *$\mathscr{R}$-commutant of the core* to denote the set of operators in an nsva $\mathscr{A}$ which commute with the core modulo the radical, and the term *$\mathscr{R}$-center* will denote the set of operators in $\mathscr{A}$ that commute modulo the radical with every element of $\mathscr{A}$. Here *core* denotes the von Neumann algebra generated by the nest and the central projections of the von Neumann algebra $\mathscr{B}$. While if $\mathscr{B}$ is not a factor different nests in $\mathscr{B}$ can give rise to the same nsva, the core remains the same. Moreover, since $W^*(\mathscr{A}) = \mathscr{B}$ the core is an internal aspect of $\mathscr{A}$. In [19] the $\mathscr{R}$-commutant of the core was characterized for an arbitrary nest algebra as the sum of the diagonal of the nest algebra and the radical. Also, the $\mathscr{R}$-center was characterized for nest algebras with continuous nests of multiplicity one (in particular for the Volterra nest algebra) in terms of a class of *essentially continuous functions*. This implied in particular that the $\mathscr{R}$-center was the sum of the radical and the C^*-algebra generated by the projections in the nest. It was pointed out later by J. Erdos that this $\mathscr{R}$-center decomposition can also be deduced from a result of E.C. Lance ([18], Theorem 6.2) in which he showed that the diagonal algebras related to a nest algebra have trivial center, and this decomposition is valid for arbitrary nest algebras.

In Section 2 of this paper we characterize the $\mathscr{R}$-commutant of the core for an arbitrary nsva as the sum of the diagonal and the radical of the nsva. In Section 3 we extend the notion of an essentially continuous function to that of essentially continuous operator with respect to a nest. We then show that the C^*-algebra generated by the nest has a representation as the essentially continuous operators with respect to that nest. We use this result and those of Section 2 to extend the characterization of the $\mathscr{R}$-center to nsva of factors for which the core is a normal subalgebra. In Section 4 we remove the normality condition and generalize the characterization of the $\mathscr{R}$-center to arbitrary nsva. Thus the $\mathscr{R}$-center is the sum of the C^*-algebra generated by the nest $\mathscr{N}$ and the projections from the center of the von Neumann algebra plus the radical of the nsva. In Section 5 we give an application to the theory of spectral operators. We show that a spectral operator T is the sum of a normal operator N and an operator Q where Q is in the Jacobson radical of the weakly closed algebra generated by N, Q and I.

We note that the $\mathscr{R}$-center of an abstract Banach algebra and related structural questions have been studied by several authors ([22], [28]). In particular, Vlastimil Pták has given nine equivalent conditions which determine the $\mathscr{R}$-center of a Banach algebra [21]. These all involve spectral conditions on the elements of the algebra and are not the type of determination of the $\mathscr{R}$-center we undertake here for our specific Banach algebras.

The techniques utilized here and in the attainment of the Ringrose criterion for an nsva [10] involve the direct integral decomposition of a norm closed object (the Jacobson radical) along the center of a von Neumann algebra. Such techniques are usually reserved for strongly closed sets and the difficulties that have to be overcome stem in part from the fact that a direct integral (or sum) of quasinilpotent operators need not be quasinilpotent. The kinds of problems involved here and in [10] generally involves proving a result for the general factor case and then extending via the reduction theory. On the other hand the principal difficulty encountered in [11] was quite different in that the central problem there was establishing a particular result for the type I and II factor cases.

1. PRELIMINARIES

Throughout this paper, all Hilbert spaces will be separable, all operators bounded, all subspaces closed and all projections will be selfadjoint. Let H be a Hilbert space. Then $\mathscr{L}(H)$, $\mathscr{C}(H)$ and $\mathscr{P}(H)$ will denote respectively all operators on H, the contraction operators and all projections on H. The spaces $\mathscr{C}(H)$ and $\mathscr{P}(H)$ are to be regarded as equipped with the strong operator topology and the Borel structure subordinate to it; this makes $\mathscr{C}(H)$ and $\mathscr{P}(H)$ into complete metric spaces.

For convenience we shall disregard the distinction between a subspace of H and the orthogonal projection onto it. Let $\mathscr{N}$ be a complete nest of projections. We use "$\leqslant$" to denote inclusion, "$<$" being reserved for proper inclusion. If $N \in \mathscr{N}$, $N \neq 0$, write $N_- = \vee \{M \in \mathscr{N}: M < N\}$, and set $N_- = 0$ for $N = 0$. If $N_- \neq N$ we term N_- the *immediate predecessor* of N in $\mathscr{N}$. Also, if $N \neq I$ write $N_+ = \wedge \{M \in \mathscr{N}: M > N\}$, and set $N_+ = I$ for $N = I$. If $N_+ \neq N$ we term N_+ the *immediate successor* of N. If $\mathscr{N}$ is a nest we use the notation $\mathscr{A}_{\mathscr{N}}$ to denote the *nest algebra* Alg $\mathscr{N}$. The *core* of a commutative subspace lattice $\mathscr{L}$ is the von Neumann algebra generated by $\mathscr{L}$ and is denoted by $\mathscr{C}$ or $\mathscr{C}_{\mathscr{L}}$.

The Jacobson radical of an arbitrary algebra is defined to be the intersection of the kernels of all strictly transitive representations of the algebra [24]. A right or left ideal in a Banach algebra $\mathscr{A}$ is topologically nil if each of its elements is quasinilpotent. The radical $\mathscr{R}$ of a Banach algebra $\mathscr{A}$ is a closed 2-sided topologically nil ideal which contains every topologically nil left or right ideal in $\mathscr{A}$ (cf. [24], p. 57). If $\mathscr{A}$ is a Banach algebra with identity then

$$\mathscr{R} = \{B \in \mathscr{A}: AB \text{ is quasinilpotent, } A \in \mathscr{A}\} =$$
$$= \{B \in \mathscr{A}: BA \text{ is quasinilpotent, } A \in \mathscr{A}\}.$$

Also, if $\sigma(A)$ denotes the spectrum of A in $\mathscr{A}$ then $B \in \mathscr{R}$ if and only if $\sigma(A + B) = \sigma(A)$ for all $A \in \mathscr{A}$. For ten additional spectral conditions equivalent to $B \in \mathscr{R}$ see the paper of V. Pták [21].

If $\mathscr{N}$ is a nest of projections then a projection E is an *$\mathscr{N}$-interval* if E is of the form $E = M - N$ where $M, N \in \mathscr{N}$, $M > N$. The projections M, N are called the *upper* and *lower endpoints* of E, respectively. The endpoints of a nonzero $\mathscr{N}$-interval are well defined, for suppose $E = M_i - N_i$, $i = 1,2$, and $E \neq 0$. Then $M_1 > E$ and $N_2 \perp E$ implies $M_1 > N_2$, hence $M_1 > M_2$. Similarly $M_2 > M_1$. Thus $M_1 = M_2$, so also $N_1 = N_2$. For two core projections E and F we denote $E \ll F$ when $E\mathscr{L}(H)F \subseteq \mathscr{A}$ and say that E and F are *strictly ordered*. An operator is *$\mathscr{N}$-simple* if it is a linear combination of a finite number of mutually orthogonal $\mathscr{N}$-intervals. We can make the analogous definitions as above for any subspace lattice, however the uniqueness of the endpoints of an interval no longer need hold.

In his original paper on nest algebras [25] Ringrose presented an operator-theoretic characterization of the Jacobson radical of an arbitrary nest algebra (the Ringrose criterion). Ringrose actually presented two such criteria equivalent for nest algebras. Let $\mathscr{R}$ denote the radical of $\mathscr{A}_{\mathscr{N}}$.

THEOREM. (Ringrose). *If $A \in \mathscr{A}_{\mathscr{N}}$, then $A \in \mathscr{R}$ if and only if both of the following are satisfied:*

(i) *Given* $N \in \mathcal{N}$ *with* $N \neq 0$ *and given* $\varepsilon > 0$ *there exists* $L \in \mathcal{N}$ *such that* $L < N$ *and* $\|(N - L)A(N - L)\| < \varepsilon$;

(ii) *Given* $N \in \mathcal{N}$ *with* $N \neq I$ *and given* $\varepsilon > 0$ *there exists* $M \in \mathcal{N}$ *such that* $N < M$ *and* $\|(M - N)A(M - N)\| < \varepsilon$.

THEOREM. (The Ringrose criterion). *If* $A \in \mathcal{A}_{\mathcal{N}}$, *then* $A \in \mathcal{R}$ *if and only if for each* $\varepsilon > 0$ *there exists a finite set* $\{E_i\}$ *of mutually orthogonal* $\mathcal{N}$*-intervals with* $\sum E_i = I$ *such that* $\|E_iAE_i\| < \varepsilon$, *all i.*

For a nest $\mathcal{N}$ in a von Neumann algebra $\mathcal{B}$ let $\mathcal{A}$ be the nsva $\mathcal{A}_{\mathcal{N}} \cap \mathcal{B}$. Let $\mathcal{M}_0$ be the projections in the center of $\mathcal{B}$ and $\mathcal{L}$ be commutative subspace lattice $\mathcal{M}_0 \vee \mathcal{N}$. The main result in [10] concerns the extension of the Ringrose criterion to the radical $\mathcal{R}$ of $\mathcal{A}$. This is also analogous to the characterization of the radical obtained for certain CSL algebras [14].

THEOREM 5.1. ([10]). *If* $A \in \mathcal{A}$ *then* $A \in \mathcal{R}$ *if and only if for each* $\varepsilon > 0$ *there exists a finite set* $\{E_i\}$ *of mutually orthogonal* $\mathcal{L}$*-intervals with* $\sum E_i = I$ *such that* $\|E_iAE_i\| \leqslant \varepsilon$ *for all i.*

For a nsva $\mathcal{A}$ we define the *core* of $\mathcal{A}$ to be the von Neumann algebra $\mathcal{C}$ generated by $\mathcal{N}$ and the center of $\mathcal{B}$. In [16] R.V.Kadison and I.M. Singer defined the core of a triangular algebra $\mathcal{A}$ in a factor $\mathcal{B}$ as the von Neumann algebra generated by the hulls, that is the invariant projections for $\mathcal{A}$ which lie in $\mathcal{B}$. The following lemma shows that for an nsva $\mathcal{A}$ the nest $\mathcal{N}$ is in fact the invariant projections of $\mathcal{A}$ in $\mathcal{B}$ when $\mathcal{B}$ is a factor. Hence for arbitrary von Neumann algebras $\mathcal{B}$ it follows by the results in [10] and [12] that the invariant projections for $\mathcal{A}$ in $\mathcal{B}$ are just those in $\mathcal{N} \vee \mathcal{M}_0$ where $\mathcal{M}_0$ are the central projections for $\mathcal{B}$. We call a lattice *relatively reflexive in* $\mathcal{B}$ if $\mathcal{L} = \mathrm{Lat}(\mathcal{B} \cap \mathrm{Alg}\,\mathcal{L}) \cap \mathcal{B}$. In [17] Jon Kraus calls such a lattice $\mathcal{B}$-reflexive.

LEMMA 1.1. *A complete nest* $\mathcal{N}$ *in a factor von Neumann algebra* $\mathcal{B}$ *is relatively reflexive in* $\mathcal{B}$.

Proof. Let $\mathcal{A}$ be the nsva for $\mathcal{N}$ in $\mathcal{B}$ and let $P \in \mathcal{B} \cap \mathrm{Lat}\,\mathcal{A}$. Since the set $N_+\mathcal{B}N^\perp \subseteq \mathcal{A}$ it follows that the subspace spanned by this set applied to vectors in PH (denoted $[N_+\mathcal{B}N^\perp PH]$) is contained in PH. However $[N_+\mathcal{B}N^\perp PH] = N_+[\mathcal{B}N^\perp PH]$ and since $\mathcal{B}$ is a factor $[\mathcal{B}H^\perp PH] = I$ or 0 with $[\mathcal{B}N^\perp PH] = 0$ if and only if $N^\perp P = 0$, i.e., $P \leqslant N$. Thus $P \geqslant N_+[\mathcal{B}N^\perp PH] = N_+$ whenever $N^\perp P \neq 0$. Let $N_0 = \sup\{N: N^\perp P \neq 0\}$. If $N > N_0$ then $N^\perp P = 0$ or $P \leqslant N$. Therefore $N_{0_+} \leqslant P < N$ for any $N > N_0$ and so $N_{0_+} = P \in \mathcal{N}$. ∎

We shall denote by $\mathcal{R}$-com $\mathcal{C}$ all $T \in \mathcal{L}(H)$ for which $TA - AT \in \mathcal{R}$ for all $A \in \mathcal{C}$. The following shows that $\mathcal{R}$-com $\mathcal{C}$ is contained in $\mathcal{A}_{\mathcal{N}}$.

LEMMA 1.2. $\mathcal{R}$-com $\mathcal{C} \subset A_{\mathcal{N}}$ *and so* $\mathcal{B} \cap \mathcal{R}$-com $\mathcal{C} \subset \mathcal{A}$.

Proof. Let $T \in \mathcal{R}$-com $\mathcal{C}$. It suffices to show $N^\perp TN = 0$ for all $N \in \mathcal{N}$. Since $TN - NT \in \mathcal{R}$ and $\mathcal{R}$ is an ideal in $\mathcal{A}$ then $N^\perp TN \in \mathcal{R} \subset \mathcal{A}_{\mathcal{N}}$. However $N\mathcal{L}(H)N^\perp \subset \mathcal{A}_{\mathcal{N}}$

so that $N^{\perp}TN \in \mathscr{A}_{\mathscr{N}}^{*}$ as well. Whence $N^{\perp}TN \in \mathscr{A}_{\mathscr{N}} \cap \mathscr{A}_{\mathscr{N}}^{*} = \mathscr{C}'_{\mathscr{N}}$ and so $0 = N(N^{\perp}TN) = (N^{\perp}TN)N = N^{\perp}TN$. ▨

We remark that $\mathscr{R}$-com $\mathscr{C}$ and likewise $\mathscr{R}$-com $\mathscr{A}$ need not be in $\mathscr{A}$ and in fact both contain $\mathscr{B}'$. For a nest algebra $\mathscr{A}_{\mathscr{N}}$ and an ideal $\mathscr{I}$ in $\mathscr{A}_{\mathscr{N}}$ the set $\mathscr{I}$-com $\mathscr{C}$ has been subsequently investigated (cf. [8]).

2. $\mathscr{R}$ COMMUTANT OF THE CORE

For a nest algebra $\mathscr{A}_{\mathscr{N}}$, T commutes with the core modulo the radical precisely when T decomposes into the sum of a diagonal operator plus a radical operator: that is $\mathscr{R}$-com$(\mathscr{C}_{\mathscr{N}}) = \mathscr{D}_{\mathscr{N}} + \mathscr{R}_{\mathscr{N}}$ [19]. In this section we shall extend this result to nest subalgebras of von Neumann algebras. First we shall consider the case when $\mathscr{A}$ is an nsva of a factor and then using measure theoretic techniques "lift" the result to the general nsva.

Our notation here will generally follow that in [10]. Let $\mathscr{A}$ be a nest subalgebra of $\mathscr{B}$ with respect to a nest $\mathscr{N}$ and let $\mathscr{L}_0 = \mathscr{N} \vee \mathscr{M}_0$ where $\mathscr{M}_0$ is the set of projections in the center of $\mathscr{B}$. We shall fix ψ_0 to be an expectation onto the algebra $\mathscr{C}'$ ($\mathscr{C} = W^*(\mathscr{L}_0)$) obtained via an invariant mean on the group of symmetries determined by the simple projections generated by $\mathscr{L}_0$ (cf. Section 4 of [11]). We shall call ψ_0 the *diagonal projection* since $\psi_0 \colon \mathscr{B} \to \mathscr{C}' \cap \mathscr{B} = \mathscr{D}$. As an immediate corollary of Lemma 4.1 in [11] we obtain the following useful lemma.

LEMMA 2.1. *Let $T \in \mathscr{L}(H)$ such that $\psi_0(T) = 0$. Then there exists a simple projection E so that $\|TE - ET\| \geqslant \frac{1}{2}\|T\|$.*

In the subsequent arguments we use the concept of paving number for an operator T. If $E_1, \ldots, E_n$ is a partition of the identity by intervals of $\mathscr{N}$ and $E_i \ll E_{i+1}$ we call the set $\{E_i\}$ an $\mathscr{N}$ *paving*, or if there is no ambiguity simply a paving. If T compressed to each interval E_i has norm less than or equal to ε, then $\{E_i\}_1^n$ is said to *ε-pave* T. If T has a finite ε-paving, then the minimum cardinality among all sets which ε-pave T is the *ε-paving number* for T and is denoted $p_\varepsilon(T)$. We write $p_\varepsilon(T) = \infty$ if T fails to have a finite ε-paving. The Ringrose criterion states that if $T \in \mathscr{A}_{\mathscr{N}}$ then $T \in \mathscr{R}_{\mathscr{N}}$ iff T has a finite ε-paving number for all $\varepsilon > 0$.

LEMMA 2.2. *Let $p_\varepsilon(T) = k_0 < \infty$ and $n \leqslant k_0/2$. Then there exists a paving by n intervals $E_1, \ldots, E_n$ with $\|E_iTE_i\| \geqslant \varepsilon$ for each i.*

Proof. Let $F_1, \ldots, F_{k_0}$ be an ε-paving for T. Let $E_i = F_{2i-1} + F_{2i}$ for $i = 1,2, \ldots$ (with F_{k_0} included in $E_{\frac{1}{2}(k_0-1)}$ if k_0 is odd). Thus the partition of I with $\{F_1, \ldots, F_{k_0}\}$ minus $\{F_{2i_0-1}, F_{2i_0}\}$ and with E_{i_0} is a paving with fewer than k_0

members; moreover, since each $\|F_iTF_i\| \leqslant \varepsilon$ it follows that $\|E_{i_0}TE_{i_0}\| \geqslant \varepsilon$. This is the case for each E_i so that $E_1, \ldots, E_{n_0}$ where n_0 is $\frac{1}{2} k_0$ if k_0 is even or otherwise $\frac{1}{2}(k_0 - 1)$ is a paving satisfying the lemma. ∎

The following two results are in essence the extension to an nsva of a factor the characterization of the $\mathscr{R}$ commutant of the core for a nest algebra. We use $\delta_T(A)$ to denote $TA - AT$.

LEMMA 2.3. *Let $\mathscr{A}$ be an nsva of a factor $\mathscr{B}$ and ψ_0 the diagonal projection of $\mathscr{L}(H)$ onto $\mathscr{C}'$. If $T \in \mathscr{A}$ with $\psi_0(T) = 0$ and $p_\varepsilon(T) \geqslant k$, then for $n \leqslant \frac{1}{2} k$ there exists contraction operators A, B in $\mathscr{A}$ and $C \in \mathscr{C}$ so that $\|(A\delta_T(C))^{n-1}\| \geqslant (\varepsilon/3)^{n-1}$ and $\|(\delta_T(C)B)^{n-1}\| \geqslant (\varepsilon/3)^{n-1}$.*

Proof. Using Lemma 2.2 fix a paving $E_1, \ldots, E_n$ for which $\|E_iTE_i\| \geqslant \varepsilon$. Since $\psi_0(AT) = A\psi_0(T) = \psi_0(T)A$ whenever A is in the core, the assumption that $\psi_0(T) = 0$ implies that $\psi_0(E_iTE_i) = 0$ for $1 \leqslant i \leqslant n$. Thus by Lemma 2.1 there are simple projections $F_i \leqslant E_i$ with $\|E_iTE_iF_i - F_iE_iTE_i\| \geqslant \varepsilon/2$. Let R_iA_i be the polar decomposition of $E_i(TF_i - F_iT)E_i$ and G_i the projection on $\{x \in E_i \mid \|A_ix\|^2 \geqslant \varepsilon/3\|x\|^2\}$. Since G_i is a spectral projection for A_i, $A_iG_i = G_iA_i$ and it follows that G_i and the range projection of R_iG_i are in $\mathscr{B}$ and both are contained in E_i. Using the comparison theorem for factor von Neumann algebras we can find nonzero partial isometries S_j in $\mathscr{B}$ with support in the range of $R_jA_jS_{j+1}$ and range in G_{j-1} if $j = 2, \ldots, n-1$ and S_n a partial isometry with support in $R_nA_nG_n$ and range in G_{n-1}. Recall that $E_{i-1} \ll E_i$ and since by definition S_i has support in E_i and range in $E_{i-1} \leqslant 0$ it follows that S_i is in the nest algebra $\mathscr{A}_{\mathscr{N}}$ and hence in $\mathscr{A}$.

Let $S = S_2 + \ldots + S_n$ and let $F = F_1 + \ldots + F_n$, then $F \in \mathscr{C}$ and $S \in \mathscr{A}$. A straightforward calculation shows that if $x \perp G_n$ then

$$[S(TF - FT)]^{n-1}x = 0$$

while if $x \in G_n$ then

$$[S_1(TF - FT)]^{n-1}x = S_2(TF_2 - F_2T)S_3 \ldots S_{n-1}(TF_{n-1} - F_{n-1}T)S_n(TF_n - F_nT)x.$$

Now by the definition of the partial isometries $S_2, \ldots, S_n$ it follows that

$$\|[S(TF - FT)]^{n-1}\| \geqslant (\varepsilon/3)^{n-1}.$$ ∎

With the added hypothesis that the ε-paving number for T is infinite we get the following lemma.

LEMMA 2.4. *If $p_\varepsilon(T) = \infty$ for some $T \in \mathscr{A}$, and $\psi_0(T) = 0$ then T is not in $\mathscr{R}$-com$(\mathscr{C})$.*

Proof. Using a Bolzano-Weierstrass type argument and Lemma 2.2 we can construct a nested sequence of intervals $E_1 \supset E_2 \supset \dots$ with two properties. First each compression E_iTE_i has infinite ε-paving number and secondly, if J_i and K_i are the two intervals whose union is $E_{i-1} - E_i$, the ε-paving number of T compressed to J_i or K_i is at least $2i + 1$. Now by Lemma 2.3 there is a core operator B_i and an operator A_i in $\mathscr{A}$ both reduced by $E_{i-1} - E_i$ and such that if T_i is T compressed to $E_{i-1} - E_i$ then $\|[(T_iB_i - B_iT_i)A_i]^i\| \geqslant (\varepsilon/3)^i$. Let $A = \sum A_i$ and $B = \sum B_i$. Since the compression to an interval is a homomorphism on $\mathscr{A}$

$$\|[(TB - BT)A]^i\| \geqslant \|[(T_iB_i - B_iT_i)A_i]^i\| \geqslant (\varepsilon/3)^i \quad \text{for all } i.$$

Thus $(TB - BT)A$ is not quasinilpotent and hence $TB - BT$ cannot be in the radical of $\mathscr{A}$. ▨

We use $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{R}$ to indicate respectively the core, diagonal and radical of an nsva $\mathscr{A}$. When $\mathscr{B}$ is a factor then Theorem 5.2 in [10] shows that $\mathscr{R} = \mathscr{R}_{\mathscr{N}} \cap \mathscr{B}$; moreover it is clear that in a factor $\mathscr{C} = \mathscr{C}_{\mathscr{N}}$ and $\mathscr{D} = \mathscr{D}_{\mathscr{N}} \cap \mathscr{B}$.

THEOREM 2.5. *Let $\mathscr{A}$ be an nsva of a factor $\mathscr{B}$. If $T \in \mathscr{B} \cap \mathscr{R}$-com $\mathscr{C}$ then $T = D + R$ where $D \in \mathscr{D}$ and $R \in \mathscr{R}$.*

Proof. Let ψ_0 be a diagonal projection and set $D = \psi_0(T)$ and $R = T - D$ Since $\psi_0 \colon \mathscr{B} \to \mathscr{B}$ we have $D \in \mathscr{D}$ and $R \in \mathscr{R}$-com $\mathscr{C}$ while $\psi_0(R) = 0$. The previous result shows that $p_\varepsilon(R) < \infty$ for all $\varepsilon \geqslant 0$ and thus by Theorem 5.7 in [10] R is in the radical of $\mathscr{A}$. ▨

Next we "lift" this result to the general nsva case. Let $\mathscr{A}$ be a nest subalgebra of the von Neumann algebra $\mathscr{B}$ with respect to the nest $\mathscr{N}$ in $\mathscr{B}$. Let $\mathscr{B} = \int_\Lambda^\oplus \mathscr{B}(\lambda)\mu(d\lambda)$ be the central decomposition of $\mathscr{B}$ on $H = \int_\Lambda^\oplus h(\lambda)\mu(d\lambda)$. We assume that μ is a complete regular Borel measure on a separable metric space Λ. We shall supress the symbol Λ in the notation unless it is needed for clarity. In our proofs we shall sometimes make the assumption that the spaces $h(\lambda)$ are all equal to a single h. When this is done an obvious modification adapts to the more general situation (cf. §3 in [10]). By [10] we have $\mathscr{A}$ and $\mathscr{N}$ decompose along the center of $\mathscr{B}$ as $\mathscr{A} = \int^\oplus \mathscr{A}(\lambda)\mu(d\lambda)$ and $\mathscr{N} \sim \int^\oplus \mathscr{N}(\lambda)\mu(d\lambda)$, where $\mathscr{N}(\lambda)$ is a nest in $\mathscr{B}(\lambda)$ and $\mathscr{A}(\lambda) = \mathscr{B}(\lambda) \cap \mathscr{A}_{\mathscr{N}(\lambda)}$ μ-a.e.. For notational simplicity we shall write $\mathscr{C}(\lambda)$, $\mathscr{D}(\lambda)$ and $\mathscr{R}(\lambda)$ for the core, diagonal and radical respectively of $\mathscr{A}(\lambda)$. The map $\lambda \to \mathscr{N}(\lambda)$ is then a measurable multifunction of Λ into $\mathscr{C}(h)$. By results in [3] the maps $\lambda \to N(\lambda)' \cap \mathscr{B}(\lambda) = \mathscr{D}(\lambda)$ and consequently $\lambda \to \mathscr{C}(\lambda)$ are measurable and thus attainable. It follows that

$$\mathscr{A} \cap \mathscr{A}^* = \mathscr{D} = \int^{\oplus} \mathscr{D}(\lambda)\mu(d\lambda) = \mathscr{C}' \cap \mathscr{B} \quad \text{and} \quad W^*(\mathscr{L}_0) = \int^{\oplus} \mathscr{C}(\lambda)\mu(d\lambda)$$

where $\mathscr{L}_0 = \mathscr{N} \vee \mathscr{M}_0$ and $\mathscr{M}_0$ are the projections in the center of $\mathscr{B}$ [10].

LEMMA 2.6. *Let $T \in \mathscr{B} \cap \mathscr{R}$-com$(\mathscr{C})$ where $\mathscr{A}$ is a nest subalgebra of the von Neumann algebra $\mathscr{B}$. Then $T = D + R$ where $D \in \mathscr{D}$ and $R = \int^{\oplus} R(\lambda)\mu(d\lambda)$ where $R(\lambda) \in \mathscr{R}(\lambda)$ μ-a.e. .*

Proof. Fix a Borel representative $T = \int^{\oplus} T(\lambda)\mu(d\lambda)$ of T and consider the subset of $\Lambda \times \mathscr{C}(h)$:

$$\sigma = \{(\lambda, R) \in \Lambda \times \mathscr{C}(h) : T(\lambda) - R \in \mathscr{D}(\lambda)\}.$$

Since $\psi: (\lambda, R) \to (\lambda, T(\lambda) - R)$ is a Borel map on $\Lambda \times \mathscr{C}(h)$ then $\sigma = \psi^{-1}(\mathrm{Gr}(\mathscr{D}))$, where $\mathscr{D}: \lambda \to \mathscr{D}(\lambda)$ is the diagonal map, is a Borel set in $\Lambda \times \mathscr{C}(h)$. Our object is to show that R can be taken to be in $\mathscr{R}(\lambda)$ and then using measurable selection find a measurable map $\lambda \to T(\lambda) - R(\lambda) \in \mathscr{D}(\lambda)$.

Recall that $\mathscr{I}(\lambda)$ is the strongly closed set of intervals formed from the nest $\mathscr{N}(\lambda)$ and $\mathscr{I}$ is the set of intervals formed from $\mathscr{N} \vee \mathscr{M}_0 = \mathscr{L}_0$. It is precisely these intervals $\mathscr{I}(\lambda)$ and $\mathscr{I}$ which determine the radical of $\mathscr{A}(\lambda)$ and $\mathscr{A}$ respectively [10]. By $\mathscr{I}(\lambda)^n$ we mean the n-fold product of the set $\mathscr{I}(\lambda)$ in $\mathscr{C}(h)^n$. Let $\mathscr{P}(h)$ be the projections in $\mathscr{C}(h)$ then define

$$\sigma_{nk} = \{(R, \lambda, E_1, \ldots, E_n) : \|E_i R E_i\| \leqslant 1/k, \quad E_i \in \mathscr{P}(h), \quad I = E_1 + \ldots + E_n\} \cap$$
$$\cap\, \mathscr{C}(h) \times \mathrm{Gr}(\mathscr{I}(\lambda)^n).$$

Lemma 5.5 in [10] shows that $\lambda \to \mathscr{I}(\lambda)$ is measurable and $\mathscr{I} = \int^{\oplus} \mathscr{I}(\lambda)\mu(d\lambda)$ so σ_{nk} is measurable in $X = \mathscr{C}(h) \times \Lambda \times \mathscr{P}(h)^n$; moreover by modifying $\lambda \to \mathscr{I}(\lambda)$ on a set of measure zero if necessary we may assume that σ_{nk} is a Borel set in the product Borel structure on X. Thus the set (λ, R) where $(R, \lambda) \in \prod_{\mathscr{C}(h) \times \Lambda} (\sigma_{nk})$ and denoted by $\prod_{\Lambda \times \mathscr{C}(h)} (\sigma_{nk})$ is analytic in $\Lambda \times \mathscr{C}(h)$. In particular the Ringrose Criterion is precisely the condition that

$$\mathscr{E} = \{(\lambda, \mathscr{R}) : R \in \mathscr{R}(\lambda)\} = \bigcap_k \bigcup_n \prod_{\Lambda \times \mathscr{C}(h)} (\sigma_{nk})$$

and thus $\mathscr{E}$ is an analytic subset of $\Lambda \times \mathscr{C}(h)$. Finally since $\mathscr{E}$ is an analytic and σ is a Borel set their intersection is an analytic set.

Let the operators $A_n = \int^{\oplus} A_n(\lambda)\mu(\mathrm{d}\lambda)$ be a dense set of Borel selectors for $\mathscr{A} = \int_{\Lambda}^{\oplus} \mathscr{A}(\lambda)\mu(\mathrm{d}\lambda)$, that is $\{A_n\}$ is dense in $\mathscr{A} \cap \mathscr{C}(H)$ while $\{A_n(\lambda)\}$ is dense in $\mathscr{A}(\lambda) \cap \mathscr{C}(h)$ μ-a.e. and $\lambda \to A_n(\lambda)$ are Borel maps of Λ to $\mathscr{C}(h)$. Since $\delta_T(A_n) \in \mathscr{R}$ for all n it follows that μ-a.e. $\delta_{T(\lambda)}(A_n(\lambda)) \in \mathscr{R}(\lambda)$ for all n (Lemma 5.3 in [10]).
Moreover, $\mathscr{A}(\lambda)$ is an nsva of a factor $\mathscr{B}(\lambda)$ so by (2.5) $T(\lambda) \in \mathscr{D}(\lambda) + \mathscr{R}(\lambda)$ μ-a.e.. That is $\coprod_{\Lambda} (\sigma \cap \mathscr{E})$ is of full measure in Λ. Now using measurable selection there exists a Borel function $\lambda \to R(\lambda)$ so that $(\lambda, R(\lambda)) \in \sigma \cap \mathscr{E}$, that is $R(\lambda) \in \mathscr{R}(\lambda)$ and $T(\lambda) - R(\lambda) \in \mathscr{D}(\lambda)$ μ-a.e. (cf. I, §4 in [27]). Let us denote by $R = \int^{\oplus} R(\lambda)\mu(\mathrm{d}\lambda)$ and by $D = T - R$. ▨

The following technical lemma is needed and should be compared to a similar result concerning the direct integral of quasinilpotent operators.

LEMMA 2.7. *If* $R = \int^{\oplus} R(\lambda)\mu(\mathrm{d}\lambda)$ *where* $R(\lambda) \in \mathscr{R}(\lambda)$ μ-a.e., *then there exist central projections* $E_n \to I$ *strongly so that* $RE_n \in \mathscr{R}$.

Proof. Since $R(\lambda) \in \mathscr{R}(\lambda)$, for each $\varepsilon > 0$, $p_\varepsilon(R(\lambda)) < \infty$. Let $\delta > 0$ be fixed, then there exist Borel sets Λ_{nk} with $\mu(\Lambda_{nk}) < \delta/2^n$ and so that $p_{1/n}(R(\lambda))$ is uniformly bounded μ-a.e. for $\lambda \notin \Lambda_{nk}$. Let $\Lambda_n = \bigcup_k \Lambda_{nk}$ and $E_n = \int_{\Lambda} I(\lambda)\mu(\mathrm{d}\lambda)$, then $\mu(E_n) < \delta$ so $E_n \to I$ strongly and it follows from the proof of (5.7) in [10] that $RE_n \in \mathscr{R}$. ▨

REMARK 2.8. The reference in the above proof to the proof of Theorem 5.7 in [10] will also be used below. For any operator $R = \int^{\oplus} R(\lambda)\mu(\mathrm{d}\lambda)$ in $\mathscr{A}$ one defines the $\mathscr{L}_0$-ε-paving number for R where $\mathscr{L}_0 = \mathscr{N} \vee \mathscr{M}_0$. Then the $\mathscr{L}_0$-paving number of R is precisely $\|p_\varepsilon(\cdot)\|_\infty$ as a measurable function on (Λ, μ). This follows from the proof of Theorem 5.6 in [10] and thus we could restate (5.6) in [10] to the effect that $R \in \mathscr{R}$ if and only if $\|p_\varepsilon\|_\infty < \infty$ for all $\varepsilon > 0$.

COROLLARY 2.9. *Let* $T = D + R$ *where* $D \in \mathscr{D}$, $R = \int^{\oplus} R(\lambda)\mu(\mathrm{d}\lambda)$ *and* $R(\lambda) \in \mathscr{R}(\lambda)$ μ-a.e.. *If* $\psi_0(T) = 0$ *then* $D = 0$.

Proof. Let $E_n \to I$ strongly be central projections so that $RE_n \in \mathscr{R}$. Then $0 = \psi_0(T)E_n = \psi_0(TE_n) = \psi_0(D)E_n + \psi_0(RE_n) = DE_n \to D$ strongly. ▨

THEOREM 2.10. $\mathscr{B} \cap \mathscr{R}\text{-com}\,\mathscr{C} = \mathscr{D} + \mathscr{R}$.

Proof. That $\mathscr{D} + \mathscr{R} \subset \mathscr{R}$-com$\mathscr{C}$ is clear. If $T \in \mathscr{R}$-com$\mathscr{C}$ and $T \in \mathscr{B}$ then by remarks in the preliminaries $T \in \mathscr{A}$ and set $S = T - \psi_0(T)$ where ψ_0 is the diagonal projection on $\mathscr{D} = \mathscr{A} \cap \mathscr{A}^*$. Clearly $\psi_0(S) = 0$ and $S \in \mathscr{R}$-com$\mathscr{C}$, thus by Lemma 2.6 and Corollary 2.8, $S = \int^{\oplus} S(\lambda)\,\mu(\mathrm{d}\lambda)$ where $S(\lambda) \in \mathscr{R}(\lambda)$ μ-a.e. .

If $\|p_\varepsilon\|_\infty < \infty$ for all ε, then by (2.8) $S \in \mathscr{R}$ and we are done. If not, for some $\varepsilon > 0$ we can construct using measurable selection and Lemma 2.3 a sequence of contraction operators $\{A_n\}$ in $\mathscr{A}$, core contractions $\{C_n\}$ and mutually orthogonal central projections $\{F_n\}$ so that A_n and C_n are supported on F_n and such that $\|(A_n\delta_T(C_n))^n\| \geq \varepsilon^n$. Letting $A = \sum A_n$, $C = \sum C_n$ we can conclude that $A\delta_T(C)$ is not in $\mathscr{R}$ and hence $\delta_T(C)$ is not in $\mathscr{R}$ which contradicts the fact that $T \in \mathscr{R}$-com$\mathscr{C}$. ▨

3. ESSENTIALLY CONTINUOUS OPERATORS

In this section we generalize the notion of an essentially continuous function on [0,1] to that of an essentially continuous operator with respect to a nest. The C^*-algebra generated by the nest has a representation as the algebra $\mathscr{E}.\mathscr{C}.$ of essentially continuous operators with respect to the nest. With this we show that $\mathscr{C} \cap \mathscr{R}$-center$\mathscr{A}$ for an nsva of a factor von Neumann algebra is $\mathscr{E}.\mathscr{C}.$.

Let $\mathscr{N}$ be a complete nest. For $P, Q \in \mathscr{N}$ we define the open interval $(P, Q) = \{N \in \mathscr{N} \mid P < N < Q\}$ and the projection $E((P, Q)) = Q_- - P$ for $P < Q$. Following standard measure theoretic usage we shall define the notion of null subset of $\mathscr{N}$. We shall call a countable family $\{(P_n, Q_n)\}$ of intervals a cover of a set in $\mathscr{N}$ if $S \subset \cup (P_i, Q_i)$. For convenience we drop the outer (,) in the notation $E((P, Q))$.

Definition 3.1. A Borel set S in $\mathscr{N}$ is a *null* subset of $\mathscr{N}$ if for every strong neighborhood U of zero in $\mathscr{L}(H)$ there exists a cover $\{(P_i, Q_i)\}$ of S with $\sum E(P_i, Q_i) \in U$.

Using Theorem 3.2 in [7] we can define a spectral measure $E(\,\cdot\,)$ on the space of Borel subsets of $\mathscr{N}$ in such a way that $E[P, Q] = Q - P_-$; $E(P, Q] = Q - P$ and $E[P, Q) = Q_- - P_-$. This spectral measure has the property that for a Borel set δ of $\mathscr{N}$, $E(\delta) = \int \mathscr{X}_\delta(N)\mathrm{d}E(N)$ when this is interpreted as $\langle E(\delta)x, x\rangle = \int \mathscr{X}_\delta(N)\mathrm{d}\langle E(N)x, x\rangle$ for all $x \in H$. We shall use the notation $\mu_x(\,\cdot\,)$ for the Borel measure $\langle E(\delta)x, x\rangle$.

The following lemma and theorem are essentially consequences of the results of Erdos [7].

Lemma 3.2. $E(\delta) = 0$ *if and only if* δ *is a null set as defined above.*

Proof. Let δ be a null set as defined in (3.1) and let x and ε be given. Let $U = N(0, x, \varepsilon)$ be a s.o.t. neighborhood of zero in $\mathscr{L}(H)$. There exist intervals $\{(P_i, Q_i)\}$ with $\delta \subset \cup (P_i, Q_i)$ and $\sum E(P_i, Q_i) \in U$. Thus $\|\sum E(P_i, Q_i)x\| < \varepsilon$ so $\|E(\delta)x\| < \varepsilon$ and consequently $E(\delta) = 0$. The converse uses the regularity of the measures $\mu_x(\cdot) = \|E(\cdot)x\|^2$. ▨

With the above definition of null subsets of $\mathscr{N}$ one can define $L^\infty(\mathscr{N})$ as equivalence classes of bounded complex valued Borel functions on $\mathscr{N}$ with the quotient supremum norm. There is a natural homomorphism φ of $L^\infty(\mathscr{N})$ into $\mathscr{C}_{\mathscr{N}}$ given by $\varphi(f) = \int_{\mathscr{N}} f(N)\mathrm{d}E(N)$ where this is defined via $(\varphi(f)x, x) = \int_{\mathscr{N}} f(N)\mathrm{d}\mu_x(N)$. Lemma 3.2 above and Theorems 3.2 and 3.7 in [7] show φ is a well defined $*$-isomorphism.

THEOREM 3.3. *$L^\infty(\mathscr{N})$ is isometrically isomorphic to $\mathscr{C}_{\mathscr{N}}$.*

Proof. We need to show φ is an isometry and is onto. For the latter we need only show the range of φ is closed in the strong operator topology.

Since $\langle\varphi(f)x, x\rangle = \int_{\mathscr{N}} f(N)\mathrm{d}\mu_x(N)$ it follows immediately that $\|\varphi(f)\| \leqslant \|f\|_\infty$.

Conversely, if $\alpha < \|f\|_\infty$ then the Borel set $\delta = \{N : |f(N)| > \alpha\}$ is not a null set so $E(\delta) \neq 0$, and if $x = E(\delta)(x)$ then

$$\|\varphi(f)x\|^2 = \int |f(N)|^2 \mathrm{d}\langle E(N)x, x\rangle = \int \mathscr{X}_\delta(N)\, |f(N)|^2\, \mathrm{d}\langle E(N)x, x\rangle \geqslant$$

$$\geqslant \alpha^2 \int \mathscr{X}_\delta(N)\, \mathrm{d}\langle E(N)x, x\rangle =$$

$$= \alpha^2 \int \mathrm{d}\langle E(N)x, x\rangle = \alpha^2\|x\|^2.$$

Clearly $\varphi(L^\infty(\mathscr{N}))$ contains $\mathscr{N}$ and by the above paragraph is isometrically isomorphic to $L^\infty(\mathscr{N})$. It remains to show that $\varphi(L^\infty(\mathscr{N}))$ is strongly closed. To see this let $\varphi(f_n) \to A \in \mathscr{C}_{\mathscr{N}}$ strongly. Then $\|\varphi(f_n)\| \leqslant M$ by the Principle of Uniform Boundedness and hence $\|f_n\|_\infty \leqslant M$. Now for $x_0 \in H$, $\{f_n\}$ converges in $L^2(\mathscr{N}, \mu_{x_0})$, and thus there is a Borel function f and a subsequence f_{n_k} such that $f_{n_k} \to f$ μ_{x_0}-a.e. If x_0 is a separating vector for $\mathscr{C}$ then $f_{n_k} \to f$ μ_x-a.e. for all $x \in H$. Since $\varphi(f_n) \to A$ strongly then $\int f_n(N)\mathrm{d}\langle E(N)x, x\rangle \to \langle Ax, x\rangle$ for all x and the Dominated Convergence. Theorem implies that $\int f(N)\mathrm{d}E(N) = A$. ▨

REMARK 3.4. If $A \in \mathcal{C}_{\mathcal{N}}$, then A is $\varphi(f)$ for an element in $L^\infty(\mathcal{N})$ and has a spectral representation $A = \int f(N)\mathrm{d}E(N)$ via the above theorem. We shall identify A and f in our arguments and whenever there is no confusion we shall denote by a single Borel function an element in $L^\infty(\mathcal{N})$.

For a function f in $L^\infty(\mathcal{N})$ and a fixed N_0 in $\mathcal{N}$ we define the set of *left essential values* of f at N_0 and denote them by $L(f, N_0)$. A scalar α is a left essential value of f at N_0 if for all $N_1 < N_0$ and $\varepsilon > 0$ the Borel set $\{N \in (N_1, N_0) \mid |f(N) - \alpha| < \varepsilon\}$ is not a null set in $\mathcal{N}$. Right essential values are defined in the analogous fashion. We say $L(f, N_0) = \emptyset$ if $N_{0-} \neq N_0$ and note that $L(f, N_0) \neq \emptyset$ otherwise.

LEMMA 3.5. *If $N_0 = N_{0-}$ then $L(f, N_0) \neq \emptyset$ and if $f = g$ a.e. then $L(f, N_0) = L(g, N_0)$. If $N_{0+} = N_0$ then $R(f, N_0) \neq \emptyset$ and if $f = g$ a.e. then $R(f, N_0) = R(g, N_0)$.*

Proof. If $L(f, N_0) = \emptyset$, then for all α in the closed ball B about the origin of radius $\|f\|_\infty$ there is an $\varepsilon_\alpha > 0$ and an $N_\alpha < N$ so that $\{N' \in (N_\alpha, N) \mid |f(N') - \alpha| < \varepsilon_\alpha\}$ is a null set. The sets centered at α with radius ε_α for all α in B form an open cover of B and by compactness there are a finite number centered at $\alpha_1, \ldots, \alpha_n$ which cover B. Let $N_1 = \max\{N_{\alpha_1}, \ldots, N_{\alpha_n}\}$ and since each $N_{\alpha_i} < N_0$ so is N_1. Moreover $\bigcup_{i=1}^{n} \{N \in (N_1, N_0) \mid |f(N) - \alpha_i| < \varepsilon_{\alpha_i}\}$ is a null set and is just (N_1, N_0) which is a contradiction. It is clear that if $f = g$ a.e. then $L(f, N_0) = L(g, N_0)$. ▨

LEMMA 3.6. *Let K be a compact subset of the plane such that $K \cap L(f, N_0) = \emptyset$. Then there exists $N_1 < N_0$ such that $\{N \in (N_1, N_0) \mid f(N) \in K\}$ is a null set (i.e., ess range of f on (N_1, N_0) does not intersect K).*

Proof. For each $\alpha \in K$ since $\alpha \notin L(N_0, f)$ there is a ball $B_{\varepsilon_\alpha}(\alpha)$ and an $N_\alpha < N_0$ so that $\{N \in (N_\alpha, N_0) \mid f(N) \in B_{\varepsilon_\alpha}(\alpha)\}$ is a null set. Since K is compact we can cover K by a finite number of such balls and let N_1 be the max of the corresponding N_α's. Clearly $\{N \in (N_1, N_0) \mid f(N) \in K\}$ is a null set. ▨

REMARK. This lemma can be reworded to the effect that if $\{N \in (N_1, N_0) \mid f(N) \in K\}$ is not null for all $N_1 < N_0$, then $L(f, N_0) \cap K \neq \emptyset$.

LEMMA 3.7. *Let $a \in L(f, N_0)$, $\varepsilon > 0$ and $N_1 < N_0$ be given. There is an interval (N_2, N_3) with $N_1 < N_2 < N_3 < N_0$ so that the essential range of $f(N)$ for $N \in (N_2, N_3)$ intersects $B_\varepsilon(a) = \{\alpha \mid |\alpha - a| < \varepsilon\}$.*

Proof. Since $L(f, N_0) \neq \emptyset$ we have $N_{0-} = N_0$. Let $\{N_k\} \in (N_1, N_0)$ and $N_k \to N_0$, then if $\{N \mid N \in (N_1, N_k), f(N) \in B_\varepsilon(a)\}$ is a null set for all k, then $\{N \mid N \in (N_1, N_0), f(N) \in B_\varepsilon(a)\}$ is a null set contradicting the fact that a is in $L(f, N_0)$. ▨

It is clear that $L(f, N_0)$ as well as $R(f, N_0)$ are compact subsets contained in the ball about the origin of radius $\|f\|_\infty$. Recall that the Hausdorff metric on closed subsests of a metric space X is defined by

$$d(F, G) = \max\{\sup_{x\in F} \operatorname{dist}(X, G), \sup_{y\in G} \operatorname{dist}(F, y)\}.$$

LEMMA 3.8. *If* $\|f - g\|_\infty < \varepsilon$ *for* $f, g \in L^\infty(\mathcal{N})$ *and* $N_0 \in \mathcal{N}$ *with* $N_{0-} = N_0$ *then* $d(L(f, N_0), L(g, N_0)) \leqslant 2\varepsilon$ *where* $d(\cdot, \cdot)$ *is the Hausdorff metric.*

Proof. $L(f, N_0)$ and $L(g, N_0)$ are nonempty compact subsets of the plane. Let $\alpha \in L(f, N_0)$ then for all $N_1 < N_0$ the set $\{N \in (N_1, N) \mid f(N) \in B_\varepsilon(\alpha)\}$ is not empty, thus $\{N \in (N_1, N_0) \mid g(N) \in B_{2\varepsilon}(a)\}$ is not null for all $N_1 < N$. By the remark following (3.6) $L(g, N_0) \cap \overline{B_{2\varepsilon}(\alpha)} \neq \emptyset$. Since the argument is symmetric in f and g we are done. ▨

PROPOSITION 3.9. *If* $f_n \to f$ *in* $L^\infty(\mathcal{N})$ *and* $N_0 \in \mathcal{N}$ *such that* $N_{0-} \neq N_0$, *then* $f_n(N_0) \to f(N_0)$. *If* $N_{0-} = N_0$, *then* $L(f_n, N_0) \to L(f, N_0)$ *in the Hausdorff metric. Similarly,* $R(f_n, N_0) \to R(f, N_0)$ *in the Hausdorff metric if* $N_{0+} = N_0$.

Proof. If $N_{0-} \neq N_0$ then $E[N_0, N_0] = N_0 - N_{0-}$ is not a null set so $f_n(N_0) \to f(N_0)$. The balance of the results follows from the preceding lemmas. In case $N_{0+} \neq N_0$ then $R(f_n, N_0) = R(f, N_0) = \emptyset$ however the behaviour at N_0 is determined solely by N_0 and N_{0-}. ▨

COROLLARY 3.10. *If* $f_n \to f$ *in* $L^\infty(\mathcal{N})$ *so that* $N_0 = N_{0-}$ *and* $L(f_n, N_0)$ *is a singleton for all n then* $L(f, N_0)$ *is also a singleton and* $L(f_n, N_0) \to L(f, N_0)$. *Similarly* $R(f_n, N_0) \to R(f, N_0)$ *if* $R(f_n, N_0)$ *are all one point sets.*

We say a function $f \in L^\infty(\mathcal{N})$ is *essentially continuous* if $L(f, N_0)$ and $R(f, N_0)$ are either empty or singleton sets for all $N_0 \in \mathcal{N}$. The set of essentially continuous functions in $L^\infty(\mathcal{N})$ as well as the corresponding algebra of operators in $\mathscr{C}$ will be denoted by $\mathscr{E}.\mathscr{C}.$. An operator A in $\mathscr{C}$ will be called *essentially continuous* with respect to $\mathcal{N}$ if $A = \int f(N)\, \mathrm{d}E(N)$ where $f \in L^\infty(\mathcal{N})$ is essentially continuous. From the spectral theory it follows that A is an essentially continuous operator with respect to a nest $\mathcal{N}$ precisely when the following continuity condition holds for A at all $N_0 \in \mathcal{N}$.

There exist scalars a and b depending on N_0 so that

$$\lim_{N \uparrow N_0} (A - aI)(N - N_0) = 0$$

and

$$\lim_{N \downarrow N_0} (A - bI)(N - N_0) = 0$$

where the limits are in norm.

A real valued function f on [0,1] is said to be *regulated* if $\lim_{t\to x_-} f(t)$ and $\lim_{t\to x_+} f(t)$ both exist for all x [6]. The essentially continuous functions in $L^\infty([0,1], \mu)$ where μ is a Lebesgue measure were used to describe the $\mathscr{R}$-center of a nest algebra with a continuous multiplicity one nest [19]. From the characterization given in [19] and [6] one can see that $f \in L^\infty([0,1], \mu)$ is essentially continuous if and only if there is a regulated function in the equivalence class of f. Similarly one can define regulated functions on a nest $\mathscr{N}$ and using the result below obtain an analogous result for $\mathscr{N}$. Thus it is also appropriate to call an essentially continuous operator A a *regulated* operator with respect to the nest $\mathscr{N}$.

We have defined a function f as *simple* if there is a partition of the identity $0 = P_0 < P_1 < \ldots < P_n = I$ in $\mathscr{N}$ and scalars $\alpha_0, \ldots, \alpha_{n-1}$ so that $f = \alpha_0 \chi_{P_0, P_1} + \alpha_1 \chi_{P_1, P_2} + \ldots + \alpha_{n-1} \chi_{P_{n-1}, I}$. The core operator A represented by f is just $\sum \alpha_i E_i$ where $E_0 = P_1$ and $E_i = P_i - P_{i-1}$ for $i = 2, \ldots, n$. Clearly any simple function is essentially continuous and we show they are dense in $\mathscr{E}.\mathscr{C}.$. The following theorem gives the representation of the C^*-algebra generated by the nest as the C^*-algebra $\mathscr{E}.\mathscr{C}.$.

THEOREM 3.11. *Let $\mathscr{N}$ be a nest in $\mathscr{L}(H)$ and $\mathscr{E}.\mathscr{C}.$ the essentially continuous operators with respect to $\mathscr{N}$. The C^*-algebra generated by $\mathscr{N}$ is the algebra $\mathscr{E}.\mathscr{C}.$.*

Proof. That $\mathscr{E}.\mathscr{C}.$ is closed in $L^\infty(\mathscr{N})$ follows from the preceding results. Let $f \in \mathscr{E}.\mathscr{C}.$ and $\varepsilon > 0$ be given. We shall find a simple function g so that $\|f - g\|_\infty < \varepsilon$. Let N_0 be in $\mathscr{N}$ with $N_{0-} = N_0$ then by Lemma 3.5 there is an $N_1 < N_0$ so that the essential range of f on (N_1, N_0) is in the ball $B_\varepsilon(L(f, N_0))$. That is $|f(N) - L(f, N_0)| < \varepsilon$ for almost all N in (N_1, N_0). Similarly if $N_{0+} = N_0$ then there is an N_2 so that $|f(N) - R(f, N_0)| < \varepsilon$ for almost all N in (N_0, N_2). Let Ω_1 be the set of all such intervals (N_1, N_0) and (N_1, N_2) over each $N_0 \in \mathscr{N}$. Let $\tilde{\Omega}_1$ be the union of the intervals in Ω_1.

While $\tilde{\Omega}_1$ need not equal $\mathscr{N}$ the difference $\mathscr{N} - \tilde{\Omega}_1$ must be finite. For if $\mathscr{N} - \tilde{\Omega}_1$ were infinite by compactness of $\mathscr{N}$ there would be a limit point N_0 of $\mathscr{N} - \tilde{\Omega}_1$ and a sequence M_i in $\mathscr{N} - \tilde{\Omega}_1$ converging from above or below to N_0. But in either case the corresponding interval (N_1, N_0) or (N_0, N_2) is in Ω and thus containing all but a finite number of the M_i.

Now if $\mathscr{N} - \tilde{\Omega}_1 = \{M_1, \ldots, M_k\}$ and $M_{i-} = M_i = M_{i+}$ then replace (M_{i1}, M_i) and (M_i, M_{i2}) with (M_{i1}, M_{i2}) and note that the singleton set $\{M_i\}$ is a null set. If $M_{i-} = M_i \neq M_{i+}$ replace (M_{i1}, M_i) with (M_{i1}, M_{i+}) and again note that the singleton set $\{M_i\}$ is a null set. If $M_{i-} \neq M_i = M_{i+}$ replace (M_i, M_{i2}) with (M_{i-}, M_{i+}) but now $\{M_i\} = (M_{i-}, M_i]$ is not a null set however, $(M_{i-}, M_{i+}) = (M_{i-}, M_i) \cup (M_i, M_{i1})$. Finally, if $M_{i-} \neq M_i \neq M_{i+}$ add to Ω the interval $(M_{i-}, M_{i+}) = \{M_i\}$. With these modifications Ω is now an open cover of $\mathscr{N}$ and as such has a finite subcover.

We next order the left end points of the resultant finite subcover of $\mathcal{N}$ and divide the set (M_{i-}, M_{i+}) in case it is in the cover and $M_{i-} \neq M_i = M_{i+}$ as noted above or in case $M_{i-} \neq M_i \neq M_{i+}$ as noted above. Finally, if (N_1, N_0) or (N_0, N_2) are in the cover then on (N_1, N_0) or (N_0, N_2) the essential range of f is within ε of $L(N_0, f)$ or respectively $R(N_0, f)$. In the standard fashion we can construct the projections $P_0 < P_1 < \ldots < P_n = I$ in $\mathcal{N}$ and scalars α_i so that f is within ε of $\alpha_0\chi_{[P_0, P_1]} + \alpha_1\chi_{(P_1, P_2]} + \ldots + \alpha_{n-1}\chi_{(P_{n-1}, I]}$ in $L^\infty(\mathcal{N})$. ▨

REMARK. It would be interesting to know which C^*-algebras can be represented as an algebra of essentially continuous operators with respect to a nest. Related questions arise if a nest is replaced by a commutative subspace lattice.

PROPOSITION 3.12. *Let B be a factor and assume that A is not in $\mathcal{E}.\mathcal{C}.$ with respect to a nest $\mathcal{N} \subset \mathcal{B}$. Then A is not in the $\mathcal{R}$-center of $\mathcal{A}_{\mathcal{N}} \cap \mathcal{B}$. That is ($\mathcal{R}$-center) $\cap$ $\cap$ $\mathcal{C}_{\mathcal{N}} \subset \mathcal{E}.\mathcal{C}.$* .

Proof. Let A correspond to the function $f \in L^\infty(\mathcal{N})$. There is an $N_0 \in \mathcal{N}$ so that either $N_{0-} = N_0$ and $L(f, N_0)$ is not a singleton set or $N_{0+} = N_0$ and $R(f, N_0)$ is not a singleton set. Assume the former is true let $a \neq b$ be in $L(f, N_0)$. By subtracting b and multiplying by $\dfrac{1}{a-b}$ we may without loss of generality assume that $a = 1$ and $b = 0$. Now by induction apply (3.7) first to 1 to get an interval (N_1, N_2) with $\{N \in (N_1, N_2) \mid f(N) \in B_{1/3}(1)\}$ is not a null set with $N_2 < N_0$ and then to 0 to get an interval (N_2, N_3) with $\{N \in (N_2, N_3) \mid f(N) \in B_{1/3}(0)\}$ is not a null set. Let these sets be F_1 and G_1 respectively. By induction we get an increasing sequence $N_i \to N_0$ so that there are non null Borel subsets F_i and G_i of adiacent intervals determined by the $\{N_i\}$ so that f on F_i is within 1/3 of 1 and on G_i within 1/3 of 0. These Borel subsets correspond to nontrivial core projections F_i and G_i with $F_1 \ll G_1 \ll F_2 \ll \ldots$ and $\|A|F_i - I|F_i\| \leqslant 1/3$ while $\|A|G_i - 0|G_i\| \leqslant 1/3$.

Let S^* be a partial isometry which is defined by induction so that $S^*F_1 \subset G_1$, $S^*(S^*F_1) \subset F_2$, $S^*(S^{*2}F_1) \subset G_2$ and so forth. That is S^* maps part of F_i into G_i and G_i into F_{i+1} in such a way that S^{*n} is a partial isometry. Moreover, since $\mathcal{C}_{\mathcal{N}}$ is contained in a factor von Neumann algebra $\mathcal{B}$ then S^* is obtained using comparison theorem so as to lie in $\mathcal{B}$. Finally, S is in $\mathcal{A}_{\mathcal{N}}$ since $F_1 \ll G_1 \ll F_2 \ll \ldots$ and hence in $\mathcal{A}_{\mathcal{N}} \cap \mathcal{B}$.

Now consider $SA - AS$ where A is our core operator represented by $f \in L^\infty$. On G_i we have $S: G_i \to F_i$ so A on SG_i is close to $1|SG_i$ while A on G_i is close to zero. Thus

$$\|(SA - AS)|G_i\| \geqslant 1/3$$

and in particular

$$\|(F_i + G_i)(SA - AS)(F_i + G_i)\| \geqslant 1/3.$$

However, S is zero off $F_i + G_i$ and A is invariant under $F_i + G_i$ so we have

$$\|(N_{2i+1} - N_{2i-1})(SA - AS)(N_{2i+1} - N_{2i-1})\| \geqslant 1/3$$

for $i = 1, 2, \ldots$. Using the Ringrose Criterion for inclusion in the radical we see that A is not in the $\mathscr{R}$-center of $\mathscr{A}_{\mathscr{N}} \cap \mathscr{B}$. ☐

Now it is just a formality to state the following theorem.

THEOREM 3.13. *If $\mathscr{B}$ is a factor, then the intersection of the $\mathscr{R}$-center of $\mathscr{A}_{\mathscr{N}} \cap \mathscr{B}$ with the core $\mathscr{C}_{\mathscr{N}}$ is the C^*-algebra generated by $\mathscr{N}$.*

Proof. Since the $\mathscr{R}$-center is a norm closed subalgebra of $\mathscr{A}_{\mathscr{N}}$ containing the simple operators (correspoding to simple functions), it contains the C^*-algebra they generated. By Proposition 3.12 this C^*-algebra contains the intersection of the $\mathscr{R}$-center with the core. ☐

Recall that a subalgebra $\mathscr{C}$ of a von Neumann algebra $\mathscr{B}$ is called *normal* if its relative double commutant is $\mathscr{C}$. When $\mathscr{A}$ is an nsva of a factor $\mathscr{B}$ and the core $\mathscr{C}$ is a normal subalgebra of $\mathscr{B}$, then Theorem 3.13 and Section 2 can be used to characterize the $\mathscr{R}$-center of $\mathscr{A}$.

PROPOSITION 3.14. *If $\mathscr{A}$ is an nsva of a factor $\mathscr{B}$ and $\mathscr{C}$ is a normal subalgebra of $\mathscr{B}$ then $\mathscr{R}$-center $(\mathscr{A})$ is the C^*-algebra generated by $\mathscr{N}$ plus the radical $\mathscr{R}$ of $\mathscr{A}$.*

Proof. Let $T \in \mathscr{R}$-center $\mathscr{A}$, then T is in $\mathscr{R}$-com $\mathscr{C}$ so $T = D + R$ where $D \in \mathscr{D} = \mathscr{C}' \cap \mathscr{B}$ and $R \in \mathscr{R}$. Now $D \in \mathscr{R}$-com $\mathscr{A}$ so if $\mathscr{A} \in \mathscr{D}$ then $DA - AD \in \mathscr{R}(\mathscr{A}) \cap \cap \mathscr{D} \subset \mathscr{R}(\mathscr{D})$ but $\mathscr{R}(\mathscr{D}) = \{0\}$ so $DA = AD$. Thus $D \in \mathscr{C}$ and Theorem 3.13 finishes the argument. ☐

REMARK. In particular if $\mathscr{N}$ is a nest in $\mathscr{L}(H)$, then the $\mathscr{R}$-center of $\mathscr{A}_{\mathscr{N}}$ is the C^*-algebra generated by $\mathscr{N}$ plus the radical $\mathscr{R}$ of $\mathscr{A}_{\mathscr{N}}$. Proposition 4.2 in the next section shows that if $D \in \mathscr{D}$ and $D \in \mathscr{R}$-center $\mathscr{A}$ then $D \in \mathscr{C}$. Thus the normality assumption in (3.14) is not needed.

4. $\mathscr{R}$-CENTER OF AN NSVA

In the preceeding section we described the $\mathscr{R}$-center of a nest algebra or more generally of an nsva of a factor $\mathscr{B}$ when $\mathscr{C}$ is a normal subalgebra of $\mathscr{B}$. In this section we shall present the reduction machinery necessary to remove the normality condition on $\mathscr{C}$ and subsequently lift our result to the general case. The reduction technique employed here utilizes the multiplicity theory for nest algebras as developed by J. A. Erdos in [7]. Specifically, we shall generalize Theorem 3.3 by representing the core $\mathscr{C}_{\mathscr{N}}$ as the diagonal algebra with respect to a direct integral decomposition and thus representing $\mathscr{D}_{\mathscr{N}}$ as the set of decomposable operators. This decomposition will prove useful in that many operators in $\mathscr{A}_{\mathscr{N}}$ which are not in $\mathscr{D}_{\mathscr{N}}$ can still be described in terms of the decomposition. An unfortunate semantics problem arises since the algebra $\mathscr{D}_{\mathscr{N}}$ is called the diagonal algebra with respect to the nest while $\mathscr{C}_{\mathscr{N}}$ is the diagonal algebra with respect to the decomposition we will use. Our usage will be clear from context.

The poof of the following theorem essentially consists of pulling together results and segments of proofs given by J. A. Erdos in [7] and using direct integral definitions.

THEOREM 4.1. *Let $\mathscr{N}$ be a complete nest in $\mathscr{L}(H)$. There exists a direct integral decomposition $H = \int_{\Lambda}^{\oplus} h(N)\mu(\mathrm{d}N)$ where $\Lambda = \mathscr{N}$ so that:*

i) $\mathscr{C}_{\mathscr{N}}$ consists of the diagonal algebra with respect to this decomposition and such that

ii) $N_0 \in \mathscr{N}$ is represented by $N_0 = \int^{\oplus} \chi_{[0,N_0]}(N)\mu(\mathrm{d}N)$ and

iii) the algebra $\mathscr{D}_{\mathscr{N}}$ is the algebra of decomposable operators with respect to this decomposition.

Proof. Given a nest $\mathscr{N}$ on H, Lemma 4.3 in [7] proves the existence of a family of measures $[\mu]$ and a multiplicity function $m(\,\cdot\,)$ defined on the Borel sets of $\mathscr{N}$ which are shown to be a set of unitary equivalences for the nest. Using Lemma 6.2 in [7] we let e_n be the Borel set of $\mathscr{N}$ so that the nest $\mathscr{N}$ restricted to e_n has uniform multiplicity n and as an immediate corollary of Lemma 6.2 in [7] the sets $\{e_n\}$ are a Borel partition of $\mathscr{N}$. Let $H_n = E(e_n)H$ where $E(e_n)$ is given by (3.3) and for each $N \in \mathscr{N}$ let $N_n = E([0,N] \cap e_n)$. Then $\mathscr{N}_n = \{N_n : N \in \mathscr{N}\}$ is a complete nest on H_n of uniform multiplicity n and using (3.3) one sees that $H = \sum^{\oplus} H_n$ and $N = \sum^{\oplus} N_n$ for each $N \in \mathscr{N}$.

The projection $N_n = NE(e_n)$ in $\mathscr{N}_n$ has a representation as multiplication by $\chi_{[0,N]} \otimes I_n$ on $L_2(\mathscr{N}_n, \mu_n) \otimes I_n$ where μ_n is a regular Borel measure on $\mathscr{N}$ with $\mu_n(\mathscr{N} \setminus e_n) = 0$ and $\mu_n(N_n, M_n) \neq 0$ if $N_n > M_n$ (cf. Theorem 5.2 and proof of Lemma 6.3 in [7]). That is H_n is unitarily equivalent to $L_2(\mathscr{N}_n, \mu_n) \otimes I_n$ via the map U_n and $U_n N U_n^*$ is multiplication by $\chi_{[0,N]} \otimes I_n$. As in (3.3) it follows that $\mathscr{C}_{\mathscr{N}_n} = \mathscr{C}_{\mathscr{N}} \mid E(e_n)$ is spatially isomorphic under the unitary transformation to multiplication by L^∞ functions tensor I_n on $L_2(\mathscr{N}_n, \mu_n) \otimes I_n$.

Letting $\mu = \sum_{1 \leqslant n \leqslant \infty} \mu_n$, then μ is a regular Borel measure on $\mathscr{N}$ so that $\mathscr{N}$ is the disjoint union of Borel sets e_n and $\mu_n(B) = \mu(B \cap e_n)$ for any Borel set B on $\mathscr{N}$. The direct integral of Hilbert spaces $H' = \int^{\oplus} H(N)\mu(\mathrm{d}N)$ where $h(H) = h_n$ for $N \in e_n$ is spatially isomorphic to $\sum^{\oplus} L_2(\mathscr{N}_n, \mu_n) \otimes I_n$ so that $M_{\chi_{[0,N_0]}} \otimes I_n$ on

$L_2(\mathcal{N}_n, \mu_n) \otimes I_n$ is unitarily equivalent to the diagonal operator $\int_{c_n}^{\oplus} \chi_{[0, N_0]}(N)\mu(dN)$ on H'. Composing these spatial isomorphisms we can conclude that the projection N_0 on H is equivalent to the diagonal operator $\int^{\oplus} \chi_{[0, N_0]}(N)\mu(dN)$ on H' (cf. 1.5.11 in [27]). Consequently the core $\mathscr{C}_{\mathcal{N}}$ spatially corresponds to the diagonal operators on H' and $\mathscr{D}_{\mathcal{N}}$ spatially corresponds to the decomposable operators on H'. ▨

REMARK. Any operator on $H = \int^{\oplus} h(N)\mu(dN)$ which is supported on $[N_1, N_2]$ and has range in $[N_3, N_4]$ where $N_4 < N_1$ is in $\mathscr{A}_{\mathcal{N}}$ and of course is in $\mathscr{R}_{\mathcal{N}}$. Heuristically, $A \in \mathscr{A}_{\mathcal{N}}$ if A is a diagonal or a "left translation", that is A can be viewed as mapping $h(N_0)$ into $\int_{[0, N_0]}^{\oplus} h(N)\mu(dN)$. This is the continuous analogue of the matrix representation for a nest algebra of a finite nest.

Let $\mathscr{A}$ be an nsva of a factor $\mathscr{B}$ and $T \in \mathscr{R}$-center $\mathscr{A}$, it follows from Section 2 that $T = D + R$ where $D \in \mathscr{D}$ and $R \in \mathscr{R}$ and moreover $D \in \mathscr{R}$-center $\mathscr{A}$. However, $\mathscr{D}$ is a von Neumann subalgebra of $\mathscr{A}$ and thus D is in $\mathscr{R}$-center $\mathscr{D}$ which is just the center of $\mathscr{D}$. The key result of this section shows that indeed $D \in \mathscr{C}$ so that (3.13) can be applied.

PROPOSITION 4.2. *If $D \in \mathscr{D} \cap \mathscr{D}'$ is in the $\mathscr{R}$-center $\mathscr{A}$ where $\mathscr{A}$ is an nsva of a factor $\mathscr{B}$, then $D \in \mathscr{C}$.*

Proof. Let $H = \int^{\oplus} h(N)\, d(\mu N)$ be the decomposition of H yielding the representation of $\mathscr{C}_{\mathcal{N}}$ and $\mathscr{D}_{\mathcal{N}}$ given in (4.1). Thus $D = \int^{\oplus} D(N)\, \mu(dN)$ and we may assume that $N \to D(N)$ is a Borel representative for D. Since D and consequently $D(N)$ (μ-a.e.) are normal operators, to show $D \in \mathscr{C}$ we need only show $\sigma(D(N))$ is a singleton μ-a.e..

It follows from [2] that the set $\delta \subset \mathcal{N}$ on which $\sigma(D(N))$ is not a singleton is a Borel set. If $\mu(\delta) = 0$ we are done and thus we assume that $\mu(\delta) > 0$. First we consider the case where $N_0 \in \delta$ and N_0 is a point mass for the measure μ, that is, $E(N_0) = E(\{N_0\}) \neq 0$. Since $D(N_0) = DE(N_0)$ thus $D(N_0) \in \mathscr{D}$ and applying the spectral theorem for $D(N_0)$ we can obtain a nontrivial decomposition of $D(N_0)$. That is $D(N_0) = D_1 + D_2 + D_3$ where D_1, D_2 and D_3 are $D(N_0)$ times respectively orthogonal nonzero spectral projections so that (D_3 may be zero) the distance between $\sigma(D_1)$ and $\sigma(D_2)$ is positive. Now the comparison theorem for factors implies that there is a partial isometry $S(N_0)$ in $\mathscr{B}$ with support in the support of D_1 and range in the support of D_2. Since N_0 is a point mass of μ it follows from (4.1) that $S(N_0)$

is also in $\mathscr{D}$ yet

$$DS(N_0) - S(N_0)D = D(N_0)S(N_0) - S(N_0)D(N_0) =$$

$$= D_2S(N_0) - S(N_0)D_1 \neq 0$$

since $\sigma(D_1) \cap \sigma(D_2) = \emptyset$. This contradicts the fact that $D \in \mathscr{D} \cap \mathscr{D}'$ and thus we may assume that δ contains no point masses.

We are assuming that $\mu(\delta) > 0$ so there is an $\varepsilon > 0$ so that $\delta_\varepsilon = \{N \mid \text{diameter } \sigma(D(N)) \geqslant 5\varepsilon\}$ has positive measure [2]. Now we may partition δ_ε into a countable number of disjoint sets δ_i each of positive measure so that $\delta_i \subset (N_{i+1}, N_i)$ for a decreasing sequence of points $N_1 > N_2 > N_3 > \ldots$ in $\mathscr{N}$. For each i there exist complex numbers α_i and β_i for which $|\alpha_i - \beta_i| \geqslant 4\varepsilon$ and if $F(\cdot)$ is the spectral measure for D, then both $F(B_\varepsilon(\alpha_i)) \cap \sigma(D \mid E(\delta_i)H) \neq \emptyset$ and $F(B_\varepsilon(\beta_i)) \cap \cup\, \sigma(D \mid E(\delta_i)H) = \emptyset$. Now as in the previous sections we are in a position to construct an operator $A \in \mathscr{A}$ so that $\delta_D(A) \notin \mathscr{R}$.

Also as in previous arguments the operator A is defined by a straightforward induction argument which we only indicate. Given α_i either $|\alpha_i - \alpha_{i+1}|$ or $|\alpha_i - \beta_{i+1}| > 2\varepsilon$ and for β_i the similar inequality holds. Thus $B_\varepsilon(\alpha_i)$ and $B_\varepsilon(\beta_i)$ each respectively do not intersect one of the balls $B_\varepsilon(\beta_{i+1})$, $B_\varepsilon(\alpha_{i+1})$. Choose by induction a sequence $\gamma_1, \gamma_2, \ldots,$ where $\gamma_i = \alpha_i \cap \beta_i$ and $B_\varepsilon(\gamma_i) \cap B_\varepsilon(\gamma_{i+1}) = \emptyset$. Now consider the operators $D_i = \int_{\gamma_i}^{\oplus} D(N)F(B_\varepsilon(\gamma_i))(N)\mu(\mathrm{d}N)$ which are just $DF(B_\varepsilon(\gamma_i))$ and are in $\mathscr{D}$. Using the comparison theorem for factors and induction in the manner we have done in previous arguments we construct partial isometries S_i in $\mathscr{B}$ whose initial domain are in $F(B_\varepsilon(\gamma_i))$ and final domains in $F(B_\varepsilon(\gamma_{i+1}))$. These isometries have the property that $\|\delta_D(S_i)\| \geqslant \varepsilon$ and moreover each S_i is also in $\mathscr{A}_{\mathscr{N}}$ (remark following (4.1)). Let $S = \sum_{i=1}^{\infty} S_{2i}$ we may conclude using the Ringrose Criterion ((5.1) and (5.2) in [10]) that $\delta_D(S) \notin \mathscr{R}$. This contradicts our assumption so therefore $\sigma(D(N))$ must be a singleton for μ-almost all N and thus $D \in \mathscr{C}$ by (4.1). ▨

Before proving the main theorem we need two technical lemmas concerning the central reduction of an nsva $\mathscr{A}$ in an arbitrary von Neumann algebra $\mathscr{B}$. Let $\mathscr{A} = \int^{\oplus} \mathscr{A}(\lambda)\mu(\mathrm{d}\lambda)$ be the central decomposition of $\mathscr{A}$ on $H = \int_{\Lambda}^{\oplus} h(\lambda)\mu(\mathrm{d}\lambda)$ (§4 in [10]).

LEMMA 4.3. *Let* $T = \int^{\oplus} T(\lambda)\mu(\mathrm{d}\lambda)$ *be in the* $\mathscr{R}$*-center of an nsva* $\mathscr{A}$. *Then* $T(\lambda) \in \mathscr{A}$-center $\mathscr{A}(\lambda)$ μ-a.e..

Proof. Since the map $\lambda \to p_\varepsilon(A(\lambda))$ is measurable whenever $A \in \mathscr{A}$, (2.8), the map $(\lambda, A) \to p_\varepsilon(\delta_{T(\lambda)}(A))$ is a measurable map on $\Lambda \times \mathscr{C}(h)$. Let

$$\sigma_n := \{(\lambda, A) \in \Lambda \times \mathscr{C}(h) : P_\varepsilon(\delta_{T(\lambda)}(A)) \geqslant n\} \cap \mathrm{Gr}(\mathscr{A}).$$

For ε fixed let Λ_n be the projection on Λ of σ_n and using measurable selection there is an $A \in \mathscr{A}$ so that $(\lambda, A(\lambda)) \in \sigma_n$ if $\lambda \in \Lambda_n$ and $A(\lambda) = 0$ if $\lambda \notin \Lambda_n$. However, $p_\varepsilon(\delta_T(A)) = \|p_\varepsilon(\delta_{T(\lambda)}(A(\lambda))\|_\infty$ and if the decreasing sets Λ_n have positive measure for all n then T cannot be in $\mathscr{R}$-center $\mathscr{A}$. Consequently for $\varepsilon = 1/k$ there exists an n_k so that Λ_{n_k} is a set of full measure. Letting $\Lambda_0 = \cup \Lambda_{n_k}$ then Λ_0 is a set of full measure and for $\lambda \in \Lambda_0$ and $A \in \mathscr{A}(\lambda) \cap \mathscr{C}(h)$, $T(\lambda)A - AT(\lambda)$ has a finite ε-paving number for all $\varepsilon > 0$. Thus $T(\lambda) \in \mathscr{R}$-center $\mathscr{A}(\lambda)$ μ-a.e.. ▨

For the following lemma we recall that an operator C is an *$\mathscr{N}$-simple operator* if $C = \sum_1^n \alpha_i E_i$ where $\{E_i\}$ is a finite $\mathscr{N}$-paving. By (3.11) and (3.13) these operators are dense in the C^*-algebra generated by $\mathscr{N}$. A $\mathscr{L}$-simple operator is similarly defined for any commutative subspace lattice $\mathscr{L}$.

LEMMA 4.4. *Let C be a core operator in the $\mathscr{R}$-center for an nsva $\mathscr{A}$ of a factor $\mathscr{B}$. Suppose C cannot be approximated within ε by an $\mathscr{N}$-simple operator with $2n$ intervals, then there is an $A \in \mathscr{A}$ so that $\|\delta_C(A)^n\| \geqslant \varepsilon^n$.*

Proof. We first decompose C according to (4.1) then there is an $L^\infty(\mathscr{N})$ function $g(N)$ so that $C = \int^\oplus g(N)I(N)\mu(\mathrm{d}N)$. Let $k > 2n$ be the smallest integer so that C can be approximated by an $\mathscr{N}$-simple operator $\sum_{i=1}^k \alpha_i E_i$. Such a k exists by Theorems 3.11 and 3.13. Combining pairs of E_i much in the same fashion as in the proof of (2.2) we see that the essential diameter of the range of $g(N)$ on $E_{2i-1} \cup E_{2i}$ for each $i = 1, 2, \ldots, n$ is greater than ε as is the diameter of the essential range of $g(N)$ on $E_{2i} \cup E_{2i+1}$ for each $i < n$. ▨

By induction and the comparison theorem for factors we can find subintervals F_j, $j = 1, \ldots, n$ of at least every other E_i and partial isometries S_j from F_j to F_{j-1} with the following properties. The product $S_1 S_2 \ldots S_n$ is also a nonzero partial isometry and the Euclidean distance between $\sigma(C \mid F_j H)$ and $\sigma(C \mid F_{j-1}H)$ is at least ε. Thus $\|CS_j - S_jC\| > \varepsilon$ while each S_j as well as $S = S_1 + \ldots + S_n$ is in $\mathscr{A}$. By our construction we now have $\|\delta_C(S)^n\| \geqslant \varepsilon^n$.

LEMMA 4.5. *Let C be a core operator for an nsva $\mathscr{A}$ of an von Neumann algebra $\mathscr{B}$. Suppose C cannot be approximated within ε by an $\mathscr{L}_0$-simple operator with $2n_0$ steps; then there is a contraction $A \in \mathscr{A}$ so that $\|\delta_C(A)^{n_0}\| \geqslant \varepsilon_0^{n_0}$.*

Proof. Let $\mathscr{A} = \int^{\oplus} \mathscr{A}(\lambda)\mu(\mathrm{d}\lambda)$ be the central decomposition of $\mathscr{A}$ and $C = \int^{\oplus} C(\lambda)\mu(\mathrm{d}\lambda)$ be a Borel representative for C. The hypothesis implies that on a set Λ_0 of positive measure $C(\lambda)$ cannot be approximated by an $\mathscr{N}(\lambda)$-simple operator corresponding to a paving of length $2n_0$. By (4.4) for each $\lambda \in \Lambda_0$ there is a contraction operator $S \in \mathscr{A}(\lambda)$ so that $\|\delta_{C(\lambda)}(S)^{n_0}\| \geqslant \varepsilon^{n_0}$. Since $(\lambda, A) \to \delta_{C(\lambda)}(A) \to \|(\delta_{C(\lambda)}(A))^{n_0}\|$ is a measurable mapping on $\Lambda \times \mathscr{C}(h)$ we can use measurable selection to find a contraction $A = \int^{\oplus} A(\lambda)\mu(\mathrm{d}\lambda)$ in $\mathscr{A}$ with $\|(\delta_{C(\lambda)}(A(\lambda)))^{n_0}\| \geqslant \varepsilon^{n_0}$ for $\lambda \in \Lambda_0$ and consequently $\|\delta_C(A)^{n_0}\| \geqslant \varepsilon^{n_0}$. ▨

Now we come to the main theorem of this section in which we "lift" the characterization of $\mathscr{R}$-center $\mathscr{A}$ from the factor to the general nsva case.

THEOREM 4.6. *Let $\mathscr{A}$ be an nsva of an von Neumann algebra $\mathscr{B}$ with respect to a nest $\mathscr{N}$ in $\mathscr{B}$. Then $\mathscr{R}$-center $\mathscr{A}$ is the sum of the radical of $\mathscr{A}$ and the C^*-algebra generated by $\mathscr{L}_0 = \mathscr{N} \vee \mathscr{M}_0$, where $\mathscr{M}_0$ is the set of central projections of $\mathscr{B}$.*

Proof. Let $H = \int^{\oplus} h(\lambda)\mu(\mathrm{d}\lambda)$ be the decomposition of H so that $\mathscr{B} = \int^{\oplus} \mathscr{B}(\lambda)\mu(\mathrm{d}\lambda)$ is the central decomposition of $\mathscr{B}$ and $\mathscr{A} = \int^{\oplus} \mathscr{A}(\lambda)\mu(\mathrm{d}\lambda)$ and $\mathscr{N} \sim \int^{\oplus} \mathscr{N}(\lambda)\mu(\mathrm{d}\lambda)$ the corresponding decompositions of $\mathscr{A}$ and $\mathscr{N}$. If $T \in \mathscr{R}$-center $\mathscr{A}$ then T is in the $\mathscr{R}$-com $\mathscr{C}$, where $\mathscr{C}$ is the core of $\mathscr{A}$. By Theorem 2.10, $T = D + R$ where $D \in \mathscr{D}$ and $R \in \mathscr{R}$ and $D \in \mathscr{R}$-center $\mathscr{A}$. By Lemma 4.3 $D(\lambda) \in \mathscr{R}$-center $\mathscr{A}(\lambda)$ μ-a.e. so that $D(\lambda) \in \mathscr{C}(\lambda)$ and in fact $D(\lambda)$ is in the C^*-algebra generated by $\mathscr{N}(\lambda)$.

Let $\varepsilon > 0$ be fixed and consider the subset of $\Lambda \times \mathscr{C}(h)^n \times \mathbf{C}^n$:

$$\sigma_n = \{(\lambda, E_1, \ldots, E_n, \alpha_1, \ldots, \alpha_n) : \|D(\lambda) - \sum_{i=1}^{n} \alpha_i E_i\| \leqslant \varepsilon\} \cap \mathrm{Gr}(\mathscr{I}^n) \times \mathbf{C}^n$$

where $\mathrm{Gr}(\mathscr{I}^n)$ is the graph of the n-fold product of the map $\lambda \to \mathscr{I}(\lambda)$ (cf. Lemma 5.5 in [10]). The set $\mathrm{Gr}(\mathscr{I}^n)$ can be taken to be a Borel subset of $\Lambda \times \mathscr{C}(h)^n$ and since $\lambda \to D(\lambda)$ and $\|\cdot\|$ are Borel maps we conclude that σ_n is a Borel subset of $\Lambda \times \mathscr{C}(h)^n \times \mathbf{C}^n$. For each n let Λ_n be the projection of σ_n into Λ. Clearly $\Lambda_n \subseteq \Lambda_{n+1}$ and $\bigcup_{n=1}^{\infty} \Lambda_n = \Lambda$ by (3.11). Next we will show that $\Lambda - \bigcup_{n=1}^{k} \Lambda_n$ is a null set for some k.

If $\Lambda - \bigcup_{n=1}^{k} \Lambda_n$ is not a null set for all k then as expected we will construct an operator $A \in \mathscr{A}$ so that $DA - AD \notin \mathscr{R}$. Thus there exists a sequence n_i and non null sets $\tilde{\Lambda}_{n_i} = \Lambda_{n_i+1} \setminus \Lambda_{n_i}$ so that for $\lambda \in \tilde{\Lambda}_{n_i}$, $D(\lambda)$ cannot be approximated by an $\mathscr{N}(\lambda)$-simple operator corresponding to a paving of length n_i. By Lemma 4.5 there is a contraction $A_{n_i} \in \mathscr{A}$ supported on $\int^{\oplus} \chi_{\tilde{\Lambda}_{n_i}}(\lambda) I(\lambda)\mu(\mathrm{d}\lambda)$ so that $p_\varepsilon(\delta_D(A_i)) \geqslant n_i/3$. Clearly if $A = \sum A_{n_i}$ then $A \in \mathscr{A}$ and $p_\varepsilon(\delta_D(A)) = \infty$ which contradicts the fact that D is in $\mathscr{R}$-center $\mathscr{A}$. Thus there is a k so that $\Lambda - \bigcup_{n=1}^{k} \Lambda_n$ is a μ-null set.

Now using measurable selection we can find an operator which using intervals from $\mathscr{L}_0$ is of the form $\sum_1^k \int^{\oplus} \alpha_i(\lambda) E_i(\lambda)\mu(\mathrm{d}\lambda)$ where $\alpha_i(\lambda)$ are Borel functions essentially bounded by a constant K. Now each map $\alpha_i(\lambda)$ can be approximated within ε by $\sum_{j=1}^{r} C_{ji}\, \chi_{\Lambda_{ji}}(\lambda)$ where r depends on K and ε and $\{\Lambda_{ji}\}_{j=1}^{r}$ are mutually disjoint Borel sets whose union is Λ. Thus D can be approximated to within 2ε in norm by the $\mathscr{L}_0$-simple operator $\sum_{i=1}^{k} \sum_{j=1}^{r} \int^{\oplus} C_{ji}\chi_{\Lambda_{ji}}(\lambda) E_i(\lambda)\mu(\mathrm{d}\lambda)$. ▨

5. SPECTRAL OPERATORS

We noted in Section 1 the spectral relationship between an element R of the radical $\mathscr{R}$ of an algebra $\mathscr{A}$ and a member $T \in \mathscr{A}$. This relationship bears a resemblance to that of a spectral operator and its quasinilpotent part. Let $T = S + Q$ be the representation given by N. Dunford for a spectral operator T. That is, S is a scalar type operator, Q is quasinilpotent and $SQ = QS$. Furthermore, there is a normal operator N and an invertible operator C so that $S = C^{-1}NC$. It follows then that $\sigma(T) = \sigma(S + Q) = \sigma(S) = \sigma(N)$ and also the same is true for any polynomial p, i.e., $\sigma(p(T)) = \sigma(p(S))$. The same spectral type relationship would hold relative to the algebra $\mathscr{A}$ generated by T and S if Q was a member of the radical of $\mathscr{A}$ (S and Q need not commute).

As an application of the characterization of the $\mathscr{R}$-center of a nest algebra we can show that any spectral operator $T = S + Q$ has a second representation as $N_0 + R$ where N_0 is a normal operator and R is (a quasinilpotent) in the radical of the algebra generated by T and N_0. We lose the commutivity that S and Q enjoyed but N_0 is always a normal operator while S is a scalar type operator. Finally a recent result of the second author can be used to show that the spectral multiplicity of N_0 and N where $S = C^{-1}NC$ need not be the same.

THEOREM 5.1. *Let T be a spectral operator. Then $T = N_0 + R$ where N_0 is a normal operator and R is in the radical of the weakly closed algebra generated by T and N_0.*

Proof. Let $T = S + Q$ be the cannonical decomposition of T into scalar and quasinilpotent parts and let N be the normal operator similar to the scalar part, $S = C^{-1}NC$. A theorem of P. R. Halmos [13] states that $N = f(A_0)$ for a selfadjoint operator A_0 and a continuous function f. We let $\mathcal{N}$ be the nest consisting of $E(\delta)$ where $E(\cdot)$ is the spectral measure of A_0 and δ is of the form $(-\infty, a)$ or $(-\infty, a]$. Let $\mathcal{A}$ be the nest algebra Alg $\mathcal{N}$. Since $N = f(A_0)$ is in the C^*-algebra generated by $\mathcal{N}$ we have $N \in \mathcal{R}$-center $\mathcal{A}$.

We let $\hat{\mathcal{N}}$ be the nest $\{CN : N \in \mathcal{N}\}$ where CN is the orthogonal projection, on the subspace CN if N is considered as a subspace. The nest algebra $\mathcal{A}_{\hat{\mathcal{N}}} = \{C^{-1}AC$ where $A \in \mathcal{A}_{\mathcal{N}}\}$. Hence $C^{-1}NC = S$ is in the $\mathcal{R}$-center of $\mathcal{A}_{\hat{\mathcal{N}}}$ as the radical of $\mathcal{A}_{\hat{\mathcal{N}}}$ is just $\{C^{-1}BC : B \in \operatorname{rad} A_{\mathcal{N}}\}$. Hence $S = N_0 + Q_1$ where N_0 is a normal operator and $Q_1 \in \operatorname{rad} \mathcal{A}_{\hat{\mathcal{N}}}$. Now $T = N_0 + Q_1 + Q$ and next we show that $Q \in \mathcal{A}_{\hat{\mathcal{N}}}$. Since $QS = SQ$ it follows that $CQC^{-1}CSC^{-1} = CSC^{-1}CQC^{-1}$ so $CQC^{-1}N = NCQC^{-1}$. Thus $CQC^{-1} \in \mathcal{C}'_{\mathcal{N}} = \mathcal{D}_{\mathcal{N}}$ so $CQC^{-1} \in \mathcal{A}_{\mathcal{N}}$ and $Q \in \mathcal{A}_{\hat{\mathcal{N}}}$. Let $\mathcal{A}$ be the algebra generated by $\{N_0, Q, Q_1, I\}$. Clearly $\mathcal{A} \subset \mathcal{A}_{\hat{\mathcal{N}}}$ and we want to show $Q + Q_1 = R \in \operatorname{rad} \mathcal{A}$. Let π be the cannonical map of $\mathcal{A} \to \mathcal{A}/\operatorname{rad} \mathcal{A}$. Since $Q_1 \in \operatorname{rad} \mathcal{A}_{\hat{\mathcal{N}}}$ we have $\pi(Q_1) = 0$. Thus $\pi(\mathcal{A})$ is generated by $\pi(N_0)$, $\pi(Q_1)$ and $\pi(I)$. However, $N_0 + Q_1$ and Q commute so $\pi(N_0)$ and $\pi(Q)$ commute. Moreover $\pi(Q)$ is quasinilpotent and hence in $\operatorname{rad}(\mathcal{A}/\operatorname{rad}\mathcal{A}) = O$. Thus $\pi(Q + Q_1) = 0$ or $Q + Q_1 \in \operatorname{rad} \mathcal{A}$. The algebra $\mathcal{A}_0$ generated by T, N_0 and I is contained in $\mathcal{A}$. Since $Q + Q_1 \in \mathcal{A}_0 \cap \operatorname{rad} \mathcal{A}$ we have $Q + Q_1 \in \operatorname{rad} \mathcal{A}_0$. ▨

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