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BOUNDS ON THE NUMBER OF BOUND STATES FOR THE SCHRÖDINGER EQUATION IN ONE AND TWO DIMENSIONS

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1. INTRODUCTION

It is well known that the Birman-Schwinger method [2, 6, 8] for estimating the number of bound states of the Schrödinger equation cannot be directly applied in $\mathbf{R}$ and $\mathbf{R}^2$. The reason is that in these cases the Green's function of the Lippmann-Schwinger equation possesses no finite limit as $E \to 0$. In $\mathbf{R}$ it diverges as $|E|^{-1/2}$, and in $\mathbf{R}^2$ as $\ln|E|$. As a consequence no bound on the number of bound states is explicitly known in $\mathbf{R}^2$. In this paper we prove such bounds by a suitable modification of the Birman-Schwinger method, both for local and nonlocal potentials.

The necessary modification was, in fact, introduced by this author [5] in 1962 in a context in which its relevance to $\mathbf{R}$ and $\mathbf{R}^2$ was not recognized. The bound there derived for the number of Regge trajectories (for local central potentials in $\mathbf{R}^3$) that lead to $l = -1/2$ as $E \to 0$, was

$$n_0 \leqslant 1 + \frac{\int_0^\infty \mathrm{d}r \int_0^r \mathrm{d}r' rr' U(r) U(r') \ln(r/r')}{\int_0^\infty \mathrm{d}r \, rU(r)}, \tag{1}$$

where

$$U(x) = \sup[0, -V(x)], \quad x \in \mathbf{R}_+ . \tag{2}$$

This bound is also an upper limit for the number of rotationally invariant bound states for a local central potential in $\mathbf{R}^2$.

The method of Reference 5 is applicable whenever the kernel K of the modified Lippmann-Schwinger equation for $-U$, i.e., $K = -U^{1/2}\mathcal{G}U^{1/2}$, $\mathcal{G} = (E - H_0)^{-1}$, near $E = 0$- is of the form

$$K = \xi P + K', \tag{3}$$

where K' is self-adjoint and in the trace-class and has a finite norm-limit as $E \to 0$-, P is an orthogonal projection on a one-dimensional subspace (spanned by a unit vector φ), and ξ increases without bounds as $E \to 0$-.

Let $\alpha_n(E)$ be the eigenvalues of $K(E)$. Then the crux of the Birman-Schwinger method is the recognition that the number $n(E)$ of bound states of energies not greater than E is equal to the number of eigenvalues $\alpha_n(E)$ of $K(E)$ that are not less than 1. Therefore

$$\operatorname{tr} K(E) = \sum_n \alpha_n(E) \geqslant n(E).$$

However, since (3) implies that as $E \to 0$- the leading eigenvalue $\alpha_1 \to \infty$, this inequality is useless. Therefore it was replaced in Reference 5 by the inequality

$$\operatorname{tr} K(E) = \sum_n \alpha_n(E) \geqslant n(E) - 1 + \alpha_1(E),$$

or

$$n(E) \leqslant 1 + \operatorname{tr} K(E) - \alpha_1(E).$$

For large ξ writing $K = \xi(P + \xi^{-1}K')$ one easily calculates $\alpha_1(E)$ by perturbation theory:

$$\alpha_1(E) = \xi + (\varphi, K'\varphi) + o(1)$$

as $E \to 0$-. Since $\operatorname{tr} K = \xi + \operatorname{tr} K'$, we have in the limit as $E \to 0$-,

$$n \leqslant 1 + \operatorname{tr} K_0' - (\varphi, K_0'\varphi), \tag{4}$$

where K_0' is the limit of K' as $E \to 0$-. Note from the derivation of (4) that its right-hand side is never less than one.

2. LOCAL POTENTIALS ON R

The Green's function in one dimension is, for $E < 0$,

$$\mathcal{G}(E, x, y) = -\frac{1}{2}|k|^{-1}\exp(-|k|\,|x - y|), \quad |k| = |E|^{1/2}.$$

Therefore in this case

$$\xi = \frac{1}{2}|k|^{-1}\int_{-\infty}^{\infty} dx\, U(x)$$

$$\varphi(x) = U^{1/2}(x) \Big/ \left[\int_{-\infty}^{\infty} dy\, U(y)\right]^{1/2},$$

$$K_0'(x, y) = -\frac{1}{2} U^{1/2}(x) U^{1/2}(y) |x - y|.$$

Consequently by (4)

$$n^{(1)} \leqslant 1 + \frac{\frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, U(x)U(y)|x-y|}{\int_{-\infty}^{\infty} dz\, U(z)}. \tag{5}$$

This bound is similar to that obtained in [4].

3. LOCAL CENTRAL POTENTIALS ON $\mathbf{R}^2$

Separation of the Schrödinger equation leads to the radial equation

$$-\psi'' + \frac{\lambda^2 - 1/4}{r^2}\psi + V\psi = k^2\psi \tag{6}$$

where $\lambda = 0,1,2, \ldots$. This equation is, of course, identical with the radial equation in $\mathbf{R}^3$ if we replace λ by $l+1/2$. The difficulty at $E = 0$ arises only for $\lambda = 0$, in which case the Green's function is

$$\mathscr{G}_0(k; r, r') = -1/2\, i\pi (rr')^{1/2} H_0^{(1)}(kr_>) J_0(kr_<),$$

where $r_<$ and $r_>$ are the lesser and the greater of r and r', respectively, and $H_0^{(1)}$ and J_0 are Hankel and Bessel functions, respectively. Therefore ([3], pp. 4—8),

$$\xi = \left|\ln\left(\frac{1}{2}|k|e^{\gamma}\right)\right| \int_0^{\infty} dr\, rU(r),$$

$$\varphi(r) = r^{1/2}U^{1/2}(r) \Big/ \left[\int_0^{\infty} dr'\, r' U(r')\right]^{1/2},$$

$$K_0'(r, r') = (rr')^{1/2} \ln r_>\, U^{1/2}(r) U^{1/2}(r'),$$

where γ is Euler's constant. As a result we obtain (1). (In Reference 5 this result was derived as a limit $\lambda \to 0+$ at $E = 0$, but the limits turn out to be interchangeable.)

For $\lambda = 1, 2, \ldots$ one obtains the Bargmann bound [1,5],

$$n_\lambda \leqslant \frac{1}{2\lambda} \int_0^\infty dr\, r U(r). \tag{7}$$

If Λ is the largest integer less than or equal to $\frac{1}{2} \int_0^\infty dr\, r U(r)$, then (1) and (5) imply that the total number n^2 of bound states is limited by

$$n^{(2)} \leqslant n_0 + 2\Lambda\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\Lambda}\right). \tag{8}$$

4. LOCAL NONCENTRAL POTENTIAL ON $\mathbf{R}^2$

The results for central potentials may be applied if we define $U(r)$ by ($\hat{x} = x/|x|$)

$$U(|x|) = \sup_{\hat{x}}(0, -V(|x|\hat{x})). \tag{9}$$

However, one may also proceed directly.

The Green's function for $E < 0$ is

$$\mathscr{G}(E; x, y) = -\frac{1}{2} i\pi H_0^{(1)}(i|k|\,|x - y|).$$

Therefore, again defining $U(x)$ as in (2) but with $x \in \mathbf{R}^2$,

$$\xi = \int d^2x\, U(x) \left| \ln\left(\frac{1}{2} |k| e^\gamma\right)\right|,$$

$$\varphi(x) = U^{1/2}(x) \Big/ \left[\int d^2y\, U(y)\right]^{1/2},$$

$$K_0'(x, y) = U^{1/2}(x) U^{1/2}(y) \ln|x - y|.$$

In this case K_0' is not in the trace-class and (4) has to be modified by applying the same argument to K^2:

$$\operatorname{tr} K^2 = \sum_n \alpha_n^2 \geqslant n - 1 + \alpha_1^2,$$

or

$$n \leqslant 1 + \operatorname{tr} K^2 - \alpha_1^2.$$

But by (3)

$$\operatorname{tr} K^2 = \xi^2 + \operatorname{tr} K'^2 + 2\xi(\varphi, K'\varphi) =$$

$$= \xi^2 + 2\xi(\varphi, K_0'\varphi) + 2(\varphi, K_0''\varphi) + \operatorname{tr} K_0'^2 + o(1)$$

if

$$K' = K_0' + \xi^{-1}K_0'' + o(\xi^{-1}).$$

Second-order perturbation theory, on the other hand, leads to

$$\alpha_1 = \xi + (\varphi, K_0' \varphi) + \xi^{-1}[(\varphi, K_0''\varphi) + (\varphi, K_0'^2\varphi) - (\varphi, K_2'\varphi)^2] + o(\xi^{-1}).$$

As a result we get, as $\xi \to \infty$,

$$n \leqslant 1 + \operatorname{tr} K_0'^2 - 2(\varphi, K_0'^2\varphi) + (\varphi, K_0'\varphi)^2,$$

which leads to the inequality

$$(10) \quad n^{(2)} \leqslant 1 + \frac{\int d^2u \int d^2x \int d^2y \int d^2z\, U(u)U(x)U(y)U(z) \ln|x-y| \ln \left|\frac{x-y}{x-z}\right| \left|\frac{z-u}{y-u}\right|}{\left[\int d^2v\, U(v)\right]^2}.$$

Note that if the potential is multiplied by a strength parameter β then for small values of β the bound (8) (used by means of (9)) is linear in β, while (10) is quadratic. Thus (10) can be expected to be more restrictive (though not necessarily always so, particularly since n has to be an integer). On the other hand, for large β, (8) is of order $\beta \ln \beta$, while (10) is still quadratic. Hence for strong potentials (8) can be expected to be more restrictive. In fact, (8) is then only logarithmically weaker than the semi-classical limit [8], which is $O(\beta)$.

5. REMARK ON $\mathbf{R}^p$

For local central potentials in $\mathbf{R}^p$ the radial equation is (6), where $\psi = r^{(p-1)/2} R(r)$, $R(r)$ being the radial factor of the solution of the Schrödinger equation in $\mathbf{R}^p$, and $\lambda = m + \frac{1}{2} p - 1$, $m = 0, 1, 2, \ldots$ ([3], p. 235). The number of linearly independent spherical harmonics for a given value of m is ([3], p. 237)

$$h_m^{(p)} = 2\lambda \frac{(m+p-3)!}{(p-2)!\, m!} \equiv 2\lambda \overline{h}_m^{(p)}.$$

Therefore for $p \geqslant 3$ use of (9) and (7) leads to the bound

$$n^{(p)} \leqslant 2(h_0^{(p)} + \bar{h}_1^{(p)} + \ldots + \bar{h}_M^{(p)})A$$

where $M = A + 1 - \frac{1}{2}p$, in the notation used in (8). For large β this bound is of order β^p, as compared with the semi-classical limit [8], which is $O(\beta^{p/2})$. (See also [7] for $p = 3$.)

6. NONLOCAL POTENTIALS

It is a simple matter to apply the Birman-Schwinger method and its modification also to nonlocal potentials. Let U be a positive operator such that $V + U \geqslant 0$ in the operator sense. Then K may be taken, again, as $K = -U^{1/2}\mathcal{G}U^{1/2}$, and if the right-hand side exists, the number of bound states is limited by

$$n \leqslant \operatorname{tr} U^{1/2}H_0^{-1}U^{1/2} = \operatorname{tr} UH_0^{-1}.$$

In $\mathbf{R}^3$ this yields the bound

$$n^{(3)} \leqslant \frac{1}{4\pi}\int d^3x\, d^3y\, \frac{U(x, y)}{|x - y|}\,, \tag{11}$$

where $U(x, y)$ is the integral kernel of U. In $\mathbf{R}$ and $\mathbf{R}^2$ we use (4).

In $\mathbf{R}$,

$$\xi = \frac{1}{2}|k|^{-1}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, U(x, y),$$

$$\varphi(x) = \int_{-\infty}^{\infty} dy\, U^{1/2}(x, y)\left[\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\, U(s, t)\right]^{1/2}$$

$$K_0'(x, y) = -\frac{1}{2}\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\, U^{1/2}(x, s)U^{1/2}(t, y)\,|s - t|,$$

which gives

$$n^{(1)} \leqslant 1 + \frac{\frac{1}{2}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, U(x, y)U(z, u)\,(|y - z| - |y - x|)}{\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\, U(s, t)}\,. \tag{12}$$

In $\mathbf{R}^2$ we have

$$\xi = \left|\ln\left(\frac{1}{2}|k|e^{\gamma}\right)\right|\int d^2s\, d^2t\, U(s,t),$$

$$\varphi(x) = \int d^2y\, U^{1/2}(x,y) \Big/ \left[\int d^2s\, d^2t\, U(s,t)\right]^{1/2}$$

$$K_0'(x,y) = \int d^2s \int d^2t\, U^{1/2}(x,s) U^{1/2}(t,y) \ln|s-t|.$$

The resulting bound is the same as in $\mathbf{R}$, except for the replacement of $|x-y|$ by $-2\ln|x-y|$;

$$n^{(2)} \leqslant 1 + \frac{\int d^2u\, d^2x\, d^2y\, d^2z\, U(x,y)U(z,u)\ln(|y-x|/|y-z|)}{\int d^2s\, d^2t\, U(s,t)}, \tag{13}$$

povided that these integrals exist.

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