J. OPERATOR THEORY
10(1983), 141–152

COMPLETELY BOUNDED MAPS OF C^*-ALGEBRAS

TADASI HURUYA and JUN TOMIYAMA

1. INTRODUCTION

In recent developments of the theory of operator algebras, it has been recognized ([6], [8], [25]) that appropriate positive maps associated with the (so-called) matricial order structure of C^*-algebras are completely positive maps. It will then be naturally supposed that the appropriate class of linear maps attached to C^*-algebras beyond completely positive maps is the class of maps compatible with matricial structure, that is, completely bounded maps. A completely positive map is of course completely bounded and a derivation in a C^*-algebra is an example of a completely bounded map which is not completely positive, whereas in case of another type of derivations (from a C^*-algebra acting on a Hilbert space H into $L(H)$) their property of completely boundedness is known to be equivalent to the assertion that they are inner [6].

This paper originates from the basic study of completely bounded maps of C^*-algebras. One of our results (Theorem 3) then determines the class of C^*-algebras where every bounded linear map between them becomes necessarily completely bounded. The result has been expected in literature for some years (see, for example, [13]). On the other hand, if every completely bounded map were written as a linear combination of completely positive maps we could pass over study of completely bounded maps, and Wittstock [25] has recently shown that this is the case where the image algebra is injective. We shall show however (Proposition 7 and Example 12) that in principle this can not be expected even in the commutative case as well as the case of von Neumann algebras. We shall also prove a converse of Wittstock's theorem for a separable C^*-algebra as an image algebra (Theorem 11), but in a more restricted form, that is, under the assumption of the decomposability of any bounded linear map into a linear combination of positive maps. This extends however a result of Tsui [24] to the case of separable C^*-algebras.

2. PRELIMINARIES

We denote by M_n the $n \times n$ matrix algebra over the complex number field $\mathbf{C}$. If D is a C^*-algebra, we mean by $D \otimes M_n = M_n(D)$ the C^*-algebra of all $n \times n$ matrices $a = [a_{ij}]$ with entries in D. Let A and B be C^*-algebras. For a linear map φ of A into B, we define the multiplicity map $\varphi \otimes \mathrm{id}_n$: $M_n(A) \to M_n(B)$ by $\varphi \otimes \mathrm{id}_n[a_{ij}] = [\varphi(a_{ij})]$. The map φ is then said to be completely positive if all maps $\varphi \otimes \mathrm{id}_n$'s are positive and to be completely bounded if $\sup_n \|\varphi \otimes \mathrm{id}_n\| < \infty$ [2]. If φ is completely bounded, we put the norm $\|\varphi\|_{\mathrm{cb}} = \sup_n \|\varphi \otimes \mathrm{id}_n\|$. It is known that a completely positive map φ is completely bounded and $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$. Every bounded linear map of a C^*-algebra into a commutative C^*-algebra is completely bounded [13, Theorem A]. A linear map φ of A into B is said to have a positive (resp. completely positive) decomposition if there exist positive (resp. completely positive) maps φ_1, φ_2, φ_3 and φ_4 of A into B such that $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$.

Let $A \odot B$ denote the algebraic tensor product of A and B. In general there exist many distinct C^*-norms on $A \odot B$. If β is a C^*-norm on $A \odot B$, denote by $A \otimes_\beta B$ the completion of $A \odot B$ with respect to the norm β. Two such C^*-tensor products are of particular interest: the projective C^*-tensor product $A \otimes_{\max} B$ [20, Chapter IV, Definition 4.5] and the injective C^*-tensor product $A \otimes_{\min} B$ [20, Chapter IV, Definition 4.8].

For a compact Hausdorff space X, let $C(X)$ be the C^*-algebra of all continuous functions on X. Let φ be a continuous map of a compact Hausdorff space S into another compact Hausdorff space T. Let φ^0 denote the map φ^0: $C(T) \to C(S)$ defined by $\varphi^0(f)(t) = f(\varphi(t))$ for f in $C(T)$ and t in S.

For a Hilbert space H, denote by $L(H)$ the C^*-algebra of all bounded linear operators on H.

3. BOUNDED AND COMPLETELY BOUNDED MAPS

Let A and B be C^*-algebras. Loebl [13, Theorem E] has shown that for a fixed C^*-algebra A every bounded linear map of A into an arbitrary C^*-algebra B is completely bounded if and only if A is finite-dimensional. Tomiyama [23, Theorem 1.3] has also proved that for a fixed C^*-algebra B every bounded linear map of an arbitrary C^*-algebra A into B is completely bounded if and only if every irreducible representation of B is finite-dimensional with bounded degree.

In this section we clear up the situation where every bounded linear map of A into B is completely bounded.

The following lemma is derived from Lanford's example ([14], [24]).

LEMMA 1. *Let A be a C^*-algebra containing a sequence $\{a_i\}_{i=1}^\infty$ of positive elements with $\|a_i\| = 1$ and $a_ia_j = a_ja_i = 0$ $(i \neq j)$ and let B and C denote the C^*-sub-*

algebras generated by $\{a_i\}_{i=1}^{n}$ *and* $\{a_i\}_{i=n+1}^{\infty}$ *respectively, where n is a positive integer. Then for the integer* $m = 2^n$ *there exist an element* b_n *in* $B \otimes M_m$ *and a linear map* Φ_n *of A into* M_m *such that* $\|b_n\| \leq 1$, $\|\Phi_n\| \leq 1$, $\Phi_n \mid C = 0$ *and* $\|(\Phi_n \otimes \mathrm{id}_m)(b_n)\| \geqslant (n/2)^{1/2}$.

Proof. (i) We first assume that $A = \ell^\infty(n)$ and choose elements $c_1, \ldots, c_n$ of M_m such that $\|c_i\| = 1$ and $\left\|\sum_{i=1}^{n} \alpha_i c_i\right\| \leqslant \left(2 \sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2}$ for any α_i in $\mathbf{C}$, $1 \leqslant i \leqslant n$ (cf. [14, p. 122], [24, Lemma 1.3.2]). We define the map $\varphi_n : \ell^\infty(n) \to M_m$ by

$$\varphi_n(x) = \sum_{i=1}^{n} (x(i)/(2n)^{1/2}) c_i, \quad x \in \ell^\infty(n),$$

where $x(i)$ means the i'th component of x. By the above inequality,

$$\|\varphi_n(x)\| \leqslant \left(2 \sum_{i=1}^{n} |x(i)|^2/(2n)\right)^{1/2} \leqslant \|x\|,$$

so that $\|\varphi_n\| \leqslant 1$.

Let $d_n = \sum_{i=1}^{n} \delta_i \otimes c_i$, where δ_i is the element of A such that $\delta_i(k) = \delta_{ik}$. Then $\|d_n\| = \sup_i \|\delta_i \otimes c_i\| = 1$ and

$$(\varphi_n \otimes \mathrm{id}_m)(d_n) = \sum_{i=1}^{n} \varphi_n(\delta_i) \otimes c_i = (1/2n)^{1/2} \sum_{i=1}^{n} c_i \otimes c_i.$$

Here we make use of the unit vector z in $\mathbf{C}^m \otimes \mathbf{C}^m$ such that $(c_i \otimes c_i)(z) = z$ for all i constructed in Loebl [14, pp. 123—125]. Hence

$$\|\varphi_n\|_{\mathrm{cb}} \geqslant \|(\varphi_n \otimes \mathrm{id}_m)(d_n)\| \geqslant |((\varphi_n \otimes \mathrm{id}_m)(d_n)z|z)| = (n/2)^{1/2},$$

as desired.

(ii) Let $\{g_i\}_{i=1}^{n}$ be a family of states on A such that $g_i(a_j) = \delta_{ij}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j < \infty$. We define completely positive maps $\varphi : \ell^\infty(n) \to B$ and $\psi : A \to \ell^\infty(n)$ by

$$\varphi(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i a_i \quad \text{and} \quad \psi(a) = (g_1(a), \ldots, g_n(a)).$$

Both maps have norm one and $\psi\varphi$ is the identity on $\ell^\infty(n)$. With φ_n and d_n as in (i), let $\Phi_n = \varphi_n \psi$ and $b_n = (\varphi \otimes \mathrm{id}_m)(d_n) \in B \otimes M_m$. Then $\|b_n\| \leqslant 1$, $\|\Phi_n\| \leqslant \|\varphi_n\| \|\psi\| \leqslant 1$, $\Phi_n | C = 0$ and

$$(n/2)^{1/2} \leqslant \|(\varphi_n \otimes \mathrm{id}_m)(d_n)\| = \|((\varphi_n \psi \varphi) \otimes \mathrm{id}_m)(d_n)\| =$$
$$= \|((\varphi_n \psi) \otimes \mathrm{id}_m)(\varphi \otimes \mathrm{id}_m)(d_n)\| = \|(\Phi_n \otimes \mathrm{id}_m)(b_n)\|.$$

The following is a slight modification of Smith [18, Lemma 2.7].

LEMMA 2. *Suppose that a C^*-algebra B has an irreducible representation π on a Hilbert space H_π with* $\dim H_\pi \geqslant n$, *a positive integer. For a positive number* $\varepsilon > 0$, *there exist then completely positive contractive maps* $\varphi: M_n \to B$ *and* $\psi: B \to M_n$ *such that* $\|\psi\varphi - \mathrm{id}\| < \varepsilon$ *on* M_n.

Proof. If B is unital, such φ and ψ are given in [18, Lemma 2.7]. Suppose that B is non-unital, let $\tilde{B}$ denote the C^*-algebra adjoined a unit to B. The representation π has, by [20, Chapter III, Lemma 2.2], a w^*-continuous extension to a representation $\tilde{\pi}: (\tilde{B})^{**} \to L(H_\pi)$. Let $\{\xi_i\}_{i=1}^n$ be an orthnormal system in H_π and define the map $\psi: \tilde{B} \to M_n$ by $\psi(x)_{ij} = (\tilde{\pi}(x)\xi_j|\xi_i)$, the (i,j) entry of $\tilde{\pi}(x)$ in $L(H_\pi)$ with respect to the system $\{\xi_i\}_{i=1}^n$. By the proof of [18, Lemma 2.7] there exists a unital completely positive map φ_1 of M_n into $\tilde{B}$ such that $\|\psi\varphi_1 - \mathrm{id}\| < \varepsilon$ on M_n.

Let $\{u_\lambda\}$ be an approximate unit for B. The net $\{\pi(u_\lambda)\}$ converges strongly to the identity on H_π. For each (i,j), the net of functionals: $x \to (\pi(u_\lambda x u_\lambda)\xi_j|\xi_i)$ on $\tilde{B}$ converges uniformly to the functional: $x \to (\tilde{\pi}(x)\xi_j|\xi_i)$. Put $\varphi_\lambda(x) = u_\lambda\varphi_1(x)u_\lambda$ in B for x in M_n. From the definition of the map φ_λ we see that $\{\psi\varphi_\lambda\}$ converges to $\psi\varphi_1$ on M_n. We then obtain the desired map φ of M_n into B.

THEOREM 3. *Let A be an infinite-dimensional C^*-algebra and let B be a C^*-algebra satisfying* $\sup \dim H_\pi = \infty$ *for the family of its irreducible representations* $\pi: B \to L(H_\pi)$. *Then there exists a bounded linear map Φ of A into B having two properties;*

(1) *Φ is not completely bounded.*

(2) *Φ has no positive decomposition.*

Proof. Let a be a self-adjoint element of A with infinite spectrum [15] and let $C^*(a)$ be the C^*-subalgebra generated by a. By an elementary spectral argument we can then find a commuting sequence $\{a_{ij}\}_{i,j=1}^\infty$ of positive elements in $C^*(a)$ with norm one and having disjoint supports. Let f and g be functions on $\mathbf{N}$, the set of positive integers, defined by $f(n) = n^3$ and $g(n) = 2^{f(n)}$. For each n let A_n be the C^*-subalgebra generated by $\{a_{nj}\}_{j=1}^{f(n)}$. By Lemma 1 there exist an element $b_{f(n)}$ in $A_n \otimes M_{g(n)}$ and also a linear map $\Phi_{f(n)}$ of A into $M_{g(n)}$ with $\|b_{f(n)}\| \leqslant 1$, $\|\Phi_{f(n)}\| \leqslant 1$, $\Phi_{f(n)}|A_k = 0$ $(n \neq k)$ and $\|(\Phi_{f(n)} \otimes \mathrm{id}_{g(n)})(b_{f(n)})\| \geqslant (f(n)/2)^{1/2}$.

For any k, there exists an irreducible representation of B on a Hilbert space H with $\dim H \geqslant k$. It follows from Lemma 2 that we obtain completely positive contractive maps $\varphi_n: M_{g(n)} \to B$ and $\psi_n: B \to M_{g(n)}$ such that

$$\|\psi_n\varphi_n - \mathrm{id}_{g(n)}\| < (1/g(n)^2)(f(n)^{1/2}/2).$$

We have now

$$\|\varphi_n\Phi_{f(n)}|A_n\|_{\mathrm{cb}} \geqslant \|\psi_n\varphi_n\Phi_{f(n)}|A_n\|_{\mathrm{cb}} \geqslant$$

$$\geqslant \|(\Phi_{f(n)} \otimes \mathrm{id}_{g(n)})(b_{f(n)})\| - \|((\mathrm{id}_{g(n)} - \psi_n\varphi_n)\Phi_{f(n)} \otimes \mathrm{id}_{g(n)})(b_{f(n)})\| \geqslant$$

$$\geqslant (f(n)/2)^{1/2} - g(n)^2\|\mathrm{id}_{g(n)} - \psi_n\varphi_n\| \geqslant$$

$$\geqslant (f(n)/2)^{1/2} - (f(n)^{1/2}/2) = ((2^{1/2} - 1)/2)n^{3/2}.$$

Let $\Phi = \sum_{n=1}^{\infty} (1/n)^{5/4} \varphi_n\Phi_{f(n)}$. Then Φ is a bounded linear map of A into B.

(1) If Φ is completely bounded, we have $\|\Phi\|_{\mathrm{cb}} \geqslant \|\Phi|C^*(a)\|_{\mathrm{cb}}$. We show therefore that $\Psi = \Phi|C^*(a)$ is not completely bounded. Suppose that Ψ is completely bounded. By the above result, we have

$$\|\Psi\|_{\mathrm{cb}} \geqslant \|\Phi|A_n\|_{\mathrm{cb}} = \|(1/n)^{5/4}\varphi_n\Phi_{f(n)}|A_n\|_{\mathrm{cb}} \geqslant$$

$$\geqslant (1/n)^{5/4}((2^{1/2} - 1)/2)n^{3/2} = ((2^{1/2} - 1)/2)n^{1/4}.$$

Hence, Ψ is not completely bounded.

(2) Suppose that Φ has a positive decomposition $\Phi = \Phi_1 - \Phi_2 + \mathrm{i}(\Phi_3 - \Phi_4)$. Then $\Psi = \Phi_1|C^*(a) - \Phi_2|C^*(a) + \mathrm{i}(\Phi_3|C^*(a) - \Phi_4|C^*(a))$ and as the alg bra $C^*(a)$ is commutative, all maps $\Phi_i|C^*(a)$'s are completely positive (cf. [20, Chapter IV, Corollary 3.5]). Hence Ψ is completely bounded. This contradicts the result mentioned in (1).

Combining (1) of Theorem 3 with [13, Theorem E] and [23, Theorem 1.3], we can affirmatively answer a question of Loebl [13, Conjecture 2].

COROLLARY 4. *Let A and B be C^*-algebras. The following assertions are equivalent:*

(1) *Every bounded linear map of A into B is completely bounded.*

(2) *Either A is finite-dimensional or every irreducible representation of B is finite-dimensional with bounded degree, say n, that is, B is a C^*-subalgebra of a matrix C^*-algebra $M_n(C)$ for a commutative C^*-algebra C.*

Theorem 3(1) and Corollary 4 have been also proved in Smith [18, Theorem 2.8] by a different formulation with the same idea.

4. POSITIVE AND COMPLETELY POSITIVE DECOMPOSITIONS

It has been known ([10], [18], [24]) that there exist bounded linear maps between commutative C^*-algebras which have no positive decompositions. On the other hand, Wittstock [25] has proved that every completely bounded linear map of a unital C^*-algebra into an injective C^*-algebra has a completely positive decomposition. Another proof of this result is given by Paulsen [16].

In this section we show first that if $LC(H)_1$ denotes the C^*-algebra generated by all compact linear operators and the identity operator on an infinite-dimensional Hilbert space H, then there exists a completely bounded linear map of $LC(H)_1 \otimes_{\min} LC(H)_1$ into $LC(H)_1$ which has no completely positive decomposition. This may be regarded as a non-commutative version of Kaplan-Tsui [10, 24]. We also show general impossibility of positive decompostions in commutative C^*-algebras. Next, for a fixed separable C^*-algebra B we show that every bounded linear map of any C^*-algebra into B has a positive decomposition if and only if B is finite-dimensional. Using De Cannière and Haagerup [4], we finally construct a completely bounded map of a C^*-algebra into a von Neumann algebra which has no completely positive decomposition.

We begin with a slight modification of [24, 1.3.4, Example II].

The authors are indebted to the referee for pointing out an error of our first definition of the map Ψ in the proof of Lemma 5, replacing with the correct one for which the rest of the proof remains valid.

LEMMA 5. *Let* $\alpha\mathbf{N}$ *denote the one point compactification of the set of natural numbers. Then there exists a bounded self-adjoint linear map* Ψ *of* $C(\alpha\mathbf{N}) \otimes_{\min} C(\alpha\mathbf{N})$ *into* $C(\alpha\mathbf{N})$ *which has no positive decomposition.*

Proof. Let $\alpha(\mathbf{N}\times\mathbf{N})$ denote the one point compactification of $\mathbf{N}\times\mathbf{N}$. Let ∞ and ω be the points at infinity of $\alpha\mathbf{N}$ and $\alpha(\mathbf{N}\times\mathbf{N})$ respectively. Then $\alpha(\mathbf{N}\times\mathbf{N})$ is homeomorphic to $\alpha\mathbf{N}$ and $C(\alpha\mathbf{N}) \otimes_{\min} C(\alpha\mathbf{N})$ is $*$-isomorphic to $C(\alpha\mathbf{N}\times\alpha\mathbf{N})$ [20, Chapter IV, Theorem 4.14]. It then suffices to construct a bounded self-adjoint linear map of $C(\alpha\mathbf{N}\times\alpha\mathbf{N})$ into $C(\alpha(\mathbf{N}\times\mathbf{N}))$ which has no positive decomposition.

Define $\Psi\colon C(\alpha\mathbf{N}\times\alpha\mathbf{N}) \to C(\alpha(\mathbf{N}\times\mathbf{N}))$ by

$$\Psi(f)(m,n) :=$$

$$=: f(m,n) - f(m,n+1) + f(m+1,n+1) - f(m+1,n) \quad (m,n) \text{ in } \mathbf{N}\times\mathbf{N},$$

$$\Psi(f)(\omega) = 0$$

for f in $C(\alpha\mathbf{N}\times\alpha\mathbf{N})$. We assert that $\Psi(f) \in C(\alpha(\mathbf{N}\times\mathbf{N}))$, that is, it is continuous at ω. In fact, given $\varepsilon > 0$, there exists n_0 such that

$$|f(m,n) - f(\infty,\infty)| < \varepsilon/4 \quad \text{for all } m,n > n_0$$

$$|f(m,n) - f(m,\infty)| < \varepsilon/4 \quad \text{for all } 1 \leqslant m \leqslant n_0,\ n > n_0$$

$$|f(m,n) - f(\infty,n)| < \varepsilon/4 \quad \text{for all } m > n_0,\ 1 \leqslant n \leqslant n_0.$$

Hence for all (m,n) outside the finite set $\{(m,n) : m,n = 1,\ldots,n_0\}$, we have $|\Psi(f)(m, n)| < \varepsilon$. It is easy to check that Ψ is bounded and self-adjoint.

Suppose that Ψ has a positive decomposition $\Psi = \Psi^+ - \Psi^-$. Let $\psi = \Psi^+$. Then we have $\psi(f)(m, n) \geqslant f(m, n)$ for all $f \geqslant 0$ in $C(\alpha\mathrm{N} \times \alpha\mathrm{N})$ and (m, n) in $\mathrm{N} \times \mathrm{N}$. For each m let e_m be the characteristic function of the set $\{(m, x) \mid x \in \alpha\mathrm{N}\}$. Then

$$\lim_n \psi(e_m)(m, n) = \psi(e_m)(\omega) \geqslant e_m(m, \infty) = 1.$$

We have that $1 \geqslant \sum_{m=1}^{k} e_m$ for every k, whence

$$\psi(1)(\omega) \geqslant \psi\left(\sum_{m=1}^{k} e_m\right)(\omega) = \sum_{m=1}^{k} \psi(e_m)(\omega) \geqslant k.$$

This shows the unboundedness of $\Psi^+ = \psi$.

PROPOSITION 6. *For an infinite-dimensional Hilbert space H, let $LC(H)_1$ denote the C^*-algebra generated by all compact linear operators and the identity operator on H. Then there exists a completely bounded map of $LC(H)_1 \otimes_{\min} LC(H)_1$ into $LC(H)_1$ which has no positive decomposition.*

Proof. Let $\{p_n\}$ be a sequence of pairwise orthogonal minimal projections in $LC(H)_1$. For each n let $\varphi_n(x) = p_n x p_n$ for x in $LC(H)_1$. Put $\Phi = \sum_{n=1}^{\infty} \varphi_n$ and $A = \Phi(LC(H)_1)$. Then Φ is a completely positive map satisfying $\Phi(x) = x$ for all x in A and we can identify A with $C(\alpha\mathrm{N})$. By [20, Chapter IV, Proposition 4.23] there exists a unique completely positive map $\Phi \otimes \Phi$ of $LC(H)_1 \otimes_{\min} LC(H)_1$ into $A \otimes_{\min} A$. With Ψ as in Lemma 5, let $\Phi_1 = \Psi(\Phi \otimes \Phi)$. Since Ψ is completely bounded, so is Φ_1 [13, Theorem A].

Suppose that there exist positive maps Φ_1^+, Φ_1^- of $LC(H)_1 \otimes_{\min} LC(H)_1$ into $LC(H)_1$ such that $\Phi_1 = \Phi_1^+ - \Phi_1^-$. The equality

$$\Phi(\Phi_1 | A \otimes_{\min} A) = \Phi(\Phi_1^+ | A \otimes_{\min} A) - \Phi(\Phi_1^- \mid A \otimes_{\min} A)$$

implies that Ψ has a positive decomposition. This is a contradiction.

Smith [18, Example 2.1] constructed a bounded linear map in the C^*-algebra $C[0, 1]$ of all continuous functions on the unit interval which has no positive decomposition. The following proposition illustrates the general situation.

PROPOSITION 7. *Let T be an uncountable compact metric space, then there exists a bounded linear map of $C(T)$ into itself which has no positive decomposition.*

Proof. Let K be the Cantor set. Since $\alpha\mathrm{N} \times \alpha\mathrm{N} \times K$ is a compact totally disconnected perfect metric space, by [11, Chapter 2, §6, Theorem 1] it is homeomorphic to K. On the other hand, the space T contains a homeomorphic image of K [3, Lemma 5.6]. Hence, there exist homeomorphic imbeddings $\varphi: \alpha N \times \alpha N \to T$ and $\psi: \alpha N \to T$. As in Section 2, we define the $*$-homomorphisms $\varphi^0: C(T) \to C(\alpha N \times \alpha N) = C(\alpha N) \otimes_{\min} C(\alpha N)$ and $\psi^0: C(T) \to C(\alpha N)$ by

$$\varphi^0(f)(s) = f(\varphi(s)) \quad \text{for } f \text{ in } C(T) \text{ and } s \text{ in } \alpha N \times \alpha N$$

and

$$\psi^0(g)(t) = g(\psi(t)) \quad \text{for } g \text{ in } C(T) \text{ and } t \text{ in } \alpha N.$$

It follows from [3, Theorem 3.11] that there exist positive maps $\nu_\varphi: C(\alpha N) \otimes_{\min} C(\alpha N) \to C(T)$ and $\nu_\psi: C(\alpha N) \to C(T)$ such that $\varphi^0 \nu_\varphi$ is the identity on $C(\alpha N) \otimes_{\min} C(\alpha N)$ and $\psi^0 \nu_\psi$ is the identity on $C(\alpha N)$.

With the self-adjoint linear map Ψ as in Lemma 5, we consider the map $\nu_\psi \Psi \varphi^0: C(T) \to C(T)$. Suppose that $\nu_\psi \Psi \varphi^0$ has a positive decomposition $\nu_\psi \Psi \varphi^0 = \psi_1 - \psi_2$. We have then a positive decomposition $\Psi = \psi^0 \nu_\psi \Psi \varphi^0 \nu_\varphi = \psi^0 \psi_1 \nu_\varphi - \psi^0 \psi_2 \nu_\varphi$. This is a contradiction.

REMARK 8. Every bounded linear map φ of $C(\alpha N)$ into itself with $\varphi(\infty) = 0$ is determined by an infinite matrix [21, Theorem 4.51-C]. Using this representation we have a positive decomposition of φ.

PROPOSITION 9. *Let T be a compact Hausdorff space which contains a convergent sequence of distinct points. Then there exists a bounded linear map of $C(T) \otimes_{\min} C(T)$ into $C(T)$ which has no positive decompostion.*

Proof. By the assumption, there exists a homeomorphic imbedding $\varphi: \alpha N \to T$. By [3, Theorem 3.11] we have a positive map ν of $C(\alpha N)$ into $C(T)$ such that $\varphi^0 \nu$ is the identity on $C(\alpha N)$. Let Ψ be the self-adjoint linear map as in Lemma 5. A similar argument as in Proposition 7 shows that $\nu \Psi(\varphi^0 \otimes \varphi^0)$ has no positive decomposition.

Since a separable unital commutative C^*-algebra is $*$-isomorphic to $C(X)$ for a compact metric space, the next lemma follows from Proposition 9.

LEMMA 10. *Let B be a separable unital commutative C^*-algebra. If every bounded linear map of any C^*-algebra A into B has a positive decomposition, then B is finite-dimensional.*

Tsui proved in [24, Theorem 1.4.6] that every bounded linear map of any C^*-algebra into a von Neumann algebra M has a positive decomposition if and only if $M = \sum_i^{\oplus} R_i$, where each R_i is of type $I_{n(i)}$ and $\sup_i n(i) < \infty$. The corresponding result for C^*-algebras has not been known yet but we prove the problem in the case of separable C^*-algebras. This may be also regarded as a limited converse of Wittstock's theorem [25, Satz 4.5].

THEOREM 11. *Let B be a separable C^*-algebra. Every bounded linear map of any C^*-algebra into B has a positive decomposition if and only if B is finite-dimensional.*

Proof. We only need to establish the theorem in one direction; assume therefore every bounded linear map of any C^*-algebra A into B has a positive decomposition. The proof is divided into several steps.

(i) We first show that B is unital. Suppose that B is non-unital. We may assume that B acts on a Hilbert space such that an approximate unit $\{u_n\}$ for B converges strongly to the identity operator 1. Let B_1 be the C^*-algebra generated by B and 1, so that B is an ideal of B_1. We define the map $\nu: B_1 \to B$ by $x + \alpha 1 \to x$. Suppose that ν has a positive decomposition $\nu = \nu_1 - \nu_2$, where ν_1 and ν_2 are positive maps of B_1 into B. For each u_n, we have that

$$u_n = \nu(u_n) \leqslant \nu_1(u_n) \leqslant \nu_1(1).$$

Hence $1 \leqslant \nu_1(1)$, so that $\nu_1(1)$ is invertible and $\nu_1(1) \notin B$. This is a contradiction Therefore B is unital.

(ii) By Theorem 3 every irreducible representation of B is finite-dimensional with bounded degree. Let $n(1) < n(2) < \ldots < n(k)$ be the dimensions of irreducible representations of B and let I_i be the intersection of kernels of irreducible representations of B with dimension less than or equal to $n(i)$.

It then suffices to show that the dual space $B^\wedge$ is a finite set.

(iii) We prove that $(B/I_1)^\wedge = X$ is a finite set. We first notice that since B/I_1 is an $n(1)$-homogeneous C^*-algebra, X is a compact Hausdorff space and the center of B/I_1 is regarded as $C(X)$. The map $\psi_1: x \to \{\mathrm{Tr}(\rho(x))\}_{\rho \in X}$ is, by [7, Proposition 3.6.4] a projection of norm one from B/I_1 onto $C(X)$, where Tr denotes the canonical trace on $M_{n(1)}$ such that $\mathrm{Tr}(1_{n(1)}) = 1$. By [5, Corollary 3.11] we see that the injection $\tau_1: x \to x$ of $C(X)$ into B/I_1 has a completely positive lifting map τ_2 of $C(X)$ into B such that $\pi_1\tau_2 = \tau_1$, where π_1 denotes the quotient map $B \to B/I_1$.

Let A be a C^*-algebra and let ν be a bounded linear map of A into $C(X)$. The map $\tau_2\nu$ has a positive decomposition $\tau_2\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. We then have a positive decomposition $\nu = \psi_1\pi_1\nu_1 - \psi_1\pi_1\nu_2 + i(\psi_1\pi_1\nu_3 - \psi_1\pi_1\nu_4)$. It follows from Lemma 10 that X is a finite set.

(iv) Assume that $(B/I_{j-1})^\wedge$ is a finite set. We assert that $(B/I_j)^\wedge$ is a finite set. Since $(B/I_j)^\wedge = (B/I_{j-1})^\wedge \cup (I_{j-1}/I_j)^\wedge$, it suffices to show that $(I_{j-1}/I_j)^\wedge$ is a finite set. Thus, suppose that $(I_{j-1}/I_j)^\wedge$ is an infinite set. Since B/I_{j-1} is finite-dimensional, we have a finite set $\{a_i\}_{i=1}^n$ of B such that for x in B there exist an a in I_{j-1} and comdlex numbers $\{\alpha_i\}_{i=1}^n$ with $x = a + \sum_{i=1}^{n} \alpha_i a_i$. Let Tr be the trace on $M_{n(j)}$ such that $\mathrm{Tr}(1_{n(j)}) = 1$. The set $\{\mathrm{Tr}(\rho(a_i))\}_{\rho \in (I_{j-1}/I_j)^\wedge}$ is bounded. Since $(I_{j-1}/I_j)^\wedge$ is a metric space, let $\{\pi_k\}_{k=1}^\infty$ (resp. D) be a discrete sequence or a convergent sequence of distinct points of $(I_{j-1}/I_j)^\wedge$ (resp. $I_{j-1}+\mathbf{C}1$ or I_{j-1}) according as $(I_{j-1}/I_j)^\wedge$ is non-compact or compact. Passing to subsequences we may assume that there exists the limit $\lim_k \mathrm{Tr}(\pi_k(a_i))$ for each $i \leqslant n$. The we have

$$\lim_k \mathrm{Tr}(\pi_k(x)) = \lim_k \mathrm{Tr}(\pi_k(a)) + \sum_{i=1}^{n} \alpha_i \lim_k \mathrm{Tr}(\pi_k(a_i)).$$

Let J be the intersection of kernels of $\{\pi_k\}_{k=1}^{\infty}$. Then $(D/J)^\wedge$ is homeomorphic to $\alpha\mathbf{N}$. Hence the center of D/J is regarded as $C(\alpha\mathbf{N})$ by [17, Corollary 4.4.8]. Furthermore the above argument shows that the map $\psi_2 : x \to \{\mathrm{Tr}(\pi_k(x))\}_{k=1}^{\infty}$ is a projection of norm one of B/J onto $C(\alpha\mathbf{N})$. The injection $\tau_3 : x \to x \in B/J$ has, by [5, Corollary 3.11], a completely positive lifting $\tau_4 : C(\alpha\mathbf{N}) \to B$ such that $\pi_2\tau_4 = \tau_3$, where π_2 denotes the quotient map $B \to B/J$. Therefore the same argument as in (iii) shows that every bounded linear map of any C^*-algebra into $C(\alpha\mathbf{N})$ has a positive decomposition. This contradicts Lemma 5. Hence $(I_{j-1}/I_j)^\wedge$ is a finite set.

Consequently the space $B^\wedge$ consists of finite points. This completes the proof.

So far we have been considering (completely) bounded maps and the problem of their (completely) positive decompositions in the category of rather proper C^*-algebras as range algebras. We shall however finally construct an example which shows that completely positive decompositions may not be expected in general even within the category of von Neumann algebras.

Let φ be a completely bounded map of a C^*-algebra A into a C^*-algebra B. If C is a C^*-algebra acting on a Hilbert space H, let $\{p_\lambda\}$ be a net of finite rank projections in $L(H)$ which converges strongly to the identity operator and let $\{\psi_\lambda\}$ denote the net of compression maps by $\{p_\lambda\}$. Considering the diagram

$$A \otimes_{\min} L(H) \xrightarrow{\mathrm{id}\otimes\psi_\lambda} A \otimes_{\min} L(p_\lambda(H)) \xrightarrow{\varphi\otimes\mathrm{id}} B \otimes_{\min} L(p_\lambda(H)) \subseteq B \otimes_{\min} L(H),$$

we have the map $\varphi\otimes\psi_\lambda : A\otimes_{\min} L(H) \to B \otimes_{\min} L(H)$ with $\|\varphi\otimes\psi_\lambda\| \leqslant \|\varphi\|_{cb}$. By a standard argument [22, Theorem 5.1] we also obtain the map $\varphi\otimes\mathrm{id} : A \otimes_{\min} C \to B \otimes_{\min} C$. Moreover we note that if φ has a completely positive decomposition, we have a unique linear map $\varphi \otimes \mathrm{id}$ of $A \otimes_{\max} C$ into $B \otimes_{\max} C$ [20, Chapter IV, Proposition 4.23].

Now let A be the C^*-algebra generated by the left regular representation of the free group F_2 on two generators on $\ell^2(F_2)$ and let B denote the enveloping von Neumann algebra of A. Recently De Cannière and Haagerup [4] has shown that there exists a sequence $\{\varphi_n\}$ of linear maps of finite rank of A into itself with $\|\varphi_n\|_{cb} \leqslant 1$, which converges to the identity map in the pointwise norm topology. Let M be the direct sum of a sequence of copies of B.

EXAMPLE 12. With the above notation the map $\varphi = \sum_{n=1}^{\infty} {}^{\oplus} \varphi_n$ is a completely bounded map of A into M which has no completely positive decomposition.

Proof. It is easily seen that $\|\varphi\|_{cb} \leqslant 1$. Suppose that it has a completely positive decomposition. By the above remark there exists a unique bounded linear map $\varphi \otimes \mathrm{id}$ of $A \otimes_{\max} A$ into $M \otimes_{\max} A$. Let ψ_n be the $*$-homomorphism of M onto B such that $\psi_n\left(\sum_{i=1}^{\infty} {}^{\oplus} x_i\right) = x_n$ for $\sum_{i=1}^{\infty} {}^{\oplus} x_i$ in M and let $\psi_n \otimes \mathrm{id}$ be the $*$-homomorphism of $M \otimes_{\max} A$ onto $B \otimes_{\max} A$. Since φ_n is of finite rank, we have the bounded linear map $\varphi_n \otimes \mathrm{id}$ of $A \otimes_{\max} A$ into $B \otimes_{\max} A$. Then we have $(\psi_n \otimes \mathrm{id})(\varphi \otimes \mathrm{id}) = \varphi_n \otimes \mathrm{id}$. Hence, $\|\varphi_n \otimes \mathrm{id}\| \leqslant \|\varphi \otimes \mathrm{id}\|$ for all n. Therefore we have that $\lim_n (\varphi_n \otimes \mathrm{id})(x) = x$ for all x in $A \otimes_{\max} A$.

Now by [1, Theorem 2] there exists a C^*-norm β on $B \odot A$ such that $B \otimes_\beta A \supseteq A \otimes_{\max} A$, that is, the restriction of the norm β to the subalgebra $A \otimes_{\max} A$ agrees with the maximal C^*-norm. Let ρ be the canonical homomorphism of $B \otimes_{\max} A$ to $B \otimes_\beta A$. We notice that $\rho(\varphi_n \otimes \mathrm{id})$ maps $A \otimes_{\max} A$ into the subalgebra $A \otimes_{\max} A$ in $B \otimes_\beta A$ and as the family $\{\varphi_n \otimes \mathrm{id}\}$ is uniformly bounded, we have that $\lim_n \rho(\varphi_n \otimes \mathrm{id})(x) = x$ for all x in $A \otimes_{\max} A$. Let f be a state on $A \otimes_{\max} A$ and $\tilde{f}$ be a state extension of f to $B \otimes_\beta A$. Note that the functional $\tilde{f}\rho(\varphi_n \otimes \mathrm{id})$ on $A \otimes_{\max} A$ belongs to $A^* \odot A^*$, the algebraic tensor product of the conjugate spaces. Let a be an element of the kernel of the homomorphism: $A \otimes_{\max} A \to A \otimes_{\min} A$. As in the proof of [12, Theorem 3.4] we have then

$$0 = \lim_n \tilde{f}\rho(\varphi_n \otimes \mathrm{id})(a^*a) = \tilde{f}(a^*a) = f(a^*a)$$

and $a^*a = 0$, that is, $a = 0$. Since $A \otimes_{\max} A$ is different from $A \otimes_{\min} A$ [19, pp. 119—121], this is a contradiction.

REFERENCES

1. ARCHBOLD, R. J., On the centre of a tensor product of C^*-algebras, *J. London Math. Soc.*, (2), **10**(1975), 257—262.
2. ARVESON, W. B., Subalgebras of C^*-algebras, *Acta Math.*, **123**(1969), 141—224.
3. BADE, W. G., *The Banach space C(S)*, Aarhus University Lectures Notes Series, No. 26, 1971.
4. DE CANNIÈRE, J.; HAAGERUP, U., Multipliers of the Fourier algebras on semisimple Lie groups and their discrete subgroups, preprint.
5. CHOI, M.-D.; EFFROS, E. G., The completely positive lifting problem for C^*-algebras, *Ann. of Math.*, **104**(1976), 585—609.
6. CHRISTENSEN, E., Extensions of derivations. II, preprint.
7. DIXMIER, J., *Les C^*-algèbres et leurs représentations*, 2^e éd., Gauthier-Villars, Paris, 1969.

8. HAAGERUP, U., Solution of the similarity problem for cyclic representations of C^*-algebras, preprint.
9. HADWIN, D. W., Dilations and Hahn decompositions for linear maps, *Canad. J. Math.*, **33**(1981) 826–839.
10. KAPLAN, S., An example in the space of bounded operators from $C(X)$ to $C(Y)$, *Proc. Amer. Math. Soc.*, **38**(1973), 595–597.
11. LACEY, H. E., *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin, 1974.
12. LANCE, C., On nuclear C^*-algebras, *J. Functional Analysis*, **12**(1973), 157–176.
13. LOEBL, R. I., Contractive linear maps on C^*-algebras, *Michigan Math. J.*, **22**(1975), 361–366.
14. LOEBL, R. I., A Hahn decomposition for linear maps, *Pacific J. Math.*, **65**(1976), 119–133.
15. OGASAWARA, T., Finite dimensionality of certain Banach algebras, *J. Sci. Hiroshima Univ.*, **17**(1954), 359–364.
16. PAULSEN, V. I., A new proof of a result of Wittstock with an application to invariant operator ranges, preprint.
17. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, New York, 1979.
18. SMITH, R. R., Completely bounded maps between C^*-algebras, preprint.
19. TAKESAKI, M., On the cross-norm of the direct product of C^*-algebras, *Tôhoku Math. J.*, **16**(1964), 111–122.
20. TAKESAKI, M., *Theory of operator algebras*. I, Springer-Verlag, Berlin, 1979.
21. TAYLOR, A. E., *Introduction to functional analysis*, Wiley, New York, 1958.
22. TOMIYAMA, J., *Tensor products and projections of norm one in von Neumann algebras*, Lecture Notes (mineographed), University of Copenhagen, 1970.
23. TOMIYAMA, J., Notes on the transpose map of matrix algebras, preprint.
24. TSUI, S.-K. J., Decompositions of linear maps, *Trans. Amer. Math. Soc.*, **230**(1977), 87–112.
25. WITTSTOCK, G., Ein operatorwertinger Hahn-Banach Satz, *J. Functional Analysis*, **40**(1981), 127–150.

TADASI HURUYA
Faculty of Education,
Niigata University,
Niigata, 950–21,
Japan.

JUN TOMIYAMA
Faculty of Science,
Niigata University,
Niigata, 950–21,
Japan.

Received April 29, 1982.