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ERGODIC THEORY AND THE FUNCTIONAL EQUATION $(I - T)x = y$

MICHAEL LIN and ROBERT SINE

The problem of solving the functional equation $(I - T)x = y$, for a given linear operator T on a Banach space X and a given $y \in X$, appears in many areas of analysis and probability. The well-known Neumann series gives $(I - T)^{-1}$ when $\|T\|<1$. When $\|T\|=1$, the problem is first to know if $y \in (I-T)X$, and then to find the solution x. The solution is usually found using an iterative procedure (see [4], [5], [6], [16]). We are interested in the convergence of $x_n := n^{-1} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^j y$ to the solution x, and obtain the precise *necessary and sufficient* conditions (Corollary 3). The necessa y condition $\sup_{k\geq 1}\left\| \sum_{j=0}^{k-1} T^j y \right\| < \infty$ is shown to be sufficient if T^m (for some $m > 0$) is weakly compact. An example shows that otherwise the condition need not be sufficient. The reflexive case appears in [1], [2], [3].

We then solve the problem of existence in the case of a dual operator on a dual space, obtaining as a corollary an application to Markov operators.

Next, we look at the same problem for $Tf(s) = f(\theta s)$, where T is induced on a suitable function space by a measurable map θ. A new "ergodic" proof for θ a minimal continuous map of a Hausdorff space is given.

Finally, we obtain results for positive conservative contractions (Markov operators) on $L_1(S, \Sigma, \mu)$. In that case we look also at solutions which are finite a.e., though not necessary in L_1.

For the general Banach space approach, we need the *mean ergodic theorem*:

If $T^n/n \to 0$ *strongly, and* $\sup_n \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j \right\| < \infty$, *then*

$$\left\{ x : \frac{1}{n} \sum_{j=0}^{n-1} T^j x \ \textit{converges} \right\} = \{y : Ty = y\} \oplus \overline{(I - T)X}.$$

We call T mean ergodic if the above subspace is all of X. We mention the *uniform ergodic theorem* [19]:

$$(I - T)X \text{ is closed} \Leftrightarrow n^{-1} \sum_{k=0}^{n-1} T^k \text{ converges uniformly.}$$

In that case, $I - T$ is invertible on $(I - T)X$, and $\frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^j$ converges *uniformly* to $(I - T)^{-1}$ (on $(I - T)X$), which is a generalization of the Neumann series theorem.

THEOREM 1. *Let T be mean ergodic. The following conditions are equivalent for $y \in X$:*

(i) $y \in (I - T)X$;

(ii) $x_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^j y$ *has a weakly convergent subsequence;*

(iii) $\{x_n\}$ *converges strongly (and $x = \lim x_n$ satisfies $(I - T)x = y$).*

Proof. (i) $\Rightarrow$ (iii). Let $y = (I - T)x'$. By the mean ergodic theorem, $x' = x + z$, with $x \in \overline{(I - T)X}$ and $(I - T)z = 0$. Hence $y = (I - T)x$ with $x \in \overline{(I - T)X}$.

$$x_n = n^{-1} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^j(I - T)x = n^{-1} \sum_{k=1}^{n} (I - T^k)x = x - n^{-1} \sum_{k=1}^{n} T^k x.$$

But $\left\| n^{-1} \sum_{k=1}^{n} T^k x \right\| \to 0$, since $x \in \overline{(I - T)X}$, so $\|x_n - x\| \to 0$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let $x_{n_i} \to x$ weakly. Then

$$(I - T)x = \lim (I - T)x_{n_i} = \lim n_i^{-1} \sum_{k=1}^{n_i} (I - T^k)y = y - \lim n_i^{-1} \sum_{k=1}^{n_i} T^k y.$$

By the mean ergodic theorem the limit satisfies

$$Ex_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^j Ey = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} Ey = \frac{(n+1)}{2} Ey$$

so $Ex_{n_i} \to Ex$ is possible only if $Ey = 0$. Hence $(I - T)x = y$.

REMARK. The solution x of $(I-T)x=y$, obtained in (iii), is always in $\operatorname{clm}\{T^j y : j > 0\}$.

COROLLARY 2. *Let T be power-bounded, and assume that for some $m > 0$, T^m is weakly compact. Let $y \in X$. Then the condition* (iv) *below is equivalent to the three conditions of Theorem* 1*:*

(iv) $\sup_{k>0} \left\| \sum_{j=0}^{k-1} T^j y \right\| < \infty.$

Proof. (i) $\Rightarrow$ (iv).

$$y = (I-T)x \Rightarrow \left\| \sum_{j=0}^{k-1} T^j y \right\| = \|(I-T^k)x\| \leq \|x\|(1+\sup\|T^n\|).$$

(iv) $\Rightarrow$ (i). By (iv), $\left\| \frac{1}{k} \sum_{j=0}^{k-1} T^j y \right\| \to 0$. We restrict ourselves to $\operatorname{clm}\{T^j y : j \geq 0\}$, on which T is now mean ergodic (in fact, T is mean ergodic on X). By (iv) and weak compactness of T^m, $\left\{ \sum_{j=0}^{k-1} T^j(T^m y) \right\}$ is weakly sequentially compact, and so is $z_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^j T^m y$, so, by Theorem 1 (iii), $z_n \to z$ which satisfies $(I-T)z = T^m y$. Now $x = z + \sum_{j=0}^{m-1} T^j y$ satisfies $(I-T)x = y$.

EXAMPLE 1. T may be a mean ergodic contraction, but, in general, (iv) does not imply the conditions of Theorem 1.

Let Y be a non-reflexive Banach space and T a contraction which is not mean ergodic (e.g., $Y = \ell_1$, T the shift to the right). Take $z \in Y$ such that $n^{-1} \sum_{j=0}^{n-1} T^j z$ does not converge (i.e., $z \notin \overline{(I-T)Y} \oplus \{Tx = x\}$). Let $y = (I-T)z$, and $X = \operatorname{clm}\{T^j y : j \geq 0\}$. X is an invariant subspace for T, and T on X is mean ergodic (with no fixed points). Clearly y satisfies (iv). If there were $x \in X$ with $(I-T)x = y$, then

$$(I-T)(z-x) = 0, \quad \text{so} \quad n^{-1} \sum_{k=1}^{n} T^k z = n^{-1} \sum_{k=1}^{n} T^k(z-x) + n^{-1} \sum_{k=1}^{n} T^k x \to z - x,$$

contradicting the choice of z. Hence $(I-T)x = y$ has no solution in X.

REMARK. The previous example shows also that without ergodicity in Theorem 1, (i) need not imply (ii): The $\{x_n\}$ is always in $\overline{(I-T)Y}$ (in fact, in X), while the solution is in Y, and if x_{n_i} converges weakly, the limit must be a solution. Hence $\{x_n\}$ has no weakly convergent subsequence.

COROLLARY 3. *Let T satisfy:*

(a) $\sup_N \left\| N^{-1} \sum_{i=0}^{n-1} T^i \right\| < \infty$;

(b) $T^n/n \to 0$ *strongly.*

Then the following conditions are equivalent for $y \in X$:

(i) $y \in (I - T)\overline{(I - T)X}$;
(ii) *as in Theorem* 1;
(iii) *as in Theorem* 1.

Proof. Let $Y = \overline{(I - T)X}$. On Y, T is mean ergodic.

(i) $\Rightarrow$ (iii). $y = (I - T)x$, with $x \in Y$.
(iii) follows from Theorem 1, applied in Y.
(ii) $\Rightarrow$ (i). If $x_{n_i} \xrightarrow{w} x$, the computation in the proof of Theorem 1 yields

$$n_i^{-1} \sum_{k=1}^{n_i} T^k y \xrightarrow{w} y - (I - T)x.$$

Hence $y \in Y \oplus \{Tz = z\} \equiv Z$. Apply Theorem 1 to T on Z to obtain $y \in (I - T)Z = (I - T)Y$.

COROLLARY 4. *Let T be as in Corollary 2. Then the following conditions are equivalent for* $y \in X$:

(1) $\sum_{j=0}^{k} T^j y$ *converges weakly (to $x \in X$, and then $(I - T)x = y$)*;

(2) $T^n y \xrightarrow{w} 0$, *and* $\liminf_{k \to \infty} \left\| \sum_{j=0}^{k} T^j y \right\| < \infty$.

Proof. (1) $\Rightarrow$ (2) is easy.

(2) $\Rightarrow$ (1). If $\left\| \sum_{j=0}^{k_i} T^j y \right\| \leqslant M$, then $\sum_{j=0}^{k_i} T^j T^m y$ is weakly sequentially compact. Take a subsequence of $\{k_i\}$ (called still $\{k_i\}$) with $\sum_{j=0}^{k_i} T^j T^m y \xrightarrow{w} z$. Then

$$(I - T)z = T^m y - \lim_{i \to \infty} T^{m+k_i+1} y = T^m y.$$

Hence $x = z + \sum_{j=0}^{m-1} T^j y$ is in $\text{clm}\{T^n y\}$ with $(I - T)x = y$. Now also $T^n x \to 0$ weakly, so (1) holds.

REMARK. For strong convergence in (1) we put strong convergence in (2). If we *know* that $y\in(I-T)X$ and T^ny converges (necessarily to 0) then $\sum_{j=0}^{k} T^jy$ will converge to x (in the same topology that $T^ny\to 0$), assuming only mean ergodicity, instead of weak compactness, for T power-bounded (see also [2]). However, (2) does not imply that $y\in(I-T)X$ (even when $\|T^ny\|\to 0$): see the beginning of Example 3.

EXAMPLE 2. The condition that $\left\{\sum_{j=0}^{k-1} T^jy\right\}_{k=1}^{\infty}$ be weakly sequentially compact, though sufficient to imply the other conditions in Theorem 1, is not necessary.

In [17] there is an example of a real Banach space X and an isometry T, such that for some vector $x_0\in X$ we have $\sup_{\|x\|=1}\frac{1}{N}\sum_{k=0}^{N-1}|\langle x^*,T^kx_0\rangle|\to 0$, but for no subsequence n_j does $T^{n_j}x_0$ converge weakly to 0. Since clearly $\left\|\frac{1}{N}\sum_{k=1}^{N}T^kx_0\right\|\to 0$, by restricting ourselves to $\operatorname{clm}\{T^jx_0 : j\geqslant 0\}$ we have T mean ergodic. Let $y=(I-T)x_0$. Then $\sum_{j=0}^{k-1}T^jy=x_0-T^kx_0$. The choice of x_0 shows that 0 is in the weak closure of $\{T^kx_0\}$. If this closure were weakly compact, some subsequence of $\{T^kx_0\}$ would converge weakly to zero, (since the weak topology on a weakly compact set in a separable Banach space is metrizable [7, V.6.3]) — a contradiction. Hence the closure is not weakly compact, and $\{T^kx_0\}$ is not weakly sequentially compact [7, V.6.1].

REMARKS. 1. Examples 1 and 2 show that we cannot, in general, reverse any of the implications $\left\{\sum_{j=0}^{k-1} T^jy\right\}_{k\geqslant 1}$ is w.s. compact $\Rightarrow y\in(I-T)X\Rightarrow\left\{\sum_{j=0}^{k-1} T^jy\right\}_{k\geqslant 1}$ bounded. Example 2 is new, and shows how remarks on compactness made by previous authors should be understood in relation to Theorem 1. Special examples of the kind of Example 1, for the shift in ℓ_∞, appear in [10] (expressed in different terms).

Corollary 2 improves the result of Butzer and Westphal [3] (for Cesáro averages). In that connection they too consider the linear manifold $(I-T)\overline{(I-T)X}$. However, Corollary 3 is new. Theorem 1 is essentially given in [4].

In many cases, we may have to identify if $y^*\in(I-T)X^*$ when T is a contraction on X. Here condition (iv) works, because of weak-* compactness. For completeness, we repeat the first author's proof from [17].

THEOREM 5. *Let* $\sup\|T^n\|<\infty$. *The following conditions are equivalent for* $y^*\in X^*$.

(i) $y^* \in (I - T^*)X^*$;

(ii) $\sup_{k>0} \left\| \sum_{j=0}^{k-1} T^{*j} y^* \right\| < \infty$.

Proof. (ii) $\Rightarrow$ (i). Let $x_n^* = n^{-1} \sum_{k=1}^{n} \sum_{j=0}^{k-1} T^{*j} y^*$. Then $\{x_n^*\}$ is bounded, hence is relatively compact in the weak-* topology. Let x^* be a weak-* closure point of $\{x_n^*\}$. For $y \in X$ there is a sequence $\{n_j\}$ with

$$\langle (I - T^*)x^*, y\rangle = \langle x^*, (I - T)y\rangle = \lim \langle x_{n_j}^*, (I - T)y\rangle =$$

$$= \lim \langle (I - T^*)x_{n_j}^*, y\rangle = \lim \langle y^* - n_j^{-1} \sum_{k=1}^{n} T^{*k} y^*, y\rangle = \langle y^*, y\rangle .$$

Hence $(I - T^*)x^* = y^*$.

As an application of Theorem 5 we have the following corollary, which, in the measure preserving case, was proved by Browder [1, Theorem 2] by using a different method.

COROLLARY 6. *Let (S, Σ, μ) be a σ-finite measure space, and θ a non-singular measurable transformation of S. Then $f \in L_\infty$ is of the form $f(s) = g(s) - g(\theta s)$, with $g \in L_\infty$, if and only if* $\sup_{k\geq 1} \left\| \sum_{j=0}^{k-1} f \circ \theta^j \right\|_\infty < \infty$.

Proof. On $X = M(S, \Sigma, \mu)$, the space of finite signed measures absolutely continuous with respect to μ, define $T\nu$ by $T\nu(A) = \nu(\theta^{-1}A)$. Then $X^* = L_\infty$, and $T^*f(s) = f(\theta s)$, and Theorem 5 applies.

The following result was conjectured by M. Keane and J. Aaronson for T positive.

THEOREM 7. *Let (S, Σ, μ) be a σ-finite measure space, and let T be a contraction on $L_1(S, \Sigma, \mu)$. Then $f \in L_1$ is of the form $f = (I - T)g$ with $g \in L_1$ if and only if* $\sup_{k\geq 1} \left\| \sum_{j=0}^{k-1} T^j f \right\|_1 < \infty$.

Proof. We identify $L_1(S, \Sigma, \mu)$, via the Radon-Nikodym theorem, with the space $M(S, \Sigma, \mu)$ of countably additive measures $\ll \mu$. Then we have $\sup_{k\geq 1} \left\| \sum_{j=0}^{k-1} T^j \nu \right\| < \infty$, with $d\nu = f d\mu$.

T^{**} acts on $L_\infty(S, \Sigma, \mu)^* = \text{ba}(S, \Sigma, \mu)$, the space of bounded finitely additive measures (= charges). By Theorem 5 (applied to ν in L_∞^* and T^{**}), there exists $\eta \in \text{ba}(S, \Sigma, \mu)$ with $(I - T^{**})\eta = \nu$. Decompose [21] $\eta = \eta_1 + \eta_2$, with η_1

countably additive and η_2 a pure charge (i.e., $|\eta_2|$ does not bound any non-negative measure). Then

$$\nu = (I - T^{**})\eta = \eta_1 - T^{**}\eta_1 + \eta_2 - T^{**}\eta_2 .$$

Since $T^{**}\eta_1 = T\eta_1 \in M(S, \Sigma, \mu)$, we obtain that $\nu_1 = \eta_2 - T^{**}\eta_2$ is countably additive. Hence $\|\eta_2\| \geqslant \|T^{**}\eta_2\| = \|\eta_2 - \nu_1\| = \|\eta_2\| + \|\nu_1\|$ since $\|T^{**}\| \leqslant 1$, while η_2 (a pure charge) and ν_1 (a measure) are mutually singular [21]. Thus $\nu_1 = 0$ and $\nu = (I - T^{**})\eta_1 = (I - T)\eta_1$, yielding $g = \frac{d\eta_1}{d\mu}$ as a required solution.

In the next proposition, Theorem 5 cannot be applied, since the space $B(S, \Sigma)$ of bounded measurable functions is not a dual space, in general.

PROPOSITION 8. *Let (S, Σ) be a measurable space, and θ a measurable transformation of S into itself. Then $f \in B(S, \Sigma)$ is of the form $f(s) = g(s) - g(\theta s)$, with $g \in B(S, \Sigma)$, if and only if* $\sup_{k>1} \left\| \sum_{j=0}^{k-1} f(\theta^j s) \right\| < \infty$.

Proof. For f satisfying the condition, define

$$g(s) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} f(\theta^j s).$$

Since $\left\| \frac{1}{n} \sum_{j=0}^{n-1} f \circ \theta^j \right\|_\infty \to 0$, we obtain

$$g(\theta s) = g(s) - f(s).$$

REMARKS. 1. The previous proof gives also a direct proof of Corollary 6.
2. In Corollary 6, if θ is recurrent, a function g can be obtained by setting $g(s) = \sup_{k\geqslant 0} \sum_{j=0}^{k} f(\theta^j s)$ (see the first and last paragraphs of the proof of Theorem 9).

EXAMPLE 3. There exists a compact metric space S, a uniquely ergodic continuous map φ such that $\varphi^n s$ converges for every $s \in S$, and a function $f \in C(S)$ with $\sup_k \left\| \sum_{j=0}^{k-1} f(\varphi^j s) \right\|_\infty < \infty$, such that for every $g \in C(S)$, $g(s) - g(\varphi s) \not\equiv f(s)$.

Proof. Let T' be an operator as in Example 1, on Y. Let $T = \frac{1}{2}(I + T')$. Then $I - T = \frac{1}{2}(I - T')$, so T is mean ergodic too, on X, and T^n converges strongly to zero on X ($\|T^n(I-T)\| = \|2^{-n+1}(I+T')^n(I-T')\| \to 0$). Now T yields also an example of (iv) $\not\Rightarrow$(i). Let S be the unit ball of X^* and the weak-* topology, φ is the restriction of T^* to S and for $s \in S \subset X^*$, $f(s) := \langle s, y\rangle$, where y satisfies (iv). Hence $\sup \left|\sum_{j=0}^{k-1} f(\varphi^j s)\right| = \sup \left\|\sum_{j=0}^{k-1} T^j y\right\| < \infty$. Now $\|T^n x\| \to 0$ for every $x \in X$ yields $\varphi^n(s) \to 0$ for every $s \in S$. Hence φ is uniquely ergodic and the operator $Ah(s) = h(\varphi s)$ is mean ergodic on $C(S)$, since $A^n h \to h(0)$ weakly (= pointwise). If $f \in (I-A)C(S)$, we must have, by Theorem 1(iii), that $g_n = n^{-1}\sum_{k=1}^{n}\sum_{j=0}^{k-1} A^j f$ converges strongly. But

$$g_n(s) = n^{-1}\sum_{k=1}^{n}\sum_{j=0}^{k-1} f(\varphi^j s) = n^{-1}\sum_{k=1}^{n}\sum_{j=0}^{k-1}\langle y, T^{*j}s\rangle =$$

$$= \left\langle n^{-1}\sum_{k=1}^{n}\sum_{j=0}^{k-1} T^j y, s\right\rangle,$$

and the right-hand side does not converge uniformly on S, by the choice of T and y. Hence $f \notin (I-A)C(S)$.

THEOREM 9. *Let φ be a continuous map of a topological Hausdorff space S into itself, such that $\{\varphi^n s : n > 0\}$ is dense in S for every $s \in S$. Then $f \in C(S)$ is of the form $f(s) = g(s) - g(\varphi s)$, with $g \in C(S)$, if and only if* $\sup_{k>0}\left|\sum_{j=0}^{k} f(\varphi^j s)\right| < \infty$.

Proof. We have to prove only the "if" part. Define $g(s) = \sup_{k\geq 0} \sum_{j=0}^{k} f(\varphi^j s)$. Then

$$g(\varphi s) = \sup_{k\geq 0}\sum_{j=1}^{k+1} f(\varphi^j s) = \sup_{k\geq 1}\sum_{j=0}^{k} f(\varphi^j s) - f(s).$$

If $g(s) = f(s)$, then $g(\varphi s) \leq 0$, so $g^+(\varphi s) = 0 = g(s) - f(s)$. If $g(s) > f(s)$, then $g(\varphi s) = g(s) - f(s) > 0$, so in any case we have $g^+(\varphi s) = g(s) - f(s)$.

Our purpose now is to show the continuity of g. We say that a function h has a jump of at least δ at s_0 if for every $\varepsilon > 0$ and U open containing s_0 there are s', s'' in U with $|h(s') - h(s'')| > \delta - \varepsilon$. If $J_\delta(h)$ is the set of points where h has a jump of least δ, then $J_\delta(h)$ is clearly closed. It is easy to show that $J_\delta(h^+) \subset J_\delta(h)$.

CLAIM 1. $\varphi(J_\delta(g)) \subset J_\delta(g)$.

We show that for $s_0 \in J_\delta(g)$, $\varphi s_0 \in J_\delta(g^+)$, which is enough. Let U be open with $\varphi s_0 \in U$, and let $\varepsilon > 0$. Since f is continuous, there is V open with $|f(s) - f(s_0)| < \dfrac{\varepsilon}{4}$ for $s \in V$. Let $W = \varphi^{-1}(U) \cap V$. It contains s_0, so there are s',

s'' in W with $|g(s')-g(s'')|>\delta-\frac{\varepsilon}{2}$. But $\varphi s'$ and $\varphi s''$ are in U, and using $g^+(\varphi s)=g(s)-f(s)$ we obtain

$$|g^+(\varphi s')-g^+(\varphi s'')|=|g(s')-g(s'')-[f(s)-f(s'')]|>\delta-\frac{\varepsilon}{2}-2\frac{\varepsilon}{4}=\delta-\varepsilon.$$

CLAIM 2. $J_\delta(g)=\emptyset$.

By Claim 1 J_δ is closed invariant for φ. If $J_\delta\neq\emptyset$, there is $s_0\in J_\delta$ and $\{\varphi^n s_0\}\subset J_\delta$, so $J_\delta=S$. By definition, g is lower semicontinuous, i.e., $\{g>\alpha\}$ is open for every α. Let $\alpha_0=\inf\{g(s):s\in S\}$, $0<\beta<\delta$. If $J_\delta=S$, then every open set $\neq\emptyset$ contains two points s', s'' with $|g(s')-g(s'')|>\beta$. Now $\{g>\alpha_0\}$ is open and non-empty (or $g\equiv\alpha_0$ and $J_\delta=\emptyset$). Hence there are points $s',s''\in\in\{g>\alpha_0\}$. Hence $\{g>\alpha_0+\beta\}$ is not empty. Similarly $\{g>\alpha_0+n\beta\}\neq\emptyset$ for every n, contradicting the boundedness of g.

We have $J_\delta(g)=\emptyset$ for every $\delta>0$, hence g is continuous. Now $g(\varphi s)\leqslant\leqslant g^+(\varphi s)=g(s)-f(s)$, so that $h(s)\equiv g(s)-g(\varphi s)-f(s)\geqslant 0$ is continuous non-negative. But

$$\sum_{j=0}^{k}h(\varphi^j s)=g(s)-g(\varphi^{k+1}s)-\sum_{j=0}^{k}f(\varphi^j s),$$

so that $\sum_{j=0}^{\infty}h(\varphi^j s)<\infty$ for every $s\in S$. But our condition on φ implies that $\varphi^n s$ enters every non-empty open set infinitely many times. If $\left\{h>\frac{1}{n}\right\}$ is entered infinitely many times, $\sum_{j=0}^{\infty}h(\varphi^j s)=\infty$, a contradiction. Hence $\left\{h>\frac{1}{n}\right\}=\emptyset$ and $h\equiv 0$, so that $f(s)=g(s)-g(\varphi s)$.

COROLLARY 10. *Let φ be as in the previous theorem and $f\in C(S)$. If $\sup_{k\geqslant 0}\left|\sum_{j=0}^{k}f(\varphi^j s_0)\right|\neq\infty$, then there is a $g\in C(S)$ with $f(s)=g(s)-g(\varphi s)$.*

Proof. We prove $\sup_{k\geqslant 0}\left\|\sum_{j=0}^{k}f(\varphi^j s)\right\|<\infty$. Let $s_0\in S$ satisfy $\sup_{k\geqslant 0}\left|\sum_{j=0}^{k}f(\varphi^j s_0)\right|==\alpha<\infty$. Then, for every m and n, we have $\left|\sum_{j=m}^{n}f(\varphi^j s_0)\right|\leqslant 2\alpha$. Now $\left\{s\in S:\sup_{n,m}\left|\sum_{j=m}^{n}f(\varphi^j s)\right|\leqslant\alpha\right\}$ is closed, φ-invariant, and non-empty. Hence it is all of S.

REMARKS. 1. Theorem 9 for the compact case appears in Gottschalk and Hedlund [15, 14.11] with a different proof. Browder [1] generalized their approach in order to obtain it in the general case treated here. The problem is treated (in disguise) also by Furstenberg [10, p. 162].

2. A result of Gottschalk [14] shows that if S is locally compact and φ is minimal, then in fact S must be compact.

3. Corollary 10 for the compact case, with a proof which generalizes that of [15], appears in Furstenberg, Keynes and Shapiro [13, Lemma 2.2], and in Shapiro 20, Theorem 2.3].

4. Our proof is more direct, since it is based on the fact that if $f(s) = g(s) - g(\varphi s)$, with $\inf\{g(s) : s \in S\} = 0$, then the minimality of φ implies that

$$\max_{0 \leqslant k \leqslant n} \sum_{j=0}^{k} f(\varphi^j s) = \max_{0 \leqslant k \leqslant n} [g(s) - g(\varphi^{k+1}s)] = g(s) - \min_{0 \leqslant k \leqslant n} g(\varphi^{k+1}s)$$

must converge everywhere to g. If S is compact the convergence is uniform, by Dini's theorem.

Claim 1 in our proof of continuity in Theorem 9 is a simplification of a method used by Furstenberg [11] for a different functional equation (which he attributes to Kakutani in [12]). Claim 2 avoids Baire's theorem (used in [11]), and allows general spaces.

The analogue of the previous corollary for non-singular transformations is easier:

THEOREM 11. *Let (S, Σ, μ) be a σ-finite measure space, and θ a non-singular transformation of S, which is conservative and ergodic* (i.e., *$\theta(A) \subset A$ implies $\mu(A) = 0$ or $\mu(S \setminus A) = 0$). If f is* a.e. *finite and satisfies $\mu\left\{s : \sup_{k \geqslant 0} \left|\sum_{j=0}^{k} f(\theta^j s)\right| < \infty\right\} > 0$, then there is a $g \in L_\infty$ with $f(s) = g(s) - g(\theta s)$* a.e. *(hence $f \in L_\infty$).*

Proof. Let $g_k(s) = \sum_{j=0}^{k-1} f(\theta^j s)$. We show $\sup_{k \geqslant 1} |g_k(s)|$ finite a.e. .

Let $A = \{s : \sup_k |g_k(s)| < \infty\}$. Then $g_k(\theta s) = g_{k+1}(s) - f(s)$ shows that $\theta s \in A$ for $s \in A$, and $\mu(A) > 0$ implies $\mu(S \setminus A) = 0$. Hence for a.e. s, $g_k(s)$ is bounded, yieldin g$k^{-1}g_k(s) \to 0$ a.e. . Now let $g(s) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_k(s)$. Then

$$g(\theta s) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_k(\theta s) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_{k+1}(s) - f(s) =$$

$$= g(s) - f(s) + \lim_{n\to\infty} [g_{n+1}(s) - g(s)]/n = g(s) - f(s).$$

We have to show now that g is bounded. We have $g(s)-g(\theta^k s)=$ $=\sum_{j=0}^{k-1} f(\theta^j s)$, hence $\sup_k |g(\theta^k s)| < \infty$ a.e..

Let $A_N = \{s : \sup_{k\geqslant 0} | g(\theta^k s) | \leqslant N\}$. Then $S = \bigcup_{N=1}^{\infty} A_N$ (mod μ), and $\mu(A_N) > 0$ for some N. But $\theta(A_N) \subset A_N$, hence $S = A_N$ (mod μ), and $|g(s)| \leqslant N$ a.e..

REMARKS. 1. The previous theorem may fail for a general conservative and ergodic Markov operator on L_∞. Let $\mu(S) = 1$, and define $Tf = \int f \mathrm{d}\mu$, for $f \in L_1$. If $f \in L_1$ with $\int f \mathrm{d}\mu = 0$, then $\left|\sum_{j=0}^{k-1} T^j f\right| = |f|$. But we may take $f \notin L_\infty$.

2. If θ is only conservative (i.e., $\theta^{-1}(A) \supset A \Rightarrow \theta^{-1}(A) = A$), the theorem may fail. (Examples are easy to construct.)

For the general set-up of Theorem 1, if $(I-T)x_i = y$, then $T(x_1-x_2) = $ $=x_1-x_2$, so uniqueness of solutions in the Banach space depends on the fixed points of T. We now look at a Markov operator on L_1, and study the finite solutions (not necessarily integrable) in a special case (see [8] for the extension of T).

DEFINITION. A positive contraction of $L_1(S, \Sigma, \mu)$ is called *conservative* if for $u > 0$ a.e., $u \in L_1$, we have $\sum_{j=0}^{\infty} T^j u(s) = \infty$ a.e..

THEOREM 12. *Let T be a conservative positive contraction on $L_1(S, \Sigma, \mu)$, and let $f \in L_1$. Let g_1 and g_2 be* a.e. *finite (measurable) functions satisfying $(I-T)g_i = f$. If*

$$(*) \qquad \lim_{n\to\infty} \frac{T^n |g_i|}{\sum_{j=0}^{n} T^j u} = 0 \quad \text{a.e.} \quad \textit{for some } 0 < u \in L_1,$$

then

$$T(g_2-g_1)^{\pm} = (g_2-g_1)^{\pm}, \quad \textit{and} \quad \left\| n^{-1} \sum_{j=0}^{n-1} T^j f \right\|_1 \to 0.$$

Proof. Let $g = g_2 - g_1$. Then $Tg = g$, hence $T|g| \geqslant |g|$, and $Tg^{\pm} \geqslant g^{\pm}$. (Since g need not be integrable, we cannot conclude equality immediately.)

Then $Tg^+ = g^+ + h$, with $0 < h < \infty$ a.e.. (By assumption, $T^n|g| < \infty$ a.e. for every n.) Hence also $Tg^- = g^- + h$.

Without loss of generality, we may and do assume $\mu(S) = 1$.

$$\sum_{i=0}^{n-1} T^i g^+ + \sum_{i=0}^{n-1} T^i h = \sum_{i=1}^{n} T^i g^+ \Rightarrow \sum_{i=0}^{n-1} T^i h + g^+ = T^n g^+.$$

We take the $u \in L_1$ with $u > 0$ a.e., for which (*) holds. Then

$$\frac{g^+ + \sum_{i=0}^{n-1} T^i h}{\sum_{i=0}^{n} T^i u} = \frac{T^n g^+}{\sum_{i=0}^{n} T^i u} \to 0 \quad \text{a.e.}\,.$$

Let $\Sigma_I = \{A \in \Sigma : T^* 1_A = 1_A \text{ a.e.}\}$. By the Chacon-Ornstein theorem (see [8])

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} T^i v}{\sum_{i=0}^{n} T^i u} = \frac{E(v \mid \Sigma_I)}{E(u \mid \Sigma_I)} \quad \text{a.e.,} \qquad \text{for } v \in L_1,$$

and therefore also for any finite $v \geqslant 0$. We conclude that $\dfrac{E(h \mid \Sigma_I)}{E(u \mid \Sigma_I)} = 0$, so $h = 0$ a.e., since $h \geqslant 0$. Hence $Tg^{\pm} = g^{\pm}$.

Now $(I - T)g_1 = f$ implies, using (*), that

$$0 = \lim_{n\to\infty} \frac{g - T^{n+1} g}{\sum_{i=0}^{n} T^j u} = \lim_{n\to\infty} \frac{\sum_{i=0}^{n} T^j f}{\sum_{i=0}^{n} T^i u} = \frac{E(f \mid \Sigma_I)}{E(g \mid \Sigma_I)}\,.$$

Hence $E(f \mid \Sigma_I) = 0$. Since all T^*-invariant functions in the conservative case are Σ_I-measurable, f is orthogonal to all T^*-invariant functions, hence is in $\overline{(I - T)L_1}$. Thus $\left\| n^{-1} \sum_{j=0}^{n-1} T^j f \right\| \to 0$.

COROLLARY 13. *Let T be as above. Let $f \in L_1$ satisfy* $\left\| n^{-1} \sum_{j=0}^{n-1} T^j f \right\|_1 \to 0$. *If $g_1 \geqslant 0$, $g_2 \geqslant 0$ satisfy $(I - T)g_i = f$, then $T(g_1 - g_2)^{\pm} = (g_1 - g_2)^{\pm}$.*

Proof. We show that (*) is satisfied for g_i:

$$\frac{g_i - T^n g_i}{\sum_{j=0}^{n-1} T^j u} = \frac{\sum_{j=0}^{n-1} T^j f}{\sum_{j=0}^{n-1} T^j u} \underset{n\to\infty}{\longrightarrow} \frac{E(f \mid \Sigma_I)}{E(u \mid \Sigma_I)} = 0.$$

(Since $E(f \mid \Sigma_I)$ must be zero.)

REMARKS. 1. Condition $(*)$ in Theorem 12 is a necessary and sufficient condition for obtaining $T|g| = |g|$ from $Tg = g$, for T conservative. If $T|g| = |g|$, then the proof of Corollary 13 shows that $(*)$ holds. The following example shows that $Tg = g$ does not imply $T|g| = |g|$. Define T on $\ell_1(Z)$ by $(Tu)_i = \frac{1}{2}(u_{i-1} + u_{i+1})$. Then $g_i = i$ defines an invariant function, but $T|g| \neq |g|$, since $(T|g|)_0 = 1$.

2. In Corollary 13 we have looked at the uniqueness of positive solutions g to $(I - T)g = f$, when $f \in \overline{(I - T)L_1}$. Fong and Sucheston [9, Theorem 2.4] proved that (in the conservative case) positive *integrable* solutions exist for a dense subset of $\overline{(I - T)L_1}$.

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MICHAEL LIN
Ben-Gurion University,
Beer-Sheva,
Israel.

ROBERT SINE
Department of Mathematics,
University of Rhode-Island,
Kingston, RI 02881,
U.S.A.

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