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QUASITRIANGULAR EXTENSIONS OF C^*-ALGEBRAS AND PROBLEMS ON JOINT QUASITRIANGULARITY OF OPERATORS

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1. INTRODUCTION

In this paper we shall be concerned with the question of when an n-tuple of operators on Hilbert space is jointly quasitriangular.

We recall that an n-tuple $(T_1, \ldots, T_n)$ of operators on a Hilbert space $\mathscr{H}$ is said to be jointly quasitriangular (cf. [5], [2], [19]) if there exists an increasing sequence $\{P_m\}$ of finite rank projections on $\mathscr{H}$, that tends strongly to the identity operator $1_{\mathscr{H}}$, and such that $\lim_{m\to\infty}\|(1 - P_m)T_kP_m\| = 0$, for $1 \leqslant k \leqslant n$. Since the separability of $\mathscr{H}$ is implicit in this definition, and if $\mathscr{H}$ is finite dimensional all n-tuples of operators on $\mathscr{H}$ are jointly quasitriangular, we assume throughout the paper that $\mathscr{H}$ is an infinite dimensional, separable Hilbert space. The set of all jointly quasitriangular n-tuples of operators on $\mathscr{H}$ will be denoted by $\mathbf{QT}_n$.

One of our main objectives in the present paper is to generalize, to the case on n-tuples of operators, various results obtained in [2], [3], and [4] for the case of single operators. This program was started in [29], where attention was focused in proving invariant subspace theorems for commuting n-tuples of operators. The n-tuples that we consider there are not necessarily commuting, and, since our aim in this paper is different, we obtain other generalizations that were not given in [29].

Given an operator T on $\mathscr{H}$, let $\omega_-(T)$ be the set of those complex numbers λ such that $T-\lambda$ is essentially left invertible and of negative Fredholm index. Alternatively, $\lambda \in \omega_-(T)$ if and only if $T - \lambda$ is essentially left invertible but $T-\lambda + K$ is right invertible for no compact operator K. It was shown by Douglas and Pearcy ([16], Theorem 2.2) that $\omega_-(T) = \emptyset$ whenever $T \in \mathbf{QT}_1$. Apostol, Foiaş and Voiculescu in ([2], Theorem 5.4) completed the spectral characterization of quasitriangularity by proving that the converse of this result is valid. They also succeeded in computing the distance from a non quasitriangular operator T to $\mathbf{QT}_1$ in terms of "thickness" of $\omega_-(T)$. It is reasonable to ask what are the natural generalizations of these results when one considers (not necessarily commuting) n-tuples $\mathbf{t} = (T_1, \ldots, T_n)$ of operators on $\mathscr{H}$. The extension of the notion of quasitriangularity to n-tuples of operators is straightforward and is already mentioned at the beginning of this paper. What is not so obvious is what should be the substitute for the set $\omega_-(\cdot)$ in the case of n-tuples of operators. One may probably need to extend the following three notions:

(#) The left essential spectrum.

(# #) The index outside the left essential spectrum.

(# # #) A certain order structure on the set of indices, so that one may speak of negative indices (or equivalently positive indices).

An alternative approach is to use the equivalent definition of $\omega_-(T)$ given previously, in which case one also needs to extend the notion of right spectrum. An important class of n-tuples on which this program could be tested is the class of essentially commuting n-tuples of essentially normal operators.

In Section 2 we introduce various notions of left and right (essential) spectra of n-tuples of operators and in Section 3 we prove a generalization of a result of Douglas and Pearcy using the alternative definition of $\omega_-(\cdot)$ mentioned previously (see Corollary 3.15). However, this is not the natural generalization that we are looking for, because, as Theorem 6.11 shows, the converse of Corollary 3.15 can not be valid for essentially commuting pairs of essentially normal operators. On the other hand we also prove in Theorem 6.4 (see also Remark 6.5) using the Hartogs's phenomenon of several complex variables, that the standard notion of index outside the essential spectrum (as defined in [13]) for essentially commuting pairs of essentially normal operators is not appropriate to reflect obstruction to quasitriangularity.

Inspired by the work of Brown, Douglas and Fillmore and its application to the classification of essentially commuting n-tuples of essentially normal operators (see [8], [9]) one is led to consider extension theory of C^*-algebras for the appropriate generalizations of the notions (#), (# #) and (# # #). Thus, in Section 5, we introduce the semigroup $\mathrm{Ext}_{qt}(\mathscr{A}; \mathscr{B})$ of equivalence classes of C^*-extensions of the ideal of compact operators by the C^*-algebras $\mathscr{A}$ which are quasitriangular with respect to the Banach subalgebra $\mathscr{B}$ of $\mathscr{A}$ (see the definition after Lemma 5.7). The general program is to characterize $\mathrm{Ext}_{qt}(\mathscr{A}; \mathscr{B})$ in terms of $\mathscr{A}$ and $\mathscr{B}$. To this end, we also introduce the subsemigroup $\mathrm{K}_1(\mathscr{A}; \mathscr{B})$ of $\mathrm{K}_1(\mathscr{A})$ (see the definition after Theorem 5.10). Using index theory, as is done in [9], one can define a homomorphism $\varkappa: \mathrm{Ext}(\mathscr{A}) \to \mathrm{Hom}(\mathrm{K}_1(\mathscr{A}); \mathbf{Z})$. The result of Douglas and Pearcy generalizes as follows: If $[\tau] \in \mathrm{Ext}_{qt}(\mathscr{A}; \mathscr{B})$ then $\varkappa[\tau]$ is non-negative on $\mathrm{K}_1(\mathscr{A}; \mathscr{B})$ (see Lemma 5.12). A natural question is in what cases the converse is valid. An equivalent formulation of this problem for essentially commuting pairs of essentially normal operators is the following: Assume that (T_1, T_2) is an essentially commuting pair of essentially normal operators such that every 2×2 matrix whose entries are polynomials in T_1, T_2 is quasitriangular; does it follow that $(T_1, T_2) \in \mathbf{QT}_2$? We conjecture that the answer to this problem is affirmative and we present some partial results in Section 6. However, the solution to this problem is far from being complete. We should also point out that if one drops the assumption that (T_1, T_2) is an essentially commuting pair of essentially normal operators then the question has a negative answer. For instance, take an operator T on $\mathscr{H}$ such that $(T, T^*) \notin \mathbf{QT}_2$ (i.e. T is non-quasidiagonal) and such that every $n \times n$ matrix whose entries are polynomials in T, T^* is quasitriangular (see Remark 5.13, Part A).

In order to make this paper more accessible to readers in Operator Theory, we have adopted an expository style. In §4, for instance, we have stated several facts which are classic results in the theory of functions of several complex variables, and we have made some remarks, which are perhaps less standard, that are needed in the proof of Theorem 4.7 and in §6.

The organization of the paper is as follows. We start, in §2, with some remarks concerning the joint left essential spectrum of an n-tuple of operators. In this section, we study certain properties of the left resolvent functions $\delta(\cdot)$ and $\delta_e(\cdot)$, which we use later, in §3, to estimate the distance from an arbitrary n-tuple of operators to $\mathbf{QT}_n$. We devote §3 to study the various moduli of quasitriangularity $q(\cdot), q'(\cdot)$ and $q^{\#}(\cdot)$. These quantities represent the distance from an n-tuple of operators to $\mathbf{QT}_n$, in various equivalent norms. The main results, in this section, are Theorem 3.9 and Theorem 3.14. In §4, we introduce the pseudoconvex hull (also called Stein hull) of a compact subset of $\mathbf{C}^n$ and we obtain a refinement of Theorem 3.9 (see Theorem 4.7) employing a modification of an argument due to Voiculescu (see [29]). We reserve §5 to present some general facts concerning extensions of the ideal of compact operators on $\mathscr{H}$ by a separable C^*-algebra $\mathscr{A}$. We introduce the notion of quasitriangular extensions with respect to a closed subalgebra of $\mathscr{A}$, and we show that the natural topology of $\mathrm{Ext}(\mathscr{A})$ is given by a pseudometric. We also prove that the closure of the neutral element of $\mathrm{Ext}(\mathscr{A})$ is always a group (see Theorem 5.3). It should be pointed out that an example of a separable C^*-algebra $\mathscr{A}$ for which $\mathrm{Ext}(\mathscr{A})$ is not a group was given in [1]. In §6 we prove the main results of the paper. Using Theorem 4.7 and Hartogs' theorem (i.e. Theorem 4.1) we show that if an essentially commuting n-tuple $(T_1, \ldots, T_n)$ of essentially normal operators satisfies the property that its joint essential spectrum X is connected, and is contained in a contractible compact subset of $\mathbf{C}^n$ whose boundary is contained in X, then $(T_1, \ldots, T_n)$ is jointly quasitriangular (see Theorem 6.2). We then use this result and Hartogs' theorem to show that the n-tuple of Toeplitz operators associated with the coordinate functions on a bounded strongly pseudoconvex contractible domain with a smooth boundary is jointly quasitriangular (see Theorem 6.4). We further observe (see

Remark 6.5) that there are Fredholm n-tuples of Toeplitz operators of this kind with arbitrary index.

There are many questions left unanswered in the present paper, and they are pointed out throughout. The most important one, perhaps, is posed in Remark 6.8(a) and is concerned with the characterization of jointly quasitriangular, essentially commuting n-tuples of essentially normal operators.

We are indebted to Norberto Kerzman for some helpful conversations concerning the results in the theory of Several Complex Variables that we present in Section 4, and, we would like to thank Ronald Douglas for his useful suggestions concerning the material of Section 6. He pointed out to us the possibility of using Extension Theory to attack the classification problem of joint quasitriangular n-tuples of essentially commuting essentially normal operators. Also, we are indebted to the referee for his many helpful comments.

Finally, we would like to introduce some terminology which will be used throughout the paper. In what follows, $\mathscr{L}$ will denote the algebra of (bounded, linear) operators on $\mathscr{H}$, and $\mathscr{K}$ will denote the ideal of compact operators on $\mathscr{H}$. The Calkin algebra $\mathscr{L}/\mathscr{K}$ will be denoted by Q and π will denote the canonical quotient map. Elements in the space $\mathscr{L}^n$ $(= \mathscr{L} \otimes \mathbf{C}^n)$ will be denoted by $\mathbf{t}$ and it will be understood that $\mathbf{t} = (T_1, \ldots, T_n)$. Given A and B in $\mathscr{L}$ we let $A\mathbf{t}$ and $\mathbf{t}B$ be the n-tuples $(AT_1, \ldots, AT_n)$ and $(T_1B, \ldots, T_nB)$, respectively. We denote by $\mathbf{t}^*$ the n-tuple $(T_1^*, \ldots, T_n^*)$. An n-tuple $(\lambda_1, \ldots, \lambda_n)$ in $\mathbf{C}^n$ will be denoted by λ, and we let $|\lambda| = \left(\sum_{k=1}^{n} |\lambda_k|^2\right)^{1/2}$. We shall also identify λ with the n-tuple of operators $(\lambda_1 1_{\mathscr{H}}, \ldots$ $\ldots, \lambda_n 1_{\mathscr{H}})$. The Banach spaces $\mathscr{L}^n$ and Q^n will be provided with the norms $\|\mathbf{t}\| = \max_{1 \leqslant k \leqslant n} \|T_k\|$, $\|\mathbf{t}\|_e = \max_{1 \leqslant k \leqslant n} \|\pi T_k\|$, respectively, which converts them into C^*-algebras.

2. SOME REMARKS ON THE JOINT LEFT ESSENTIAL SPECTRUM

We begin this section by defining two bounded, linear transformations, canonically associated with an n-tuple of operators.

DEFINITION. Given $\mathbf{t} \in L^n$, let $D_\mathbf{t} \colon \mathscr{H} \to \mathscr{H} \otimes \mathbf{C}^n$, $D_\mathbf{t}^\# \colon \mathscr{H} \otimes \mathbf{C}^n \to \mathscr{H}$ be the bounded linear transformations defined by $D_\mathbf{t}(x) = (T_1(x), T_2(x), \ldots, T_n(x))$, $x \in \mathscr{H}$, $D_\mathbf{t}^\#(x_1, \ldots, x_n) = \sum_{k=1}^{n} T_k(x_k)$, $(x_1, \ldots, x_n) \in \mathscr{H} \otimes \mathbf{C}^n$, respectively.

The following properties follow immediately from the definition given above. For every $\mathbf{t} \in \mathscr{L}^n$ we have

$$(D_\mathbf{t})^* = D_{\mathbf{t}^*}^\#,$$

$$|D_\mathbf{t}| = (D_\mathbf{t}^* D_\mathbf{t})^{1/2} = (D_{\mathbf{t}^*}^\# D_\mathbf{t})^{1/2} = \left(\sum_{k=1}^{n} T_k^* T_k\right)^{1/2},$$

$$\|\mathbf{t}\| \leqslant \|D_\mathbf{t}\| \leqslant \sqrt{n}\,\|\mathbf{t}\|,$$

and

$$\|D_t\| = \| \, |D_t| \, \| = \sup\{\alpha : \alpha \in \sigma(|D_t|)\}.$$

We shall denote by $\|D_t\|_e = \sup\{\alpha : \alpha \in \sigma_e(|D_t|)\}$. Here, and in what follows, $\sigma(T)$ and $\sigma_e(T)$ denote the spectrum and the essential spectrum of the operator T, respectively.

DEFINITION. Given $\mathbf{t} \in \mathscr{L}^n$, we define

$$\delta(\mathbf{t}) = \inf\{\alpha : \alpha \in \sigma(|D_t|)\} \quad \text{and} \quad \delta_e(\mathbf{t}) = \inf\{\alpha : \alpha \in \sigma_e(|D \, |)\}.$$

The proof of the following lemma is elementary, and can be obtained by applying [17, Theorem 1.1] to the operator $|D_t|$.

LEMMA 2.1. *Let* $\mathbf{t} \in \mathscr{L}^n$.

(a) $\delta(\mathbf{t}) = \inf\limits_{\substack{\|x\|=1 \\ x \in \mathscr{H}}} \|D_t(x)\|$. *In particular there exists a sequence of unit vectors* $\{x_m\} \subseteq \mathscr{H}$ *such that* $\delta(\mathbf{t}) = \lim\limits_{m\to\infty} \|D_t(x_m)\|$, *and for every sequence of unit vectors* $\{y_m\} \subseteq \mathscr{H}$ *we have* $\delta(\mathbf{t}) \leqslant \liminf\limits_{m\to\infty} \|D_t(y_m)\|$.

(b) $\delta_e(\mathbf{t}) = \sup\limits_{\substack{\mathscr{M} \subseteq \mathscr{H} \\ \operatorname{codim} \mathscr{M} < \infty}} \delta(\mathbf{t}|\mathscr{M})$. *Here (onwards),* $\mathbf{t}|\mathscr{M}$ *is the n-tuple of bounded linear transformations from the subspace* $\mathscr{M}$ *into* $\mathscr{H} \otimes \mathbf{C}^n$ *given by* $\mathbf{t}|\mathscr{M} = (T_1|\mathscr{M}, \ldots, T_n|\mathscr{M})$. *In particular there exists an orthonormal sequence* $\{x_m\} \subseteq \mathscr{H}$ *such that* $\delta_e(\mathbf{t}) = \lim\limits_{m\to\infty} \|D_t(x_m)\|$, *and for every orthonormal sequence* $\{y_m\} \subseteq \mathscr{H}$, $\delta_e(\mathbf{t}) \leqslant \liminf\limits_{m\to\infty} \|D_t(y_m)\|$.

LEMMA 2.2. *Let* $\mathbf{t} \in \mathscr{L}^n$. *Then* $0 \leqslant \delta(\mathbf{t}) \leqslant \delta_e(\mathbf{t}) \leqslant \|D_t\|_e \leqslant \|D_t\|$. *Furthermore,*

(a) *if* $\dim(\ker D_t) \leqslant \dim(\ker D_t^*)$, *then there exists* $\mathbf{k} \in \mathscr{K}^n$ *such that* $\delta_e(\mathbf{t}) = \delta(\mathbf{t} + \mathbf{k})$;

(b) *there exists* $\mathbf{k}' \in \mathscr{K}^n$ *such that* $\|D_t\|_e = \|D_{\mathbf{t}+\mathbf{k}'}\|$.

Proof. The first inequalities are obvious. To prove (a) we observe that we may assume $\delta_e(\mathbf{t}) > 0$. Since $0 \notin \sigma_e(|D_t|)$, we know that $|D_t|$ is Fredholm and hence D_t is left semi-Fredholm. Let $E(\,\cdot\,)$ be the spectral measure associated with $|D_t|$ on $\mathscr{H}$ and let $\alpha = \delta_e(\mathbf{t})$. This means that $(\alpha - |D_t|)E([0, \alpha])$ is in $\mathscr{K}$. Because of the assumption, we see that there exists $\mathbf{v} \in \mathscr{L}^n$ such that D_v is an isometry and $D_t = D_v|D_t|$. Let $\mathbf{k} = \mathbf{v}(\alpha - |D_t|)E([0, \alpha])$. Then $\mathbf{k} \in \mathscr{K}^n$ and for every $x \in \mathscr{H}$, $\|x\| = 1$, we have

$$\|D_{\mathbf{t}+\mathbf{k}}(x)\| = \|[D_{\mathbf{v}|D_t|} + D_{\mathbf{v}(\alpha - |D_t|)E([0,\alpha])}](x)\| =$$

$$= \|D_v[|D_t|E([\alpha, \|D_t\|]) + \alpha E([0, \alpha))](x)\| \geqslant \alpha\|x\| = \delta_e(\mathbf{t}).$$

So $\delta(\mathbf{t} + \mathbf{k}) \geqslant \delta_e(\mathbf{t})$. Now for the proof of the last part of the statement we let $\beta = \|D_t\|_e$. It follows that $(|D_t| - \beta)E((\beta, \|D_t\|]) \in \mathscr{K}$. Let $D_t = D_w|D_t|$ be the polar decomposition of D_t where D_w is a partial isometry, $w \in \mathscr{L}^n$, and let $k' = w(|D_t| - \beta)E((\beta, \|D_t\|])$. Then

$$\|D_{t+k'}\| = \|D_w[|D_t| E([0,\beta]) + \beta E((\beta, \|D_t\|])]\| \leqslant$$

$$\leqslant \|[|D_t| E([0, \beta]) + \beta E((\beta, \|D_t\|])]\| \leqslant \beta = \||D_t|\|_e .$$

REMARK 2.3. From the upper semi-continuity of the spectrum and the essential spectrum we see that the functions $\lambda \to \delta(t - \lambda)$ and $\lambda \to \delta_e(t - \lambda)$, $\lambda \in \mathbf{C}^n$ are lower semi-continuous. We also note that if $\|\lambda\| (= \|D_\lambda\|) \geqslant \|D_t\|$, then for every unit vector $x \in \mathscr{H}$ we have

$$\|D_{t-\lambda}(x)\| \geqslant \|\lambda\| - \|D_t(x)\| \geqslant \|\lambda\| - \|D_t\|,$$

so $\|\lambda\| - \|D_t\| \leqslant \delta(t - \lambda)$. We deduce that the functions $\delta(t - \lambda)$ and $\delta_e(t - \lambda)$, $\lambda \in \mathbf{C}^n$ achieve their minimum value on non-empty compact subsets of $\mathbf{C}^n$. However, these minimum values may be greater than zero. Indeed, let

$$T_1 = \begin{pmatrix} 0 & 1_{\mathscr{H}} \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 \\ 1_{\mathscr{H}} & 0 \end{pmatrix}$$

and $t = (T_1, T_2)$. An elementary calculation shows that

$$1/2 = \delta(t - (1/2, 1/2)) = \delta_e(t - (1/2, 1/2)) = \inf_{\lambda \in \mathbf{C}^2} \delta_e(t - \lambda).$$

DEFINITION. Given $t \in \mathscr{L}^n$ we define the (possibly empty) compact subsets of $\mathbf{C}^n$,

$$\sigma_l(t) = \{\lambda \in \mathbf{C}^n : \delta(t - \lambda) = 0\},$$

and

$$\sigma_{le}(t) = \{\lambda \in \mathbf{C}^n : \delta_e(t - \lambda) = 0\}.$$

Also, we let

$$\sigma_r(t) = \{\lambda \in \mathbf{C} : \bar{\lambda} \in \sigma_l(t^*)\},$$

and

$$\sigma_{re}(t) = \{\lambda \in \mathbf{C} : \bar{\lambda} \in \sigma_{le}(t^*)\}.$$

The proof of the following proposition follows from Lemma 2.1 and the fact that the bounded linear transformation D_t is (essentially) left invertible if and only if the same holds for the operator $|D_t|$ in $\mathscr{L}$ (see also [11, Theorem 4.1]). It also serves as a characterization of the sets $\sigma_l(t)$ and $\sigma_{le}(t)$ when they are not empty.

LEMMA 2.4. *Let* $\mathbf{t} \in \mathscr{L}^n$.

(a) *The following conditions are equivalent:*

(a_1) $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_l(\mathbf{t})$;

(a_2) $D_{\mathbf{t}-\lambda}$ *is not left invertible;*

(a_3) *there is no* $\mathbf{s} = (S_1, \ldots, S_n) \in \mathscr{L}^n$, *such that*

$$\sum_{k=1}^{n} S_k(T_k - \lambda_k) = 1_{\mathscr{H}};$$

(a_4) *there exists a state* φ *on the* C^**-algebra* $C^*(1, T_1, \ldots, T_n)$ *such that* $\varphi(T_k^* T_k) = \varphi(T_k^*)\varphi(T_k)$ and $\varphi(T_k) = \lambda_k$, $1 \leqslant k \leqslant n$; *and*

(a_5) *there exists a state* θ *on* $\mathscr{L}$ *such that* $\theta(AT_k) = \lambda_k \theta(A)$, *for al* $A \in \mathscr{L}$, $1 \leqslant k \leqslant n$.

(b) *The following conditions are equivalent:*

(b_1) $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_{le}(\mathbf{t})$;

(b_2) *either* $\operatorname{Ran} D_{\mathbf{t}-\lambda}$ *is not closed or* $\dim(\ker D_{\mathbf{t}-\lambda}) = \infty$;

(b_3) *there is no* $\mathbf{s} = (S_1, \ldots, S_n) \in \mathscr{L}^n$ *such that*

$$1_{\mathscr{H}} - \sum_{k=1}^{n} S_k(T_k - \lambda_k) \in \mathscr{K};$$

(b_4) *there exists a state* ψ *on the* C^**-algebra* $C^*(1, \pi(T_1), \ldots, \pi(T_n))$ *such that* $\psi(\pi(T_k)^*\pi(T_k)) = \psi(\pi(T_k)^*)\psi(\pi(T_k))$ *and* $\psi\pi(T_k) = \lambda_k$, $1 \leqslant k \leqslant n$, *where* $\pi: \mathscr{L} \to Q$ *is the Calkin map;*

(b_5) *there exists a state* η *on the Calkin algebra* Q *such that* $\eta(a\pi(T_k)) = \lambda_k \eta(a)$, *for every* $a \in Q$, $1 \leqslant k \leqslant n$;

(b_6) *there exists a projection* P *in* $\mathscr{L}$ *of infinite rank and nullity such that* $\pi(T_k) = \lambda_k \pi(P)$, $1 \leqslant k \leqslant n$;

(b_7) *for every* $\varepsilon > 0$, *there exists a projection* P_ε *in* $\mathscr{L}$ *of infinite rank and nullity such that* $\|(\mathbf{t} - \lambda)P_\varepsilon\| < \varepsilon$;

(b_8) *for every* $\varepsilon > 0$, *there exists a projection* P_ε *in* $\mathscr{L}$ *of infinite rank and nullity such that* $(T_k - \lambda_k)P_\varepsilon$ *is of trace class and* $\|(T_k - \lambda_k)P_\varepsilon\|_1 < \varepsilon$, $1 \leqslant k \leqslant n$, *where* $\|\cdot\|_1$ *denotes the trace norm.*

REMARK 2.5. (a) It follows that $\sigma_l(\mathbf{t}) \subseteq \prod_{k=1}^{n} \sigma_l(T_k)$ and $\sigma_{le}(\mathbf{t}) \subseteq \prod_{k=1}^{n} \sigma_{le}(T_k)$.

(b) If $\mathbf{t}$ is an n-tuple of essentially commuting operators in $\mathscr{L}$, i.e. $T_jT_k - T_kT_j \in \mathscr{K}$, for all $j, k \in \{1, \ldots, n\}$, it is well known that $\sigma_{le}(\mathbf{t})$ (and hence $\sigma_l(\mathbf{t})$) is nonempty. A direct proof can be given employing Lemma 2.4, the Gelfand-Naimark theorem, the G.N.S. construction and the fact that maximal left ideals in $\mathscr{L}$ are left kernels of states on $\mathscr{L}$ [14]. One can proceed as follows. First one proves that $\sigma_l(\cdot)$ enjoys the projection property for commuting operators acting on a (non-necessarily separable) Hilbert space $\mathcal{G}$. This means that if $S_1, \ldots, S_{n+1}$ are operators on $\mathcal{G}$ and $(\lambda_1, \ldots, \lambda_n) \in \sigma_1(S_1, \ldots, S_n)$, then there exists $\lambda_{n+1} \in \sigma_1(S_{n+1})$ such that $(\lambda_1, \ldots, \lambda_{n+1}) \in \sigma_1(S_1, \ldots, S_{n+1})$ (see [10]). One then represents the Calkin algebra as a C^*-algebra of operators on some Hilbert space $\mathcal{G}$, and apply the above result to conclude that the projection property also holds for $\sigma_{1e}(\cdot)$. The fact that $\sigma_{1e}(\mathbf{t})$ is not empty, for every essentially commuting n-tuple $\mathbf{t}$, follows from the fact that $\sigma_{1e}(T) \neq \emptyset$, for every $T \in \mathcal{L}$.

(c) We point out that for every $T \in \mathcal{L}$ the sets $\mathrm{R}(T), \mathrm{R}_e(T)$ as defined in [25] are given by $\mathrm{R}(T) = \{\lambda \in \mathbf{C} : (\lambda, \bar{\lambda}) \in \sigma_1(T, T^*)\}$ and $\mathrm{R}_e(T) = \{\lambda \in \mathbf{C} : (\lambda, \bar{\lambda}) \in \sigma_{1e}(T, T^*)\}$.

LEMMA 2.6. (a) *Assume that* $\sigma_1(\mathbf{t}) \neq \emptyset$ *and let* $\rho_1(\mathbf{t}) = \sup_{\lambda \in \sigma_1(\mathbf{t})} |\lambda|$.

(i) $\rho_1(\mathbf{t}) \leqslant \|D_{\mathbf{t}}\|$,

(ii) $\delta(\mathbf{t} - \lambda) \leqslant \inf_{\lambda' \in \sigma_1(\mathbf{t})} |\lambda - \lambda'|$, *for all* $\lambda \in \mathbf{C}^n$.

(a′) *Assume* $\sigma_{1e}(\mathbf{t}) \neq \emptyset$, *and let* $\rho_{1e}(\mathbf{t}) = \sup_{\lambda \in \sigma_{1e}(\mathbf{t})} |\lambda|$.

(i′) $\rho_{1e}(\mathbf{t}) \leqslant \|D_{\mathbf{t}}\|_e$,

(ii′) $\delta_e(\mathbf{t} - \lambda) \leqslant \inf_{\lambda' \in \sigma_{1e}(\mathbf{t})} |\lambda - \lambda'|$, *for all* $\lambda \in \mathbf{C}^n$.

Proof. Because of Lemma 2.2, (a′) ((i′)) follows from (a) ((i)). Also, employing Lemma 2.2 we observe that (a) ((ii)) and (a′) ((ii′)) have similar proofs. So we shall prove (a) ((i)) and (a′) ((ii′)). Let $\mu \in \sigma_1(\mathbf{t})$, $\nu \in \sigma_{1e}(\mathbf{t})$ be such that $|\mu| = \sup_{\lambda' \in \sigma_1(\mathbf{t})} |\lambda'|$, and $|\nu - \lambda| = \inf_{\lambda' \in \sigma_{1e}(\mathbf{t})} |\lambda - \lambda'|$. Because of Lemma 2.1 there exists a sequence of unit vectors $\{x_m\}$ and an orthonormal sequence $\{y_m\}$ in $\mathcal{H}$ such that $\lim_{m\to\infty} \|D_{\mathbf{t}-\mu}(x_m)\| = 0$, $\lim_{m\to\infty} \|D_{\mathbf{t}-\nu}(y_m)\| = 0$. Therefore

$$\delta_e(\mathbf{t} - \lambda) \leqslant \liminf_{m\to\infty} \|D_{\mathbf{t}-\lambda}(y_m)\| =$$

$$= \liminf_{m\to\infty} \|D_{\mathbf{t}-\nu+\nu-\lambda}(y_m)\| \leqslant$$

$$\leqslant \limsup_{m\to\infty} \|D_{\mathbf{t}-\nu}(y_m)\| + \limsup_{m\to\infty} \|D_{\lambda-\nu}(y_m)\| = |\lambda - \nu|,$$

and

$$\rho_1(\mathbf{t}) = |\mu| = \lim_{m\to\infty} \|D_\mu(x_m)\| = \lim_{m\to\infty} \|D_{\mu-\mathbf{t}+\mathbf{t}}(x_m)\| \leqslant$$

$$\leqslant \limsup_{m\to\infty} \|D_{\mathbf{t}-\mu}(x_m)\| + \limsup_{m\to\infty} \|D_{\mathbf{t}}(x_m)\| \leqslant \|D_{\mathbf{t}}\|.$$

QUESTION. If $\mathbf{t} \in \mathscr{L}^n$ consists of commuting hyponormal operators, is $\delta(\mathbf{t}-\lambda) = \inf_{\lambda' \in \sigma_1(\mathbf{t})} |\lambda - \lambda'|$, $\rho_1(\mathbf{t}) = \|D_{\mathbf{t}}\|$?

The following result gives a partial positive answer to this question.

LEMMA 2.7. *Let* $\mathbf{t} \in \mathscr{L}^n$ *be a double commuting n-tuple of quasinormal operators* i.e. $T_kT_j - T_jT_k = 0$, $T_jT_k^* - T_k^*T_j = 0$, $j \neq k$ *(double commuting property) and* $T_k(T_k^*T_k) = (T_k^*T_k)T_k$ *(quasinormality)*, $1 \leqslant k \leqslant n$. *Then*

(a) $\rho_1(\mathbf{t}) = \|D_{\mathbf{t}}\|$,

(b) $\delta(\mathbf{t}) = \inf_{\lambda \in \sigma_1(\mathbf{t})} |\lambda|$.

Proof. We recall (see [20, Problem 108]) that quasinormality implies that for $1 \leqslant k \leqslant n$ there exists a hyponormal partial isometry $W_k \in \mathscr{L}$ that commutes with $P_k = (T_k^*T_k)^{1/2}$ and such that $T_k = W_kP_k$. Now, the double commuting property implies that $P_jP_k = P_kP_j$, $P_jW_k = W_kP_j$, $W_jW_k = W_kW_j$, $j, k = 1, \ldots, n$. Let $\alpha = \inf\{\gamma: \gamma \in \sigma(|D_{\mathbf{t}}|)\}$, $\beta = \sup\{\gamma : \gamma \in \sigma(|D_{\mathbf{t}}|)\}$. By the standard functional calculus on the abelian C^*-algebra $C^*(1, P_1, \ldots, P_n)$ we can find characters φ and ψ on it such that $\alpha = \sum_{k=1}^{n} [\varphi(P_k)]^2$, $\beta = \sum_{k=1}^{n} [\psi(P_k)]^2$. Now, by the projection property of $\sigma_1(\cdot)$ [10, Proposition 1], and by [10, Proposition 7] we can find characters θ, η on $C^*(1, P_1, \ldots, P_n, W_1, \ldots, W_n)$ such that $\theta(P_k) = \varphi(P_k)$ and $\eta(P_k) = \psi(P_k)$, $1 \leqslant k \leqslant n$. Since $C^*(1, T_1, \ldots, T_n) \subseteq C^*(1, P_1, \ldots, P_n, W_1, \ldots, W_n)$ we have $\mu = \theta(\mathbf{t}) = \theta(\mathbf{wp}) = \theta(\mathbf{w})\theta(\mathbf{p})$, $\nu = \eta(\mathbf{t}) = \eta(\mathbf{wp}) = \eta(\mathbf{w})\psi(\mathbf{p})$. In particular, $|\mu| = \alpha$, $|\nu| = \beta$. Hence $\|D_{\mathbf{t}}\| = \|\,|D_{\mathbf{t}}|\,\| = \alpha = |\mu| \leqslant \rho_1(\mathbf{t})$ and $\delta(\mathbf{t}) = \beta = |\nu| \geqslant \inf_{\lambda \in \sigma_1(\mathbf{t})} |\lambda|$.

The following strengthening of Lemma 2.1 will be needed in the sequel.

LEMMA 2.8. *Let* $\mathbf{t} \in \mathscr{L}^n$, $\lambda^1, \ldots, \lambda^m \in \mathbf{C}^n$ *and* $\varepsilon > 0$. *Then there exists subspaces* $\mathscr{M}_j \subseteq \mathscr{H}$, $1 \leqslant j \leqslant m$ *such that* $\dim \mathscr{M}_j = \infty$, $\|D_{\mathbf{t}-\lambda^j}|\mathscr{M}_j\| < \delta_e(\mathbf{t}-\lambda^j) + \varepsilon$, $1 \leqslant j \leqslant m$ *and such that subspaces* $\mathscr{M}_i + T_k(\mathscr{M}_i)$ *and* $\mathscr{M}_j + T_k(\mathscr{M}_j)$ *are orthogonal* $i \neq j$ *and* $1 \leqslant k \leqslant n$.

Proof. Let $\{\mu_k^l : l = 1,2, \ldots\} \subseteq \mathbf{C}$, $1 \leqslant k \leqslant n$ be sequences $\mu_k^{(i-1)m+j} = \lambda_k^j$, for $i = 1,2, \ldots$, $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$. By Lemma 2.1, we can find inductively an orthonormal sequence $\{x_i\} \subseteq \mathscr{H}$ such that $\|D_{\mathbf{t}-\mu^j}(x_i)\| \leqslant \delta_e(\mathbf{t}-\mu^j) + \varepsilon$, $\langle T_k(x_i), x_{j+1}\rangle = \langle T_k^*(x_i), x_{j+1}\rangle = \langle T_k^*T_k(x_i), x_{j+1}\rangle = 0$ for $1 \leqslant i \leqslant j$, $1 \leqslant k \leqslant n$. Indeed, using Lemma 2.1 we find a unit vector $x_1 \in \mathscr{H}$ such that $\|D_{\mathbf{t}-\mu^1}(x_1)\| \leqslant \delta_e(\mathbf{t}-\mu^1) + \varepsilon$. Having defined $x_i \in \mathscr{H}$, $1 \leqslant i \leqslant j$, satisfying the above property, let $\mathscr{N}$ = closure of the span of $\{x_i, T_k(x_i), T_k^*(x_i), T_k^*T_k(x_i) : 1 \leqslant i \leqslant j, 1 \leqslant k \leqslant n\}$. Since $\dim \mathscr{N} < \infty$, by virtue of Lemma 2.1 we can choose a unit vector x_{j+1} in $\mathscr{N}$ such that $\|D_{\mathbf{t}-\mu^{j+1}}(x_{j+1})\| \leqslant \delta_e(\mathbf{t}-\mu^{j+1}) + \varepsilon$. Now we define $\mathscr{M}_j = [\mathrm{sp}\{x_{(i-1)m+j} : i = 1,2, \ldots\}]$. By construction $[\mathscr{M}_i + T_k(\mathscr{M}_i)]^-$ and $[\mathscr{M}_j + T_k(\mathscr{M}_j)]^-$

are orthogonal for $i \neq j$ and $1 \leqslant k \leqslant n$. If $z_j \in \mathscr{M}_j$, then

$$\|D_{\mathbf{t}-\lambda^j}(z_j)\|^2 = \sum_{k=1}^{n} \|(T_k - \lambda_k^j)z_j\|^2 = \sum_{k=1}^{n} \left\|(T_k - \lambda_k^j) \sum_{i=1}^{\infty} \langle z_j, x_{(i-1)m+j}\rangle x_{(i-1)m+j}\right\|^2 =$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{\infty} |\langle z_j, x_{(i-1)m+j}\rangle|^2 \|(T_k - \mu_k^{(i-1)m+j})x_{(i-1)m+j}\|^2 =$$

$$= \sum_{i=1}^{\infty} |\langle z_j, x_{(i-1)m+j}\rangle|^2 \|D_{\mathbf{t}-\mu^{(i-1)m+j}}(x_{(i-1)m+j})\|^2 \leqslant$$

$$\leqslant \sum_{i=1}^{\infty} (|\langle z_j, x_{(i-1)m+j}\rangle|^2 [\delta_e(\mathbf{t} - \mu^{(i-1)m+j}) + \varepsilon]^2) =$$

$$= (\delta_e(\mathbf{t} - \lambda^j) + \varepsilon)^2 \|z_j\|^2.$$

Hence, $\|D_{\mathbf{t}-\lambda^j} | \mathscr{M}_j\| \leqslant \delta_e(\mathbf{t} - \lambda^j) + \varepsilon$ and the proof is complete.

DEFINITION. Given $\mathbf{t} \in \mathscr{L}^n$, and employing the notation at the beginning of §2, we define

$$\delta^{\#}(\mathbf{t}) = \inf\{\alpha : \alpha \in \sigma(|D_{\mathbf{t}}^{\#}|)\}$$

and

$$\delta_e^{\#}(\mathbf{t}) = \inf\{\alpha : \alpha \in \sigma_e(|D_{\mathbf{t}}^{\#}|)\}.$$

Also, we let

$$\sigma_1^{\#}(\mathbf{t}) = \{\lambda \in \mathbf{C}^n : \delta^{\#}(\mathbf{t} - \lambda) = 0\},$$

and

$$\sigma_{1e}^{\#}(\mathbf{t}) = \{\lambda \in \mathbf{C}^n : \delta_e^{\#}(\mathbf{t} - \lambda) = 0\}.$$

REMARK 2.9. We observe that $|D_{\mathbf{t}}^{\#}|$ can be identified with the square root of the n by n operator matrix acting on $\mathscr{H} \otimes \mathbf{C}^n = \mathscr{H} \oplus \ldots \oplus \mathscr{H}$ (n times) whose (i, j)th entry is $T_i^* T_j$, $1 \leqslant i, j \leqslant n$. Thus, for every $\xi = (x_1, \ldots, x_n)$ belonging to $\mathscr{H} \otimes \mathbf{C}^n$, we have $\| |D_{\mathbf{t}}^{\#}| (\xi)\| = \|D_{\mathbf{t}}^{\#}(\xi)\|$. Thus, a result similar to Lemma 2.1 also holds for $\delta^{\#}(\mathbf{t})$ and $\delta_e^{\#}(\mathbf{t})$. On the other hand we point out that the sets $\sigma_1^{\#}(\mathbf{t})$ and $\sigma_{1e}^{\#}(\mathbf{t})$ are rather large. In fact, it is easy to show that $\sigma_1^{\#}(\mathbf{t}) \supseteq \prod_{k=1}^{n} \sigma_1(T_k)$ and $\sigma_{1e}^{\#}(\mathbf{t}) \supseteq \prod_{k=1}^{n} \sigma_{1e}(T_k)$.

Although the proof of the following lemma is similar to that of Lemma 2.2, we include it here because this result is central to our purpose.

LEMMA 2.10. *Let* $\mathbf{t} \in \mathscr{L}^n$. *Then* $0 \leqslant \delta^{\#}(\mathbf{t}) \leqslant \delta_e^{\#}(\mathbf{t})$. *Further, if* $\dim\ker D_{\mathbf{t}}^{\#} \leqslant \leqslant \dim\ker(D_{\mathbf{t}}^{\#})^*$, *then there exists* $\mathbf{k} \in \mathscr{K}^n$ *such that* $\delta^{\#}(\mathbf{t}+\mathbf{k}) = \delta_e^{\#}(\mathbf{t})$.

Proof. The first inequality is obvious. Thus, to prove the second assertion we can assume $\alpha = \delta_e^{\#}(\mathbf{t}) > 0$. Let $E(\cdot)$ be the spectral measure of $|D_{\mathbf{t}}^{\#}|$ on $\mathscr{H} \otimes \mathbf{C}^n$. Since $\alpha > 0$, it follows that $(\alpha - |D_{\mathbf{t}}^{\#}|)E([0,\alpha)) \in \mathscr{K}(\mathscr{H}\otimes\mathbf{C}^n)$. Because of the assumption, there exists $\mathbf{v} \in \mathscr{L}^n$ such that $D_{\mathbf{v}}^{\#}$ is an isometry and $D_{\mathbf{t}}^{\#} = D_{\mathbf{v}}^{\#}|D_{\mathbf{t}}^{\#}|$. Let $\mathbf{k} \in \mathscr{L}^n$ be such that $D_{\mathbf{k}}^{\#} = D_{\mathbf{v}}^{\#} \circ [(\alpha - |D_{\mathbf{t}}^{\#}|)E([0,\alpha))]$. Then $\mathbf{k} \in \mathscr{K}^n$ and for every $\xi = (x_1, \ldots, x_n) \in \mathscr{H} \otimes \mathbf{C}^n$ we have

$$\|D_{\mathbf{t}+\mathbf{k}}^{\#}(\xi)\| = \|D_{\mathbf{v}}^{\#} \circ [|D_{\mathbf{t}}^{\#}| + (\alpha - |D_{\mathbf{t}}^{\#}|)E([0,\alpha))]\,\xi\| =$$

$$= \|[|D_{\mathbf{t}}^{\#}|E([\alpha,\|D_{\mathbf{t}}^{\#}\|]) + \alpha E([0,\alpha))]\,\xi\| \geqslant \alpha\|\xi\| = \delta_e^{\#}(\mathbf{t})\|\xi\|.$$

This completes the proof of the lemma.

3. THE MODULUS OF QUASITRIANGULARITY

Throughout the rest of the paper $\mathscr{P}$ will denote the directed set of all finite rank projections in $\mathscr{L}$ and will be endowed with the metric $d(P,Q) = \min\{1, \|P-Q\|\}$, $P, Q \in \mathscr{P}$.

The following result is essentially contained in [19], and was explicitly proved in [26].

LEMMA 3.1. (cf. [26, Lemma 2.4]). *If* $v\colon \mathscr{P} \to \mathbf{R}$ *is a bounded uniformly continuous function, then for every* $\{P_m\} \subseteq \mathscr{P}$ *such that* $P_m \xrightarrow{s} I$ *we have* $\liminf\limits_{P\in\mathscr{P}} v(P) \leqslant \leqslant \liminf\limits_{m\to\infty} v(P_m)$. *Further, there exists an increasing* $\{P_m\} \subseteq \mathscr{P}$ *such that* $P_m \xrightarrow{s} I$ *and* $\liminf\limits_{P\in\mathscr{P}} v(P) = \lim\limits_{m\to\infty} v(P_m)$.

REMARK 3.2. Given $\mathbf{t} \in \mathscr{L}^n$, let $v_{\mathbf{t}}(P) = \|(1-P)\mathbf{t}P\|$ and $v'_{\mathbf{t}}(P) = \|D_{(1-P)\mathbf{t}P}\|$, $P \in \mathscr{P}$. Then

$$|v_{\mathbf{t}}(P) - v_{\mathbf{t}}(Q)| \leqslant \max_{1\leqslant k\leqslant n} |\,\|(1-P)T_kP\| - \|(1-Q)T_kQ\|\,| \leqslant$$

$$\leqslant \max_{1\leqslant k\leqslant n} \|(1-P)T_kP - (1-Q)T_kQ\| \leqslant$$

$$\leqslant \|(P-Q)\mathbf{t}\| + \|(1-Q)\mathbf{t}(P-Q)\| \leqslant 2\|\mathbf{t}\|\,\|P-Q\|.$$

Likewise,

$$|v'_{\mathbf{t}}(P) - v'_{\mathbf{t}}(Q)| = |\,\|D_{(1-P)\mathbf{t}P}\| - \|D_{(1-Q)\mathbf{t}Q}\|\,| \leqslant \|D_{(1-P)\mathbf{t}P} - D_{(1-Q)\mathbf{t}Q}\| \leqslant$$

$$\leqslant 2\,\|D_{\mathbf{t}}\|\,\|P-Q\|, \quad P, Q \in \mathscr{P}.$$

Observe that $0 \leqslant \nu_{\mathbf{t}}(P) \leqslant \|\mathbf{t}\|$, $0 \leqslant \nu'_{\mathbf{t}}(P) \leqslant \|D_{\mathbf{t}}\|$, and also $\nu_{\mathbf{t}}(P) \leqslant \nu'_{\mathbf{t}}(P) \leqslant \sqrt{n}\, \nu_{\mathbf{t}}(P)$, for every $P \in \mathscr{P}$. Therefore, $\nu_{\mathbf{t}}$ and $\nu'_{\mathbf{t}}$ are bounded and uniformly continuous functions on $\mathscr{P}$.

DEFINITION. Given $\mathbf{t} \in \mathscr{L}^n$ we define

$$q(\mathbf{t}) = \liminf_{P \in \mathscr{P}} \nu_{\mathbf{t}}(P), \quad \text{and} \quad q'(\mathbf{t}) = \liminf_{P \in \mathscr{P}} \nu'_{\mathbf{t}}(P).$$

LEMMA 3.3. *Given* $\mathbf{t} \in \mathscr{L}^n$, *the following conditions are equivalent.*

(a) $\mathbf{t}$ *is jointly quasitriangular*, i.e. *there exists an increasing* $\{P_m\} \subseteq \mathscr{P}$ *such that* $P_m \xrightarrow{s} I$ *and* $\lim_{m \to \infty} \|(1 - P_m)T_k P_m\| = 0$, *for* $1 \leqslant k \leqslant n$.

(b) $q(\mathbf{t}) = 0$.

(c) $q'(\mathbf{t}) = 0$.

(d) *there exists* $\{P_m\} \subseteq \mathscr{P}$ *such that* $P_m \xrightarrow{s} I$ *and* $\lim_{m \to \infty} \|(1 - P_m)T_k P_m\| = 0$ *for* $1 \leqslant k \leqslant n$.

(e) *There exists* $\mathbf{t}' = (T'_1, \ldots, T'_n) \in \mathscr{L}^n$ *and an increasing sequence* $\{P_m\} \subseteq \mathscr{P}$ *such that* $P_m \xrightarrow{s} I$, $T'_k P_m = P_m T'_k P_m$, $1 \leqslant k \leqslant n$, $m = 1,2 \ldots$ *and* $\mathbf{t} - \mathbf{t}' \in \mathscr{K}^n$

Proof. By virtue of Lemma 3.1 (e) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). To show that (a) $\Rightarrow$ (e) we need only choose an increasing $\{P_m\} \subseteq \mathscr{P}$, $P_m \xrightarrow{s} I$ such that $\sum_{m=1}^{\infty} \|(1 - P_m)T_k P_m\| < \infty$, for $1 \leqslant k \leqslant n$.

In the next two lemmas we describe some elementary properties of the functions q and q'.

LEMMA 3.4. *For every* $\mathbf{t} \in \mathscr{L}^n$ *we have*,

$$q(\mathbf{t}) \leqslant q'(\mathbf{t}) \leqslant \sqrt{n}\, q(\mathbf{t}), \quad 0 \leqslant q(\mathbf{t}) \leqslant \|t\|_e, \quad 0 \leqslant q'(\mathbf{t}) \leqslant \|D_{\mathbf{t}}\|_e$$

and for every $\mathbf{t}' \in \mathscr{L}^n$ *we also have*,

$$|q(\mathbf{t}) - q(\mathbf{t}')| \leqslant \|\mathbf{t} - \mathbf{t}'\|_e, \quad |q'(\mathbf{t}) - q'(\mathbf{t}')| \leqslant \|D_{\mathbf{t}} - D_{\mathbf{t}'}\|_e \leqslant \sqrt{n}\, \|\mathbf{t} - \mathbf{t}'\|,$$

and hence $q : \mathscr{L}^n \to \mathbf{R}$, $q' : \mathscr{L}^n \to \mathbf{R}$ *are uniformly continuous. In particular* $\mathbf{QT}_n$ *is closed in* $\mathscr{L}^n$.

Proof. The only non-trivial parts of the first assertion are the inequalities $q(\mathbf{t}) \leqslant \|\mathbf{t}\|_e$, and $q'(\mathbf{t}) \leqslant \|D_{\mathbf{t}}\|_e$. To prove these facts we note that

$$q(\mathbf{t}) = \liminf_{P \in \mathscr{P}} \|(1 - P)\mathbf{t}P\| \leqslant \limsup_{P \in \mathscr{P}} \|(1 - P)\mathbf{t}\| = \inf_{\mathbf{k} \in \mathscr{K}^n} \|\mathbf{t} - \mathbf{k}\| = \|\mathbf{t}\|_e.$$

Also, by Lemma 2.2 (b)

$$q'(\mathbf{t}) = \liminf_{P \in \mathscr{P}} \|D_{(1-P)\mathbf{t}P}\| \leqslant \limsup_{P \in \mathscr{P}} \|D_{(1-P)\mathbf{t}}\| = \inf_{\mathbf{k} \in \mathscr{K}^n} \|D_{\mathbf{t}+\mathbf{k}}\| = \|D_{\mathbf{t}}\|_e .$$

To prove the second assertion we observe that for every $P \in \mathscr{P}$,

$$\nu_{\mathbf{t}}(P) \leqslant \nu_{\mathbf{t}-\mathbf{t}'}(P) + \nu_{\mathbf{t}'}(P), \quad \nu'_{\mathbf{t}}(P) \leqslant \nu'_{\mathbf{t}-\mathbf{t}'}(P) + \nu'_{\mathbf{t}'}(P).$$

Thus, we have

$$q(\mathbf{t}) = \liminf_{P \in \mathscr{P}} \nu_{\mathbf{t}}(P) \leqslant \limsup_{P \in \mathscr{P}} \nu_{\mathbf{t}-\mathbf{t}'}(P) + \liminf_{P \in \mathscr{P}} \nu_{\mathbf{t}'}(P) \leqslant \|\mathbf{t} - \mathbf{t}'\|_e + q(\mathbf{t}').$$

Analogously, $q'(\mathbf{t}) \leqslant \|D_{\mathbf{t}} - D_{\mathbf{t}'}\|_e + q'(\mathbf{t}')$. Interchanging the roles of $\mathbf{t}$ and $\mathbf{t}'$ we obtain the desired inequalities.

Since the set of essentially commuting n-tuples is closed in $\mathscr{L}^n$, it follows from Lemma 3.4 that its intersection with $\mathbf{QT}_n$ is also closed. This result was already obtained in [24] employing a more complicated approach. In [24], it is also shown that the distance from essentially commuting n-tuple $\mathbf{t}$ to the set just mentioned is $q(\mathbf{t})$. In part (c) of the following lemma, we use a different method of proof to show that a similar result holds for arbitrary n-tuples.

LEMMA 3.5. *Let* $\mathbf{t} \in \mathscr{L}^n$.

(a) *For every* $\{P_m\} \subseteq \mathscr{P}$ *such that* $P_m \xrightarrow{s} I$ *we have*

$$(q\mathbf{t}) \leqslant \liminf_{m \to \infty} \nu_{\mathbf{t}}(P_m), \quad q'(\mathbf{t}) \leqslant \liminf_{m \to \infty} \nu'_{\mathbf{t}}(P_m).$$

Further, there exist increasing sequences $\{P_m\}$, $\{P'_m\} \subseteq \mathscr{P}$, *such that* $P_m \xrightarrow{s} I$, $P'_m \xrightarrow{s} I$ *and*

$$\lim_{m \to \infty} \nu_{\mathbf{t}}(P_m) = q(\mathbf{t}), \quad \lim_{m \to \infty} \nu'_{\mathbf{t}}(P'_m) = q'(\mathbf{t}).$$

(b) *If* $\mathbf{t}' \in \mathscr{L}^n$ *and* $\mathbf{t} - \mathbf{t}' \in \mathscr{K}^n$, *then* $q(\mathbf{t}) = q(\mathbf{t}')$, *and* $q'(\mathbf{t}) = q'(\mathbf{t}')$.

(c)

(*) $$q(\mathbf{t}) = \inf_{\mathbf{s} \in \mathbf{QT}_n} \|\mathbf{t} - \mathbf{s}\|,$$

(**) $$q'(\mathbf{t}) = \inf_{\mathbf{s} \in \mathbf{QT}_n} \|D_{\mathbf{t}} - D_{\mathbf{s}}\|.$$

Proof. Part (a) is a consequence of Lemma 3.1 and Remark 3.2, and (b) follows from (a). To prove (c), we first observe that $\leqslant$ in both (*) and (**) is a direct consequence of Lemma 3.4. For the opposite inequality we use the Arveson distance formula for quasitriangular algebras [5, Theorem 2.2]. We recall that if $\{Q_m\} \subseteq \mathscr{P}$ is an increasing sequence such that $Q_m \xrightarrow{s} I$, and $\mathscr{B} = \{S \in \mathscr{L} : (1 - Q_m)SQ_m = 0, m = 1, 2, \ldots\}$, then $\mathscr{B} + \mathscr{K}$ is norm closed in $\mathscr{L}$ and $\inf_{S \in \mathscr{B}+\mathscr{K}} \|T - S\| = \limsup_{m\to\infty} \|(1 - Q_m)TQ_m\|$, for every $T \in \mathscr{L}$ (Arveson's formula). We shall identify $\mathscr{L} \otimes \mathscr{M}_n$ and $\mathscr{K} \otimes \mathscr{M}_n$ with the algebra of n by n matrices whose entires are in $\mathscr{L}$ and $\mathscr{K}$, respectively. Here onwards $\mathscr{M}_n$ is the algebra of n by n complex matrices and 1_n is the identity matrix in $\mathscr{M}_n$. Given $\mathbf{t} \in \mathscr{L}^n$, we let $\tilde{\mathbf{t}}$ be the diagonal matrix in $\mathscr{L} \otimes \mathscr{M}_n$ whose k-th diagonal entry is T_k and we let $\tilde{\tilde{\mathbf{t}}}$ be the matrix in $\mathscr{L} \otimes \mathscr{M}_n$ all of whose entries are zero except for the first column whose k-th entry is T_k. We note that $\|\tilde{\mathbf{t}}\| = \|\mathbf{t}\|$ and $\|\tilde{\tilde{\mathbf{t}}}\| = \|D_{\mathbf{t}}\|$. Now, for a fixed $\mathbf{t} \in \mathscr{L}^n$ we let $\{P_m\}$ and $\{P'_m\} \subseteq \mathscr{P}$ be increasing sequences such that $P_m \xrightarrow{s} I$, $P'_m \xrightarrow{s} I$ and such that $q(\mathbf{t}) = \lim_{m\to\infty} \|(1 - P_m)\mathbf{t}P_m\|$, $q'(\mathbf{t}) = \lim_{m\to\infty} \|(1 - P'_m)\mathbf{t}P'_m\|$. We denote by $\mathscr{A}$ and $\mathscr{A}'$ the algebras $\mathscr{A} = \{S \in \mathscr{L} : (1 - P_m)SP_m = 0, m = 1, 2, \ldots\}$ and $\mathscr{A}' = \{S' \in \mathscr{L} : (1 - P'_m)S'P'_m = 0, m = 1, 2, \ldots\}$. We observe that

$$\mathscr{A} \otimes \mathscr{M}_n = \{R \in \mathscr{L} \otimes \mathscr{M}_n : [(1 - P) \otimes 1_n]R(P_m \otimes 1_n) = 0, \ m = 1, 2, \ldots\},$$

$$\mathscr{A}' \otimes \mathscr{M}_n = \{R' \in \mathscr{L} \otimes \mathscr{M}_n : [(1 - P'_m) \otimes 1_n]R'(P'_m \otimes 1_n) = 0, m = 1, 2, \ldots\}.$$

By the Arveson distance formula we have

$$\inf_{\mathbf{s} \in (\mathscr{A}+\mathscr{K})^n} \|\mathbf{t} - \mathbf{s}\| = \inf_{\mathbf{s} \in (\mathscr{A}+\mathscr{K})^n} \|\tilde{\mathbf{t}} - \tilde{\mathbf{s}}\| = \inf_{S \in (\mathscr{A}+\mathscr{K}) \otimes \mathscr{M}_n} \|\tilde{\mathbf{t}} - S\| =$$

$$= \limsup_{m\to\infty} \|[(1 - P_m)\mathbf{t}P_m]^{\sim}\| = \limsup_{m\to\infty} \|(1 - P_m)\mathbf{t}P_m\| = q(\mathbf{t}).$$

Analogously

$$\inf_{\mathbf{s}' \in (\mathscr{A}'+\mathscr{K})^n} \|D_{\mathbf{t}} - D_{\mathbf{s}'}\| = \inf_{\mathbf{s}' \in (\mathscr{A}'+\mathscr{K})^n} \|\tilde{\tilde{\mathbf{t}}} - \tilde{\tilde{\mathbf{s}}}'\| = \inf_{S' \in (\mathscr{A}'+\mathscr{K}) \otimes \mathscr{M}_n} \|\tilde{\tilde{\mathbf{t}}} - S'\| =$$

$$= \limsup_{m\to\infty} \|[(1 - P'_m) \otimes 1_n]\tilde{\tilde{\mathbf{t}}}(P'_m \otimes 1_n)\| = \limsup_{m\to\infty} \|D_{(1-P'_m)\mathbf{t}P'_m}\| = q'(\mathbf{t}).$$

Since $(\mathscr{A} + \mathscr{K})^n \subseteq \mathbf{QT}_n$ and $(\mathscr{A}' + \mathscr{K})^n \subseteq \mathbf{QT}_n$, the proof of the lemma is complete.

The following two properties were already observed in [29], for the function q.

LEMMA 3.6. *Let* $\mathbf{t} \in \mathcal{L}^n$ *and suppose there exists a directed family* $\{\mathcal{M}_\alpha \mid \alpha \in A\}$ *of invariant subspaces for* $\mathbf{t} = (T_1, \ldots, T_n)$ *such that* $\operatorname{span}_{\alpha \in A}\{\mathcal{M}_\alpha\}$ *is dense in* $\mathcal{H}$. *Then* $q'(\mathbf{t}) \leqslant \liminf_{\alpha \in A} q'(\mathbf{t} \mid \mathcal{M}_\alpha)$. *In particular, if* $\mathbf{t}$ *has a system of common eigenvalues which is total in* $\mathcal{H}$, *then* $\mathbf{t} \in \mathbf{QT}_n$.

Proof. Let P_α be the projection of $\mathcal{H}$ onto $\mathcal{M}_\alpha$, $\alpha \in A$. Then $P_\alpha \xrightarrow{s} I$. By the definition of lim inf, there exists an increasing sequence $\{P_{\alpha_m}\}$ such that $P_{\alpha_m} \xrightarrow{s} I$ and $\lim_{m \to \infty} q'(\mathbf{t}|\mathcal{M}_{\alpha_m}) = \liminf_{\alpha \in A} q'(\mathbf{t}|\mathcal{M}_\alpha)$. Let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}$ and for each $m = 1,2, \ldots$, choose a finite rank subprojection Q_m of P_{α_m} such that

$$\|(1 - Q_m)e_j\| < \|(1 - P_{\alpha_m})e_j\| + \frac{1}{m}, \quad 1 \leqslant j \leqslant m$$

and

$$\|D_{(P_{\alpha_m} - Q_m)\mathbf{t}Q_m}\| \leqslant q'(\mathbf{t} \mid \mathcal{M}_{\alpha_m}) + \frac{1}{m}.$$

Then, since $Q_m \xrightarrow{s} I$, we have

$$q'(\mathbf{t}) \leqslant \liminf_{m \to \infty} \|D_{(1 - Q_m)\mathbf{t}Q_m}\| = \liminf_{m \to \infty} \|D_{(P_{\alpha_m} - Q_m)\mathbf{t}Q_m}\| \leqslant$$

$$\leqslant \lim_{m \to \infty} q'(\mathbf{t}|\mathcal{M}_{\alpha_m}) = \liminf_{\alpha \in A} q'(\mathbf{t}|\mathcal{M}_\alpha).$$

The proof of the lemma is complete.

In what follows, $P_{\mathcal{N}}$ denotes the projection onto the subspace $\mathcal{N}$ of $\mathcal{H}$.

LEMMA 3.7. *If* $\mathcal{M} \subset \mathcal{H}$ *is an invariant subspace for* $\mathbf{t} \in \mathcal{L}^n$, *then*

$$q'(\mathbf{t}) \leqslant \max\{q'(\mathbf{t} \mid \mathcal{M}),\ q'(P_{\mathcal{M}^\perp}\mathbf{t} \mid \mathcal{M}^\perp)\}.$$

Proof. Given $P_1, P_2 \in \mathcal{P}$, $P_1 \leqslant P_{\mathcal{M}}$, $P_2 \leqslant P_{\mathcal{M}^\perp}$ we can find $Q_1, Q_2 \in \mathcal{P}$ such that $P_1 \leqslant Q_1 \leqslant P_{\mathcal{M}}$, $P_2 \leqslant Q_2 \leqslant P_{\mathcal{M}^\perp}$ and such that $P_{\mathcal{M}} T_k Q_2(\mathcal{H}) \subseteq Q_1(\mathcal{H})$, $1 \leqslant k \leqslant n$, $\|D_{(P_{\mathcal{M}} - Q_1)\mathbf{t}Q_1}\| \leqslant q'(\mathbf{t} \mid \mathcal{M}) + \varepsilon$, $\|D_{(P_{\mathcal{M}^\perp} - Q_2)\mathbf{t}Q_2}\| \leqslant q'(P_{\mathcal{M}^\perp}\mathbf{t} \mid \mathcal{M}^\perp) + \varepsilon$, where $\varepsilon > 0$ is given. With $Q = Q_1 + Q_2$, it follows easily that

$$\|D_{(1-Q)tQ}\| = \|D_{(P_{\mathcal{M}}-Q_1+P_{\mathcal{M}^\perp}-Q_2)t(Q_1+Q_2)}\| =$$

$$= \|D_{(P_{\mathcal{M}}-Q_1)tQ_1} + D_{(P_{\mathcal{M}^\perp}-Q_2)tQ_2}\| \leqslant \max\{\|D_{(P_{\mathcal{M}}-Q_1)tQ_1}\|, \|D_{(P_{\mathcal{M}^\perp}-Q_2)tQ_2}\|\} \leqslant$$

$$\leqslant \{\max q'(\mathbf{t}\,|\,\mathcal{M}) + \varepsilon, q'(P_{\mathcal{M}^\perp}\mathbf{t}\,|\,\mathcal{M}^\perp) + \varepsilon\}.$$

Since ε is arbitrary the proof of the lemma is complete.

LEMMA 3.8. *Let $\mathcal{H}_j$, $1 \leqslant j \leqslant m$ be separable Hilbert spaces, let $\mathbf{t} \in \mathcal{L}^n$ and let $\lambda^j \in \mathbf{C}^n$, $1 \leqslant j \leqslant m$. Then*

$$q'(\mathbf{t}) \leqslant q'\left(\mathbf{t} \oplus \left(\bigoplus_{j=1}^{m} \lambda^j 1_{\mathcal{H}_j}\right)\right) + \max_{1\leqslant j\leqslant m} \delta_e(\mathbf{t} - \lambda^j).$$

Proof. Let $\varepsilon > 0$ and let $\mathcal{M}_j$, $1 \leqslant j \leqslant m$ be as in Lemma 2.8. If $\mathcal{X}$ is a finite dimensional subspace of $\mathcal{H}$, we can find a finite dimensional subspace $\mathcal{R} \subset \mathcal{H} \oplus \oplus\left(\bigoplus_{j=1}^{m} \mathcal{H}_j\right)$ such that $\|D_{P^*_{\mathcal{R}^\perp}\left(\mathbf{t}\oplus\left(\bigoplus_{j=1}^{m}\lambda^j\right)\right)P_{\mathcal{R}}}\| < q'\left(\mathbf{t} \oplus \left(\bigoplus_{j=1}^{m} \lambda^j\right)\right) + \varepsilon$. Let $\mathcal{Y} \subset \mathcal{H}$, $\mathcal{Y}_j \subset \mathcal{H}_j$, $1 \leqslant j \leqslant m$ be finite dimensional subspaces such that $\mathcal{R} \subset \mathcal{Y} \oplus \oplus\left(\bigoplus_{j=1}^{m} \mathcal{Y}_j\right)$ and let P_j, $0 \leqslant j \leqslant m$ be the projections in $\mathcal{L}\left(\mathcal{H} \oplus \left(\bigoplus_{j=1}^{m} \mathcal{H}_j\right)\right)$ defined by

$$P_0 = P_{\mathcal{H} \oplus \left(\bigoplus_{i=1}^{m} \{0\}\right)},$$

$$P_m = P_{\left(\bigoplus_{i=0}^{m-1} \{0\}\right) \oplus \mathcal{H}_m},$$

$$P_j = P_{\left(\bigoplus_{i=0}^{j-1} \{0\}\right) \oplus \mathcal{H}_j \oplus \left(\bigoplus_{i=j+1}^{m} \{0\}\right)}, \quad 1 \leqslant j \leqslant m-1.$$

Since $\dim(\mathcal{Y}) < \infty$, $\dim(\mathcal{M}_j) = \infty$, and we can assume without loss of generality that $\dim(\mathcal{H}_j) = \infty$, we can find a unitary transformation $U\colon \mathcal{H} \oplus \left(\bigoplus_{j=1}^{m} \mathcal{H}_j\right) \to \mathcal{H}$ such that $U\left(x \oplus \left(\bigoplus_{j=1}^{m} 0\right)\right) = x$, for every $x \in \left[\mathcal{Y} + \sum_{k=j}^{n} T_k\mathcal{Y}\right]$, and $\mathrm{Ran}(UP_j) \subseteq \mathcal{M}_j$, for $1 \leqslant j \leqslant m$. Now for every $z \in \mathcal{R}$ there exists $y_z \in \mathcal{Y}$ such that $P_0 z = y_z \oplus \left(\bigoplus_{j=1}^{m} 0\right)$. It follows that $UP_0 z = y_z$, $U^* T_k U P_0 z = U^*\left(T_k y_z \oplus \left(\bigoplus_{j=1}^{m} 0\right)\right) = P_0\left(T_k \oplus \left(\bigoplus_{j=1}^{m} \lambda^j_k\right)\right)z$, so we obtain

$$\|P_{U\mathscr{R}^{\perp}}T_kUz\|^2 = \left\|UP_{\mathscr{R}^{\perp}}U^*T_k\left(UP_0z + \sum_{j=1}^{m} UP_jz\right)\right\|^2 =$$

$$= \left\|UP_{\mathscr{R}^{\perp}}\left[P_0\left(T_k \oplus \left(\bigoplus_{j=1}^{m} \lambda_k^j\right)\right)z + \bigoplus_{j=1}^{m} U^*T_kUP_jz\right]\right\|^2 =$$

$$= \left\|UP_{\mathscr{R}^{\perp}}\left[\left(T_k \oplus \left(\bigoplus_{j=1}^{m} \lambda_k^j\right)\right)z - \bigoplus_{j=1}^{m} \lambda_k^j P_jz + \bigoplus_{j=1}^{m} U^*T_kUP_jz\right]\right\|^2 =$$

$$= \left\|UP_{\mathscr{R}^{\perp}}\left[\left(T_k \oplus \left(\bigoplus_{j=1}^{m} \lambda_k^j\right)\right)z\right] + \sum_{j=1}^{m} UP_{\mathscr{R}^{\perp}}U^*(T_k - \lambda_k^j)UP_jz\right\|^2.$$

Summing over k, $1 \leqslant k \leqslant n$, and taking square root we obtain

$$\left\|D_{P_{U_{\mathscr{R}}^{\perp}}\mathbf{t}P_{U_{\mathscr{R}}}}(z)\right\| = \left\|D_{UP_{\mathscr{R}^{\perp}}\left[\mathbf{t}\oplus\left(\bigoplus_{j=1}^{m}\lambda^j\right)\right]P_{\mathscr{R}}}(z) + D_{UP_{\mathscr{R}^{\perp}}U^*(\mathbf{t}-\lambda^j)UP_j}(z)\right\| \leqslant$$

$$\leqslant \left\|D_{P_{\mathscr{R}^{\perp}}\left[\mathbf{t}\oplus\left(\bigoplus_{j=1}^{m} {}^{j}\right)\right]P_{\mathscr{R}}}\right\|\|z\| + \left\|\sum_{j=1}^{m} D_{(\mathbf{t}-\lambda^j)UP_j}(z)\right\| =$$

$$= \left\|D_{P_{\mathscr{R}^{\perp}}\left[\mathbf{t}\oplus\left(\bigoplus_{m=1}^{m}\lambda^j\right)\right]P_{\mathscr{R}}}\right\|\|z\| + \left[\sum_{j=1}^{m}\left\|D_{(\mathbf{t}-\lambda^j)UP_j}(z)\right\|\right]^{1/2} \leqslant$$

$$\leqslant \left(q'\left(\mathbf{t}\oplus\left(\bigoplus_{j=1}^{m}\lambda^j\right)\right)+\varepsilon\right)\|z\| + \left[\sum_{j=1}^{m}(\delta_e(\mathbf{t}-\lambda^j)+\varepsilon)^2\,\|P_j(z)\|^2\right]^{1/2} \leqslant$$

$$\leqslant \left[q'\left(\mathbf{t}\oplus\left(\bigoplus_{j=1}^{m}\lambda^j\right)\right) + \max_{1\leqslant j\leqslant m}\delta_e(\mathbf{t}-\lambda^j) + 2\varepsilon\right]\|z\|,$$

for every $z \in \mathscr{R}$. Since $U\mathscr{R} \supset \mathscr{Y} \supset \mathscr{X}$ we deduce that

$$\inf_{\substack{\mathscr{Z}\supset\mathscr{X}\\ \dim(\mathscr{Z})<\infty}} \left\|D_{P_{\mathscr{Z}^{\perp}}\mathbf{t}P_{\mathscr{Z}}}\right\| \leqslant \left\|P_{U_{\mathscr{R}}^{\perp}}\mathbf{t}U_{\mathscr{R}}\right\| \leqslant q'\left(\mathbf{t}\oplus\left(\bigoplus_{j=1}^{m}\lambda^j\right)\right) + \max_{1\leqslant j\leqslant m}\delta_e(\mathbf{t}-\lambda^j) + 2\varepsilon.$$

Now the conclusion follows from the fact that $\mathscr{X}$ is an arbitrary finite dimensional subspace of $\mathscr{H}$ and $\varepsilon > 0$ is also arbitrary.

The following result is a generalization of [3, Theorem 1.4].

THEOREM 3.9. *Let* $\mathbf{t} \in \mathscr{L}^n$ *and let* $\mathbf{n} = (N_1, \ldots, N_n)$ *be a commuting n-tuple of normal operators. Then,*

$$q'(\mathbf{t} \oplus \mathbf{n}) \leqslant q'(\mathbf{t}) \leqslant q'(\mathbf{t} \oplus \mathbf{n}) + \sup_{\lambda \in \sigma_e(\mathbf{n})} \delta_e(\mathbf{t} - \lambda).$$

Proof. We first remark that the joint essential spectrum $\sigma_e(\mathbf{n})$ coincides with $\sigma_{1e}(\mathbf{n})$ (see Lemma 2.4). By [8, Corollary 5.4] we can find an n-tuple $\mathbf{n}' = (N_1', \ldots, N_n')$ of diagonal operators having a common system of eigenvectors, each eigenvalue repeated infinitely many times, and such that $\mathbf{n} - \mathbf{n}' \in \mathscr{K}$. Then $\sigma(\mathbf{n}') = \sigma_e(\mathbf{n}') = \sigma_e(\mathbf{n})$. In particular $\mathbf{n} \in \mathbf{QT}_n$. Given $\varepsilon > 0$ it is easy to see that we can find $\lambda^j \in \mathbf{C}^n$, and reducing orthogonal infinite dimensional subspaces $\mathscr{H}_j$, $1 \leqslant j \leqslant m$ of $\mathscr{M}'$ such that $\sum_{j=1}^{m} \mathscr{H}_j = \mathscr{H}$ and $\left\| \mathbf{n}' - \bigoplus_{j=1}^{m} \lambda^j 1_{\mathscr{H}_j} \right\| < \varepsilon$. So, from Lemma 3.8 we have

$$q'(\mathbf{t}) \leqslant q'\left(\mathbf{t} \oplus \left(\bigoplus_{j=1}^{m} \lambda^j\right)\right) + \max_{1 \leqslant j \leqslant m} \delta_e(\mathbf{t} - \lambda^j) \leqslant$$

$$\leqslant q'(\mathbf{t} \oplus \mathbf{n}') + \sup_{\lambda \in \sigma(\mathbf{n}')} \delta_e(\mathbf{t} - \lambda) + \left\| \mathbf{n}' - \bigoplus_{j=1}^{m} \lambda^j 1_{\mathscr{H}_j} \right\| \leqslant$$

$$\leqslant q'(\mathbf{t} \oplus \mathbf{n}) + \sup_{\lambda \in \sigma_e(\mathbf{n})} \delta_e(\mathbf{t} - \lambda) + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, and the left hand side of the desired inequality follows from Lemma 3.7, the proof of the theorem is complete.

COROLLARY 3.10. *Let* $\mathbf{t} \in \mathscr{L}^n$ *be such that* $\sigma_{1e}(\mathbf{t}) \neq \emptyset$ *and let* $\mathbf{n} \in \mathscr{L}^n$ *be a commuting n-tuple of normal operators. Then*

$$q'(\mathbf{t} \oplus \mathbf{n}) \leqslant q'(\mathbf{t}) \leqslant q'(\mathbf{t} \oplus \mathbf{n}) + \sup_{\lambda \in \sigma_e(\mathbf{n})} \inf_{\mu \in \sigma_{1e}(\mathbf{t})} |\lambda - \mu|.$$

Proof. It is an immediate consequence of Theorem 3.9 and Lemma 2.6

REMARK 3.11. (a) Employing similar arguments to those given above one can show an analogous statement to Corollary 3.10 for the function q .

(b) We recall that if $T \in \mathscr{L}$ and $\dim \ker(T) < \dim \ker(T^*)$, then $q(T) = q'(T) > \delta(T)$ [4, Theorem 2.4]. It may be worth pointing out that the case $n > 1$ is radically different from the case $n = 1$. Indeed, if one considers the example $\mathbf{t} = (T_1, T_2)$ given in Remark 2.3 one has $\dim \ker(D_{\mathbf{t}}) = 0$, $\dim \ker(D_{\mathbf{t}}^*) = \infty$, $q'(\mathbf{t}) = 0$ but $\delta(\mathbf{t}) = 1/2$. However, as we shall see in Lemma 3.13, there is another modulus of quasitriangularity, and a different left resolvent function for which the above statement does have a generalization to the case $n > 1$.

DEFINITION. For $\mathbf{t} \in \mathcal{L}^n$ we define the following net on $\mathcal{P}$: $v_{\mathbf{t}}^{\#}(P) = \|D^{\#}_{(1-P)\mathbf{t}P}\|$, $P \in \mathcal{P}$. We also let $q^{\#}(\mathbf{t}) = \liminf_{P \in \mathcal{P}} v_{\mathbf{t}}^{\#}(P)$.

REMARK 3.12. It is easy to see that all the results at the beginning of the present section up to Lemma 3.7 are valid (with similar proofs), when the function $q'(\cdot)$ is replaced by $q^{\#}(\cdot)$.

In the following result we use the left resolvent function $\delta^{\#}(\cdot)$ introduced in the definition after Lemma 2.8.

LEMMA 3.13. *Let* $\mathbf{t} \in \mathcal{L}^n$. *If* $\dim\ker(D_{\mathbf{t}}^{\#}) < \dim\ker(D_{\mathbf{t}}^{\#})^*$, *then* $\delta^{\#}(\mathbf{t}) \leqslant q^{\#}(\mathbf{t})$.

Proof. If $\delta^{\#}(\mathbf{t}) = 0$, there is nothing to prove. Therefore assume $\alpha = \delta^{\#}(\mathbf{t}) > 0$. Then $\ker(D_{\mathbf{t}}^{\#}) = \{0\}$ and hence $\ker(D_{\mathbf{t}}^{\#})^* = \ker(D_{\mathbf{t}^*}) \neq \{0\}$. Let $x_0 \in \ker(D_{\mathbf{t}^*})$, $x_0 \neq 0$, and let $P \in \mathcal{P}$ be such that $x_0 \in \operatorname{Ran} P$. Then, using the notation of the proof of Lemma 3.5(c) we see that $\ker[(P\mathbf{t}^*P)^{\sim}|\operatorname{ran} P \times 1_n] \neq 0$. Since $P \times 1_n$ is finite dimensional we conclude that $\ker((P\mathbf{t}P)^{\sim}|\operatorname{ran} P \times 1_n) \neq \{0\}$. Let ξ_0 be a unit vector in this latter space, then

$$v_{\mathbf{t}}^{\#}(P) = \|D^{\#}_{(1-P)\mathbf{t}P}\| \geqslant \|D^{\#}_{(1-P)\mathbf{t}P}\xi_0\| = \|D_{\mathbf{t}}^{\#}\xi_0\| = \||D_{\mathbf{t}}^{\#}|\xi_0\| \geqslant \alpha,$$

as desired.

THEOREM 3.14. *Let* $\mathbf{t} \in \mathcal{L}^n$ *and assume that* $\bigcap_{\mathbf{k} \in \mathcal{K}^n} \sigma_{\mathrm{r}}(\mathbf{t}+\mathbf{k}) \neq \emptyset$. *Then,*

$$\sup_{\lambda \in \bigcap_{\mathbf{k} \in \mathcal{K}^n} \sigma_{\mathrm{r}}(\mathbf{t}+\mathbf{k})} \delta_{\mathrm{e}}^{\#}(\mathbf{t}-\lambda) \leqslant q^{\#}(\mathbf{t}).$$

Proof. Since $q^{\#}(\mathbf{t}) = q^{\#}(\mathbf{t}-\lambda)$, we can assume that $0 \in \bigcap_{\mathbf{k} \in \mathcal{K}^n} \sigma_{\mathrm{r}}(\mathbf{t}+\mathbf{k})$, and we need only prove that $\delta_{\mathrm{e}}^{\#}(\mathbf{t}) \leqslant q^{\#}(\mathbf{t})$, in the case that $\delta_{\mathrm{e}}^{\#}(\mathbf{t}) > 0$. Then, since $D_{\mathbf{t}}^{\#}$ is left semi-Fredholm we see that $(D_{\mathbf{t}}^{\#})^* = D_{\mathbf{t}^*}$ has closed range. We claim that $\dim\ker(D_{\mathbf{t}}^{\#}) < \dim\ker(D_{\mathbf{t}}^{\#})^*$. Otherwise $\dim\ker(D_{\mathbf{t}^*}) < \infty$ and hence $D_{\mathbf{t}^*}$ is left semi-Fredholm. Since $\dim\ker(D_{\mathbf{t}^*}) \leqslant \dim\ker(D_{\mathbf{t}}^{\#}) = \dim\ker(D_{\mathbf{t}^*})^*$, by Lemma 2.2 we see that there exists $\mathbf{k} \in \mathcal{K}^n$ such that $(\mathbf{t}+\mathbf{k})^*$ is left invertible, i.e. $0 \notin \sigma_{\mathrm{r}}(\mathbf{t}+\mathbf{k})$, contradicting our assumption and our claim is established. Now, since $\dim\ker(D_{\mathbf{t}}^{\#}) < \dim\ker(D_{\mathbf{t}}^{\#})^*$, it follows from Lemma 2.10 that there exists $\mathbf{k} \in \mathcal{K}^n$ such that $\delta_{\mathrm{e}}^{\#}(\mathbf{t}) = \delta^{\#}(\mathbf{t}+\mathbf{k})$. By the invariance of the index under compact perturbations and Lemma 3.13, we conclude that $\delta_{\mathrm{e}}^{\#}(\mathbf{t}) = \delta^{\#}(\mathbf{t}+\mathbf{k}) \leqslant q^{\#}(\mathbf{t}+\mathbf{k}) = q^{\#}(\mathbf{t})$.

The following result can be regarded as a generalization of [16, Theorem 2.2].

COROLLARY 3.15. *If* $\mathbf{t} \in \mathbf{QT}_n$, *then* $\bigcap_{\mathbf{k} \in \mathcal{K}^n} \sigma_{\mathrm{r}}(\mathbf{t}+\mathbf{k}) = \sigma_{\mathrm{le}}^{\#}(\mathbf{t}) \cap \sigma_{\mathrm{re}}(\mathbf{t})$ *(see the definition after Remark* 2.3 *and Lemma* 3.8*).*

Proof. Since one inclusion is obvious we need only show that if $0 \in \bigcap_{\mathbf{k} \in \mathscr{K}^n} \sigma_r(\mathbf{t}+\mathbf{k})$, then $0 \in \sigma_{le}^{\#}(\mathbf{t}) \cap \sigma_{re}(\mathbf{t})$. Since $q^{\#}(\mathbf{t}) = 0$, from Theorem 3.14 we see that $D_{\mathbf{t}}^{\#}$ is not essentially left invertible and hence $0 \in \sigma_{le}^{\#}(\mathbf{t})$. If $D_{\mathbf{t}}^{\#}$ were essentially right invertible, then $\ker(D_{\mathbf{t}^*}) = \ker(D_{\mathbf{t}}^{\#})^*$ would be finite dimensional and $\operatorname{ran}(D_{\mathbf{t}^*}) = \operatorname{ran}(D_{\mathbf{t}}^{\#})^*$ would be infinite dimensional. By Lemma 2.2(a) we would then conclude that there exists $\mathbf{k} \in \mathscr{K}^n$ such that $(\mathbf{t}+\mathbf{k})^*$ is left invertible, i.e. $0 \in \sigma_r(\mathbf{t}+\mathbf{k})$ contradicting the assumption that $0 \in \bigcap_{\mathbf{k} \in \mathscr{K}^n} \sigma_r(\mathbf{t}+\mathbf{k})$.

REMARK 3.16. (a) In Theorem 6.11 we shall see that the converse of Corollary 3.15 is not valid in general. This has a sharp contrast with the case $n = 1$ where the converse of Corollary 3.15 does hold and constitutes the celebrated characterization of quasitriangular operators obtained in [3, Theorem 5.4].

(b) The referee pointed out that the condition $\dim\ker(D_{\mathbf{t}}^{\#}) < \dim\ker(D_{\mathbf{t}}^{\#})^*$ of Lemma (3.13) is much more restrictive than the condition $\dim\ker(T) < \dim\ker(T^*)$ of Remark 3.11 (b). In fact, the former implies that $\sum_{k=1}^{n} \dim\ker(T_k) < \dim \bigcap_{k=1}^{n} \ker T_k$, while the latter is satisfied whenever $\dim\ker(T_k) < \dim\ker(T_k^*)$ for some k, with $1 \leqslant k \leqslant n$.

In the next section we shall make use of certain results of the theory of functions of several complex variables to produce an improvement of Theorem 3.9 (see Theorem 4.7).

4. THE PSEUDOCONVEX HULL OF A COMPACT SET IN $\mathbf{C}^n$

In the present section and in Section 6 we shall make repeated use of some results of [21]. For the reader's convenience we recall these results. In what follows given an open set Ω in $\mathbf{C}^n$, we denote by $A(\Omega)$ the space of all analytic complex valued functions on Ω.

THEOREM 4.1. [21, Theorem 2.3.2]. Hartogs' phenomenon. *Let Ω be an open set in $\mathbf{C}^n$, $n > 1$, and let X be a compact subset of Ω such that $\Omega \setminus X$ is connected. Then for every $u \in A(\Omega \setminus X)$ one can find $U \in A(\Omega)$ such that $u = U$ on $\Omega \setminus X$. In particular Ω must be connected and U is necessarily unique.*

The Hartogs' phenomenon for functions of several complex variables is intimately connected with the notion of domain of holomorphy (this is reflected in the equivalence of (a_1) and (a_2) of Theorem (4.2) below). The following theorem is a combination of [21, Theorem 2.5.5, Theorem 2.6.7, Theorem 2.6.11 and Theorem 4.2.8]. The group of statements (a_1), (a_2) and (a_3) represent the equivalent definitions of domain of holomorphy, and the group of assertions (b_1), (b_2) and (b_3) consist of the various equivalent definitions of open pseudoconvex subsets of $\mathbf{C}^n$.

We recall that $h: \Omega \to [-\infty, \infty)$ is plurisubharmonic on the open set $\Omega \subset \mathbf{C}^n$ if (i) h is uppersemicontinuous, (ii) the function $\tau \to h(z + \tau w)$ is subharmonic in $\tau \in \mathbf{C}$ for every $z, w \in \mathbf{C}^n$ whenever such a function is defined. The set of all plurisubharmonic functions on Ω is denoted by $P_s(\Omega)$.

THEOREM 4.2. *Let $\Omega \subseteq \mathbf{C}^n$ be open. Then the following conditions are equivalent.*

(a_1) *There exist no open sets Ω_1, Ω_2 in $\mathbf{C}^n$ with the following properties.*

(i) $\emptyset \neq \Omega_1 \subseteq \Omega_2 \cap \Omega$,

(ii) *Ω_2 is connected and $\Omega_2 \nsubseteq \Omega$,*

(iii) *For every $u \in A(\Omega)$ there exists (a uniquely determined) $U \in A(\Omega_2)$ such that $U = u$ on Ω_1.*

(a_2) *There exists a function $f \in A(\Omega)$ which cannot be contained analytically beyond Ω, that is, it is not possible to find Ω_1 and Ω_2 satisfying* (i) *and* (ii) *in* (a_1) *and $f_2 \in A(\Omega_2)$ so that $f = f_2$ in Ω_1.*

(a_3) *Given any compact $X \subseteq \Omega$, the Ω-holomorphic hull $\hat{X}_\Omega$ of X defined by $\hat{X}_\Omega = \{z \in \Omega : |f(z)| \leqslant \sup_{z' \in X} |f(z')|, \text{ for all } f \in A(\Omega)\}$ is compact in Ω.*

(b_1) *There exists a plurisubharmonic function h on Ω such that $\{z \in \Omega : h(z) \leqslant c\}$ is compact in Ω for all $c \in \mathbf{R}$.*

(b_2) *For every compact $X \subseteq \Omega$, the Ω-plurisubharmonic hull $X_\Omega^{P_s}$ of X defined by $X_\Omega^{P_s} = \{z \in \Omega : h(z) \leqslant \sup_{z' \in X} h(z'), \text{ for all } h \in P_s(\Omega)\}$ is compact in Ω.*

(b_3) *For every compact set $X \subseteq \Omega$ the set $X_\Omega^{P_s}$ is compact in Ω and further, for every open neighborhood Ω' of $X_\Omega^{P_s}$, $\Omega' \subseteq \Omega$ there exists a smooth (strictly) plurisubharmonic function on Ω such that*

(i) *$h(z) < 0$ for all $z \in X$, and $h(z) > 0$ for all $z \in \Omega \setminus \Omega'$;*

(ii) *for every $c \in \mathbf{R}$ the set $\{z \in \Omega : h(z) \leqslant c\}$ is compact in Ω.*

REMARK 4.3. (a) If $\{\Omega_\alpha : \alpha \in A\}$ is an arbitrary family of open pseudoconvex subsets of $\mathbf{C}^n$, then $\operatorname{Int}(\bigcap_{\alpha \in A} \Omega_\alpha)$ is also pseudoconvex [21, Corollary 2.5.7]. Further, if $\Omega_\alpha \cap \Omega_\beta = \emptyset$ for $\alpha \neq \beta \in A$, then $\bigcup_{\alpha \in A} \Omega_\alpha$ is open and pseudoconvex.

(b) $\Omega \subseteq \mathbf{C}^n$ is open and pseudoconvex if and only if each connected component of Ω is open and pseudoconvex.

(c) If $\Omega \subset \mathbf{C}^n$, $\Omega' \subset \mathbf{C}^{n'}$ are open, pseudoconvex and $\Phi : \Omega \to \mathbf{C}^{n'}$ is holomorphic, then the set $\{z \in \Omega : \Phi(z) \in \Omega'\}$ is also pseudoconvex [22, Theorem 2.5.14]. In particular, $\Omega \times \Omega'$ is an open, pseudoconvex subset of $\mathbf{C}^{n+n'}$.

(d) If Ω is open and pseudoconvex, and $X \subseteq \Omega$ is compact, then $\mathrm{Int}(\hat{X}_\Omega)$ is open pseudoconvex. In particular, open convex sets in $\mathbf{C}^n$ are pseudoconvex.

DEFINITION 4.4. Let $X \subseteq \mathbf{C}^n$ be compact. We define the *pseudoconvex hull* of X to be the set $\hat{X} = \bigcap_{X \subseteq \Omega} \hat{X}_\Omega$ where Ω ranges over all open, pseudoconvex sets containing X. We say that X is *pseudoconvex* if $X = \hat{X}$.

REMARK 4.5. (a) Since $\hat{X}_\Omega$ is a filter of compact sets when Ω ranges over all open pseudoconvex sets containing X it follows that $\hat{X}$ is compact. Further from Remark 4.3(c) and Theorem 4.2(b_3) we conclude that $\hat{X}$ is the intersection of all open, pseudoconvex sets containing X. In particular, compact convex subsets of $\mathbf{C}^n$ are pseudoconvex. However, if Ω is open, pseudoconvex and bounded in $\mathbf{C}^n$, $\bar{\Omega}$ may not be pseudoconvex. For instance consider $\Omega = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| < |z_2| \text{ and } \max(|z_1|, |z_2|) < 1\}$. Then Ω is pseudoconvex, but $\bar{\Omega} \neq \{(z_1, z_2) \in \mathbf{C}^2 : \max(|z_1|, |z_2|) \leqslant \leqslant 1\} = \bar{\Omega}^\wedge$. This example is known in the theory of functions of several complex variables by the Hartogs' triangle.

(b) Given an open $\Omega \subset \mathbf{C}^n$ there exists a smallest open, pseudoconvex set containing Ω (see Remark 4.3(a)) which will be denoted by $\hat{\Omega}$. Thus, $\hat{X} = \bigcap_{\Omega \supseteq X} \hat{\Omega}$, Ω ranges over all open neighborhoods of X, whenever X is a compact set in $\mathbf{C}^n$. Further, in the above intersection we need only consider a fundamental system of open neighborhoods of X. Thus, if $X_\varepsilon = \{z \in \mathbf{C}^n : \inf_{z' \in X} |z - z'| < \varepsilon\}$, then $\hat{X} = \bigcap_{\varepsilon > 0} \hat{X}_\varepsilon$.

(c) Given an open connected $\Omega \subseteq \mathbf{C}^n$, it follows from [21, Theorem 5.4.5] that there exists a Riemann domain $\mathcal{M}_\Omega$ which is a Stein manifold and a holomorphic extension of Ω. This means that $\mathcal{M}_\Omega$ is a connected n-dimensional complex manifold with the following properties:

(i) there exists a regular holomorphic map $\eta : \mathcal{M}_\Omega \to \mathbf{C}^n$,

(ii) for every compact $X \subseteq \mathcal{M}_\Omega$ the holomorphic hull $\hat{X}_{\mathcal{M}_\Omega}$ is also compact in $\mathcal{M}_\Omega$,

(iii) Ω is an open subset of $\mathcal{M}_\Omega$, the map η of (i) is the extension of the coordinate map from Ω to $\mathbf{C}^n$ and the restriction map $\rho : A(\mathcal{M}_\Omega) \to A(\Omega)$ given by $\rho(F) = F|\Omega$ is an isomorphism.

(d) Let $\Omega \subseteq \mathbf{C}^n$ be open and $\Omega_j, j \in J$ be the connected components of Ω. We define $\mathcal{M}_\Omega$ to be the complex manifold obtained by taking the disjoint union of $\mathcal{M}_{\Omega_j}$, $j \in J$. If we let $\eta : \mathcal{M}_\Omega \to \mathbf{C}^n$ be the regular map $\eta|\mathcal{M}_{\Omega_j} = \eta_j$, $j \in J$, then $(\mathcal{M}_\Omega, \eta)$ is a Riemann domain which is by definition the envelop of holomorphy of Ω Let $\tilde{\Omega} = \eta(\mathcal{M}_\Omega)$. From the regularity of η it follows that $\tilde{\Omega}$ is open in $\mathbf{C}^n$, and Ω is pseudoconvex if and only if $\tilde{\Omega} = \Omega$.

(e) Let $\Omega \subseteq \mathbf{C}^n$ be open. Let $\Omega_0 = \Omega$ and for every ordinal α we let $\Omega_{\alpha+1} = \tilde{\Omega}_\alpha$. If α is a limit ordinal we let $\Omega_\alpha = \bigcup_{\beta < \alpha} \Omega_\beta$. Since $\mathbf{C}^n$ is second countable, there exists a countable ordinal α_0 such that $\tilde{\Omega}_\alpha = \Omega_\alpha$ for all $\alpha \geqslant \alpha_0$ and it follows that $\hat{\Omega} = \Omega_{\alpha_0}$.

In what follows $L^2(\Omega)$ will denote the Hilbert space of square integrable complex functions on the open set $\Omega \subseteq \mathbf{C}^n$, with respect to the Lebesgue measure. Also, let $\mathbf{m}_\Omega$ denote the commuting normal n-tuple in $L^n[L^2(\Omega)]$ whose k-th element is the operator multiplication by the k-th coordinate on $L^2(\Omega)$.

The following lemma uses a modification of an argument given by Voiculescu in [29].

LEMMA 4.6. *Let* $\mathbf{t} \in \mathscr{L}^n$ *and let* $\Omega \subseteq \mathbf{C}^n$ *be open. Then* $q'(\mathbf{t} \oplus \mathbf{m}_\Omega) = q'(\mathbf{t} \oplus \mathbf{m}_{\hat{\Omega}})$.

Proof. Let $A^2(\Omega) = L^2(\Omega) \cap A(\Omega)$ and let $\bar{A}^2(\Omega) = \{\bar{f} : f \in A^2(\Omega)\}$. Then, $L^2(\Omega) \ominus \bar{A}^2(\Omega)$ is an invariant subspace of $\mathbf{m}_\Omega$. Let $\mathbf{m}'_\Omega$ denote the restriction of $\mathbf{m}_\Omega$ to the above invariant subspace and let $\mathbf{m}''_\Omega$ denote the compression of $\mathbf{m}_\Omega$ to $\bar{A}^2(\Omega)$. It follows from the above mentioned paper of Voiculescu ([29, page 1450]), that $\mathbf{m}'_\Omega \in \mathbf{QT}_n$. Next we show that $\sigma_1(\mathbf{m}''_\Omega) \supseteq \tilde{\Omega}$. To this end let $\omega \in \mathscr{M}_\Omega$. By [21, Lemma 5.4.1] there exists a compact $X \subseteq \Omega$ such that $|F(\omega)| \leqslant \sup_{z \in X} |F(z)|$, for all F in $A(\mathscr{M}_\Omega)$. Now, for ε sufficiently small we have $(X_{n\varepsilon})^- \subseteq \Omega$ and hence

$$|f(z)|^2 \leqslant \frac{1}{(\pi\varepsilon^2)^n} \int_{X_\varepsilon} |f(\lambda)|^2 \, d\lambda \leqslant \frac{1}{(\pi\varepsilon^2)^n} \int_{\Omega} |f(\lambda)|^2 \, d\lambda,$$

for all $f \in A(\Omega)$, and $z \in X$. By the Riesz representation theorem there exists $g_\omega \in \bar{A}^2(\Omega)$ such that $\overline{(\rho^{-1}\bar{f})}(\omega) = \langle f, g_\omega \rangle$ for all $f \in \bar{A}^2(\Omega)$. In particular, for $1 \leqslant k \leqslant n$ we have $\langle M_{\bar{z}_k} f, g_\omega \rangle = \overline{(\eta(\omega))_k} \overline{(\rho^{-1}\bar{f})}(\omega)$ for all $f \in \bar{A}^2(\Omega)$, where $M_{\bar{z}_k}$ denotes the operator multiplication by the conjugate of the k-th coordinate on $L^2(\Omega)$. Thus, for every $\omega \in \mathscr{M}_\Omega$ we have $D_{\mathbf{m}''_\Omega}(g_\omega) = \eta(\omega) g_\omega$ and hence $\eta(\omega) \in \sigma_1(\mathbf{m}''_\Omega)$. This means that $\tilde{\Omega} = \eta(\mathscr{M}_\Omega) \subseteq \sigma_1(\mathbf{m}''_\Omega)$, as claimed. Now we prove that

$$(*) \qquad q'(\mathbf{t} \oplus \mathbf{m}_\Omega) = q'(\mathbf{t} + \mathbf{m}_{\tilde{\Omega}}).$$

Since the joint spectrum $\sigma(\mathbf{m}_\Omega)$ and the joint essential spectrum $\sigma_e(\mathbf{m}_\Omega)$ coincides with $\bar{\Omega}$ and we also have $\sigma(\mathbf{m}_\Omega \otimes 1_{\mathscr{H}}) = \sigma_e(\mathbf{m}_\Omega \otimes 1_{\mathscr{H}}) = \bar{\Omega}$, there exists (see [8]) a unitary transformation $U: L^2(\Omega) \to L^2(\Omega) \otimes \mathscr{H}$ such that $U\mathbf{m}_\Omega U^* - \mathbf{m}_\Omega \otimes 1_{\mathscr{H}} \in \mathscr{K}^n(L^2(\Omega) \otimes \mathscr{H})$. Hence, from Lemma 3.7, Corollary 3.10, and the fact that $\mathbf{m}''_\Omega \otimes 1_{\mathscr{H}} \in \mathbf{QT}_n$ (because $\mathbf{m}''_\Omega$ has a total system of eigenvectors, cf. Lemma 3.6), we have

$$q'(\mathbf{t} \oplus \mathbf{m}_\Omega) = q'(\mathbf{t} \oplus (\mathbf{m}_\Omega \otimes 1_{\mathscr{H}})) \leqslant$$
$$\leqslant \max\{q'(\mathbf{t} \oplus \mathbf{m}''_\Omega \otimes 1_{\mathscr{H}}), q'(\mathbf{m}'_\Omega \otimes 1_{\mathscr{H}})\} = q'(\mathbf{t} \oplus (\mathbf{m}''_\Omega \otimes 1_{\mathscr{H}}) \oplus \mathbf{m}_{\tilde{\Omega}}) \leqslant$$
$$\leqslant \max\{q'(\mathbf{t} \oplus \mathbf{m}_{\tilde{\Omega}}), q'(\mathbf{m}''_\Omega \otimes 1_{\mathscr{H}})\} = q'(\mathbf{t} + \mathbf{m}_{\tilde{\Omega}}).$$

Here we have also used the fact that $\mathbf{m}'_\Omega \otimes 1_{\mathscr{H}}$ belongs to $\mathbf{QT}_n$ (which follows from the result of Voiculescu mentioned at the beginning of this proof). Now using the results of [8], as above, we observe that

$$q'(\mathbf{t} \oplus \mathbf{m}_{\tilde{\Omega}}) = q'(\mathbf{t} \oplus \mathbf{m}_{\tilde{\Omega}} \oplus \mathbf{m}_{\Omega}) \leqslant \max\{q'(\mathbf{t} \oplus \mathbf{m}_{\Omega}), q'(\mathbf{m}_{\tilde{\Omega}})\} = q'(\mathbf{t} \oplus \mathbf{m}_{\Omega})$$

because $\Omega \subseteq \tilde{\Omega}$. Thus, assertion (*) is established. In order to prove the statement of the lemma, let Ω_α be as in Remark 4.5(e). From (*) we see that $q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_{\alpha+1}}) = q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\alpha})$ for every ordinal α. On the other hand, if β is a limit ordinal we deduce from Lemma 3.6, Corollary 3.10 and an argument similar to one used previously, that

$$q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\beta}) = q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\beta} \oplus \mathbf{m}_{\Omega_\alpha}) \leqslant q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\alpha}) \leqslant$$

$$\leqslant q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\alpha} \oplus \mathbf{m}_{\Omega_\beta}) + \sup_{\lambda \in \Omega_\beta} \inf_{\lambda' \in \Omega_\alpha} |\lambda - \lambda'| \leqslant$$

$$\leqslant q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\beta}) + \sup_{\lambda \in \Omega_\beta} \inf_{\lambda' \in \Omega_\alpha} |\lambda - \lambda'|$$

for every $\alpha < \beta$. But since $\lim_{\alpha \to \beta} \sup_{\alpha < \beta} \sup_{\lambda \in \Omega_\beta} \inf_{\lambda' \in \Omega_\alpha} |\lambda - \lambda'| = 0$, we have $q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\beta}) = q'(\mathbf{t} \oplus \mathbf{m}_{\Omega_\alpha})$ for all $\alpha < \beta$. Now the lemma follows from the fact that $\hat{\Omega} = \Omega_{\alpha_0}$ for some countable ordinal α_0 (see Remark 4.5(e)).

The following theorem is the main result of the present section, and is a strengthening of Theorem 3.9 and [29, Theorem 2.3].

THEOREM 4.7. *Let* $\mathbf{t} \in \mathscr{L}^n$ *and let* $\mathbf{n} \in \mathscr{L}^n$ *be a commuting normal n-tuple such that* $\sigma_e(\mathbf{n}) \subseteq \hat{X}$, *where* X *is a given non-empty compact subset of* $\mathbf{C}^n$. *Then* $q'(\mathbf{t} \oplus \mathbf{n}) \leqslant q'(\mathbf{t}) \leqslant q'(\mathbf{t} \oplus \mathbf{n}) + \sup_{\lambda \in X} \delta_e(\mathbf{t} - \lambda)$. *In particular, if* $\sigma_{le}(\mathbf{t}) \neq \emptyset$ *and* $\sigma_e(\mathbf{n}) \subseteq [\sigma_{le}(\mathbf{t})]^\wedge$, *then* $q'(\mathbf{t}) = q'(\mathbf{t} \oplus \mathbf{n})$.

Proof. Let Ω be an open neighborhood of X such that $\sup_{\lambda \in \Omega} \inf_{\mu \in X} |\lambda - \mu| < \varepsilon/2$, where ε is a given positive number. Then there exists a commuting normal n-tuple $\mathbf{m} \in \mathscr{L}^n[L^2(\Omega)]$ such that $\|\mathbf{m} - \mathbf{m}_\Omega\| < \varepsilon$ and $\sigma(\mathbf{m}) = \sigma_e(\mathbf{m}) = X$. Then by Theorem 3.9 and Lemma 4.6, we have

$$q'(\mathbf{t}) \leqslant q'(\mathbf{t} \oplus \mathbf{m}) + \sup_{\lambda \in X} \delta_e(\mathbf{t} - \lambda) \leqslant q'(\mathbf{t} \oplus \mathbf{m}_\Omega) + \varepsilon + \sup_{\lambda \in X} \delta_e(\mathbf{t} - \lambda) =$$

$$= q'(\mathbf{t} \oplus \mathbf{m}_{\hat{\Omega}}) + \sup_{\lambda \in X} \delta_e(\mathbf{t} - \lambda) + \varepsilon =$$

$$= q'(\mathbf{t} \oplus \mathbf{m}_{\hat{\Omega}} \oplus \mathbf{n}) + \sup_{\lambda \in X} \delta_e(\mathbf{t} - \lambda) + \varepsilon \leqslant q'(\mathbf{t} \oplus \mathbf{n}) + \sup_{\lambda \in X} \delta_e(\mathbf{t} - \lambda) + \varepsilon.$$

Since ε is arbitrary and the other inequality is already known, the proof of the theorem is complete.

COROLLARY 4.8. *Let* $\mathbf{t} \in \mathscr{L}^n$ *be such that* $\sigma_{\mathrm{le}}(\mathbf{t}) \neq \emptyset$, *let* X *be a nonempty compact subset of* $\mathbf{C}^n$, *and* $\mathbf{n}$ *be a commuting normal n-tuple such that* $\sigma_{\mathrm{e}}(\mathbf{n})$ *is contained in* $\hat{X}$. *Then* $q'(\mathbf{t} \oplus \mathbf{n}) \leqslant q'(\mathbf{t}) \leqslant q'(\mathbf{t} \oplus \mathbf{n}) + \sup_{\lambda \in X} \inf_{\mu \in \sigma_{\mathrm{le}}(\mathbf{t})} |\mu - \lambda|$.

Proof. It is an immediate consequence of Theorem 4.7 and Lemma 2.6(a').

In the next section we provide the necessary background from the theory of extensions of C^*-algebras for the application of Theorem 4.7 to the case of essentially commuting n-tuples of essentially normal oprators given in Section 6 (see Theorem 6.2).

5. QUASITRIANGULAR EXTENSIONS

In what follows, unless otherwise specified, $\mathscr{A}$ and $\mathscr{A}'$ will denote unital, separable C^*-algebras. Let $\mathrm{CP}(\mathscr{A}, \mathscr{A}')$ be the set of all unital completely positive maps $\varphi: \mathscr{A} \to \mathscr{A}'$. We recall that $\varphi \in \mathrm{CP}(\mathscr{A}, \mathscr{A}')$ if and only if the naturally induced map $\varphi_n: \mathscr{A} \otimes \mathscr{M}_n \to \mathscr{A}' \otimes \mathscr{M}_n$ is positive for $n = 1,2, \ldots$. In [6] it was observed that $\mathrm{CP}(\mathscr{A}, \mathscr{A}')$ is a complete metric space with the strong operator topology. A metric d on $\mathrm{CP}(\mathscr{A}, \mathscr{A}')$ can be defined as follows. Let $\{a_k\} \subseteq \mathscr{A}$ be dense, $a_k \neq 0$, $k = 1, 2, \ldots$, and let $\varphi, \psi \in \mathrm{CP}(\mathscr{A}, \mathscr{A}')$. Then $d(\varphi, \psi) = \sum_{k=1}^{\infty} \frac{\|(\varphi - \psi)(a_k)\|}{2^k \|a_k\|}$. A change in the dense set $\{a_k\}$ produces an equivalent metric on $\mathrm{CP}(\mathscr{A}, \mathscr{A}')$. Let LCP $(\mathscr{A}, Q)$ be the set of all liftable maps from $\mathscr{A}$ into the Calkin algebra Q, i.e. $\mathrm{LCP}(\mathscr{A}, Q) = \{\pi\varphi : \varphi \in \mathrm{CP}(\mathscr{A}, \mathscr{L})\}$. From [6, Theorem 6], it follows that $\mathrm{LCP}(\mathscr{A}, Q)$ is a closed subset of $\mathrm{CP}(\mathscr{A}, Q)$. Let $E(\mathscr{A})$ be the set of all unital $*$-monomorphisms $\tau: \mathscr{A} \to Q$. These τ are called extensions of $\mathscr{K}$ by $\mathscr{A}$ after [8], [9]. It is clear that $E(\mathscr{A})$ is a closed subset of $\mathrm{CP}(\mathscr{A}, Q)$. Thus, the set $E_{-1}(\mathscr{A}) = E(\mathscr{A}) \cap \mathrm{LCP}(\mathscr{A}, Q)$ is also closed. Let $\mathscr{U}$ be the group of unitary elements in Q. If $\tau \in E(\mathscr{A})$ we denote by $u^*\tau u$ the conjugate of τ by an element $u \in \mathscr{U}$, i.e. $(u^*\tau u)(a) = u^*\tau(a)u$, $a \in \mathscr{A}$. We define an equivalence relation on $E(\mathscr{A})$ by $\tau \sim \tau'$ if there exists $u \in \mathscr{U}$ such that $u^*\tau u = \tau'$. It follows from [30, Theorem 1.3] that if $E_0(\mathscr{A})$ denotes the set of trivial extensions in $E(\mathscr{A})$ (i.e. $\tau_0 \in E_0(\mathscr{A})$ if and only if there exists a unital $*$-monomorphism $\sigma: \mathscr{A} \to \mathscr{L}$ such that $\pi \circ \sigma = \tau_0$), then $\tau \oplus \tau_0 \sim \tau$ for every $\tau \in E(\mathscr{A})$ and $\tau_0 \in E_0(\mathscr{A})$. Here we have chosen a canonical identification of $Q \oplus Q$ as a C^*-subalgebra of Q. Following [9] we denote by $\mathrm{Ext}(\mathscr{A})$ the set of all equivalence classes $[\tau]$ of elements τ in $E(\mathscr{A})$, and we define the abelian operation on $\mathrm{Ext}(\mathscr{A})$ induced by the direct sum of elements of $E(\mathscr{A})$, i.e. $[\tau] + [\tau'] = [\tau \oplus \tau']$. The celebrated result of Voiculescu described above implies that $E_0(\mathscr{A})$ is contained in a single equivalence class which serves as the neutral element of the abelian semigroup structure of $\mathrm{Ext}(\mathscr{A})$.

Let $\mathrm{Ext}(\mathscr{A})$ be endowed with the quotient topology induced by that of $E(\mathscr{A})$. Our next task is to observe that this topology is induced by a pseudometric on $\mathrm{Ext}(\mathscr{A})$.

DEFINITION. Given a subset $\Sigma \subset E(\mathscr{A})$ we let $R(\Sigma) := \{\tau \in E(\mathscr{A}) : \tau \sim \sigma$ for some $\sigma \in \Sigma\}$. Given $[\tau], [\tau'] \in \mathrm{Ext}(\mathscr{A})$, let $\tilde{d}([\tau], [\tau']) = \inf_{\substack{\tau_1 \in R(\{\tau\}) \\ \tau_1' \in R(\{\tau'\})}} d(\tau_1, \tau_1')$.

LEMMA 5.1. *$\tilde{d}$ is a pseudometric on* $\mathrm{Ext}(\mathscr{A})$. *Further*

(a) *the topology on* $\mathrm{Ext}(\mathscr{A})$ *coincides with that induced by $\tilde{d}$ on* $\mathrm{Ext}(\mathscr{A})$;

(b) *if* $[\tau_j], [\tau_j'] \in \mathrm{Ext}(\mathscr{A})$, $j = 1,2$, *then*

$$\tilde{d}([\tau_1] + [\tau_2], [\tau_1'] + [\tau_2']) \leqslant \tilde{d}([\tau_1], [\tau_1']) + \tilde{d}([\tau_2], [\tau_2']);$$

(c) *if* $[\tau] \in \mathrm{Ext}(\mathscr{A})$ *has an additive inverse, then*

$$\tilde{d}([\tau_1] + [\tau], [\tau_2] + [\tau]) = \tilde{d}([\tau_1], [\tau_2])$$

and in particular

$$\tilde{d}([\tau], [\tau_0]) = \tilde{d}(-[\tau], [\tau_0]), \quad \textit{for every } \tau_0 \in E_0(\mathscr{A}).$$

Proof. We first note the following elementary properties of the metric d on $E(\mathscr{A})$:

(i) for every $u \in \mathscr{U}$, $\tau, \tau' \in E(\mathscr{A})$, we have

$$d(u^*\tau u, \tau') = \sum_{k=1}^{\infty} \frac{\|(u^*\tau u - \tau')a_k\|}{2^k\|a_k\|} = \sum_{k=1}^{\infty} \frac{\|(\tau - u\tau' u^*)a_k\|}{2^k\|a_k\|} = d(\tau, u\tau' u^*);$$

(ii) for every $\tau_j, \tau_j' \in E(\mathscr{A})$, $j = 1,2$, we have

$$d(\tau_1 \oplus \tau_2, \tau_1' \oplus \tau_2') = \sum_{k=1}^{\infty} \frac{\|[(\tau_1 \oplus \tau_2) - (\tau_1' \oplus \tau_2')]a_k\|}{2^k\|a_k\|} \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} \frac{\|(\tau_1 - \tau_1')a_k\|}{2^k\|a_k\|} + \sum_{k=1}^{\infty} \frac{\|(\tau_2 - \tau_2')a_k\|}{2^k\|a_k\|} = d(\tau_1, \tau_1') + d(\tau_2, \tau_2').$$

We now prove the first assertion. It is obvious that $\tilde{d}([\tau], [\tau]) = 0$ and that $\tilde{d}([\tau], [\tau']) = \tilde{d}([\tau'], [\tau])$. Also, from (i) it follows that for any $[\tau_j], j = 1,2,3$, we have

$$\tilde{d}([\tau_1], [\tau_2]) = \inf_{u_1, u_2 \in \mathscr{U}} d(u_1^*\tau_1 u_1, u_2^*\tau_2 u_2) \leqslant \inf_{u_1, u_2 \in \mathscr{U}} [d(u_1^*\tau_1 u_1, \tau_3) + d(u_2^*\tau_2 u_2, \tau_3)] =$$

$$= \inf_{u_1 \in \mathscr{U}} d(u_1^*\tau_1 u_1, \tau_3) + \inf_{u_2 \in \mathscr{U}} d(u_2^*\tau_2 u_2, \tau_3) = \tilde{d}([\tau_1], [\tau_3]) + \tilde{d}([\tau_2], [\tau_3]).$$

In order to prove (a) we first show that if $\Omega \subseteq E(\mathscr{A})$ is open, then $R(\Omega)$ is also open. It clearly suffices to prove that $R(B_\varepsilon(\tau))$ is open in $E(\mathscr{A})$ for every $\varepsilon > 0$ and $\tau \in E(\mathscr{A})$ where $B_\varepsilon(\tau) = \{\tau' \in E(\mathscr{A}) : d(\tau, \tau') < \varepsilon\}$. But, if $\tau' \in R(B_\varepsilon(\tau))$, then by (i) $\tau' \in B_\varepsilon(u^*\tau u)$ for some $u \in \mathscr{U}$. Again by (i) it follows that $R(B_\varepsilon(u^*\tau u)) = R(B_\varepsilon(\tau))$ which contains $B_\varepsilon(u^*\tau u)$ and hence $R(B_\varepsilon(\tau))$ is open, as desired. Since

$$\{[\tau'] \in \mathrm{Ext}(\mathscr{A}) : \tilde{d}([\tau'], [\tau]) < \varepsilon\} = \{[\tau'] \in \mathrm{Ext}(\mathscr{A}) : \tau' \in R(B_\varepsilon(\tau))\},$$

it readily follows that the topology induced by d on $\mathrm{Ext}(\mathscr{A})$ coincides with the quotient topology, and (a) follows. To show (b) we use the properties (i), and (ii). Indeed, for $\tau_j, \tau_j' \in E(\mathscr{A})$, $j = 1,2$, we have

$$\tilde{d}([\tau_1] + [\tau_2], [\tau_1'] + [\tau_2']) = \inf_{u \in \mathscr{U}(\mathscr{H} \oplus \mathscr{H})} d(u^*(\tau_1 \oplus \tau_2)u, \tau_1' \oplus \tau_2') \leqslant$$

$$\leqslant \inf_{u_1, u_2 \in \mathscr{U}} d((u_1 \oplus u_2)^*(\tau_1 \oplus \tau_2)(u_1 \oplus u_2), \tau_1' \oplus \tau_2') \leqslant$$

$$\leqslant \inf_{u_1 \in \mathscr{U}} d(u_1^*\tau_1 u_1, \tau_1') + \inf_{u_2 \in \mathscr{U}} d(u_2^*\tau_2 u_2, \tau_2') = \tilde{d}([\tau_1], [\tau_1']) + \tilde{d}([\tau_2], [\tau_2']).$$

Finally, (c) is an easy consequence of (b). Indeed, since $\tilde{d}([\tau], [\tau]) = 0 = \tilde{d}(-[\tau], -[\tau])$, we have

$$\tilde{d}([(\tau_1], [\tau_2]) = \tilde{d}([\tau_1] + [\tau] - [\tau], [\tau_2] + [\tau] - [\tau]) \leqslant$$

$$\leqslant \tilde{d}([\tau_1] + [\tau], [\tau_2] + [\tau]) \leqslant \tilde{d}([\tau_1], [\tau_2]).$$

REMARK 5.2. (a) Since the topology of $\mathrm{Ext}(\mathscr{A})$is induced by a pseudometric, the closure of a subset of $\mathrm{Ext}(\mathscr{A})$ is the set of limit points of sequences in the subset.

(b) Let $\mathrm{Ext}_{-1}(\mathscr{A})$ be the group of invertible elements in $\mathrm{Ext}(\mathscr{A})$. It was observed in [6, § 4] that $[\tau] \in \mathrm{Ext}_{-1}(\mathscr{A})$ if and only if τ is liftable to an element of $\mathrm{CP}(\mathscr{A}, \mathscr{L})$, i.e. $\tau \in E_{-1}(\mathscr{A})$. Since $E_{-1}(\mathscr{A})$ is closed in $E(\mathscr{A})$, and clearly $R[E_{-1}(\mathscr{A})] = E_{-1}(\mathscr{A})$ we conclude that $\mathrm{Ext}_{-1}(A)$ is a closed subset in $\mathrm{Ext}(\mathscr{A})$. From the continuity of the operations on $\mathrm{Ext}_{-1}(\mathscr{A})$, proved in Lemma 5.1, it follows that $\mathrm{Ext}_{-1}(\mathscr{A})$ is a topological group (cf. [26, Remark, 2.8]).

From the above discussion we obtain the following result.

THEOREM 5.3. *Let* $\mathrm{Ext}_0(\mathscr{A})$ *be the closure of the neutral element in* $\mathrm{Ext}(\mathscr{A})$. *Then* $\mathrm{Ext}_0(\mathscr{A})$ *is a closed subgroup of* $\mathrm{Ext}_{-1}(\mathscr{A})$.

REMARK 5.4. Since $\tilde{d}$ restricted to $\mathrm{Ext}_{-1}(\mathscr{A})$ is translation invariant, we observe that for each $[\tau] \in \mathrm{Ext}_{-1}(\mathscr{A})$ the closure of $\{[\tau]\}$ is $[\tau] + \mathrm{Ext}_0(\mathscr{A})$. Let $q\colon \mathrm{Ext}_{-1}(\mathscr{A}) \to \mathrm{Ext}_{-1}(\mathscr{A})/\mathrm{Ext}_0(\mathscr{A})$ be the quotient map. Then $\tilde{d}([\tau_1'], [\tau_2']) = \tilde{d}([\tau_1], [\tau_2])$ for every $[\tau_1'] \in \overline{\{[\tau_1]\}}$, $[\tau_2'] \in \overline{\{[\tau_2]\}}$, $[\tau_1], [\tau_2] \in \mathrm{Ext}_{-1}(\mathscr{A})$ and hence the function ρ defined by

$\rho(q[\tau_1],q[\tau_2]) = \tilde{d}([\tau_1],[\tau_2])$ is a metric on $\mathrm{Ext}_{-1}(\mathscr{A})/\mathrm{Ext}_0(\mathscr{A})$ which is translation invariant. Further, this metric space is complete. To see this, let $\{q[\tau_m]\}$ be a Cauchy sequence. By dropping to a subseqence, if necessary, we can assume that $\rho(q[\tau_m],q[\tau_{m+1}]) = \tilde{d}([\tau_m],[\tau_{m+1}]) < \frac{1}{2^m}$, $m = 1,2,\ldots$. Inductively we can then choose a sequence $[\tau'_m\}$ in $E(\mathscr{A})$ such that $d(\tau'_m, \tau'_{m+1}) < \frac{1}{2^m}$, $m = 1,2,\ldots$, and $[\tau'_m] = [\tau_m]$. Since $E(\mathscr{A})$ is complete there exists τ in $E(\mathscr{A})$ such that $\lim_{m\to\infty} d(\tau'_m,\tau) = 0$. Hence $\lim_{m\to\infty} \rho(q[\tau_m], q[\tau]) = 0$, as desired. In particular if $\mathrm{Ext}_{-1}(\mathscr{A})/\mathrm{Ext}_0(\mathscr{A})$ is countable, then it must be discrete.

DEFINITION. Let $\mathscr{S} \subseteq \mathscr{L}$. We say that $\mathscr{S}$ is *quasitriangular* if $\mathbf{t} \in \mathbf{QT}_n$ for every $\mathbf{t} \in \mathscr{S}^n$, $n = 1,2,\ldots$.

LEMMA 5.5. *Let $\mathscr{C} \subseteq \mathscr{L}$ be countable and let $\mathscr{B}$ be the norm closed and unstarred subalgebra of $\mathscr{L}$ generated by $\mathscr{C}$. Then the following are equivalent:*

(a_1) *$\mathscr{C}$ is quasitriangular.*

(a_2) *There exists an increasing sequence $\{P_m\}$ contained in $\mathscr{P}$ such that $P_m \xrightarrow{s} I$ and $\lim_{m\to\infty} \|(1 - P_m)CP_m\| = 0$, for all $C \in \mathscr{C}$.*

(a_3) *There exists a sequence $\{P_m\} \subseteq \mathscr{P}$ such that $P_m \xrightarrow{s} I$ and $\lim_{m\to\infty} \|(1 - P_m)CP_m\| = 0$ for all $C \in \mathscr{C}$.*

(b_1) *$\mathscr{B}$ is quasitriangular.*

(b_2) *There exists an increasing sequence $\{P_m\}$ contained in $\mathscr{P}$ such that $P_m \xrightarrow{s} I$ and $\lim_{m\to\infty} \|(1 - P_m)BP_m\| = 0$, for all $B \in \mathscr{B}$.*

(b_3) *There exists a sequence $\{P_m\}$ in $\mathscr{P}$ such that $P_m \xrightarrow{s} I$ and $\lim_{m\to\infty} \|(1 - P_m)BP_m\| = 0$ for all $B \in \mathscr{B}$.*

Proof. We observe first that the following implications are obvious: (a_2) $\Rightarrow$ (a_3) $\Rightarrow$ (a_1), (b_2) $\Rightarrow$ (b_3) $\Rightarrow$ (b_1) $\Rightarrow$ (a_1), so it remains to show that (a_1) $\Rightarrow$ both (a_2) and (b_2). Let $\mathscr{C} = \{C_1, C_2, \ldots\}$. Since $\mathscr{C}$ is quasitriangular we can construct inductively an increasing sequence $\{P_m\} \subseteq \mathscr{P}$ such that $P_m \xrightarrow{s} I$ as follows. Let $\{e_m\}$ be an orthonormal basis for $\mathscr{H}$ and let $P_1 \in \mathscr{P}$ be such that $\|(1 - P_1)C_1P_1\| < 1/2$, $e_1 \in \operatorname{ran} P_1$. Having chosen $P_j \in \mathscr{P}$, $e_j \in \operatorname{ran} P_j$, $1 \leqslant j \leqslant n$, such that $P_1 \leqslant \ldots \leqslant P_n$, $\|(1 - P_j)C_iP_j\| < 1/2^j$, $1 \leqslant i \leqslant j$, $1 \leqslant j \leqslant m$, we use again the quasitriangularity of $\mathscr{C}$ to choose $P_{n+1} \in \mathscr{P}$, $P_{n+1} \geqslant P_n$, $e_{n+1} \in \operatorname{Ran} P_{n+1}$ and $\max_{1\leqslant i\leqslant n+1} \|(1 - P_{n+1})C_iP_{n+1}\| < 1/2^{n+1}$. Then, $P_m \xrightarrow{s} I$ and $\lim_{m\to\infty} \|(1 - P_m)C_jP_m\| = 0$

for every $j = 1,2,\ldots,$ and (a_2) follows. Further, by [5, §2] the set $\{T \in \mathscr{S}: \lim_{m\to\infty} \|(1 - P_m)TP_m\| = 0$ is a norm closed subalgebra of $\mathscr{L}$ and (b_2) also follows.

COROLLARY 5.6. *Let $\mathscr{S} \subseteq \mathscr{L}$ be such that the norm closed subalgebra $\mathscr{B}$ generated by $\mathscr{S}$ is separable. Then, $\mathscr{S}$ is quasitriangular if and only if $\mathscr{B}$ is.*

DEFINITION. Let $\mathscr{B}$ be a closed (not necessarily starred) subalgebra of a unital separable C^*-algebra $\mathscr{A}$. We say that τ in $E(\mathscr{A})$ is *quasitriangular with respect to* $\mathscr{B}$ if $\pi^{-1}(\tau(\mathscr{B}))$ is quasitriangular.

LEMMA 5.7. *Let τ in $E(\mathscr{A})$ be quasitriangular with respect to $\mathscr{B}$. If $\tau' \in E(\mathscr{A})$ and $\tau' \sim \tau$, then τ' is also quasitriangular with respect to $\mathscr{B}$.*

Proof. From Lemma 5.5 it follows that if $\mathscr{S} \subseteq \mathscr{L}$ and $\mathscr{S}$ is quasitriangular, then any compression of $\mathscr{S}$ to a subspace of finite codimension is also quasitriangular. From this observation and the fact that $\tau \sim \tau'$ if and only if there exists an isometry or a coisometry $V \in \mathscr{L}$ such that $(\pi V)^*\tau(\pi V) = \tau'$, the lemma follows.

DEFINITION. Given a closed subalgebra $\mathscr{B}$ of $\mathscr{A}$ we denote by $\mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})$ the set of equivalence classes of quasitriangular extensions with respect to $\mathscr{B}$.

REMARK 5.8. (a) We observe that if $\mathscr{C} \subseteq \mathscr{A}$, and $\mathscr{B}$ is the closed subalgebra of $\mathscr{A}$ generated by $\mathscr{C}$, then by Corollary 5.6, $\pi^{-1}(\tau(\mathscr{C}))$ is quasitriangular if and only if τ is quasitriangular with respect to $\mathscr{B}$.

(b) If $\mathscr{B}$ and $\mathscr{B}'$ are closed subalgebras of $\mathscr{A}$ such that $\mathscr{B} \subseteq \mathscr{B}'$, then $\mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B}') \subseteq \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})$.

(c) If $\mathscr{B}$ is a C^*-subalgebra of $\mathscr{A}$ and τ is quasitriangular with respect to $\mathscr{B}$, then τ is actually quasidiagonal with respect to $\mathscr{B}$, i.e. there exists an increasing $\{P_m\} \subset \mathscr{P}$ such that $P_m \xrightarrow{s} I$ and $\lim_{m\to\infty} \|TP_m - P_mT\| = 0$ for every $T \in \pi^{-1}(\tau(\mathscr{B}))$. Thus, when $\mathscr{B}$ is a C^*-subalgebra of $\mathscr{A}$ we shall write $\mathrm{Ext}_{\mathrm{qd}}(\mathscr{A};\mathscr{B})$ in place of $\mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})$. Notice that $\mathrm{Ext}_{\mathrm{qd}}(\mathscr{A};\mathbf{C}) = \mathrm{Ext}(\mathscr{A})$ and $\mathrm{Ext}_{\mathrm{qd}}(\mathscr{A};\mathscr{A}) = \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A})$ as defined in [26].

THEOREM 5.9. *If $\mathscr{B}$ is a closed subalgebra of $\mathscr{A}$, then $\mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})$ is closed in $\mathrm{Ext}(\mathscr{A})$.*

Proof. Let $\{[\tau_m]\} \subseteq \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})$ and assume that $\lim_{m\to\infty} \tilde{d}([\tau_m],[\tau]) = 0$ for some $[\tau] \in \mathrm{Ext}(\mathscr{A})$. Since $\{\tau' \in E(\mathscr{A}) : [\tau'] \in \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})\}$ is R-invariant, we can further assume there exists a sequence $\{\tau_m\}$ such that $\lim_{m\to\infty} d(\tau_m,\tau) = 0$, and $[\tau_m] \in \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A};\mathscr{B})$. We now recall that given $\mathbf{t} \in \mathscr{L}^n$, we have

$$\|D_{\mathbf{t}}\|_e = \sup_{\alpha\in\sigma_e(|D_{\mathbf{t}}|)} \alpha \leqslant \sum_{k=1}^{n} \|(\pi T_k)^*(\pi T_k)\|^{1/2} = \left[\sum_{k=1}^{n} \|\pi T_k\|^2\right]^{1/2}.$$

For each $k = 1,2, \ldots$ let $T_k, S_k^{(m)} \in \mathscr{L}$ be such that $\pi T_k = \tau(a_k)/\|a_k\|$, $\pi(S_k^{(m)}) = \tau_m(a_k)/\|a_k\|$, $m = 1,2, \ldots$. Here, we have chosen the dense sequence $\{a_k\}$ in $\mathscr{A}$ such that $\{a_{2k}\}$ is also a dense sequence in $\mathscr{B}$. For a fixed integer n, let $\mathbf{t} = (T_1, \ldots \ldots, T_{2n})$, and $\mathbf{s}^{(m)} = (S_1^{(m)}, \ldots, S_{2n}^{(m)})$, $m = 1,2, \ldots$. Then

$$\lim_{m\to\infty} \|D_{\mathbf{t}} - D_{\mathbf{s}^{(m)}}\|_e \leqslant 2^{2n} \lim_{m\to\infty} \frac{1}{2^{2n}} \left(\sum_{k=1}^{2n} \|\pi T_k - \pi S_k^{(m)}\|^2 \right)^{1/2} \leqslant$$

$$\leqslant 2^{2n} \lim_{m\to\infty} \sum_{k=1}^{2n} \frac{\|(\tau - \tau_m)(a_k)\|}{2^k \|a_k\|} \leqslant 2^{2n} \lim_{m\to\infty} d(\tau, \tau_m) = 0.$$

It follows from Lemma 3.4 that the n-tuple $(T_2, T_4, \ldots, T_{2n})$ is in $\mathbf{QT}_n$, $n = 1,2, \ldots$, and hence τ is quasitriangular with respect to $\mathscr{B}$.

THEOREM 5.10. *Let $\gamma: [0,1] \to \mathrm{Ext}_{-1}(\mathscr{A})$ be a continuous function. Then $\gamma(1) \in \in \{\gamma(0)\}$. In particular, if $\mathscr{B}$ is a closed subalgebra of $\mathscr{A}$, and $\gamma(0) \in \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A}; \mathscr{B}) \cap \cap \mathrm{Ext}_{-1}(\mathscr{A})$, then $\gamma(1) \in \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A}; \mathscr{B})$.*

Proof. The proof can be obtained by repeating verbatim the arguments in Theorem 2.15 of [26].

DEFINITION. Let $\mathrm{K}_1(\mathscr{A})$ be the K_1-group of $\mathscr{A}$ (also called the *stable index group* of $\mathscr{A}$ [26]). We recall that $\mathrm{K}_1(\mathscr{A})$ is the inductive limit of the index groups of $\mathscr{A} \otimes \mathscr{M}_n$, $n = 1,2, \ldots$, (see [15, Chapter 2] for the definition of the index group), where the group of invertible elements of $\mathscr{A} \otimes \mathscr{M}_n$ is considered as a subgroup of the group of the invertible elements of $\mathscr{A} \otimes \mathscr{M}_{n+1}$ via the map

$$(a_{ij}) \to \begin{pmatrix} (a_{ij}) & 0 \\ 0 & 1 \end{pmatrix}.$$

For a closed subalgebra $\mathscr{B}$ of $\mathscr{A}$ we define $\mathrm{K}_1(\mathscr{A}; \mathscr{B})$ to be the sub-semigroup of $\mathrm{K}_1(\mathscr{A})$ consisting of all the equivalence classes of invertible elements of $\mathscr{B} \otimes \mathscr{M}_n$, $n = 1,2, \ldots$.

REMARK 5.11. (a) It would be interesting to know under what conditions $\mathrm{K}_1(\mathscr{A})$ is an ordered group with $\mathrm{K}_1(\mathscr{A}; \mathscr{B})$ acting as a cone for its order, i.e. $\mathrm{K}_1(\mathscr{A}; \mathscr{B}) \cap \cap (-\mathrm{K}_1(\mathscr{A}; \mathscr{B})) = \{0\}$ and $\mathrm{K}_1(\mathscr{A}; \mathscr{B}) + (-\mathrm{K}_1(\mathscr{A}; \mathscr{B})) = \mathrm{K}_1(\mathscr{A})$. Of course, this is not always the case. For instance, if X is the suspension of the real projective plane RP_1, then it is well known that $\mathrm{K}^1(X)$ is isomorphic to $\mathrm{K}^0(RP_1) = \mathbf{Z}_2$ $(= \mathbf{Z}/2\mathbf{Z})$, and hence if $\mathscr{A}$ is the algebra $C(X)$ of continuous complex valued functions on X, one cannot have any suitable order on $\mathrm{K}_1(\mathscr{A})$ $(= \mathrm{K}^1(X))$. On the other hand if X is a compact subset of $\mathbf{C}$ and $P(X)$ is the closed subalgebra of $C(X)$ consisting of the closure of analytic polynomials, then it is an easy exercise to check that $\mathrm{K}_1(C(X); P(X))$ is a cone, and $\mathrm{K}_1(C(X))$ is ordered by this cone.

(b) We recall that the index map from Q^{-1} onto $\mathbf{Z}$ defines a natural transformation $\varkappa$: $\operatorname{Ext}(\cdot) \to \operatorname{Hom}(K_1(\cdot), \mathbf{Z})$. Indeed, if $\alpha = [(a_{ij})]$ be the equivalence class of the invertible matrix (a_{ij}) of $\mathscr{A} \otimes \mathscr{M}_n$, then for $[\tau] \in \operatorname{Ext}(\mathscr{A})$ we let $(\varkappa[\tau])[(a_{ij})] = \operatorname{ind}((\tau a_{ij}))$.

LEMMA 5.12. *Let $\mathscr{B}$ be a closed subalgebra of $\mathscr{A}$. If $[\tau] \in \operatorname{Ext}_{qt}(\mathscr{A}; \mathscr{B})$, then $\varkappa[\tau]$ is non-negative on $K_1(\mathscr{A}; \mathscr{B})$.*

Proof. We must check that given an invertible matrix (b_{ij}) in $\mathscr{B} \otimes \mathscr{M}_n$, the matrix (τb_{ij}) in $Q \otimes \mathscr{M}_n$ has non-negative index. This is a consequence of the fact that an n by n matrix of jointly quasitriangular operators in $\mathscr{L}$ is quasitriangular, and the fact that Fredholm quasitriangular operators have non-negative index (see [16, Theorem 2.2]).

REMARK 5.13. (a) The converse of Lemma 5.12 is not true in general. For instance this happens when $\mathscr{B}$ is too large, i.e. $\mathscr{B} = \mathscr{A}$. In fact, if $\mathscr{A} = C(RP_1)$, then, by a result of Larry Brown (private communication), $\operatorname{Ext}_{qt}(\mathscr{A}; \mathscr{A}) = \operatorname{Ext}_{qd}(\mathscr{A}) = 0$ while $\operatorname{Ext}(\mathscr{A}) = \mathbf{Z}_2$ [23], and therefore, since $\operatorname{Hom}(K_1(\mathscr{A}), \mathbf{Z})$ is torsion free, we have $\ker \varkappa = \operatorname{Ext}(\mathscr{A}) \neq \operatorname{Ext}_{qt}(\mathscr{A}, \mathscr{A})$. On the other hand, if $\mathscr{B}$ is too small (i.e $\mathscr{B} = \mathbf{C}$), then the converse of Lemma 5.12 holds trivially (because $K_1(\mathscr{A}; \mathbf{C}) = \{0\}$). *Question:* when does the converse of Lemma 5.12 hold?

(b) An interesting question related to the above problem is the following. When does there exist a closed subagebra $\mathscr{B}$ of $\mathscr{A}$ such that $C^*(\mathscr{B}) = \mathscr{A}$, and $\ker \varkappa \subseteq \operatorname{Ext}_{qt}(\mathscr{A}; \mathscr{B})$? Note that the example in (a) tells us that this might not happen in general.

(c) We recall that $\mathscr{A}$ is called quasidiagonal if the neutral element in $\operatorname{Ext}(\mathscr{A})$ is quasidiagonal, i.e. $0 \in \operatorname{Ext}_{qd}(\mathscr{A})$. By [30], this is equivalent to asserting the existence of a quasidiagonal trivial extension in $E(\mathscr{A})$. Furthermore, since direct sums of quasidiagonal representations of $\mathscr{A}$ are quasidiagonal, the above definition is equivalent to the existence of a faithful non-degenerate quasidiagonal representation $\rho: \mathscr{A} \to \mathscr{L}$, because for $\sigma = \rho \otimes 1_{\mathscr{H}}$, $\pi\sigma$ is a trivial extension. It may be worth mentioning that $\operatorname{Ext}_{qt}(\mathscr{A})$ might be empty, in general, if $\mathscr{A}$ is non-quasidiagonal. Indeed, this phenomenon occurs, for example, if $\mathscr{A}$ contains a non-unitary isometry (see [26, Remark 2.1]). On the other hand, as was observed in [26, Remark 2.13], if $\mathscr{A}$ is an AF C^*-algebra, then $\mathscr{A}$ is quasidiagonal and $\operatorname{Ext}(\mathscr{A}) = \operatorname{Ext}_{qd}(\mathscr{A})$.

THEOREM 5.14. *Let $\mathscr{A}$ be quasidiagonal.*

(a) *If $\mathscr{B}$ is a closed subalgebra of $\mathscr{A}$, then $\operatorname{Ext}_{qt}(\mathscr{A}; \mathscr{B})$ is a non-empty closed abelian sub-semigroup of $\operatorname{Ext}(\mathscr{A})$ containing the identity of $\operatorname{Ext}(\mathscr{A})$.*

(b) *If $\mathscr{A}'$ is another unital separable C^*-algebra, $\mathscr{B}'$ is closed sub-algebra of $\mathscr{A}'$, and $\theta: \mathscr{A} \to \mathscr{A}'$ is a $*$-homomorphism such that $\theta(\mathscr{B})$ is contained in $\mathscr{B}'$, then the induced homomorphism $\theta^*: \operatorname{Ext}(\mathscr{A}') \to \operatorname{Ext}(\mathscr{A})$ satisfies $\theta^*(\operatorname{Ext}_{qt}(\mathscr{A}'; \mathscr{B}')) \subseteq \subseteq \operatorname{Ext}_{qt}(\mathscr{A}; \mathscr{B})$. Further, $\operatorname{Ext}_{qt}(\cdot\,; \cdot)$ is a contravariant functor from the category of quasidiagonal separable unital C^*-algebras, with distinguished closed subalgebras* *into the category of abelian semi-groups with identity.*

(c) $$\mathrm{Ext}_{\mathrm{qd}}(\mathscr{A}) \cap \mathrm{Ext}_{-1}(\mathscr{A}) = \mathrm{Ext}_0(\mathscr{A}).$$

Proof. (a) Follows from Theorem 5.9 and Remark 5.8(b). To prove (b) we remind the reader that θ^* is defined as follows. Let $[\tau] \in \mathrm{Ext}(\mathscr{A}')$. Then $\theta^*([\tau]) = [\tau \circ \theta \oplus \tau_0]$ where $\tau_0 \in E_0(\mathscr{A})$. Since $\mathscr{A}$ is quasidiagonal, it follows that $\tau \circ \theta \oplus \tau_0$ is quasitriangular with respect to $\mathscr{B}$ whenever $[\tau] \in \mathrm{Ext}_{\mathrm{qt}}(\mathscr{A}'; \mathscr{B}')$. Finally, the proof of (c) is obtained by repeating word by word the arguments given in [26, Theorem 2.9].

REMARK 5.15. In [26] it was shown that $\mathrm{Ext}(\cdot)$ is a homotopy invariant functor from the category of quasidiagonal, nuclear, separable C^*-algebras into the category of abelian groups. Thus, by Theorem 5.10, it follows that $\mathrm{Ext}_{\mathrm{qt}}(\cdot\,;\cdot)$ is homotopy invariant on the category of quasidiagonal, nuclear, separable C^*-algebras with distinguished closed subalgebras. We recall that $\mathscr{A}$ is nuclear if and only if the identity map on $\mathscr{A}$ is a limit of finite rank maps in $\mathrm{CP}(\mathscr{A}, \mathscr{A})$. By [12], it follows that if $\mathscr{A}$ is nuclear, then $\mathrm{CP}(\mathscr{A}, Q) = \mathrm{LCP}(\mathscr{A}, Q)$ and hence by [6, §4] $\mathrm{Ext}_{-1}(\mathscr{A}) = \mathrm{Ext}(\mathscr{A})$ and hence $\mathrm{Ext}(\mathscr{A})$ is a group. For abelian C^*-algebras, this fact was first shown by Brown, Douglas and Fillmore in [8].

6. JOINT QUASITRIANGULARITY OF ESSENTIALLY COMMUTING n-TUPLES OF ESSENTIALLY NORMAL OPERATORS

DEFINITION. Let $\mathbf{t} \in \mathscr{L}^n$. We say that $\mathbf{t} = (T_1, \ldots, T_n)$ is an essentially commuting n-tuple of essentially normal operators (and we abbreviate this expression by e.c. n-tuple of e.n. operators) whenever $T_iT_j^* - T_j^*T_i \in \mathscr{K}$, for $1 \leqslant i, j \leqslant n$ (and hence $T_iT_j - T_jT_i \in \mathscr{K}$, $1 \leqslant i, j \leqslant n$).

In this section we turn our attention to the problem of joint quasitriangularity for the class of e.c. n-tuples of e.n. operators.

Let $X \subseteq \mathbf{C}^n$ be compact and let $\mathrm{Ext}(X) = \mathrm{Ext}(C(X))$ and $\mathrm{Ext}_{\mathrm{qt}}(X) = \mathrm{Ext}_{\mathrm{qt}}(C(X); P(X))$. As in Section 5, $C(X)$ denotes the C^*-algebra of continuous complex valued function on X and $P(X)$ denotes the closure in $C(X)$ of analytic polynomials on $\mathbf{C}^n$. Of course, $C(X)$ (respectively $P(X)$) is the C^*-algebra (respectively Banach algebra) generated by the coordinate functions $\chi_k\colon X \to \mathbf{C}$ given by $\chi_k(Z) = Z_k$, $1 \leqslant k \leqslant n$. As observed in [8] each e.c. n-tuple of e.n. operators $\mathbf{t} = (T_1, \ldots, T_n)$ on $\mathscr{H}$ such that $\sigma_e(\mathbf{t}) = X$ gives rise to an extension $\tau_{\mathbf{t}}$ in $E(C(X))$ determined (uniquely) by $\tau_{\mathbf{t}}(\chi_k) = \pi(T_k)$, $1 \leqslant k \leqslant n$. Conversely, if $\tau \in E(C(X))$ and $\mathbf{t}$ in $\mathscr{L}^n$ satisfies $\tau(\chi_k) = \pi(T_k)$, $1 \leqslant k \leqslant n$. Then $\mathbf{t}$ is an e.c. n-tuple of e.n. operators on $\mathscr{H}$ such that $\sigma_e(\mathbf{t}) = X$.

THEOREM 6.1. *Let $\mathbf{t} \in \mathscr{L}^n$ be an e.c. n-tuple of e.n. operators. Then $\mathbf{t} \in \mathbf{QT}_n$ if and only if $[\tau_{\mathbf{t}}] \in \mathrm{Ext}_{\mathrm{qt}}(X)$.*

Proof. This is an immediate consequence of Lemma 5.7 and Remark 5.8(a).

As a consequence of the above theorem and the next result one can obtain many examples of jointly quasitriangular e.c. n-tuple of e.n. operators.

THEOREM 6.2. *Let X, $Y \subseteq \mathbf{C}^n$, $n > 1$ be compact sets such that $\partial Y \subseteq X \subseteq Y$ and X be connected. If i_*: $\mathrm{Ext}(X) \to \mathrm{Ext}(Y)$ is the homomorphism induced by the inclusion map $i: X \to Y$, then $\mathrm{Ext}_{qt}(X) = \{[\tau] \in \mathrm{Ext}(X) : i_*([\tau]) \in \mathrm{Ext}_{qt}(Y)\}$. In particular if Y is contractible, then $\mathrm{Ext}_{qt}(X) = \mathrm{Ext}(X)$.*

Proof. We first claim that $Y \subseteq \hat{X}$. Since X is connected we observe that for every $\varepsilon > 0$ the open set $X_\varepsilon = \{z \in \mathbf{C}^n : \inf_{z' \in X} |z - z'| < \varepsilon\}$ is also connected. By Remark 4.5(b), $\hat{X} = \bigcap_{\varepsilon > 0} \hat{X}_\varepsilon$. Hence, to prove our claim it suffices to show that if $\Omega' \subseteq \mathbf{C}^n$ is open, pseudoconvex and connected, and $X \subseteq \Omega'$, then $Y \subseteq \Omega'$. Let $\Omega = \mathrm{Int}\, Y$. Since $\partial\Omega \subseteq \partial Y \subseteq X \subseteq \Omega'$ we need only show that $\Omega \subset \Omega'$. Further, the above chain of inclusions implies that $\Omega' \cup (\mathbf{C}^n - \Omega)$ is open and hence $Z = \Omega - \Omega'$ is compact. We next observe that $\Omega \cup \Omega' = \Omega' \cup Z$ is connected. Indeed, let Ω_1 and Ω_2 be two disjoint open sets in $\mathbf{C}^n$ such that $\Omega \cup \Omega' = \Omega_1 \cup \Omega_2$. Since Ω' is connected, either Ω_1 or Ω_2 (say Ω_2) is contained in $Z = \Omega - \Omega'$. The fact that Z is compact implies that $\bar{\Omega}_2 \subseteq Z \subseteq \Omega \cup \Omega'$ and since Ω_2 is relatively closed in $\Omega \cup \Omega'$ we conclude that $\Omega_2 = (\Omega \cup \Omega') \cap \bar{\Omega}_2 = \bar{\Omega}_2$ which is impossible unless Ω_2 is empty. Now, it follows from Theorems 4.1 and 4.2 that $\Omega \cup \Omega' \subseteq \Omega'$ and our claim is established. For $[\tau] \in \mathrm{Ext}(X)$, by definition, $i_*([\tau]) = [\tau \circ i^* \oplus \tau_0]$ where τ_0 is a trivial extension for $C(Y)$ and i^*: $C(Y) \to C(X)$ is defined by composition with i. Let $\mathbf{t}, \mathbf{n} \in \mathscr{L}^n$ be such that $\mathbf{t}$ is an e.c. n-tuple of e.n. operators and $\mathbf{n}$ is a commuting n-tuple of normal operators such that $\tau(\chi_k) = \pi(T_k)$, $\tau_0(\chi_k) = \pi(N_k)$, $1 \leqslant k \leqslant n$. It follows that $\sigma_e(\mathbf{t}) = X$, $\sigma_e(\mathbf{n}) = Y \subseteq \hat{X}$, so by Theorem 4.7 we deduce that $q'(\mathbf{t} \oplus \mathbf{n}) = q'(\mathbf{t})$. Hence, $\mathbf{t} \in \mathbf{QT}_n$ if and only if $\mathbf{t} \oplus \mathbf{n} \in \mathbf{QT}_n$. It follows that $i_*([\tau]) = [\tau_{\mathbf{t} \oplus \mathbf{n}}] \in \mathrm{Ext}_{qt}(Y)$ if and only if $[\tau] \in \mathrm{Ext}_{qt}(X)$, as desired. The last assertion is a consequence of the homotopy invariance of $\mathrm{Ext}(\cdot)$, i.e. if Y is contractible, then $\mathrm{Ext}(Y) = \{0\}$ and hence $\mathrm{Ext}_{qt}(X) = \mathrm{Ext}(X)$.

REMARK 6.3. (a) Let Ω be a non-empty open subset of $\mathbf{C}^n$ and let $\rho: \mathbf{C}^n \to \mathbf{R}$ be defined as follows: $\rho(z) = -\inf_{w \in \mathbf{C}^n - \Omega} |w - z| + \inf_{w \in \Omega} |w - z|$. Notice that $\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\}$ and $\partial\Omega = \{z \in \mathbf{C}^n : \rho(z) = 0\}$. We shall say that $\partial\Omega$ is smooth if ρ is a smooth function on an open neighborhood of $\partial\Omega$ and $\mathrm{grad}\, \rho(z) \neq 0$ for all $z \in \partial\Omega$. We point out that if Ω is connected, bounded and psendoconvex, then by Theorem 4.1 it follows that $\partial\Omega$ is connected. If, in addition, $\partial\Omega$ is smooth, then the condition $\mathrm{grad}\, \rho(z) \neq 0$ for every $z \in \partial\Omega$, in the above definition, forces $\partial\Omega$ to be also connected. Moreover, if Ω is contractible, then Ω is also contractible, and hence, $X = \partial\Omega$, $Y = \bar{\Omega}$ satisfy the hypothesis of Theorem 6.2. We recall that from [21, Theorem 2.6.12] it follows that if Ω is open in $\mathbf{C}^n$ with a smooth boundary, then Ω is pseudoconvex if and only if for every $z \in \partial\Omega$ we have $\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geqslant 0$ for every $w \in \mathbf{C}^n$ such that $\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j = 0$ (Levi's condition).

(b) Let $\Omega \subseteq \mathbf{C}^n$ be a bounded, connected, open set such that $\partial\Omega$ is smooth and strongly pseudoconvex (i.e. the quadratic form in the Levi's condition is strictly positive). Also, let T_{z_k} be the restriction to $A^2(\Omega)$ of the operator of multiplication by z_k on $L^2(\Omega)$, $1 \leqslant k \leqslant n$. Further, let $\mathbf{t}_\Omega = (T_{z_1}, T_{z_2}, \ldots, T_{z_n})$. Then, $C^*(1, \mathbf{t}_\Omega)$ is the C^*-algebra $\mathscr{T}_\Omega$ of Toeplitz operators on Ω as described in [27]. In that paper we showed that certain regularity properties of the Berkman kernel of Ω can be used to prove that the following sequence is exact $0 \to \mathscr{K}(L^2(\Omega)) \to \mathscr{T}_\Omega \to C(\partial\Omega) \to 0$. Norberto Kerzman has kindly communicated to us that recent work by C. Fefferman and others imply that the same regularity conditions of the Berkman kernel of Ω also hold when Ω is an open connected pseudoconvex subset of $\mathbf{C}^n$ with real analytic boundary. It readily follows that the following sequence is also exact.

$$0 \to \mathscr{K}(L^2(\Omega)) \otimes \mathscr{M}_m \to \mathscr{T}_\Omega \otimes \mathscr{M}_m \to C(\partial\Omega) \otimes \mathscr{M}_m \to 0.$$

In particular, if $\varphi: \{1, \ldots, n\} \to \{1, *\}$ and $\mathbf{t}_\Omega^\varphi = (T_{z_1}^{\varphi(1)}, \ldots, T_{z_n}^{\varphi(n)})$, then $\mathbf{t}_\Omega^\varphi$ is an e.c. n-tuple of e.n. operators. Another important example of a set Ω satisfying these properties is obtained by taking Ω to be any bounded open convex set with real analytic boundary. A typical case is when Ω is the open unit ball in $\mathbf{C}^n$.

THEOREM 6.4. *Let $\Omega \subseteq \mathbf{C}^n$, $n > 1$ be a bounded, connected, contractible open set and assume $\partial\Omega$ is smooth and strongly pseudoconvex. Then for every $\varphi: \{1, \ldots, n\} \to \{1, *\}$ we have $\mathbf{t}_\Omega^\varphi \in \mathbf{QT}_n$.*

Proof. For each function φ as above we let $F_\varphi: \mathbf{C}^n \to \mathbf{C}^n$ be given by $F_\varphi(z) = (z_1^{\varphi(1)}, \ldots, z_n^{\varphi(n)})$, where $z_k^* = \bar{z}_k$, $1 \leqslant k \leqslant n$. Since $\partial\Omega$ is connected and $\bar{\Omega}$ is contractible by Remark 6.3(a) we see that $F_\varphi(\Omega)$ has the same properties. By the standard continuous functional calculus for commuting normal n-tuples and Remark 6.3(b) we deduce that $\sigma_e(\mathbf{t}_\Omega^\varphi) = F_\varphi(\sigma_e(\mathbf{t}_\Omega)) = F_\varphi(\partial\Omega) = \partial F_\varphi(\Omega)$. Thus, the theorem follows from Theorem 6.2 and Remark 6.3(a).

REMARK 6.5. Let $\Omega \subseteq \mathbf{C}^2$ be as in Theorem 6.4 and assume that $0 \in \Omega$. By Remark 6.3(b) it follows that $0 \notin \sigma_e(\mathbf{t}_\Omega)$ and hence $\mathbf{t}_\Omega$ is a pair of Fredholm operators in the sense of [13], i.e. the matrix

$$(\mathbf{t}_\Omega)^\wedge = \begin{pmatrix} T_{z_1} & T_{z_2} \\ -T_{z_2}^* & T_{z_1}^* \end{pmatrix}$$

is Fredholm. The index of $\mathbf{t}_\Omega$ is then defined as $\operatorname{ind}(\mathbf{t}_\Omega) = \operatorname{ind}((\mathbf{t}_\Omega)^\wedge)$. Using the results of [7] (see also [28]) one can show that $\operatorname{ind}((\mathbf{t}_\Omega)) \neq 0$. Since $\operatorname{ind}((\mathbf{t}_\Omega)^{\wedge *}) = -\operatorname{ind}((\mathbf{t}_\Omega)^\wedge)$, we see that $\mathbf{t}_\Omega^\varphi \in \mathbf{QT}_n$ for every function $\varphi: \{1, 2\} \to \{1, *\}$ and there exist two functions φ_1, φ_2 of this kind such that $\operatorname{ind}(\mathbf{t}_\Omega^{\varphi_1}) > 0$ and $\operatorname{ind}(\mathbf{t}_\Omega^{\varphi_2}) < 0$.

THEOREM 6.6. *Let $X \subseteq \mathbf{C}^n$, $Y \subseteq \mathbf{C}^m$ be non-empty compact sets and $f: X \to Y$ be such that $f \in P(X)$.*

(a) *If $f' \in C(X)$, $\operatorname{Ran} f' \subseteq Y$ and f' is homotopic to f, then*

$$f_* \mid \operatorname{Ext}_{\mathrm{qt}}(X) = f'_* \mid \operatorname{Ext}_{\mathrm{qt}}(X) \text{ and } f_*(\operatorname{Ext}_{\mathrm{qt}}(X)) \subseteq \operatorname{Ext}_{\mathrm{qt}}(Y).$$

(b) *Let $g: Y \to X$ be such that $g \in P(Y)$ and assume that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y (i.e. X and Y are polynomially homotopic), then $f_*: \operatorname{Ext}_{\mathrm{qt}}(X) \to \operatorname{Ext}_{\mathrm{qt}}(Y)$, $g_*: \operatorname{Ext}_{\mathrm{qt}}(Y) \to \operatorname{Ext}_{\mathrm{qt}}(X)$ are isomorphisms and $g_* = (f_*)^{-1}$.*

Proof. Part (a) is an easy consequence of Theorem 5.14(b) and Theorem 5.10. Part (b) follows from (a).

COROLLARY 6.7. *Let $X \subset \mathbf{C}^n$ be compact, let $\mathrm{K}_P^1(X) = \mathrm{K}_1(C(X); P(X))$ (see the definition after Theorem 5.10), and let $\varkappa_X: \operatorname{Ext}(X) \to \mathrm{K}_P^1(X)$ be as in Remark 5.11(b).*

(a) *If $[\tau] \in \operatorname{Ext}_{\mathrm{qt}}(X)$ then $\varkappa_X[\tau]$ is non-negative on $\mathrm{K}_P^1(X)$.*

(b) *If X is polynomially homotopic to a subset of the plane then the converse of* (a) *holds.*

Proof. In view of Lemma 5.12 and Theorem 6.6, it suffices to show that the part (b) of the corollary holds for $X \subseteq \mathbf{C}$. But we know from [8, §11] that for an essentially normal operator T the fact that T is quasitriangular is equivalent to the condition $\operatorname{ind}(T - \lambda) \geqslant 0$ for every $\lambda \notin \sigma_e(T)$. This observation proves the corollary.

REMARK 6.8. (a) Is the converse of Corollary 6.7(a) valid in general? In particular is $\ker \varkappa_X$ always contained in $\operatorname{Ext}_{\mathrm{qt}}(X)$? In connection with the last assertion of Theorem 6.2, we point out that if $X, Y \subseteq \mathbf{C}^n$, $n > 1$, are compact, Y is contractible and $\partial Y \subseteq X \subseteq Y$, then $\mathrm{K}_P^1(X) = \{0\}$. To see this, assume that $(p_{ij}) \in P(X) \otimes \mathscr{M}_m$ is invertible in $C(X) \otimes \mathscr{M}_m$ and let $P = \det(p_{ij})$. Then p is invertible in $C(X)$. We claim that P is actually invertible in $C(Y)$. Indeed, let Ω be a connected component of $\operatorname{Int}(Y)$. Since $\partial\Omega \subseteq \partial Y \subseteq X$, it suffices to show that $p(z) \neq 0$ for all $z \in \Omega$. Let $Z = \{z \in \Omega: p(z) = 0\}$. We know that Z is a compact subset of Ω. By the Riemann extension theorem (cf. [18, page 20]), $\Omega \setminus Z$ is connected and hence by Theorem 4.1 $Z = \emptyset$, and our claim is established. It follows that (p_{ij}) is invertible in $C(Y) \otimes \mathscr{M}_m$ and since Y is contractible (p_{ij}) is homotopic to 1 in the invertible group of $C(Y) \otimes \mathscr{M}_m$. Thus, (p_{ij}) is a product of exponentials of elements in $C(X) \otimes \mathscr{M}_m$ and therefore each of these factors has a logarithm in $C(X) \otimes \mathscr{M}_m$ and hence (p_{ij}) is homotopic to 1 in the group of invertible elements of $C(X) \otimes \mathscr{M}_m$.

(b) We learned from a private communication that R. G. Douglas conjectured that if a compact subset X of $\mathbf{C}^2$ is homeomorphic to a subset of $\mathbf{R}^3$ (case in which $\varkappa_X$ is injective), then $[\tau] \in \mathrm{Ext}_{qt}(X)$ if and only if $\mathrm{ind}(\tau f) \geqslant 0$ for every $f \in P(X)$ which is invertible in $C(X)$. An affirmative answer to this conjecture would imply that for such sets X the converse of Corollary 6.7(a) is valid. Gail Kaplan, in her thesis [24], studied this problem for certain subsets X of $\mathbf{R}^3$ that do not necessarily satisfy the hypothesis of Corollary 6.7 (b). She considered $\mathrm{Ext}_{qt}(X)$ when $X \subseteq S^1 \times [0,1]$. This would solve the problem of joint quasitriangularity for a pair of essentially commuting operators, one of which is essentially unitary and the other is selfadjoint. Another important case to consider is when $X \subseteq S^1 \times S^1$. This situation arises when considering the problem of joint quasitriangularity of an essentially commuting pair of essentially unitary operators.

The following result might be familiar to the experts in the field but we include it here in order to show what kind of arguments are needed to characterize $\mathrm{Ext}_{qt}(X)$.

THEOREM 6.9. *Let* $\mathbf{t} = (T_1, T_2)$ *be an essentially commuting pair of essentially unitary operators such that* $\sigma_e(\mathbf{t}) = S^1 \times S^1$. *Then,* $\mathbf{t} \in \mathbf{QT}_2$ *if and only if* $\mathrm{ind}(T_1) \geqslant 0$ *and* $\mathrm{ind}(T_2) \geqslant 0$.

Proof. Let $\Sigma = (S^1 \times \{1\}) \cup (\{1\}) \times S^1)$ and let $\Pi = S^1 \times S^1 - (\{\lambda \in S^1 : |\lambda + 1| < 1\} \times \{\lambda \in S^1 : |\lambda + 1| < 1\})$. Then there exists a strong homotopic retract $r : \Pi \to \Sigma$ so that if $i : \Sigma \to \Pi$ is the inclusion map, then $r \circ i = \mathrm{id}_\Sigma$ and $i \circ r$ is homotopic to id_Π. Let Δ be the boundary of Π and let $j: \Delta \to \Pi$, $k: \Pi \to S^1 \times S^1$ be the inclusion maps. Then the sequence $\Delta \xrightarrow{j} \Pi \xrightarrow{k} S^1 \times S^1$ induces (cf. [9, § 2]) the exact sequence $\mathrm{Ext}(\Delta) \xrightarrow{i_*} \mathrm{Ext}(\Pi) \xrightarrow{k_*} \mathrm{Ext}(S^1 \times S^1) \xrightarrow{\partial} \mathrm{Ext}(S\Delta)$, where $S\Delta$ is the suspension of Δ and ∂ is the boundary map. Note that $\Delta \sim S^1$ and hence $\Delta S \sim S^2$. We also observe that $j_* = 0$. Indeed, to see this it is enough to show that $r_* j_* = (r \circ j)_* = 0$ because r_* is an isomorphism. But $(r \circ j)_* = 0$ because if z traces Δ, $(r \circ j)(z)$ traces Σ 2-times both in opposite directions. Thus, from the exactness of the above sequence we conclude that the following sequence is exact: $0 \to \mathrm{Ext}(\Pi) \xrightarrow{k_*} \mathrm{Ext}(S^1 \times S^1) \to 0$. Since $i_*: \mathrm{Ext}(\Sigma) \to \mathrm{Ext}(\Pi)$ is an isomorphism we then conclude that $(k \circ i)_*: \mathrm{Ext}(\Sigma) \to \mathrm{Ext}(S^1 \times S^1)$ is an isomorphism. This proves, in particular, the well known fact (cf. [23]) that $\mathrm{Ext}(S^1 \times S^1)$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ (because Σ is a wedge of two circles). Let $\chi_k: S^1 \times S^1 \to \mathbf{C}$ be the coordinate functions for $k = 1,2$ as above and let M_k be multiplication by χ_k on $L^2(S^1 \times S^1)$. Note that $M_1 = B \otimes 1$ and $M_2 = 1 \otimes B$, where B is the multiplication operator by the coordinate function on $L^2(S^1)$ (i.e. B is the bilateral shift). It follows that a system of generators $[\tau_1], [\tau_2]$ for $\mathrm{Ext}(S^1 \times S^1)$ is given by $\tau_1(\chi_1) = U_+ \oplus M_1$, $\tau_1(\chi_2) = I \oplus M_2$, $\tau_2(\chi_1) = I \oplus M_1$ and $\tau_2(\chi_2) = U_+ \oplus M_2$. Now, let $\mathbf{t} = (T_1, T_2)$ be an essentially commuting pair of essentially unitary operators such that $\sigma_e(\mathbf{t}) = S^1 \times S^1$ and let $m_j = \mathrm{ind}(T_j)$, $j = 1,2$. Also, let $\tau'_{\mathbf{t}}$ be the extension defined as follows:

$$\tau'_{\mathbf{t}}(\chi_1) = U_+^{m_1} \oplus M_1, \quad \tau'_{\mathbf{t}}(\chi_2) = U_+^{m_2} \oplus M_2,$$

where we have denoted U_+^n the operator $U_+ \oplus \ldots \oplus U_+$ $(-n)$-times if $n < 0$ and $U_+^* \oplus \ldots \oplus U_+^*$ n-times if $n > 0$ and $U_+^0 = I$. If $\tau_{\mathbf{t}}$ is the extension determined by $\mathbf{t}$, then it follows from the above discussion that $\tau'_{\mathbf{t}} \sim \tau'_{\mathbf{t}}$. Hence, $\mathbf{t} \in \mathbf{QT}_2$ if and only if $\tau'_{\mathbf{t}} \in \mathrm{Ext}_{\mathrm{qt}}(S^1 \times S^1)$. It is easy to check now that this last condition holds if and only if $m_j = \mathrm{ind}\, T_j \geqslant 0$, $j = 1,2$.

REMARK 6.10. (a) From the fact that $S^1 \otimes S^1$ is pseudoconvex (see Remark 4.3(c)) it follows that if $X \subseteq S^1 \times S^1$ is compact, then $\hat{X} \subseteq S^1 \times S^1$.

(b) If $\mathbf{t}$ is an e.c. n-tuple of e.n. operators, then $\sigma_{\mathrm{re}}(\mathbf{t}) = \sigma_{\mathrm{e}}(\mathbf{t}) = \sigma_{\mathrm{le}}(\mathbf{t}) \subseteq \subseteq \prod_{k=1}^{n} \sigma_{\mathrm{le}}(T_k) \subseteq \sigma_{\mathrm{le}}^{\#}(\mathbf{t})$.

The following result should be compared with Corollary 3.15.

THEOREM 6.11. *If $\mathbf{t} = (T_1, T_2)$ is an essentially commuting pair of essentially normal operators, then* $\sigma_{\mathrm{e}}(\mathbf{t}) = \bigcap_{\mathbf{k} \in \mathscr{K}^2} \sigma_{\mathrm{r}}(\mathbf{t} + \mathbf{k}) = \bigcap_{\mathbf{k} \in \mathscr{K}^2} \sigma_{\mathrm{l}}(\mathbf{t} + \mathbf{k})$.

Proof. Since $\sigma_{\mathrm{le}}(\mathbf{t}) \subseteq \bigcap_{\mathbf{k} \in \mathscr{K}^2} \sigma_{l}(\mathbf{t} + \mathbf{k})$ and $\sigma_{\mathrm{re}}(\mathbf{t}) \subseteq \bigcap_{\mathbf{k} \in \mathscr{K}^2} \sigma_{r}(\mathbf{t} + \mathbf{k})$, from Remark 6.10(b), we see that one inclusion is obvious. To prove the other inclusion assume that $0 \notin \sigma_{\mathrm{e}}(\mathbf{t})$. Let $D'_{\mathbf{t}} = D_{(-T_2, T_1)}$. We observe that the hypothesis that $\mathbf{t}$ is an essentially commuting pair implies that $D_{\mathbf{t}}^{\#} \circ D'_{\mathbf{t}} \in \mathscr{K}$. Also, since $|D'_{\mathbf{t}}| = |D_{\mathbf{t}}|$ and $0 \notin \sigma_{\mathrm{le}}(\mathbf{t})$ we see that $D'_{\mathbf{t}}$ is essentially left invertible. Let $Q \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ be the projection onto the kernel of $D_{\mathbf{t}}^{\#}$. Since $0 \notin \sigma_{\mathrm{re}}(\mathbf{t})$ we see that $\mathrm{Ran}\, D_{\mathbf{t}}^{\#}$ is closed, and hence, reasoning as in [13, Proposition 6.1] there exists $\mathbf{t}' \in \mathscr{L}^2$ such that $D_{\mathbf{t}'} \circ D_{\mathbf{t}}^{\#} = 1 - Q$. It follows that $(1 - Q)D'_{\mathbf{t}} = D_{\mathbf{t}'} \circ D_{\mathbf{t}}^{\#} \circ D'_{\mathbf{t}}$ is compact. Let $s \in \mathscr{L}^2$ be such that $D_{\mathbf{s}} = QD'_{\mathbf{t}}$. Then $D'_{\mathbf{t}} - D_{\mathbf{s}} = (1 - Q)D'_{\mathbf{t}}$ is compact and hence ($D'_{\mathbf{t}}$ being essentially left invertible) we see that $D_{\mathbf{s}}$ has closed range and $\dim\ker D_{\mathbf{s}} < \infty$. It follows that $\mathrm{Ran}\, D_{\mathbf{s}}$ is infinite dimensional and since by construction $\mathrm{Ran}\, D_{\mathbf{s}} \subseteq \ker D_{\mathbf{t}}^{\#}$, we also deduce that the latter subspace is infinite dimensional. Now, we use again the fact that $0 \notin \sigma_{\mathrm{re}}(\mathbf{t})$, so that $\mathrm{Ran}\, D_{\mathbf{t}}^{\#}$ is closed and $\dim\ker(D_{\mathbf{t}}^{\#})^* = \dim\ker D_{\mathbf{t}^*} < \infty$, to deduce that $D_{\mathbf{t}^*}$ is essentially left invertible and $(\mathrm{Ran}\, D_{\mathbf{t}^*})^{\perp} = \ker D_{\mathbf{t}}^{\#}$ is infinite dimensional. By Lemma 2.2(a) there exists $\mathbf{k} \in \mathscr{K}^2$ such that $0 \notin \sigma_{\mathrm{l}}(\mathbf{t}^* + \mathbf{k})$ so that $0 \notin \bigcap_{\mathbf{k} \in \mathscr{K}^2} \sigma_{\mathrm{r}}(\mathbf{t} + \mathbf{k})$. Now, using the fact that $0 \notin \sigma_{\mathrm{e}}(\mathbf{t}^*)$, and the above proof, we also conclude that $0 \notin \bigcap_{\mathbf{k} \in \mathscr{K}^2} \sigma_{\mathrm{l}}(\mathbf{t} + \mathbf{k})$, as required.

REMARK 6.12. The above theorem implies that certain natural generalizations to the case of essentially commuting pairs of essentially normal operators of [2, Theorem 5.4] are not possible. Finally, we would like to point out another interesting phenomenon that occurs when considering essentially commuting pairs of essentially normal operators. Let X be a compact subset of $\mathbf{C}^2$. Then, a result of [22] states that there exist natural homomorphisms $\gamma_1 : \mathrm{Ext}(X) \to {}^s\tilde{H}_1(X)$, and $\gamma_3 : \mathrm{Ext}(X) \to {}^s\tilde{H}_3(X)$, where ${}^s\tilde{H}_1(X)$ and ${}^s\tilde{H}_3(X)$ are the first and third reduced Steenrod homology groups of X, so that $\gamma_1 \oplus \gamma_3$: $\mathrm{Ext}(X) \to {}^s\tilde{H}_1(X) \oplus {}^s\tilde{H}_3(X)$ is an isomorphism. We claim that $\mathrm{Ext}_{qt}(X) \supset \ker \gamma_1$. Indeed, we recall that the isomorphism $\gamma_1 \oplus \gamma_3$ is functorial in the following sense: if $Y \subset \mathbf{C}^2$ is another compact set and $f: X \to Y$ is a continuous function, then the corresponding induced homomorphisms f_* in Ext and Steenrod homology levels commute with the maps γ_1 and γ_3. For the proof of our claim we reason as in Theorem 6.2. Let $i: X \to \hat{X}$ be the inclusion map. It follows from Theorem 4.1 that the complement of every bounded pseudoconvex open set in $\mathbf{C}^n$ has no bounded components, so that an immediate consequence of Definition 4.4 yields that the complement of $\hat{X}$ has no bounded components. Since, for every $Y \subset \mathbf{C}^2$, ${}^s\tilde{H}_3(Y)$ is isomorphic to the free abelian group consisting of the locally constant integer valued functions on the bounded components of Y^c, we deduce that ${}^s\tilde{H}_3(\hat{X}) = 0$. Let $[\tau] \in \mathrm{Ext}(X)$, and assume that $\gamma_1([\tau]) = 0$. Then, $i_* \circ \gamma_1([\tau]) = 0 = \gamma_1 \circ i_*([\tau])$, and since $\gamma_3 \circ i_*([\tau]) = 0$, we see that $i_*([\tau]) = 0$. This means that $i_*([\tau]) \in \mathrm{Ext}_{qt}(\hat{X})$, and by Theorem 4.7 it follows that $[\tau] \in \mathrm{Ext}_{qt}(X)$. We conclude that the obstruction that an essentially commuting pair $\mathbf{t} = (T_1, T_2)$ of essentially normal operators to be jointly quasitriangular lies solely on the nonvanishing of γ_1 at $[\tau_t]$, where $[\tau_t]$ is as in Theorem 6.1.

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