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ISOMETRIC DILATIONS OF COMMUTING CONTRACTIONS. I

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INTRODUCTION

Let $\mathscr{H}$ be a Hilbert space and $\{T_i\}_{i\in I}$ be a system of commuting contractions $T_i \in L(\mathscr{H})$, $i \in I$ (as usual, for $\mathscr{H}$, $\mathscr{H}'$ Hilbert spaces, $L(\mathscr{H}, \mathscr{H}')$ denotes the algebra of all (linear, bounded) operators from $\mathscr{H}$ to $\mathscr{H}'$ and $L(\mathscr{H})$ denotes $L(\mathscr{H}, \mathscr{H})$). One calls an *isometric dilation* of the system $\{T_i\}_{i\in I}$ any system $\{V_i\}_{i\in I}$ of commuting isometries V_i, $i \in I$, acting on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ as a (linear, closed) subspace such that

$$T_{i_1}^{n_1} \ldots T_{i_k}^{n_k} = P_{\mathscr{H}}^{\mathscr{K}} V_{i_1}^{n_1} \ldots V_{i_k}^{n_k} \Big| \mathscr{H},$$

for $n_j \geqslant 0$, $0 \leqslant j \leqslant k$ and for any finite subset $\{i_j\}$ of I ($P_{\mathscr{H}}^{\mathscr{K}}$ denoting, for any subspace $\mathscr{H}$ of a Hilbert space $\mathscr{K}$, the orthogonal projection of $\mathscr{K}$ on $\mathscr{H}$). If the system $\{V_i\}_{i\in I}$ consists of unitary operators then it is called a *unitary dilation* of $\{T_i\}_{i\in I}$. It is known that for any pair of commuting contractions on $\mathscr{H}$ there exists an isometric dilation (see [1]) and that for a system of commuting contractions consisting of more than two contractions an isometric dilation, in general, does not exist (see [9]). This paper is concerned with the isometric dilations of a pair of commuting contractions. The first construction of such a dilation is due to T. Ando (see [1]). Another known construction is based on the theorem of B. Sz.-Nagy and C. Foiaş on the existence of a contractive dilation for any commutant of a given contraction (see [11]). But, none of these constructions can give all the isometric dilations of a pair of commuting contractions (as some very simple examples can easily show it). The aim of this paper is to provide a construction of isometric dilations of a pair of commuting contractions which produces all the dilations (Section 2). Using this construction we establish in Section 3 a necessary and sufficient condition for uniqueness of the Ando dilation "containing" a given dilation of the commutant, which permits to obtain directly the known result concerning the uniqueness of the Ando dilation of a pair of commuting contractions (see [3] and [6]). In the particular case when the pair consists of the null contractions we express the uniqueness condition in terms of the symbol of the Toeplitz operator which describes the dilation of the commutant and we make some connections with the structure of the representing measures for the bidisc algebra (Section 4).

1.

Let T, S be two commuting contractions on a Hilbert space $\mathscr{H}$ (for an operator T, respectively for a pair $[T, S]$, acting on $\mathscr{H}$ we shall use the notation $(\mathscr{H}, T)$, respectively $(\mathscr{H}, [T, S])$). By an *Ando dilation* of the pair $(\mathscr{H}, [T, S])$ we mean a pair $(\mathscr{K}, [U, V])$, where U, V are commuting isometries acting on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ (as a subspace) and satisfying

$$P^{\mathscr{K}}_{\mathscr{H}} U^n V^m | \mathscr{H} = T^n S^m \quad (n, m \geqslant 0) \tag{1.1}$$

and

$$\mathscr{K} = \bigvee_{n, m \geqslant 0} U^n V^m \mathscr{H}. \tag{1.2}$$

Notice that, since $\mathscr{K}$ satisfies the condition (1.2) of minimality, the condition (1.1) is equivalent to the following

$$P^{\mathscr{K}}_{\mathscr{H}} U = T P^{\mathscr{K}}_{\mathscr{H}}, \quad P^{\mathscr{K}}_{\mathscr{H}} V = S P^{\mathscr{K}}_{\mathscr{H}}. \tag{1.1'}$$

We say that two Ando dilations $(\mathscr{K}, [U, V])$ and $(\mathscr{K}', [U', V'])$ of $(\mathscr{H}, [T, S])$ *coincide* if there exists a unitary operator $X \in L(\mathscr{K}, \mathscr{K}')$ such that

$$\begin{cases} X | \mathscr{H} = I_{\mathscr{H}} \\ XU = U'X, \quad XV = V'X. \end{cases} \tag{1.3}$$

We also recall some simple but useful facts concerning the Ando dilations. First, notice that, for any Ando dilation $(\mathscr{K}, [U, V])$ of $(\mathscr{H}, [T, S])$, denoting

$$\mathscr{K}_0 = \bigvee_{n \geqslant 0} U^n \mathscr{H}, \quad U_0 = U | \mathscr{K}_0, \tag{1.4}$$

the space $\mathscr{K}_0$ reduces U, and U_0 is an identification of the minimal isometric dilation of T (unique, up to an isomorphism, see [12]); moreover the operator

$$V_0 = P^{\mathscr{K}}_{\mathscr{K}_0} V | \mathscr{K}_0 \tag{1.5}$$

is a *contractive dilation of the commutant* S of T (this means that V_0 is a contraction on $\mathscr{K}_0$ commuting with U_0 and satisfying $P^{\mathscr{K}_0}_{\mathscr{H}} V_0 = S P^{\mathscr{K}_0}_{\mathscr{H}}$). We say that an Ando dilation $(\mathcal{K}, [U, V])$ *crosses through* $(\mathcal{K}_0, [U_0, V_0])$ if $\mathcal{K}_0, U_0, V_0$ are attached to $(\mathcal{K}, [U, V])$ by (1.4), (1.5). Thus we have:

REMARK 1.1. Any Ando dilation of $(\mathcal{H}, [T, S])$ crosses through a pair consisting of an identification of the minimal isometric dilation of T and a contractive dilation of the commutant S of T. (Obviously the roles of T and S can be interchanged.)

Since for any contractive commutant of a given contraction there exists a contractive dilation (see [11]), from Remark 1.1 it follows that we can produce an Ando dilation of the pair $(\mathcal{H}, [T, S])$ taking, firstly, $(\mathcal{K}_0, U_0)$ the minimal isometric dilation of T and V_0 a contractive dilation of S and, secondly, putting $(\overset{\circ}{\mathcal{K}}_0, \overset{\circ}{V}_0)$ the minimal isometric dilation of V_0 and $\overset{\circ}{U}_0$ the (unique) isometric extension of U_0 commuting with $\overset{\circ}{V}_0$. Obviously $(\overset{\circ}{\mathcal{K}}_0, [\overset{\circ}{U}_0, \overset{\circ}{V}_0])$ is an Ando dilation crossing through $(\mathcal{K}_0, [U_0, V_0])$ which will be called the *distinguished* Ando dilation of $(\mathcal{H}, [T, S])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$. Note that an Ando dilation obtained in this way depends effectively on the order of the contractions in the pair $[T, S]$. Also, notice that the distinguished Ando dilation $(\overset{\circ}{\mathcal{K}}_0, [\overset{\circ}{U}_0, \overset{\circ}{V}_0])$ is not, in general, the only Ando dilation crossing through $(\mathcal{K}_0, [U_0, V_0])$; more precisely, an Ando dilation $(\mathcal{K}, [U, V])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$ coincides with the distinguished Ando dilation $(\overset{\circ}{\mathcal{K}}_0, [\overset{\circ}{U}_0, \overset{\circ}{V}_0])$ if and only if the space $\mathcal{K}_0$ is semi-invariant to V, i.e. $P^{\mathcal{K}}_{\mathcal{K}_0} V|(\mathcal{K} \ominus \mathcal{K}_0) = 0$ (see [6]).

Our intended construction of arbitrary Ando dilations is based on the Remark 1.1 and on the matricial form of an isometry acting on a direct sum of two Hilbert spaces (following directly from the matricial form of a contraction, see [10] and [4]); more specifically it is based on the following.

LEMMA 1.1. *Let $(\mathcal{K}, [U, V])$ be an Ando dilation crossing through $(\mathcal{K}_0, [U_0, V_0])$. Then $(\mathcal{K}, [U, V])$ coincides with an Ando dilation $(\tilde{\mathcal{K}}_1, [\tilde{U}_1, \tilde{V}_1])$ (also crossing through $(\mathcal{K}_0, [U_0, V_0])$) of the form:*

$$(1.6)\qquad \begin{cases} \tilde{\mathcal{K}}_1 = \mathcal{K}_0 \oplus \mathcal{G}'_0, \quad \mathcal{G}'_0 = \mathcal{D}_{V_0} \oplus \mathcal{G}_1 \\[2ex] \tilde{U}_1 = \begin{bmatrix} U_0 & 0 \\ 0 & \tilde{Y}_0 \end{bmatrix}, \quad \tilde{V}_1 = \begin{bmatrix} V_0 & D_{V_0^*}\tilde{C}_0 \\ D_{V_0} & -V_0^*\tilde{C}_0 + Z_0 D_{\tilde{C}_0} \end{bmatrix}, \end{cases}$$

where $\tilde{Y}_0$ is an isometry on $\mathcal{G}'_0$, $\tilde{C}_0$ is a contraction from $\mathcal{G}'_0$ to the space $\mathcal{D}_{V_0^*} \ominus \overline{D_{V_0^*}\mathcal{H}}$ and Z_0 is a unitary operator from $\mathcal{D}_{C_0}$ onto $\mathcal{G}_1$. Moreover the isometry $\tilde{Y}_0$ satisfies

$$(1.7)\qquad \tilde{Y}_0 D_{V_0} = D_{V_0} U_0$$

and it is uniquely determined by $\tilde{C}_0$ and Z_0. (Note that, as usual, for a contraction C on a Hilbert space $\mathscr{H}$, D_C and $\mathscr{D}_C$ denote the defect operator and the defect space of C, respectively, i.e. $D_C = (I - C^*C)^{1/2}$, $\mathscr{D}_C = \overline{D_C\mathscr{H}}$.)

Proof. Denote $\mathscr{G}_0 = \mathscr{H} \ominus \mathscr{H}_0$. From the matricial form of a contraction acting on $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{G}_0$ (see [4]) it follows, in particular, that the isometry V has the matrix

$$(1.8) \qquad V = \begin{bmatrix} V_0 & D_{V_0^*}\tilde{C}_0' \\ W_0 D_{V_0} & -W_0 V_0^* \tilde{C}_0' + Z_0' D_{\tilde{C}_0'} \end{bmatrix} : \begin{matrix} \mathscr{H}_0 \\ \oplus \\ \mathscr{G}_0 \end{matrix} \to \begin{matrix} \mathscr{H}_0 \\ \oplus \\ \mathscr{G}_0 \end{matrix}$$

where $\tilde{C}_0'$ is a contraction from $\mathscr{G}_0$ to $\mathscr{D}_{V_0^*}$, W_0 is an isometry from $\mathscr{D}_{V_0}$ to $\mathscr{G}_0$ and Z_0' is an isometry from $\mathscr{D}_{\tilde{C}_0'}$ to Ker W_0^*. Denoting $\mathscr{G}_1 = \mathscr{G}_0 \ominus W_0\mathscr{D}_{V_0}$, $\mathscr{G}_0' = \mathscr{D}_{V_0} \oplus \mathscr{G}_1$ and $\tilde{\mathscr{H}}_1 = \mathscr{H}_0 \oplus \mathscr{G}_0'$, it is clear that the operator

$$(1.9) \qquad \begin{cases} \tilde{W}_0 : \mathscr{H} = \mathscr{H}_0 \oplus W_0\mathscr{D}_{V_0} \oplus \mathscr{G}_1 \to \tilde{\mathscr{H}}_1 = \mathscr{H}_0 \oplus \mathscr{D}_{V_0} \oplus \mathscr{G}_1 \\ \tilde{W}_0 = I_{\mathscr{H}_0} \oplus (W_0^* | W_0\mathscr{D}_{V_0}) \oplus I_{\mathscr{G}_1} \end{cases}$$

is unitary and the operator $\tilde{V}_1 = \tilde{W}_0 V \tilde{W}_0^{-1}$ is an isometry on $\tilde{\mathscr{H}}_1 (= \mathscr{H}_0 \oplus \mathscr{G}_0')$ of the form given in (1.6) where $\tilde{C}_0 = \tilde{W}_0 \tilde{C}_0' \tilde{W}_0^{-1} | \mathscr{G}_0' = \tilde{C}_0' \tilde{W}_0^{-1} | \mathscr{G}_0'$ and $Z_0 = \tilde{W}_0 Z_0' \tilde{W}_0^{-1} | \mathscr{G}_0' = Z_0' \tilde{W}_0^{-1} | \mathscr{G}_0'$. Since $P^{\mathscr{K}}_{\mathscr{H}} V = S P^{\mathscr{K}}_{\mathscr{H}}$, it results $P^{\mathscr{K}_1}_{\mathscr{H}} D_{V_0^*} \tilde{C}_0 = P^{\mathscr{K}_1}_{\mathscr{H}} \tilde{V}_1 | \mathscr{G}_0' = P^{\mathscr{K}}_{\mathscr{H}} V \tilde{W}_0^{-1} | \mathscr{G}_0' = 0$, thus $\tilde{C}_0 \mathscr{G}_0' \subset \mathscr{D}_{V_0^*} \ominus \overline{D_{V_0^*}\mathscr{H}}$. Also, since the minimality of the space $\mathscr{K}$ implies the minimality of the space $\tilde{\mathscr{K}}_1$, it results that Z_0 is a unitary operator from $\mathscr{D}_{\tilde{C}_0}$ onto $\mathscr{G}_1$.

On the other hand any isometry U on $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{G}_0$ for which $\mathscr{H}_0$ is a reducing subspace has the form (with respect to the above decomposition of $\mathscr{H}$): $U = U_0 \oplus \tilde{Y}_0'$, where $U_0 = U | \mathscr{H}_0$ and $\tilde{Y}_0' = U | \mathscr{G}_0$. Then, denoting $\tilde{U}_1 = \tilde{W}_0 U \tilde{W}_0^{-1}$, it is clear that U is an isometry on $\tilde{\mathscr{H}}_1$ having (with respect to the decomposition $\tilde{\mathscr{H}}_1 = \mathscr{H}_0 \oplus \mathscr{G}_0'$) the matricial form given in (1.6) where $\tilde{Y}_0 = \tilde{W}_0 \tilde{Y}_0' \tilde{W}_0^{-1} | \mathscr{G}_0'$. Obviously, $(\tilde{\mathscr{H}}_1, [\tilde{U}_1, \tilde{V}_1])$ is an Ando dilation of $(\mathscr{H}, [T, S])$ crossing through $(\mathscr{H}_0, [U_0, V_0])$ and from the relation $\tilde{U}_1 \tilde{V}_1 = \tilde{V}_1 \tilde{U}_1$ it follows, in particular, the relation (1.7).

Now, in order to show that $\tilde{Y}_0$ is uniquely determined by $\tilde{C}_0$ and Z_0, it is sufficient to note that the isometry $\tilde{V}_1$ defined in (1.6) can be written in the form:

$$\tilde{V}_1 = \begin{bmatrix} V_0 & C \\ D_{V_0} & Z D_C \end{bmatrix} : \begin{matrix} \mathscr{H}_0 \\ \oplus \\ \mathscr{G}_0' \end{matrix} \to \begin{matrix} \mathscr{H}_0 \\ \oplus \\ \mathscr{G}_0' \end{matrix}$$

where C is the contraction in $L(\mathscr{G}_0', \tilde{\mathscr{K}}_1 \ominus \mathscr{H})$ defined by $C = D_{V_0^*}\tilde{C}_0$ and Z is the isometry in $L(\mathscr{D}_C, \mathscr{G}_0')$ defined by $Z D_C = -V_0^* \tilde{C}_0 + Z_0 D_{\tilde{C}_0}$. Then, from the fact that $\tilde{U}_1$ and $\tilde{V}_1$ commute it follows that the isometry $\tilde{Y}_0$ satisfies $\tilde{Y}_0 D_{V_0} = D_{V_0} U_0$ and $\tilde{Y}_0(ZD_C) = (ZD_C)\tilde{Y}_0$, hence

$$\tilde{Y}_0(ZD_C)^n D_{V_0} k_0 = (ZD_C)^n D_{V_0} U_0 k_0 \quad (k_0 \in \mathscr{K}_0,\ n \geqslant 0),$$

wherefrom, since the minimality of the space $\tilde{\mathscr{K}}_1$ implies that

$$\mathscr{G}_0' = \bigvee_{n \geqslant 0} (ZD_C)^n \mathscr{D}_{V_0},$$

it follows that $\tilde{Y}_0$ is uniquely determined by C and Z, thus by $\tilde{C}_0$ and Z_0. This finishes the proof.

Considering Lemma 1.1, it is clear that the free parameters in the construction of $(\mathscr{K}, [U, V])$ are given by the free parameters of $\tilde{C}_0$ and Z_0. In the next section we give a recurrent construction of $(\mathscr{K}, [U, V])$, based on the succesive application of Lemma 1.1, which also describes the constraints in choosing.

2.

Let T, S be two commuting contractions on $\mathscr{H}$. In the sequel, using Remark 1.1 and Lemma 1.1, we shall give a recurrent construction of Ando dilations of $(\mathscr{H}, [T, S])$. To this aim we introduce the following definition:

A sequence $\{\mathscr{K}_n, [U_n, V_n]\}_{n \geqslant 0}$, where $\mathscr{K}_n$ is a Hilbert space and U_n, V_n are contractions on $\mathscr{K}_n$ $(n \geqslant 0)$ is called a *$[T, S]$-adequate sequence of succesive dilations* if $(\mathscr{K}_0, U_0)$ is (an identification of) the minimal isometric dilation of T, V_0 is a contractive dilation of the commutant S of T and, for $n \geqslant 1$, $\mathscr{K}_n$, U_n, V_n are recurrently defined by

$$(2.1)_n \qquad \begin{cases} \mathscr{K}_n = \mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}} \\ U_n | \mathscr{K}_{n-1} = U_{n-1} \\ V_n = \begin{bmatrix} V_{n-1} & D_{V_{n-1}^*} C_{n-1} \\ D_{V_{n-1}} & -V_{n-1}^* C_{n-1} \end{bmatrix} \end{cases}$$

where C_{n-1} acts from $\mathscr{D}_{V_{n-1}}$ to $\mathscr{D}_{V_{n-1}^*} \ominus \overline{D_{V_{n-1}^*}\mathscr{H}}$, such that

$$(2.2)_n \qquad P_{\mathscr{K}_0}^{\mathscr{K}_n} U_n = U_n P_{\mathscr{K}_0}^{\mathscr{K}_n}$$

$$(2.3)_n \qquad (V_n U_n - U_n V_n) | \mathscr{K}_{n-1} = 0.$$

REMARK 2.1. The conditions of the definition of a $[T, S]$-adequate sequence of succesive dilations, for $n \geqslant 1$, are equivalent to the following: $\mathscr{K}_n$ is defined as in $(2.1)_n$, V_n has the form:

$$(2.4)_n \qquad V_n = \Omega(V_{n-1})(I \oplus C_{n-1}),$$

where C_{n-1} is a contraction from $\mathscr{D}_{V_{n-1}}$ to $\mathscr{D}_{V^*_{n-1}} \ominus \overline{D_{V^*_{n-1}}\mathscr{H}}$ and $\Omega(V_{n-1})$ is the unitary operator

$$\Omega(V_{n-1}) = \begin{bmatrix} V_{n-1} & D_{V^*_{n-1}} \\ D_{V_{n-1}} & -V^*_{n-1} \end{bmatrix} : \begin{matrix} \mathscr{K}_{n-1} \\ \oplus \\ \mathscr{D}_{V^*_{n-1}} \end{matrix} \to \begin{matrix} \mathscr{K}_{n-1} \\ \oplus \\ \mathscr{D}_{V_{n-1}} \end{matrix},$$

and U_n has (with respect to the decomposition $(2.1)_n$ of $\mathscr{K}_n$) the form:

$$(2.5)_n \qquad U_n = \begin{bmatrix} U_{n-1} & \Gamma_{n-1} \\ 0 & Y_{n-1}D_{\Gamma_{n-1}} \end{bmatrix},$$

such that Γ_{n-1} defined by

$$(2.6)_{n-1} \qquad \Gamma_{n-1}D_{V_{n-1}} = V_{n-1}U_{n-1} - U_{n-1}V_{n-1}$$

is a contraction from $\mathscr{D}_{V_{n-1}}$ to $\operatorname{Ker} U^*_{n-1} \cap (\mathscr{K}_{n-1} \ominus \mathscr{K}_0)$ and Y_{n-1} defined by

$$2.7)_{n-1} \qquad Y_{n-1}D_{\Gamma_{n-1}}D_{V_{n-1}} = D_{V_{n-1}}U_{n-1}$$

is an isometry from $\mathscr{D}_{\Gamma_{n-1}}$ to $\mathscr{D}_{V_{n-1}}$. Moreover U_n is an isometry on $\mathscr{K}_n$, uniquely determined by $\{V_k\}_{0 \leqslant k \leqslant n-1}$ (for $n \geqslant 1$), thus by $\{C_k\}_{0 \leqslant k \leqslant n-2}$ (for $n \geqslant 2$).

Indeed, for $n = 1$, since $U_1 \mid \mathscr{K}_0 = U_0$ and $\mathscr{K}_0$ reduces U_1 it follows that (on $\mathscr{K}_1 = \mathscr{K}_0 \oplus \mathscr{D}_{V_0}$) U_1 has the form $(2.5)_1$ where $\Gamma_0 D_{V_0} = V_0U_0 - U_0V_0 = 0$, hence $\Gamma_0 = 0$ and, according to $(2.3)_1$, Y_0 is defined by $(2.7)_0$. Moreover, since $U_0V_0 = V_0U_0$, Y_0 is an isometry (on $\mathscr{D}_{V_0}$), so that U_1 is an isometry (uniquely determined by V_0) on $\mathscr{K}_1$.

Now, assume that, on $\mathscr{K}_n = \mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}}$, V_n is defined by $(2.4)_n$ (or equivalently by $(2.1)_n$) and U_n is defined by $(2.5)_n$, where Γ_{n-1} is defined by $(2.6)_{n-1}$ and is a contraction satisfying

$$(2.9)_{n-1} \qquad P^{\mathscr{K}_{n-1}}_{\mathscr{K}_0}\Gamma_{n-1} = 0$$

and Y_{n-1} is defined by $(2.7)_{n-1}$ and is an isometry. Then, obviously, U_n is also an isometry, $(n \geqslant 2)$. By virtue of [10], it is known that any contraction U_{n+1} on $\mathscr{K}_{n+1} = \mathscr{K}_n \oplus \mathscr{D}_{V_n}$ satisfying $U_{n+1} \mid \mathscr{K}_n = U_n$ and $(2.2)_{n+1}$ has the form $(2.5)_{n+1}$, where $\Gamma_n (= D_{U^*_n}\Gamma_n)$ is a contraction from $\mathscr{D}_{V_n}$ to $\operatorname{Ker} U^*_n (= \mathscr{D}_{U^*_n})$ satisfying the condition $(2.9)_n$ (which is equivalent to $(2.2)_{n+1}$) and Y_n is a contraction from $\mathscr{D}_{\Gamma_n}$ to $\mathscr{D}_{V_n}$. On the other hand, $(2.3)_{n+1}$ (where V_{n+1} is defined by $(2.1)_{n+1}$ or, equivalently, by $(2.4)_{n+1}$) implies that Γ_n and Y_n satisfy $(2.6)_n$ and $(2.7)_n$, respectively. Moreover, we obviously have

$$(2.8)_n \qquad U_n^* V_n U_n = V_n$$

from which it follows

$$\|D_{\Gamma_n} D_{V_n} k_n\| = \|D_{V_n} U_n k_n\| \quad (k_n \in \mathscr{K}_n),$$

so that Y_n is an isometry and U_{n+1} is an isometry, too. Obviously, U_{n+1} depends (for U_0, V_0 fixed) only on $\{C_k\}_{0 \leq k \leq n-1}$. Conversely, it is clear that if U_n is defined on $\mathscr{K}_n = \mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}}$ by $(2.5)_n$, $(2.6)_{n-1}$, $(2.7)_{n-1}$, with $\Gamma_{n-1} \in L(\mathscr{D}_{V_{n-1}}, \operatorname{Ker} U_{n-1}^* \cap \cap (\mathscr{K}_{n-1} \ominus \mathscr{K}_0))$ a contraction, $Y_{n-1} \in L(\mathscr{D}_{\Gamma_{n-1}}, \mathscr{D}_{V_{n-1}})$ an isometry and V_n is defined by $(2.4)_n$, with C_n a contraction, then the conditions $(2.1)_n$, $(2.2)_n$, $(2.3)_n$ are fulfilled ($n \geq 1$).

Now, we establish the following

LEMMA 2.1. *Let $[T, S]$ be a pair of commuting contractions on $\mathscr{H}$. Then any $[T, S]$-adequate sequence of succesive dilations $\{\mathscr{K}_n, [U_n, V_n]\}_{n \geq 0}$ produces an Ando dilation $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ of the pair $(\mathscr{H}, [T, S])$, which crosses through $(\mathscr{K}_0, [U_0, V_0])$.*

Proof. For the ascendent sequence of Hilbert spaces $\{\mathscr{K}_n\}_{n \geq 0}$, define

$$(2.10) \qquad \mathscr{K}_\infty = \overline{\bigcup_{n \geq 0} \mathscr{K}_n} = \mathscr{K}_0 \oplus \mathscr{D}_{V_0} \oplus \mathscr{D}_{V_1} \oplus \dots$$

and $P_n = P_{\mathscr{K}_n}^{\mathscr{K}_\infty}$; obviously, $\{P_n\}_{n \geq 0}$ converges (strongly) to $I_{\mathscr{K}_\infty}$. Since $U_{n+1} | \mathscr{K}_n = U_n$ it is clear that there exists

$$(2.11) \qquad U_\infty = \operatorname*{s-lim}_{n \to \infty} U_n P_n$$

which defines, since U_n is an isometry and $U_\infty | \mathscr{K}_n = U_n$ ($n \geq 0$), an isometry on $\mathscr{K}_\infty$. On the other hand, it is clear that $V_{n+1} | \mathscr{K}_n$ is an isometry and $V_{n+2} | \mathscr{K}_n = V_{n+1} | \mathscr{K}_n$ ($n \geq 0$), so that there exists

$$(2.12) \qquad V_\infty = \operatorname*{s-lim}_{n \to \infty} V_n P_n$$

and (2.12) defines, since $V_\infty | \mathscr{K}_n = V_{n+1} | \mathscr{K}_n$ ($n \geq 0$), an isometry on $\mathscr{K}_\infty$. Moreover, by $(2.3)_n$ ($n \geq 1$), it is obvious that $V_\infty U_\infty = U_\infty V_\infty$. Also, since $(\mathscr{K}_0, U_0)$ is the minimal isometric dilation of T and (according to $(2.2)_n$, $n \geq 1$) $\mathscr{K}_0$ reduces U_∞, it results $P_{\mathscr{H}}^{\mathscr{K}_\infty} U_\infty = T P_{\mathscr{H}}^{\mathscr{K}_\infty}$. Finally since V_0 is a contractive dilation of S and C_n (from the definition of V_{n+1}) takes its values in $\mathscr{D}_{V_{n-1}^*} \ominus \overline{D_{V_{n-1}^*} \mathscr{H}}$ ($n \geq 0$), it follows recurrently:

$$P_{\mathscr{H}}^{\mathscr{K}_\infty} V_n P_n = P_{\mathscr{H}}^{\mathscr{K}_\infty} V_{n-1} P_{n-1} = \dots = P_{\mathscr{H}}^{\mathscr{K}_\infty} V_0 P_0 = S P_{\mathscr{H}}^{\mathscr{K}_\infty},$$

whence $P_{\mathscr{H}}^{\mathscr{K}_\infty} V_\infty = S P_{\mathscr{H}}^{\mathscr{K}_\infty}$. Thus $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ is an Ando dilation of $(\mathscr{H}, [T, S])$ which, obviously, crosses through $(\mathscr{K}_0, [U_0, V_0])$.

By Lemma 2.1 we obtain a map

(2.13) $$\varphi = \varphi_{(S,T)} = (\{\mathcal{K}_n, [U_n, V_n]\}_{n\geqslant 0} \mapsto (\mathcal{K}_\infty, [U_\infty, V_\infty]))$$

from the set of all $[T, S]$-adequate sequences of succesive dilations to the set of Ando dilations of $(\mathcal{H}, [T, S])$, which crosses through the first term of its corresponding adequate sequence.

Now we construct the inverse map of φ.

PROPOSITION 2.1. *Let $T, S \in L(\mathcal{H})$ be two commuting contractions and $(\mathcal{K}, [U, V])$ be an Ando dilation of $(\mathcal{H}, [T, S])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$. Then there exists a $[T, S]$-adequate sequence of succesive dilations $\{\mathcal{K}_n, [U_n, V_n]\}_{n\geqslant 0}$, such that the Ando dilation $\varphi(\{\mathcal{K}_n, [U_n, V_n]\}_{n\geqslant 0})$ coincides with $(\mathcal{K}, [U, V])$.*

Proof. Let $\mathcal{K}_0, U_0, V_0$ be defined by (1.4), (1.5). Then, according to Lemma 1.1, $(\mathcal{K}, [U, V])$ coincides, by the unitary operator (1.9), with the Ando dilation $(\tilde{\mathcal{K}}_1, [\tilde{U}_1, \tilde{V}_1])$ (crossing through $(\mathcal{K}_0, [U_0, V_0])$) defined by (1.6). Notice that, since $\tilde{Y}_0$ satisfies (1.7), the subspace $\mathcal{K}_0 \oplus \mathcal{D}_{V_0}$ of $\tilde{\mathcal{K}}_1$ is invariant to $\tilde{U}_1$. Denote

(2.14)$_1$ $$\mathcal{K}_1 = \mathcal{K}_0 \oplus \mathcal{D}_{V_0}, \quad U_1 = \tilde{U}_1 \mid \mathcal{K}_1, \quad V_1 = P^{\tilde{\mathcal{K}}_1}_{\mathcal{K}_1} \tilde{V}_1 \mid \mathcal{K}_1.$$

Then, U_1 has the form $(2.5)_1$, where $\Gamma_0 = 0$ and Y_0 is the isometry defined by

(2.15)$_0$ $$Y_0 = \tilde{Y}_0 \mid \mathcal{D}_{V_0},$$

so that $(2.6)_0$ and $(2.7)_0$ are fulfilled and V_1 has the form $(2.4)_1$, where

(2.16)$_0$ $$C_0 = \tilde{C}_0 \mid \mathcal{D}_{V_0}$$

is a contraction from $\mathcal{D}_{V_0}$ to $\mathcal{D}_{V_0^*} \ominus \overline{D_{V_0^*}\mathcal{H}}$.

Now, assume that we constructed a string of unitary operators $\{\tilde{W}_{k-1}\}_{1\leqslant k\leqslant n}$ and a string of Ando dilations $\{(\tilde{\mathcal{K}}_k, [\tilde{U}_k, \tilde{V}_k])\}_{1\leqslant k\leqslant n}$ crossing through $(\mathcal{K}_0, [U_0, V_0])$, for $n \geqslant 1$, where

(2.17)$_k$ $$\tilde{\mathcal{K}}_k = \tilde{W}_{k-1}\tilde{\mathcal{K}}_{k-1}, \quad \tilde{U}_k = \tilde{W}_{k-1}\tilde{U}_{k-1}\tilde{W}_{k-1}^{-1}, \quad \tilde{V}_k = \tilde{W}_{k-1}\tilde{V}_{k-1}\tilde{W}_{k-1}^{-1},$$

$1 \leqslant k \leqslant n$ (for $k = 1$, having $\tilde{\mathcal{K}}_0 = \mathcal{K}$, $\tilde{U}_0 = U$, $\tilde{V}_0 = V$ and W_0 in (1.9)) such that:

(2.18)$_k$ $$\tilde{\mathcal{K}}_{k-1} = \mathcal{K}_{k-1} \oplus \mathcal{G}_{k-1}, \quad \tilde{\mathcal{K}}_k = \mathcal{K}_{k-1} \oplus \mathcal{D}_{V_{k-1}} \oplus \mathcal{G}_k, \quad \mathcal{G}_k \subset \mathcal{G}_{k-1},$$

(2.19)$_{k-1}$ $$\begin{cases} \tilde{W}_{k-1} : \tilde{\mathcal{K}}_{k-1} = \mathcal{K}_{k-1} \oplus (\mathcal{G}_{k-1} \ominus \mathcal{G}_k) \oplus \mathcal{G}_k \to \tilde{\mathcal{K}}_k = \\ \qquad = \mathcal{K}_{k-1} \oplus \mathcal{D}_{V_{k-1}} \oplus \mathcal{G}_k \\ \tilde{W}_{k-1} \mid \mathcal{K}_{k-1} = I_{\mathcal{K}_{k-1}}, \quad \tilde{W}_{k-1} \mid \mathcal{G}_k = I_{\mathcal{G}_k} \end{cases}$$

(thus $\tilde{W}_{k-1} | (\mathscr{G}_{k-1} \ominus \mathscr{G}_k)$ is unitary from $\mathscr{G}_{k-1} \ominus \mathscr{G}_k$ onto $\mathscr{D}_{V_{k-1}}$) and $\tilde{U}_k$, $\tilde{V}_k$ have, with respect to the decomposition $\tilde{\mathscr{K}}_k = \mathscr{K}_{k-1} \oplus \mathscr{G}'_{k-1}$, where $\mathscr{G}'_{k-1} = \mathscr{D}_{V_{k-1}} \oplus \mathscr{G}_k$, the matricial forms

$$(2.20)_k \quad \tilde{U}_k = \begin{bmatrix} U_{k-1} & \tilde{\Gamma}_{k-1} \\ 0 & \tilde{Y}_{k-1} D_{\tilde{\Gamma}_{k-1}} \end{bmatrix}, \quad \tilde{V}_k = \begin{bmatrix} V_{k-1} & D_{V^*_{k-1}} \tilde{C}_{k-1} \\ D_{V_{k-1}} & -V^*_{k-1}\tilde{C}_{k-1} + Z_{k-1} D_{\tilde{C}_{k-1}} \end{bmatrix},$$

where $\tilde{\Gamma}_{k-1}$ is a contraction from $\mathscr{G}'_{k-1}$ to $\operatorname{Ker} U^*_{k-1}$, $\tilde{Y}_{k-1}$ is an isometry from $\mathscr{D}_{\tilde{\Gamma}_{k-1}}$ to $\mathscr{G}'_{k-1}$, $\tilde{C}_{k-1}$ is a contraction from $\tilde{\mathscr{G}}_{k-1}$ to $\mathscr{D}_{V^*_{k-1}} \ominus \overline{D_{V^*_{k-1}} \mathscr{H}}$ and Z_{k-1} is an isometry from $\mathscr{D}_{\tilde{C}_{k-1}}$ to $\mathscr{G}_k$ ($1 \leqslant k \leqslant n$). Moreover, in the above construction it is assumed that $\mathscr{K}_{k-1}$, U_{k-1}, V_{k-1} (for $2 \leqslant k \leqslant n$) are defined by

$$(2.14)_{k-1} \quad \mathscr{K}_{k-1} = \mathscr{K}_{k-2} \oplus \mathscr{D}_{V_{k-2}}, \quad U_{k-1} = \tilde{U}_{k-1} | \mathscr{K}_{k-1}, \quad V_{k-1} = P^{\tilde{\mathscr{K}}_{k-1}}_{\mathscr{K}_{k-1}} \tilde{V}_{k-1} | \mathscr{K}_{k-1}$$

and satisfy the conditions of Remark 2.1.

Since $(\tilde{\mathscr{K}}_n, [\tilde{U}_n, \tilde{V}_n])$ is an Ando dilation crossing through $(\mathscr{K}_0, [U_0, V_0])$ it follows (from $\tilde{U}_n \tilde{V}_n = \tilde{V}_n \tilde{U}_n$)

$$(2.21)_{n-1} \quad \tilde{\Gamma}_{n-1} D_{V_{n-1}} = V_{n-1} U_{n-1} - U_{n-1} V_{n-1}, \quad \tilde{Y}_{n-1} D_{\tilde{\Gamma}_{n-1}} D_{V_{n-1}} = D_{V_{n-1}} U_{n-1}$$

and (from the fact that $\mathscr{K}_0$ reduces $\tilde{U}_n$)

$$(2.22)_{n-1} \qquad P^{\mathscr{K}_{n-1}}_{\mathscr{K}_0} \tilde{\Gamma}_{n-1} = 0.$$

Notice also that the second relation of $(2.21)_{n-1}$ implies $\tilde{U}_n(\mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}}) \subset \subset (\mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}})$. Then, by $(2.14)_n$, we define (on $\mathscr{K}_n = \mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}}$) an isometry U_n and a contraction V_n having the matricial forms $(2.5)_n$ and $(2.4)_n$ respectively, where, obviously,

$$(2.23)_{n-1} \qquad \Gamma_{n-1} = \tilde{\Gamma}_{n-1} | \mathscr{D}_{V_{n-1}}$$

is a contraction in $L(\mathscr{D}_{V_{n-1}}, \operatorname{Ker} U^*_{n-1})$ satisfying $(2.6)_{n-1}$ and $(2.9)_{n-1}$, Y_{n-1} defined by

$$(2.15)_{n-1} \qquad Y_{n-1} D_{\Gamma_{n-1}} = \tilde{Y}_{n-1} D_{\tilde{\Gamma}_{n-1}} | \mathscr{D}_{V_{n-1}}$$

is an isometry in $L(\mathscr{D}_{\Gamma_{n-1}}, \mathscr{D}_{V_{n-1}})$ satisfying $(2.7)_{n-1}$ and

$$(2.16)_{n-1} \qquad C_{n-1} = \tilde{C}_{n-1} \mid \mathscr{D}_{V_{n-1}}$$

is a contraction in $L(\mathscr{D}_{V_{n-1}}, \mathscr{D}_{V^*_{n-1}} \ominus \overline{D_{V^*_{n-1}}\mathscr{H}})$.

On the other hand (according to [4] and [10]), the isometries $\tilde{U}_n$, $\tilde{V}_n$ have, with respect to the decomposition $\tilde{\mathscr{K}}_n = \mathscr{K}_n \oplus \mathscr{G}_n$, the matricial forms

$$(2.24)_n \qquad \tilde{V}_n = \begin{bmatrix} V_n & D_{V^*_n}\tilde{C}'_n \\ W_n D_{V_n} & -W_n V^*_n \tilde{C}'_n + Z'_n D_{\tilde{C}'_n} \end{bmatrix}$$

(where $W_n \in L(\mathscr{D}_{V_n}, \mathscr{G}_n)$ is an isometry, $\tilde{C}'_n \in L(\mathscr{G}_n, \mathscr{D}_{V^*_n} \ominus \overline{D_{V^*_n}\mathscr{H}})$ is a contraction and $Z'_n \in L(\mathscr{D}_{\tilde{C}'_n}, \operatorname{Ker} W^*_n)$ is an isometry) and

$$(2.25)_n \qquad \tilde{U}_n = \begin{bmatrix} U_n & \tilde{\Gamma}'_n \\ 0 & \tilde{Y}'_n D_{\tilde{\Gamma}'_n} \end{bmatrix}$$

(where $\tilde{\Gamma}'_n \in L(\mathscr{G}_n, \operatorname{Ker} U^*_n)$ is a contraction and $\tilde{Y}'_n \in L(\mathscr{D}_{\tilde{\Gamma}'_n}, \mathscr{G}_n)$ is an isometry). Then, denoting $\mathscr{G}_{n+1} = \mathscr{G}_n \ominus W_n \mathscr{D}_{V_n}$ and considering the spaces defined in $(2.19)_{n+1}$, it is clear that

$$(2.26)_n \qquad \tilde{W}_n = I_{\mathscr{K}_n} \oplus (W^*_n \mid W_n \mathscr{D}_{V_n}) \oplus I_{\mathscr{G}_{n+1}}$$

is a unitary operator satisfying $(2.19)_n$. Moreover, defining $\tilde{\mathscr{K}}_{n+1}$, $\tilde{U}_{n+1}$ and $\tilde{V}_{n+1}$ by $(2.17)_{n+1}$, it is obvious that $(\tilde{\mathscr{K}}_{n+1}, [\tilde{U}_{n+1}, \tilde{V}_{n+1}])$ is an Ando dilation crossing through $(\mathscr{K}_0, [U_0, V_0])$.

Consequently, we constructed, by induction, a $[T, S]$-adequate sequence of succesive dilations $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geq 0}$ and, also, a sequence $\{\tilde{\mathscr{K}}_n, [\tilde{U}_n, \tilde{V}_n]\}_{n\geq 0}$ (where $(\tilde{\mathscr{K}}_0, [\tilde{U}_0, \tilde{V}_0])$ is $(\mathscr{K}, [U, V])$) of Ando dilations of $(\mathscr{H}, [T, S])$ crossing through $(\mathscr{K}_0, [U_0, V_0])$ and a sequence of unitary operators from $\tilde{\mathscr{K}}_{n-1}$ onto $\tilde{\mathscr{K}}_n$, $n \geq 0$, $\{\tilde{W}_n\}_{n\geq 0}$, such that (according to $(2.17)_n$) $(\tilde{\mathscr{K}}_n, [\tilde{U}_n, \tilde{V}_n])$ coincides with $(\tilde{\mathscr{K}}_{n-1}, [\tilde{U}_{n-1}, \tilde{V}_{n-1}])$ $(n \geq 1)$. Then, it follows recurrently that $(\tilde{\mathscr{K}}_n, [\tilde{U}_n, \tilde{V}_n])$ coincides (by the unitary operator $(\tilde{W}_{n-1} \ldots \tilde{W}_0)$) with $(\mathscr{K}, [U, V])$. On the other hand, denoting

$$(2.27) \qquad \mathscr{K}_\infty = \mathscr{K}_0 \oplus \mathscr{D}_{V_0} \oplus \mathscr{D}_{V_1} \oplus \ldots,$$

it is easy to see, by definitions $(2.17)_n$ and $(2.26)_{n-1}$, $n \geqslant 1$, that there exists

$$W_\infty = \operatorname*{s-lim}_{n\to\infty}(\tilde{W}_{n-1} \dots \tilde{W}_0) \tag{2.28}$$

and that (2.28) defines a unitary operator from $\mathscr{K}$ to $\mathscr{K}_\infty$. Also, denoting $(\mathscr{K}_\infty, [U_\infty, V_\infty]) = \varphi(\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0})$ the Ando dilation corresponding (by Lemma 2.1) to $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0}$, from $(2.14)_n$, $n\geqslant 1$, it follows immediately that

$$U_\infty = W_\infty U W_\infty^{-1}, \quad V_\infty = W_\infty V W_\infty^{-1}, \tag{2.29}$$

therefore $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ coincides (by the unitary operator W_∞ which satisfies $W_\infty \mid \mathscr{K}_0 = I_{\mathscr{K}_0}$) with $(\mathscr{K}, [U, V])$. The proof of the proposition is complete.

Now we introduce an equivalence relation on the set of all $[T, S]$-adequate sequences of succesive dilations, namely, we say that two adequate sequences of succesive dilations $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0}$ and $\{\mathscr{K}'_n, [U'_n, V'_n]\}_{n\geqslant 0}$ *coincide* if there exists a sequence $\{X_n\}_{n\geqslant 0}$ of unitary operators X_n from $\mathscr{K}_n$ onto $\mathscr{K}'_n$, $n \geqslant 0$, such that

$$\begin{cases} X_0 \mid \mathscr{H} = I_{\mathscr{H}}, & X_{n+1} \mid \mathscr{K}_n = X_n, \\ X_n U_n = U'_n X_n, & X_n V_n = V'_n X_n, \quad n \geqslant 0. \end{cases} \tag{2.30}$$

We are able now to establish the following

THEOREM 2.1. *Let $[T, S]$ be a pair of commuting contractions on $\mathscr{H}$. There exists a one-to-one correspondence (induced by* (2.13)*) between the set of all coinciding $[T, S]$-adequate sequences of succesive dilations and the set of all coinciding Ando dilations of $(\mathscr{H}, [T, S])$.*

Proof. First notice that if $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0}$ and $\{\mathscr{K}'_n, [U'_n, V'_n]\}_{n\geqslant 0}$ are two coinciding sequences of succesive dilations and $\{X_n\}_{n\geqslant 0}$ is the sequence of unitary operators establishing this coincidence then, defining $\mathscr{K}_\infty$, $\mathscr{K}'_\infty$ by (2.10), there exists

$$X_\infty = \text{s-lim}\, X_n P_n$$

(where $P_n = P^{\mathscr{K}_\infty}_{\mathscr{K}_n}$, $n \geqslant 0$) and X_∞ is a unitary operator from $\mathscr{K}_\infty$ onto $\mathscr{K}'_\infty$ (satisfying $X_\infty \mid \mathscr{H} = I_{\mathscr{H}}$) which establishes a coincidence between the Ando dilations $(\mathscr{K}_\infty, [U_\infty, V_\infty]) = \varphi(\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0})$ and $(\mathscr{K}'_\infty, [U'_\infty, V'_\infty]) = \varphi(\{\mathscr{K}'_n, [U'_n, V'_n]\}_{n\geqslant 0})$ (defined in Lemma 2.1).

Now, taking into account Lemma 2.1, Proposition 2.1 and the above remark, it remains to show that for two coinciding (by a unitary operator X) Ando dilations $(\mathscr{K}, [U, V])$ (crossing through $(\mathscr{K}_0, [U_0, V_0])$) and $(\mathscr{K}', [U', V'])$ (crossing through $(\mathscr{K}'_0, [U'_0, V'_0])$) their corresponding $[T, S]$-adequate sequences of succesive dilations $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0}$ and $\{\mathscr{K}'_n, [U'_n, V'_n]\}_{n\geqslant 0}$ respectively (see Proposition 2.1) also coincide. Denoting W the unitary operator (satisfying $W \mid \mathscr{K}_0 = I_{\mathscr{K}_0}$) which establishes the coincidence between $(\mathscr{K}, [U, V])$ and $(\mathscr{K}_\infty, [U_\infty, V_\infty]) = \varphi(\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0})$ and W' the unitary operator (satisfying $W' \mid \mathscr{K}'_0 = I_{\mathscr{K}'_0}$) which establishes the coincidence between $(\mathscr{K}', [U', V'])$ and $(\mathscr{K}'_\infty, [U'_\infty, V'_\infty]) = \varphi(\{\mathscr{K}'_n, [U'_n, V'_n]\}_{n\geqslant 0})$ and putting

$$\tilde{X} = W'XW^{-1},$$

it is clear that $\tilde{X}$ is a unitary operator (satisfying $\tilde{X} \mid \mathscr{H} = I_{\mathscr{H}}$) which establishes the coincidence between $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ and $(\mathscr{K}', [U'_\infty, V'_\infty])$. From the definitions (1.4), (1.5) of $(\mathscr{K}_0, [U_0, V_0])$, respectively of $(\mathscr{K}'_0, [U'_0, V'_0])$ it follows that $X_0 = \tilde{X} \mid \mathscr{K}_0$ is a unitary operator from $\mathscr{K}_0$ onto $\mathscr{K}'_0$ satisfying $X_0U_0 = U'_0X_0$, $X_0V_0 = V'_0X_0$ and, therefore, $\mathscr{D}_{V'_0} = \tilde{X}\mathscr{D}_{V_0}$. Since, for $n \geqslant 1$, we have (see Lemma 2.1) $\mathscr{K}_n = \mathscr{K}_{n-1} \oplus \mathscr{D}_{V_{n-1}}$, $U_n = U_\infty \mid \mathscr{K}_n$, $V_n = P^{\mathscr{K}_\infty}_{\mathscr{K}_n} V_\infty \mid \mathscr{K}_n$ and, analogously, for $\mathscr{K}'_n$, U'_n, V'_n, it follows recurrently that $X_n = \tilde{X} \mid \mathscr{K}_n$ is a unitary operator from $\mathscr{K}_n$ onto $\mathscr{K}'_n$ satisfying $X_nU_n = U'_nX_n$, $X_nV_n = V'_nX_n$ and, also, $\mathscr{D}_{V'_n} = \tilde{X}\mathscr{D}_{V_n}$. Moreover, it is obvious that $X_0 \mid \mathscr{H} = I_{\mathscr{H}}$ and $X_{n+1} \mid \mathscr{K}_n = X_n$ $(n \geqslant 0)$, so that we obtain a sequence of unitary operators $\{X_n\}_{n\geqslant 0}$ which establishes the coincidence between $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0}$ and $\{\mathscr{K}'_n, [U'_n, V'_n]\}_{n\geqslant 0}$ and thus the proof of Theorem 2.1 is complete.

We conclude this section with the construction of a special $[T, S]$-adequate sequence of succesive dilations $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geqslant 0}$, namely the one which corresponds (by virtue of Theorem 2.1) to the distinguished Ando dilation $(\mathring{\mathscr{K}}_0, [\mathring{U}_0, \mathring{V}_0])$ crossing through $(\mathscr{K}_0, [U_0, V_0])$. More precisely, denoting by $\{\mathscr{K}_n(0), [U_n(0), V_n(0)]\}_{n\geqslant 0}$ the sequence starting with a given pair $(\mathscr{K}_0, [U_0, V_0]) = (\mathscr{K}_0(0), [U_0(0), V_0(0)])$ consisting of a minimal isometric dilation $(\mathscr{K}_0, U_0)$ of T and a contractive dilation V_0 of S, where $\mathscr{K}_n(0)$ is defined by $(2.1)_n$, $V_n(0)$ is defined by $(2.4)_n$ with $C_{n-1} = 0$ and $U_n(0)$ is defined by $(2.5)_n$ with $\Gamma_{n-1} = 0$ and Y_{n-1} given by $(2.7)_{n-1}$, we have the following

COROLLARY 2.1. *The sequence* $\{\mathscr{K}_n(0), [U_n(0), V_n(0)]\}$ *is a* $[T, S]$*-adequate sequence of succesive dilations uniquely determined for* $n \geqslant 1$, *(by Remark* 2.1*) by the sequence of contractions* $\{C_n = 0\}_{n\geqslant 0}$; *moreover, its corresponding Ando dilation* $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ *(given by* (2.13)*) coincides with the distinguished Ando dilation* $(\mathring{\mathscr{K}}_0, [\mathring{U}_0, \mathring{V}_0])$ *crossing through* $(\mathscr{K}_0, [U_0, V_0])$.

Proof. It is obvious that, for $C_n = 0$, $V_{n+1}(0)$ (defined by $(2.4)_{n+1}$) is a partial isometry. Moreover, defining $U_{n+1}(0)$ by $(2.5)_{n+1}$ (with $\Gamma_n = 0$ and Y_n given by $(2.7)_n$), we obtain recurrently that Y_n is an isometry and $U_{n+1}(0)V_{n+1}(0) = V_{n+1}(0)U_{n+1}(0)$, $n \geqslant 0$. Thus (see Remark 2.1), $\{\mathscr{K}_n(0), [U_n(0), V_n(0)]\}_{n\geqslant 0}$ is a $[T, S]$-adequate sequence of succesive dilations. In fact, identifying recurrently

$\mathscr{D}_{V_{n+1}} = \mathscr{D}_{V_n} = \ldots = \mathscr{D}_{V_0}$ and $Y_{n+1} = Y_n = \ldots = Y_0$, we have

$$\mathscr{K}_{n+1} = \mathscr{K}_0 \oplus \underbrace{\mathscr{D}_{V_0} \oplus \ldots \oplus \mathscr{D}_{V_0}}_{n\text{-times}},$$

$$(2.31)_n \qquad V_{n+1} = \begin{bmatrix} V_0 & 0 & & 0 \\ D_{V_0} & 0 & & 0 \\ 0 & I & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & I & 0 \end{bmatrix}, \quad U_{n+1} = \begin{bmatrix} U_0 & & & \\ & \ddots & & \\ 0 & & Y_0 & 0 \\ & & & \ddots \\ & & & & Y_0 \end{bmatrix},$$

$n \geqslant 0$. Therefore, defining $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ by (2.10), (2.11), (2.12), it is clear that $(\mathscr{K}_\infty, V_\infty)$ is (an identification of) the minimal isometric dilation of V_0, and U_∞ is the unique isometric extension of U_0 commuting with V_∞, i.e. $(\mathscr{K}_\infty, [U_\infty, V_\infty])$ is the distinguished Ando dilation crossing through $(\mathscr{K}_0, [U_0, V_0])$.

3.

Let $[T, S]$ be a pair of commuting contractions on a Hilbert space $\mathscr{H}$. In this section we shall give a necessary and sufficient condition for the uniqueness (modulo a coincidence) of Ando dilations of $(\mathscr{H}, [T, S])$. For the pair $(\mathscr{H}, [T, S])$ we fix a pair $(\mathscr{K}_0, [U_0, V_0])$ consisting of an identification $(\mathscr{K}_0, U_0)$ of the minimal isometric dilation of T and a contractive dilation V_0 of the commutant S of T.

For $(\mathscr{K}_0, U_0)$ there exists a unique (up to a unitary equivalence) minimal unitary extension $(\tilde{\mathscr{K}}_0, \tilde{U}_0)$, which is an identification of the minimal unitary dilation of T (see [12]), i.e. a unitary operator $\tilde{U}_0$ on a Hilbert space $\tilde{\mathscr{K}}_0$ containing $\mathscr{K}_0$ (as a subspace) such that $\mathscr{K}_0$ is invariant for $\tilde{U}_0$ and

$$(3.1) \qquad \begin{cases} \tilde{\mathscr{K}}_0 = \bigvee_{n \in \mathbf{Z}} \tilde{U}_0^n \mathscr{H} = \bigvee_{n \geqslant 0} \tilde{U}_0^{*n} \mathscr{K}_0 \\ \tilde{U}_0 \mid \mathscr{K}_0 = U_0. \end{cases}$$

Also, for V_0, there exists a unique operator $\tilde{V}_0$ on $\tilde{\mathscr{K}}_0$ satisfying

$$(3.2) \qquad \|\tilde{V}_0\| \leqslant 1, \quad \tilde{V}_0 \tilde{U}_0 = \tilde{U}_0 \tilde{V}_0, \quad \tilde{V}_0 \mid \mathscr{K}_0 = V_0,$$

which is defined by

$$(3.3) \qquad \tilde{V}_0 = \text{s-}\lim_{n \to \infty} \tilde{U}_0^{*n} V_0 P_{\mathscr{K}_0}^{\tilde{\mathscr{K}}_0} \tilde{U}_0^n$$

(see [8]) and it is called the *Douglas extension* of V_0.

On the other hand, it is known that for any Ando dilation $(\mathcal{K}, [U, V])$ of $(\mathcal{H}, [T, S])$ there exists a unique (up to a unitary equivalence) minimal unitary extension $(\tilde{\mathcal{K}}, [\tilde{U}, \tilde{V}])$, which is a minimal unitary dilation of $(\mathcal{H}, [T, S])$ (see [9]), i.e. a pair $[\tilde{U}, \tilde{V}]$ of commuting unitary operators $\tilde{U}$, $\tilde{V}$ on a Hilbert space $\tilde{\mathcal{K}}$ containing $\mathcal{K}$ (as a subspace) such that $\mathcal{K}$ is invariant for $\tilde{U}$ and $\tilde{V}$ and

$$(3.4)\qquad \begin{cases} \tilde{\mathcal{K}} = \bigvee_{n, m \in \mathbf{Z}} \tilde{U}^n \tilde{V}^m \mathcal{H} = \bigvee_{n, m \geq 0} \tilde{U}^{*n} \tilde{V}^{*m} \mathcal{K} \\ \tilde{U} \mid \mathcal{K} = U, \quad \tilde{V} \mid \mathcal{K} = V. \end{cases}$$

Now, let $(\mathcal{K}, [U, V])$ be an Ando dilation of $(\mathcal{H}, [T, S])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$ and let $(\tilde{\mathcal{K}}, [\tilde{U}, \tilde{V}])$ be the minimal unitary extension of $(\mathcal{K}, [U, V])$. Denoting

$$(3.5)\qquad \begin{cases} \tilde{\mathcal{K}}_0 = \bigvee_{n \in \mathbf{Z}} \tilde{U}^n \mathcal{H} = \bigvee_{n \geq 0} \tilde{U}^{*n} \mathcal{K}_0 \\ \tilde{U}_0 = \tilde{U} \mid \tilde{\mathcal{K}}_0, \end{cases}$$

it is clear that $\tilde{\mathcal{K}}_0$ is a subspace of $\tilde{\mathcal{K}}$ which reduces $\tilde{U}$ and $(\tilde{\mathcal{K}}_0, \tilde{U}_0)$ coincides (up to an isomorphism) with the minimal unitary extension of $(\mathcal{K}_0, U_0)$ (thus, with the minimal unitary dilation of $(\mathcal{H}, T)$). Moreover, we have

LEMMA 3.1. i) *The compression of* $\tilde{V}$ *on* $\tilde{\mathcal{K}}_0$:

$$(3.6)\qquad \tilde{V}_0 = P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} \tilde{V} | \tilde{\mathcal{K}}_0$$

is the Douglas extension of V_0.

ii) *If* $(\mathcal{K}, [U, V])$ *coincides with the distinguished Ando dilation* $(\mathring{\mathcal{K}}_0, [\mathring{U}_0, \mathring{V}_0])$ *crossing through* $(\mathcal{K}_0, [U_0, V_0])$ *then* $(\tilde{\mathcal{K}}, \tilde{V})$ *is the minimal unitary dilation of* $(\tilde{\mathcal{K}}_0, \tilde{V}_0)$.

Proof. i) Obviously, (3.6) defines a contraction on $\tilde{\mathcal{K}}_0$ which, since $\tilde{U}\tilde{V} = \tilde{V}\tilde{U}$ and $P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} \tilde{U} = \tilde{U}_0 P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0}$, commutes with $\tilde{U}_0$. Moreover, since $\mathcal{K}_0$ reduces U and $\tilde{U}$ is an extension of U, we have $\tilde{U}^{*n} \mathcal{K}_0 \perp \mathcal{K} \ominus \mathcal{K}_0$ $(n \geq 0)$, thus $\tilde{\mathcal{K}}_0 \perp \mathcal{K} \ominus \mathcal{K}_0$, from which it follows

$$\tilde{V}_0 | \mathcal{K}_0 = P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} \tilde{V} | \mathcal{K}_0 = P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} V | \mathcal{K}_0 = P^{\mathcal{K}}_{\mathcal{K}_0} V | \mathcal{K}_0 = V_0 .$$

ii) If, in addition, $(\mathcal{K}, [U, V])$ coincides with $(\mathring{\mathcal{K}}_0, [\mathring{U}_0, \mathring{V}_0])$ we have $P^{\mathcal{K}}_{\mathcal{K}_0} V = V_0 P^{\mathcal{K}}_{\mathcal{K}_0}$, therefore (using again $\tilde{\mathcal{K}}_0 \perp \mathcal{K} \ominus \mathcal{K}_0$) it follows

$$P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} \tilde{V}^n \tilde{U}_0^{*m} k_0 = P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} \tilde{U}^{*m} \tilde{V}^n k_0 = \tilde{U}_0^{*m} P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} V^n k_0 =$$

$$= \tilde{U}_0^{*m} P^{\mathcal{K}}_{\mathcal{K}_0} V^n k_0 = \tilde{U}_0^{*m} V_0^n k_0 = \tilde{U}_0^{*m} \tilde{V}_0^n k_0 = \tilde{V}_0^n \tilde{U}_0^{*m} k_0$$

$(k_0 \in \mathcal{K}_0,\ m \geq 0)$, thus $P^{\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}_0} \tilde{V}^n | \tilde{\mathcal{K}}_0 = \tilde{V}_0^n$ $(n \geq 0)$.

Now, recall that if $(\mathscr{K}_0, U_0)$ is the minimal isometric dilation of $(\mathscr{H}, T)$ and $(\tilde{\mathscr{K}}_0, \tilde{U}_0)$ is the minimal unitary extension of $(\mathscr{K}_0, U_0)$ (and also the minimal unitary dilation of $(\mathscr{H}, T)$), then

$$(3.7)\qquad \mathscr{K}_{*0} = \bigvee_{n\geq 0} \tilde{U}_0^{*n}\mathscr{H}, \quad U_{*0} = \tilde{U}_0^{*}|\mathscr{K}_{*0}$$

the space $\mathscr{K}_{*0}$ being invariant for $\tilde{U}_0^{*}$) is the minimal isometric dilation of $(\mathscr{H}, T^*)$ see [12]). Moreover, there exists a one-to-one correspondence between the set of all contractive dilations V_0 of the commutant S of T and the set of all contractive dilations V_{*0} of the commutant S^* of T^* given by

$$(3.8)\qquad V_{*0} = \tilde{V}_0^{*}\,|\,\mathscr{K}_{*0},$$

where $\tilde{V}_0$ is the Douglas extension of V_0, for which the space $\mathscr{K}_{*0}$ is invariant (see [2]). Using these facts, we obtain at once the following.

LEMMA 3.2. *There exists a one-to-one correspondence between the set of all coinciding Ando dilations $(\mathscr{K}, [U, V])$ of $(\mathscr{H}, [T, S])$ crossing through $(\mathscr{K}_0, [U_0, V_0])$ and the set of all coinciding Ando dilations $(\mathscr{K}_*, [U_*, V_*])$ of $(\mathscr{H}, [T^*, S^*])$ crossing through $(\mathscr{K}_{*0}, [U_{*0}, V_{*0}])$, which is given by*

$$(3.9)\qquad \mathscr{K}_* = \bigvee_{n,\, m\geq 0} \tilde{U}^{*n}\tilde{V}^{*m}\mathscr{H}, \quad U_* = \tilde{U}^*|\mathscr{K}_*, \quad V_* = \tilde{V}^*|\mathscr{K}_*$$

(where $(\tilde{\mathscr{K}}, [\tilde{U}, \tilde{V}])$ is the minimal unitary extension of the Ando dilation $(\mathscr{K}, [U, V])$). Moreover, by the above correspondence, to the distinguished Ando dilation $(\mathring{\mathscr{K}}_0, [\mathring{U}_0, \mathring{V}_0])$ crossing through $(\mathscr{K}_0, [U_0, V_0])$ corresponds the distinguished Ando dilation $(\mathring{\mathscr{K}}_{*0}, [\mathring{U}_{*0}, \mathring{V}_{*0}])$ crossing through $(\mathscr{K}_{*0}, [U_{*0}, V_{*0}])$.

Proof. It is obvious that $(\tilde{\mathscr{K}}, [\tilde{U}^*, \tilde{V}^*])$ is the minimal unitary dilation of $(\mathscr{H}, [T^*, S^*])$ and that the space $\mathscr{K}_*$ (defined in (3.9)) is invariant for $\tilde{U}^*$ and $\tilde{V}^*$, so that (3.9) defines an Ando dilation $(\mathscr{K}_*, [U_*, V_*])$ of $(\mathscr{H}, [T^*, S^*])$. Also the subspace $\mathscr{K}_{*0}\,(= \bigvee_{n\geq 0} U_*^n\mathscr{H})$ of $\mathscr{K}_*\,(= \bigvee_{n\geq 0} V_*^n\mathscr{K}_{*0})$ reduces U_* and $U_*|\mathscr{K}_{*0} = U_{*0}$; moreover, $P^{\mathscr{K}_*}_{\mathscr{K}_{*0}} V_*\,|\,\mathscr{K}_{*0} = P^{\mathscr{K}_*}_{\mathscr{K}_{*0}}\tilde{V}^*|\mathscr{K}_{*0} = \tilde{V}_0^*|\mathscr{K}_{*0} = V_{*0}$. Thus, $(\mathscr{K}_*, [U_*, V_*])$ crosses through $(\mathscr{K}_{*0}, [U_{*0}, V_{*0}])$. Since (3.8) defines a bijective correspondence it is obvious that (3.9) defines also a bijective correspondence. In particular, if $(\mathscr{K}, [U, V]$ coincides with $(\mathring{\mathscr{K}}_0, [\mathring{U}_0, \mathring{V}_0])$, then, by Lemma 3.1 ii), $(\tilde{\mathscr{K}}, \tilde{V})$ is the minimal unitary dilation of $(\tilde{\mathscr{K}}_0, \tilde{V}_0)$, thus $(\tilde{\mathscr{K}}, \tilde{V}^*)$ is the minimal unitary dilation of $(\tilde{\mathscr{K}}_0, \tilde{V}_0^*$ and, consequently $(\mathscr{K}_*, V_*)$ is the minimal isometric dilation of $(\mathscr{K}_{*0}, V_{*0})$, so that $(\mathscr{K}_*, [U_*, V_*])$ coincides with $(\mathring{\mathscr{K}}_{*0}, [\mathring{U}_{*0}, \mathring{V}_{*0}])$, and the lemma is proved.

Now, for the pair $(\mathscr{H}, [T, S])$ of commuting contraction on $\mathscr{H}$, let $(\mathscr{K}_0, [U_0, V_0])$ be a fixed pair as in the beginning of this section and let $(\mathscr{K}_{*0}, [U_{*0}, V_{*0}])$ be defined by (3.7), (3.8) (where $\tilde{\mathscr{K}}_0$, $\tilde{U}_0, \tilde{V}_0$ are defined by (3.1), (3.3), respectively). The $[T, S]$-adequate sequence of succesive dilations $\{\mathscr{K}_n(0), [U_n(0), V_n(0)]\}_{n\geq 0}$ (starting with $(\mathscr{K}_0, [U_0, V_0])$ and corresponding, according to Remark 2.1, to the sequence of contractions $\{C_n = 0\}_{n\geq 0}$), defined in Corollary 2.1, will be called the *distinguished* $[T, S]$-adequate sequence of succesive dilations starting with $(\mathscr{K}_0, [U_0, V_0])$. Analogously, we shall define the distinguished $[T^*, S^*]$-adequate sequence of succesive dilations $\{\mathscr{K}_{*n}(0), [U_{*n}(0), V_{*n}(0)]\}_{n\geq 0}$ starting with $(\mathscr{K}_{*0}, [U_{*0}, V_{*0}])$.

Let us also consider the operators

$$
(3.10) \qquad \begin{cases} Y_0 : \mathscr{D}_{V_0} \to \mathscr{D}_{V_0}, & Y_0 D_{V_0} = D_{V_0} U_0 \\ Y_{*0} : \mathscr{D}_{V_{*0}} \to \mathscr{D}_{V_{*0}}, & Y_{*0} D_{V_{*0}} = D_{V_{*0}} U_{*0} \end{cases}
$$

(which, by commutativity relations of U_0, V_0 and U_{*0}, V_{*0}, respectively, are isometries) and their Wold decompositions

$$
(3.11) \qquad \begin{cases} \mathscr{D}_{V_0} = \mathscr{R}_0 \oplus (\bigoplus_{n\geq 0} Y_0^n \mathscr{L}_0) \\ \mathscr{D}_{V_{*0}} = \mathscr{R}_{*0} \oplus (\bigoplus_{n\geq 0} Y_{*0}^n \mathscr{L}_{*0}), \end{cases}
$$

where $\mathscr{R}_0$ (resp. $\mathscr{R}_{*0}$) is the residual subspace and $\mathscr{L}_0$ (resp. $\mathscr{L}_{*0}$) is the wandering subspace of Y_0 (resp. Y_{*0}), namely:

$$
(3.12) \qquad \begin{aligned} \mathscr{R}_0 &= \bigcap_{n\geq 0} Y_0^n \mathscr{D}_{V_0}, \quad \mathscr{L}_0 = \mathscr{D}_{V_0} \ominus Y_0 \mathscr{D}_{V_0} = \mathscr{D}_{V_0} \ominus \overline{D_{V_0} U_0 \mathscr{K}_0} \\ \mathscr{R}_{*0} &= \bigcap_{n\geq 0} Y_{*0} \mathscr{D}_{V_{*0}}, \quad \mathscr{L}_{*0} = \mathscr{D}_{V_{*0}} \ominus Y_{*0} \mathscr{D}_{V_{*0}} = \mathscr{D}_{V_{*0}} \ominus \overline{D_{V_{*0}} U_{*0} \mathscr{K}_{*0}}. \end{aligned}
$$

With these notations we state the following

LEMMA 3.3. i) *If* $\mathscr{L}_0 = \{0\}$, *then there exists a unique (up to a coincidence)* $[T, S]$*-adequate sequence of succesive dilations starting with* $(\mathscr{K}_0, [U_0, V_0])$ *(namely, the distinguished* $[T, S]$*-adequate sequence* $\{\mathscr{K}_n(0), [U_n(0), V_n(0)]\}_{n\geq 0}$*).*

ii) *If* $\mathscr{L}_{*0} = \{0\}$, *then there exists a unique (up to a coincidence)* $[T^*, S^*]$*-adequate sequence of succesive dilations starting with* $(\mathscr{K}_{*0}, [U_{*0}, V_{*0}])$ *(namely, the distinguished* $[T^*, S^*]$*-adequate sequence* $\{\mathscr{K}_{*n}(0), [U_{*n}(0), V_{*n}(0)]\}_{n\geq 0}$*).*

Proof. i) Assume that $\mathscr{L}_0 = \{0\}$ and let $\{\mathscr{K}_n, [U_n, V_n]\}_{n\geq 0}$ be a $[T, S]$-adequate sequence of succesive dilations (starting with $(\mathscr{K}_0, [U_0, V_0])$), which is determined (according to Remark 2.1) by a sequence of contraction $\{C_n\}_{n\geq 0}$. Since $\mathscr{K}_0$ reduces U_n, from $(2.3)_2$ it follows: $P^{\mathscr{K}_1}_{\mathscr{K}_0}U_1V_1 = P^{\mathscr{K}_1}_{\mathscr{K}_0}V_1U_1$ wherefrom we obtain

$$(3.13)_0 \qquad D_{V_0^*}C_0Y_0 = U_0D_{V_0^*}C_0.$$

From $(3.13)_0$, since the range of C_0 lies in $\mathscr{D}_{V_0^*} \ominus \overline{D_{V_0^*}\mathscr{H}}$, it follows that

$$D_{V_0^*}C_0\mathscr{R}_0 \subset \bigcap_{n=0}^{\infty} D_{V_0^*}C_0Y_0^n(\mathscr{D}_{V_0}) = \bigcap_{n=0}^{\infty} U_0^nD_{V_0}C_0(\mathscr{D}_{V_0}) \subset \bigcap_{n\geq 0} U_0^n(\mathscr{K}_0 \ominus \mathscr{H}) = \{0\},$$

thus (since $D_{V_0^*}$ is injective on $\mathscr{D}_{V_0^*}$ and $\mathscr{R}_0 = \mathscr{D}_{V_0}$ by assumption) $C_0 = 0$. Then V_1 is the partial isometry defined by $(2.28)_1$ and we have $U_1V_1 = V_1U_1$.

Now, assume that $C_k = 0$ and $\Gamma_k = 0$, for $0 \leq k \leq n$, $n \geq 1$. Then identifying $\mathscr{D}_{V_n} = \mathscr{D}_{V_{n-1}} = \ldots = \mathscr{D}_{V_0}$, we have $Y_n = Y_{n-1} = \ldots = Y_0$. Consequently U_{n+1} and V_{n+1} have the matricial form $(2.31)_{n+1}$, and $\mathscr{D}_{V_{n+1}}$ is identified with $\mathscr{D}_{V_0}$, Y_{n+1} is identified with Y_0 and $D_{V_{n+1}^*}$ is the projection onto $\operatorname{Ker} V_1^*$. Obviously $U_{n+1}V_{n+1} = V_{n+1}U_{n+1}$, hence $\Gamma_{n+1} = 0$. Moreover, since Y_0 is a unitary operator (by $\mathscr{L}_0 = \{0\}$), it follows that $\mathscr{K}_{n+1}$ reduces U_{n+2}, from which we obtain

$$P^{\mathscr{K}_{n+2}}_{\mathscr{K}_{n+1}}U_{n+2}V_{n+2} = P^{\mathscr{K}_{n+2}}_{\mathscr{K}_{n+1}}V_{n+2}U_{n+2},$$

so that it results

$$(3.13)_{n+1} \qquad C_{n+1}Y_0 = U_{n+1}C_{n+1}.$$

From $(3.13)_{n+1}$, since the range of C_{n+1} lies in $[\operatorname{Ker} V_1^* \cap \mathscr{K}_1 \ominus \mathscr{H})] \oplus \{0_{\mathscr{K}_{n+1}\ominus\mathscr{K}_1}\}$ it follows

$$P^{\mathscr{K}_n}_{\mathscr{K}_0}C_{n+1}\mathscr{R}_0 \subset \bigcap_{k>0} P_{\mathscr{K}_0}C_{n+1}Y_0^k\mathscr{D}_{V_0} = \bigcap_{k=0} U_0^kP_{\mathscr{K}_0}C_{n+1}\mathscr{D}_{V_0} \subset \bigcap_{k>0} U_0^k(\mathscr{K}_0 \ominus \mathscr{H}) = \{0\},$$

so that, since an element $k_{n+1} \in \mathscr{D}_{V_{n+1}^*}$ $(= \operatorname{Ker} V_1^* \oplus \{0_{\mathscr{K}_{n+1}\ominus\mathscr{K}_0}\})$ is 0 iff $P^{\mathscr{K}_{n+1}}_{\mathscr{K}_0}k_{n+1} = 0$ and, by assumption, $\mathscr{D}_{V_0} = \mathscr{R}_0$, we have $C_{n+1} = 0$.

One proves the assertion ii) analogously.

We state now the main result of this section.

THEOREM 3.1. *Let $(\mathscr{H}, [T, S])$ be a pair of commuting contractions on $\mathscr{H}$ and let $(\mathscr{K}_0, [U_0, V_0])$ be a pair consisting of an identification $(\mathscr{K}_0, U_0)$ of the minimal isometric dilation of T and a contractive dilation V_0 of the commutant S of T. There exists a unique (up to a coincidence) Ando dilation of $(\mathscr{H}, [T, S])$ crossing* *through* $(\mathscr{K}_0, [U_0, V_0])$ *(which coincides with the distinguished Ando dilation* $(\overset{\circ}{\mathscr{K}}_0, [\overset{\circ}{U}_0, \overset{\circ}{V}_0]))$ *if and only if*

$$\mathscr{D}_{V_0} \ominus \overline{D_{V_0} U_0 \mathscr{K}_0} = \{0\} \quad or \quad \mathscr{D}_{V_0^*} \ominus \overline{D_{V_{*0}} U_{*0} \mathscr{K}_{*0}} = \{0\} \tag{3.14}$$

(where $\mathscr{K}_{*0}$, U_{*0}, V_{*0} *are defined by* (3.7), (3.8), *respectively).*

Proof. From Lemma 3.3, Corollary 2.1, and Lemma 3.2 it follows that (3.14) is a sufficient condition for the uniqueness of Ando dilations crossing through $(\mathscr{K}_0, [U_0, V_0])$.

In order to prove that it is also a necessary condition we assume that $\mathscr{L}_0 = \mathscr{D}_{V_0} \ominus \overline{D_{V_0} U_0 \mathscr{K}_0} \neq \{0\}$ and $\mathscr{L}_{*0} = \mathscr{D}_{V_{*0}} \ominus \overline{D_{V_{*0}} U_{*0} \mathscr{K}_{*0}} \neq \{0\}$. First, we define the isometries b_0 and b_{*0} on $\mathscr{D}_{V_0}$ and $\mathscr{D}_{V_{*0}}$, respectively by

$$b_0 D_{V_0} k_0 = D_{\tilde{V}_0} k_0, \quad b_{*0} D_{V_{*0}} k_{*0} = D_{\tilde{V}_0^*} k_{*0} \quad (k_0 \in \mathscr{K}_0,\ k_{*0} \in \mathscr{K}_{*0}).$$

It is easy to see that $b_0 \mathscr{L}_0 = \tilde{\mathscr{L}}_0$ and $b_{*0} \mathscr{L}_0 = \tilde{\mathscr{L}}_{*0}$, where

$$\begin{aligned} \tilde{\mathscr{L}}_0 &= \overline{D_{\tilde{V}_0} \mathscr{K}_0} \ominus \overline{\tilde{U}_0 D_{\tilde{V}_0} \mathscr{K}_0} \\ \tilde{\mathscr{L}}_{*0} &= \overline{D_{\tilde{V}_0^*} \mathscr{K}_{*0}} \ominus \tilde{U}_0^* \overline{D_{\tilde{V}_0^*} \mathscr{K}_{*0}}. \end{aligned} \tag{3.15}$$

It results that $\tilde{\mathscr{L}}_0$, $\tilde{\mathscr{L}}_{*0}$ $(\neq \{0\})$ are wandering subspaces for $\tilde{U}_0$ and

$$\overline{D_{\tilde{V}_0} \mathscr{K}_0} = M_+(\tilde{\mathscr{L}}_0) \oplus \tilde{\mathscr{R}}_0, \quad \overline{D_{\tilde{V}_0^*} \mathscr{K}_{*0}} = M_-(\tilde{\mathscr{L}}_{*0}) \oplus \tilde{\mathscr{R}}_{*0}, \tag{3.16}$$

$$\mathscr{D}_{\tilde{V}_0} = M(\tilde{\mathscr{L}}_0) \oplus \tilde{\mathscr{R}}_0, \quad \mathscr{D}_{\tilde{V}_0^*} = M(\tilde{\mathscr{L}}_{*0}) \oplus \tilde{\mathscr{R}}_{*0}, \tag{3.17}$$

where $\tilde{\mathscr{R}}_0$, $\tilde{\mathscr{R}}_{*0}$ are the residual spaces of the Wold decompositions of the isometries $\tilde{U}_0 | \overline{D_{\tilde{V}_0} \mathscr{K}_0}$ and $\tilde{U}_0^* | \overline{D_{\tilde{V}_0^*} \mathscr{K}_{*0}}$, respectively, and as usual, for a wandering subspace $\mathscr{L}$ of a unitary operator U, we denote

$$M_+(\mathscr{L}) = \bigoplus_{n \geqslant 0} U^n \mathscr{L}, \quad M(\mathscr{L}) = \bigoplus_{n \in \mathbf{Z}} U^n \mathscr{L}.$$

Now, let $\{\tilde{\mathscr{L}}_0, \tilde{\mathscr{L}}_{*0}, \theta(\lambda)\}$ be a contractive analytic function such that $\theta(0) = 0$ and let θ be the contraction from $L^2(\tilde{\mathscr{L}}_0)$ into $L^2(\tilde{\mathscr{L}}_0)$, given by the pointwise multiplication with $\theta(e^{it})$. Define $\tilde{\mathscr{R}}_0 = \Phi_{L_0}^* \theta \Phi_{L_0} \oplus 0$ as an operator from $\mathscr{D}_{\tilde{V}_0} =$

$= M(\tilde{\mathscr{L}}_0) \oplus \tilde{\mathscr{R}}_0$ into $\mathscr{D}_{\tilde{V}_0^*} = M(\tilde{\mathscr{L}}_{*0}) \oplus \tilde{\mathscr{R}}_{*0}$, where for a wandering subspace $\mathscr{L}$ of U, $\Phi_{\mathscr{L}}$ denotes the Fourier representation of the bilateral shift $U \mid M(\mathscr{L})$, i.e. $\Phi_{\mathscr{L}} \sum_{k\in\mathbf{Z}} U^k l_k = \sum_{k\in\mathbf{Z}} e^{ikt} l_k$, $l_k \in \mathscr{L}$, $\sum_{k\in\mathbf{Z}} \|l_k\|^2 < \infty$. Clearly,

$$(3.18)_0 \qquad \tilde{U}_0\tilde{R}_0 = \tilde{R}_0\tilde{U}_0 \mid \mathscr{D}_{\tilde{V}_0}.$$

Moreover, since $\tilde{R}_0 D_{\tilde{V}_0}\mathscr{K}_0 \subset \bigoplus_{n\geqslant 1} \tilde{U}_0^n \mathscr{L}_{*0}$, using (3.15) we obtain $D_{\tilde{V}_0^*}\tilde{R}_0 D_{\tilde{V}_0}\mathscr{K}_0 \perp \mathscr{K}_{*0}$ and consequently,

$$(3.19) \qquad D_{\tilde{V}_0^*}\tilde{R}_0 D_{\tilde{V}_0}\mathscr{K}_0 \subset \mathscr{K}_0 \ominus \mathscr{H}.$$

Now, let $0 < \varepsilon < 1$ such that

$$(3.20) \qquad \varepsilon < \prod_{j\geqslant 2}(1-\varepsilon^{2^j}).$$

Clearly, we can choose the function $\theta(\lambda)$ such that

$$(3.21) \qquad 0 < \|\tilde{R}_0\| < \varepsilon^2.$$

Define $\tilde{R}_1$ on $\mathscr{D}_{\tilde{R}_0} = \mathscr{D}_{\tilde{V}_0}$ by

$$(3.22)_1 \qquad \tilde{R}_1 = D^{-1}_{\tilde{R}_0^*}\tilde{R}_0\tilde{V}_0^*\tilde{R}_0 D^{-1}_{\tilde{R}_0}$$

and for $n \geqslant 2$, define $\tilde{R}_n$ on $\mathscr{D}_{\tilde{R}_{n-1}}$ by the recurrent formula

$$(3.22)_n \qquad \tilde{R}_n = D^{-1}_{\tilde{R}_{n-1}^*}\tilde{R}_{n-1}D^{-1}_{\tilde{R}_{n-2}}\dots D^{-1}_{\tilde{R}_0}V_0\tilde{R}_0 D^{-1}_{\tilde{R}_0}\dots D^{-1}_{\tilde{R}_{n-1}}.$$

Using (3.20) and (3.21) it is easy to see that, for any $n > 0$, we have $\|\tilde{R}_n\| < \varepsilon^{n+2}$ (so, the recurrent relation $(3.22)_n$ has sense). On the other hand, using $(3.18)_0$, we obtain

$$(3.18)_n \qquad \tilde{U}_0\tilde{R}_n = \tilde{R}_n\tilde{U}_0 \mid \mathscr{D}_{\tilde{V}_0}, \quad n \geqslant 1.$$

Now, denote

$$\tilde{\mathscr{K}} = \tilde{\mathscr{K}}_0 \oplus \mathscr{D}_{\tilde{V}_0} \oplus \mathscr{D}_{\tilde{R}_0} \oplus \mathscr{D}_{\tilde{R}_1} \oplus \dots .$$

and define on $\tilde{\mathscr{K}}$ a unitary operator $\tilde{U}$, given by

$$\tilde{U} = \tilde{U}_0 \oplus \tilde{Y}_0 \oplus \tilde{Y}_0 \oplus \dots,$$

where $\tilde{Y}_0 = \tilde{U}_0 | \mathscr{D}_{\tilde{V}_0}$ and an isometry $\tilde{V}$, given by the matrix

$$\tilde{V} = \begin{bmatrix} \tilde{V}_0 & D_{\tilde{V}_0^*}\tilde{R}_0 & D_{\tilde{V}_0^*}D_{\tilde{R}_0^*}\tilde{R}_1 & D_{\tilde{V}_0^*}D_{\tilde{R}_0^*}D_{\tilde{R}_1^*}\tilde{R}_2 & \dots \\ D_{\tilde{V}_0} & -\tilde{V}_0^*\tilde{R}_0 & -\tilde{V}_0^*D_{\tilde{R}_0^*}\tilde{R}_1 & -\tilde{V}_0^*D_{\tilde{R}_0^*}D_{\tilde{R}_1^*}\tilde{R}_2 & \dots \\ 0 & D_{\tilde{R}_0} & -\tilde{R}_0^*\tilde{R}_1 & -\tilde{R}_0^*D_{\tilde{R}_1^*}\tilde{R}_2 & \dots \\ 0 & 0 & D_{\tilde{R}_1} & -\tilde{R}_1^*\tilde{R}_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

From $(3.18)_n$ it results $\tilde{U}\tilde{V} = \tilde{V}\tilde{U}$. We shall show that

$$P_{\mathscr{H}}\tilde{V}^n\tilde{U}^m h = S^nT^m h \quad (h \in \mathscr{H},\ n, m \geqslant 0), \tag{3.23}$$

which is a direct consequence of the relation

$$P_{\mathscr{H}}\tilde{V}^n | \{0\} \oplus \overline{D_{\tilde{V}_0}\mathscr{K}_0} = 0 \quad (n \geqslant 0). \tag{3.24}$$

But, applying V succesively, and using the recurrent definition $(3.22)_n$ of $\tilde{R}_n$, we obtain

$$\tilde{V}\begin{bmatrix} 0 \\ D_{\tilde{V}_0}k_0 \end{bmatrix} = \begin{bmatrix} k_1 \\ -\tilde{V}_0^*\tilde{R}_0 D_{\tilde{V}_0}k_0 \\ D_{\tilde{R}_0}D_{\tilde{V}_0}k_0 \end{bmatrix}$$

with $k_1 = D_{\tilde{V}_0^*}\tilde{R}_0 D_{\tilde{V}_0}k_0 \in \mathscr{K}_0 \ominus \mathscr{H}$ (by (3.19)),

$$\tilde{V}^2\begin{bmatrix} 0 \\ D_{\tilde{V}_0}k_0 \end{bmatrix} = \begin{bmatrix} k_2 \\ D_{V_0}k_1 \\ -D_{\tilde{R}_0}^{-1}\tilde{V}_0^*\tilde{R}_0 D_{\tilde{V}_0}k_0 \\ D_{\tilde{R}_1}D_{\tilde{R}_0}D_{\tilde{V}_0}k_0 \end{bmatrix}$$

with $k_2 = V_0 k_1 \in \mathscr{K}_0 \ominus \mathscr{H}$ and, by induction, for $n \geqslant 3$,

$$\tilde{V}^n\begin{bmatrix} 0 \\ D_{\tilde{V}_0}k_0 \end{bmatrix} = \begin{bmatrix} k_n \\ D_{\tilde{V}_0}k_{n-1} - \tilde{V}_0^*\tilde{R}_0 D_{\tilde{V}_0}k_{n-2} \\ D_{\tilde{R}_0}D_{\tilde{V}}k_{n-2} - D_{\tilde{R}_0}^{-1}\tilde{V}_0^*\tilde{R}_0 D_{\tilde{V}_0}k_{n-3} \\ D_{\tilde{R}_1}D_{\tilde{R}_0}D_{\tilde{V}_0}k_{n-3} - D_{\tilde{R}_1}^{-1}D_{\tilde{R}_0}^{-1}\tilde{V}_0^*\tilde{R}_0 D_{\tilde{V}_0}k_{n-4} \\ \vdots \\ D_{\tilde{R}_{n-3}}D_{\tilde{R}_{n-2}}\dots D_{\tilde{R}_0}D_{\tilde{V}_0}k_1 \\ -D_{\tilde{R}_{n-2}}^{-1}D_{\tilde{R}_{n-3}}^{-1}\dots D_{\tilde{R}_0}^{-1}\tilde{V}_0^*\tilde{R}_0 D_{\tilde{V}_0}k_0 \\ D_{\tilde{R}_{n-1}}D_{\tilde{R}_{n-2}}\dots D_{\tilde{R}_0}D_{\tilde{V}_0}k_0 \end{bmatrix}$$

with $k_n = V_0 k_{n-1} \oplus D_{\tilde{V}_0^*} \tilde{R}_0 D_{\tilde{V}_0} k_{n-2} \in \mathscr{K}_0 \ominus \mathscr{H}$ (by (3.19)). This proves (3.24) and consequently (3.23).

Now it is clear that, taking $\mathscr{K} = \bigvee_{n,m \geq 0} \tilde{U}^n \tilde{V}^m \mathscr{H}$, $U = \tilde{U} | \mathscr{K}$, $V = \tilde{V} | \mathscr{K}$, the obtained $(\mathscr{K}, [U, V])$ is an Ando dilation of $(\mathscr{H}, [T, S])$, crossing through $(\mathscr{K}_0, [U_0, V_0])$, which does not coincide with the distinguished one. Thus, Theorem 3.1 is completely proved.

Finally, recall that a factorization $A \cdot \Gamma$ (where A, Γ are contractions on a Hilbert space $\mathscr{H}$) is *regular* iff $\mathscr{F}(A, \Gamma) = \mathscr{D}_A \oplus \mathscr{D}_\Gamma$ (where $\mathscr{F}(A, \Gamma)$ is the space defined by (1.10)). Also, with the notations used in this section, it is easy to see that the regularity of the factorization $S \cdot T$ implies $\mathscr{D}_{V_0} \ominus \overline{D_{V_0} U_0 \mathscr{K}_0} = \{0\}$ and the regularity of the factorization $T \cdot S$, being equivalent to the regularity of $S^* \cdot T^*$, implies $\mathscr{D}_{V_{*0}} \ominus \overline{D_{V_{*0}} U_{*0} \mathscr{K}_{*0}} = \{0\}$. Then, from these facts and from Remark 1.1, Theorem 1.1 of [2] and Theorem 3.1 it follows directly the known result concerning the uniqueness of Ando dilations formulated in terms of regular factorizations (see [3]), namely:

COROLLARY 3.1. *For a pair $[T, S]$ of commuting contractions on $\mathscr{H}$ there exists a unique (up to a coincidence) Ando dilation if and only if any of the factorizations $S \cdot T$ or $T \cdot S$ (of $ST = TS$) is regular.*

4.

In this section we apply the uniqueness result concerning the Ando dilations, stated in Theorem 3.1, to obtain a uniqueness condition for the probability measures whose Fourier-Stieltjes coefficients are supported in two quadrands. To this aim we specify some notations.

Let $\mathbf{D}^2$ be the unit bidisc in $\mathbf{C}^2$ and $\mathbf{T}^2$ be the bidimensional torus. For a Borel measure μ on $\mathbf{T}^2$ one denotes by $c_{n,m}(\mu)$ the Fourier-Stieltjes coefficients of μ, i.e.

$$c_{n,m}(\mu) = \int_{\mathbf{T}^2} e^{-int} e^{-im\theta} \, d\mu(e^{it}, e^{i\theta}), \quad (n, m) \in \mathbf{Z}^2. \tag{4.1}$$

Denote by $\mathscr{M}_0$ the set of all probability measures μ on $\mathbf{T}^2$ satisfying $c_{n,m}(\mu) = 0$ for $n, m \geqslant 0$, $n + m > 0$. Thus, $\mathscr{M}_0$ is the set of all representing measures, supported in $\mathbf{T}^2$, of the point $(0, 0)$, considered as an element of the maximal ideal space of the bidisc algebra $A(\mathbf{D}^2)$. For a measure $\mu \in \mathscr{M}_0$ let $\mathscr{K} = \mathscr{K}_\mu = H^2(d\mu)$ be the closure in $L^2(d\mu)$ of $A(\mathbf{D}^2)$ and let $U = U_\mu$, $V = V_\mu$ be the operators on $\mathscr{K}$ given by the multiplication with the coordinate functions e^{it} and $e^{i\theta}$, respectively, i.e.

$$\begin{cases} (Uh)(e^{it}, e^{i\theta}) = e^{it} h(e^{it}, e^{i\theta}) \\ (Vh)(e^{it}, e^{i\theta}) = e^{i\theta} h(e^{it}, e^{i\theta}) \end{cases} \tag{4.2}$$

($h \in H^2(\mathrm{d}\mu)$). It is easy to see that $(\mathscr{K} = \mathscr{K}_\mu, [U = U_\mu, V = V_\mu])$ is an Ando dilation of the pair $(\mathscr{H}, [T, S])$, where $\mathscr{H} = \mathbf{C}$ and $T = S = 0$. Moreover, the map

$$\psi = \{\mu \mapsto (\mathscr{K}_\mu, [U_\mu, V_\mu])\} \tag{4.3}$$

from $\mathscr{M}_0$ to the set of all Ando dilations of $(\mathbf{C}, [0, 0])$ is invertible, the inverse map attaches to any Ando dilation $(\mathscr{K}, [U, V])$ of $(\mathbf{C}, [0, 0])$ the measure μ having the following Fourier-Stieltjes coefficients

$$c_{n,m} = c_{n,m}(\mu) = (\tilde{U}^n \tilde{V}^m 1, 1)_{\tilde{\mathscr{K}}}, \quad (n, m) \in \mathbf{Z}^2, \tag{4.4}$$

where $(\tilde{\mathscr{K}}, [\tilde{U}, \tilde{V}])$ is the (unique) unitary extension of $(\mathscr{K}, [U, V])$ defined by (3.4) Consequently, (4.3) defines a one-to-one correspondence between the set $\mathscr{M}_0$ and the set of all coinciding Ando dilations of $(\mathbf{C}, [0, 0])$. Notice also that, if for $(\mathscr{K} = \mathscr{K}_\mu, [U = U_\mu, V = V_\mu])$ we consider the pair $(\mathscr{K}_0, [U_0, V_0])$ (defined by (1.4) and (1.5)), then, denoting by H^2 the usual (one dimensional) Hardy space and S_+ the (unilateral) shift on H^2, there exists a unitary operator X_0 from $\mathscr{K}_0$ onto H^2 such that

$$X_0 1 = 1, \quad X_0 U_0 = S_+ X_0 . \tag{4.5}$$

Moreover, the operator $X_0 V_0 X_0^*$ is a contraction on H^2 commuting with S_+, so there exists a function $f \in H^\infty$, $\|f\| \leqslant 1$, such that

$$X_0 V_0 = T_f X_0 , \tag{4.6}$$

where T_f is the Toeplitz operator of symbol f on H^2 (see [12]). In addition, since $P_{\mathbf{C}}^{\mathscr{K}_0} V_0 \mid \mathbf{C} = 0$, we have $f(0) = 0$. We say that the measure $\mu \in \mathscr{M}_0$ *crosses through* $f \in H^\infty$ ($\|f\| \leqslant 1, f(0) = 0$) if f is attached to μ as above. On the other hand, since the minimal isometric dilation of $T = 0$ is the shift operator on H^2, any contractive dilation V_0 of the commutant $S = 0$ of T is given by a Toeplitz operator T_f with $f \in H^\infty$, $\|f\| \leqslant 1, f(0) = 0$ (see [12]). Consequently, an Ando dilation $(\mathscr{K}, [U, V])$ of $(\mathbf{C}, [0, 0])$ crosses through $(\mathscr{K}_0, [U_0, V_0])$ if and only if its corresponding measure $\mu \in \mathscr{M}_0$ crosses through the function f, given by (4.6).

Now assume that $(\mathscr{K}, [U, V])$ coincides with the distinguished Ando dilation $(\mathring{\mathscr{K}}_0, [\mathring{U}_0, \mathring{V}_0])$ of $(\mathbf{C}, [0, 0])$ crossing through $(\mathscr{K}_0, [U_0, V_0])$ and let f be the function in H^∞, $\|f\| \leqslant 1, f(0) = 0$, whose Toeplitz operator T_f defines V_0. Then to the unitary extension $(\tilde{\mathscr{K}}, [\tilde{U}, \tilde{V}])$ of $(\mathscr{K}, [U, V])$ uniquely corresponds a positive definite function $\rho = \rho_f$ on $\mathbf{Z}^2$ (see (12]), defined by:

$$\begin{cases} \rho(n, m) = \begin{cases} 0, & n > 0,\ m \geqslant 0 \\ c_n(f^{-m}), & n > 0,\ m < 0 \end{cases} \\[2ex] \rho(n, m) = \begin{cases} 1, & n = 0,\ m = 0 \\ 0, & n = 0,\ m \neq 0 \end{cases} \\[2ex] \rho(n, m) = \overline{\rho(-n, -m)}, \quad n < 0,\ m \in \mathbf{Z} \end{cases} \tag{4.7}$$

(where $c_n(f^{-m})$ denotes the n-th Fourier coefficient of f^{-m}). Then, by virtue of (4.4), the measure $\mu_0 = \mu_{f,0} \in \mathcal{M}_0$, crossing through f, which corresponds to $(\overset{\circ}{\mathcal{K}}_0, [\overset{\circ}{U}_0, \overset{\circ}{V}_0])$ is the one which has the Fourier-Stieltjes coefficients $c_{n,m}(\mu) = \rho(n, m)$, $(n, m) \in \mathbf{Z}^2$ (thus $\mu_{f,0}$ depends only on f) and we shall call it the *distinguished measure in* $\mathcal{M}_0$ *crossing through* f.

Summing up, we have

LEMMA 4.1. *There exists a one-to-one correspondence (given by the formulas (4.2) and (4.4)) between the set of all coinciding Ando dilations $(\mathcal{K}, [U, V])$ of $(\mathcal{H} = \mathbf{C}, [T = 0, S = 0])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$ and the set $\mathcal{M}_0$ of all representing measures μ of $(0, 0)$, supported in $\mathbf{T}^2$, which cross through $f \in H^\infty$, $\|f\| \leqslant 1$, $f(0) = 0$ (the connection between $(\mathcal{K}_0, [U_0, V_0])$ and f is given by (4.5) and (4.6)). Moreover, by this correspondence to the distinguished Ando dilation $(\overset{\circ}{\mathcal{K}}_0, [\overset{\circ}{U}_0, \overset{\circ}{V}_0])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$ corresponds the distinguished measure $\mu_{f,0}$ crossing through f.*

From Lemma 4.1 and Theorem 3.1 it follows

COROLLARY 4.1. *Let $f \in H^\infty$, $\|f\| \leqslant 1$, $f(0) = 0$. There exists a unique measure in $\mathcal{M}_0$ crossing through f (namely the distinguished one) if and only if*

$$\int_0^{2\pi} \log(1 - |f|^2)\, \mathrm{d}t = -\infty. \tag{4.8}$$

Proof. By virtue of Lemma 4.1, for a given function $f \in H^\infty$, $\|f\| \leqslant 1$, $f(0) = 0$, the distinguished measure $\mu_0 = \mu_{f,0}$ is the unique measure in $\mathcal{M}_0$ crossing through f if and only if $(\overset{\circ}{\mathcal{K}}_0 = \mathcal{K}_{\mu_0}, [\overset{\circ}{U}_0 = U_{\mu_0}, \overset{\circ}{V}_0 = V_{\mu_0}])$ is the unique Ando dilation of $(\mathbf{C}, [0, 0])$ crossing through $(\mathcal{K}_0, [U_0, V_0])$ (where $(\mathcal{K}_0, [U_0, V_0])$ coincides, up to a unitary equivalence defined by (4.5) and (4.6), with $(H^2, [S_+, T_f])$), therefore, by virtue of Theorem 3.1, if and only if the condition (3.14) is fulfilled. But, in our case (using the notations of the previous section), the unitary extension $(\tilde{\mathcal{K}}_0, \tilde{U}_0)$ of $(\mathcal{K}_0, U_0)$ is the multiplication by $\mathrm{e}^{\mathrm{i}t}$ on L^2 and the Douglas extension $\tilde{V}_0$ of V_0 is the multiplication by f on L^2, consequently $\mathcal{K}_{*0}$ is the subspace of L^2 generated by $\{\mathrm{e}^{-\mathrm{i}nt}\}_{n\geqslant 0}$, and U_{*0}, V_{*0} are the multiplications on $\mathcal{K}_{*0}$ by $\mathrm{e}^{-\mathrm{i}t}$ and $\bar{f}$, respectively (as usual, L^2 denotes the Hilbert space of the scalar valued measurable functions $v(t)$ on $[0, 2\pi]$ with $|v(t)|^2$ integrable). Therefore, it easily follows that the condition $\mathcal{D}_{V_0} \ominus \overline{D_{V_0} U_0 \mathcal{K}_0} = \{0\}$ is equivalent to $\mathcal{D}_{V_{*0}} \ominus \overline{D_{V_{*0}} U_{*0} \mathcal{K}_{*0}} = \{0\}$ and, furthermore, any of these equivalent conditions are equivalent to

$$\overline{D_{T_f} \mathrm{e}^{\mathrm{i}t} H^2} = \mathcal{D}_{T_f}. \tag{4.9}$$

On the other hand the equivalence of (4.9) to (4.8) is a simple consequence of Szegö theorem (see [12]). Indeed, the condition (4.8) is equivalent to the fact that there exists a sequence $\{p_n\}_{n\geqslant 0}$ of analytic polynomials vanishing at the origin which converges to 1 in $L^2(\mathrm{d}\nu)$, where $\mathrm{d}\nu = (1 - |f|^2)\,\mathrm{d}t$, i.e.

$$(4.10) \qquad 0 = \lim_{n\to\infty} \int |1 - p_n|^2 \,\mathrm{d}\nu = \lim_{n\to\infty} \|D_{T_f}(1 - p_n)\|^2$$

and this last condition is obviously equivalent to (4.9). The proof is complete.

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