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PERTURBATIONS OF C^*-ALGEBRAS AND K-THEORY

MAHMOOD KHOSHKAM

INTRODUCTION

We show that if A and B are C^*-subalgebras of a C^*-algebra C and $A \overset{\gamma}{\subset} B$ for a suitably small positive number γ, then (with a mild assumption on A) there exists a natural homomorphism τ from $K_*(A)$ into $K_*(B)$. If $d(A, B) < \gamma$, τ is an isomorphism of $K_*(A)$ onto $K_*(B)$.

To each closed two-sided ideal J in B there corresponds a unique closed two-sided ideal I in A such that $I \overset{\gamma'}{\subset} J$. Using τ we connect the six-term exact sequence of K-theory associated with $0 \to I \to A \to A/I \to 0$ to that associated with $0 \to J \to B \to B/J \to 0$ and show that the resulting diagram is commutative.

We also show that if A and B are unital C^*-subalgebras of a C^*-algebra C and $d(A, B)$ is sufficiently small, then the groups of unitaries of A and B are homotopically equivalent.

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1. PRELIMINARIES

1.1. All the tensor-products considered in this paper are the minimal tensor-product of C^*-algebras.

1.2. For C^*-subalgebras A, B of a C^* algebra C and $0 \leqslant \gamma \leqslant 1$ we use Christensen's definition [2] to say that A is γ-contained in B ($A \overset{\gamma}{\subset} B$) if for every $a \in A$ there exists an element $b \in B$ such that $\|a - b\| < \gamma\|a\|$. If $A \overset{\gamma}{\subset} B$ and $B \overset{\gamma}{\subset} A$, then $d(A, B) < 2\gamma$, where $d(A, B)$ the distance between A and B is as defined in [5].

1.3. If A is a C^*-algebra we denote by A^+ the C^*-algebra obtained from A by adjunction of an identity. We also denote by A_1 the unit ball of A.

1.4. If $A \overset{\gamma}{\subset} B$ are C^*-subalgebras of a C^*-algebra C and A is nuclear, then $A \otimes D \overset{6\gamma}{\subset} B \otimes D$ for any nuclear C^*-algebra D [2, Theorem 3.1]. We also note that $D \otimes A \overset{\gamma}{\subset} D \otimes B$ implies $D \otimes A^+ \overset{2\gamma}{\subset} D \otimes B^+$. To see this, consider the exact sequence $0 \to D \otimes C \xrightarrow{i} D \otimes C^+ \xrightarrow{p} D \to 0$. Then $j(x) = x \otimes 1$, $x \in D$ is a cross section for p. Let $x \in D \otimes A^+$ and $\|x\| \leqslant 1$. There exist $a \in D \otimes A$ and $d \in D$ such that $x = i(a) + j(d)$. Now

$$\|a\| = \|i(a)\| \leqslant \|x\| + \|j(d)\| = \|x\| + \|P(x)\| \leqslant 2\|x\|.$$

Since $D \otimes A \overset{\gamma}{\subset} D \otimes B$ we can choose an element $b \in D \otimes B$ such that $\|a - b\| < 2\gamma$. Let $y = i(b) + j(d)$. Then $\|x - y\| = \|i(a) - i(b)\| < 2\gamma$. This shows $D \otimes A^+ \overset{2\gamma}{\subset} D \otimes B^+$.

1.5. Suppose A and B are C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$. If A is nuclear one can use [2, Proposition 2.6] to show that $\pi(B)' \overset{2\gamma}{\subset} \pi(A)'$, where π is any representation of the C^*-algebra generated by A and B.

1.6. We recall the definitions of K-theory cf. [9]. Let A be a unital C^*-algebra. For $a \in M_n(A)$ and $b \in M_m(A)$, $a \oplus b$ denotes $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(A)$. Let $M_\infty(A) = \bigcup_n M_n(A)$ where $M_n(A)$ is imbedded into $M_{n+1}(A)$ by $x \to x \oplus 0$. If $p_1, p_2 \in M_\infty(A)$ are projections we set $p_1 \sim p_2$ when p_1 and p_2 are unitarily equivalent in some $M_n(A)$. Then $\sim$ is an equivalence relation on the set of projections of $M_\infty(A)$. We denote by $[p]$ the equivalence class of the projection p. The set of equivalence classes of projections in $M_\infty(A)$ equipped with the addition $[p_1] + [p_2] = [p_1 \oplus p_2]$ forms a semigroup. $K_0(A)$ is the Grothendick group of this semigroup.

Let $U_n(A)$ be the group of unitaries in $M_n(A)$. Then we set $U_\infty(A) = \varinjlim U_n(A)$ where $U_n(A)$ is imbedded into $U_{n+1}(A)$ by $u \to \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. If $U^0_\infty(A)$ denotes the component of identity of $U_\infty(A)$, then $K_1(A) = U_\infty(A)/U^0_\infty(A)$. If $u \in U_\infty(A)$ we denote by $[u]$ the equivalence class of u in $K_1(A)$. K_0 and K_1 are covariant functors.

1.7. If $f: A \to B$ is a homomorphism f_* denotes the homomorphism induced by f on the K-groups.

1.8. For a non-unital C^*-algebra A, $K_0(A)$ is defined to be the kernel of the map $\varphi_{0*}: K_0(A^+) \to \mathbf{Z} = K_0(\mathbf{C})$ and $K_1(A) = K_1(A^+)$ (as $K_1(\mathbf{C}) = 0$), where $\varphi_0: A^+ \to \mathbf{C}$ is the canonical homomorphism.

1.9. We often make use of the function $\alpha: [0, 1) \to [0, \sqrt{2})$ defined by $\alpha(t) = 2 \sin \dfrac{\arcsin t}{2}$. This is the function $\beta(k) = 2^{1/2}k(1 + (1 - k^2)^{1/2})^{-1/2}$ of [1, Lemma 2.7].

In the following Lemma 1.10 ii) we obtain a sharper constant than [8, Lemma 2.4]. This proof has been indicated to me by George Skandalis.

1.10. LEMMA. *Let A and B be unital C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$. If A and B have the same unit, then*

i) *For each unitary $u \in A$ there exists a unitary $v \in B$ such that $\|u - v\| < \alpha(\gamma)$.*

ii) *For each projection $p \in A$ there exists a projection $q \in B$ such that $\|p - q\| < \frac{\alpha(\gamma)}{2}$.*

Proof. Let $u \in A$ be a unitary. Choose an element $x \in B$ such that $\|u - x\| < \gamma$, i.e. $\|I - u^*x\| < \gamma$. If $x = v|x|$ is the polar decomposition of x, then by [1, Lemma 2.7], $\|I - u^*v\| < \alpha(\gamma)$, i.e. $\|u - v\| < \alpha(\gamma)$.

ii) Let $u = 2p - 1$. Choose a self-adjoint element $x \in B$ such that $\|u - x\| < \gamma$. If v is the unitary part of x, then by i) $\|u - v\| < \alpha(\gamma)$. Then $q = \frac{v+1}{2}$ is a projection in B and $\|p - q\| < \frac{\alpha(\gamma)}{2}$.

1.11. LEMMA. *Let A be a unital C^*-algebra.*

i) *If $p, q \in A$ are projections and $\|p - q\| < 1$, then p and q are unitarily equivalent in A.*

ii) *If $u, v \in A$ are unitaries with $\|u - v\| < 2$, then u and v belong to the same connected component of the group of unitaries of A.*

Proof. i) See [4, Lemma 1.3].

ii) Since $\|I - u^*v\| < 2$, $-1 \notin \sigma(u^*v)$. Then $x(t) = u\exp(t \log u^*v)$ defines a homotopy between u and v.

§ 2

For a map $f: A \to D$ we set $G_f = \{(a, f(a)) : a \in A\} \subset A \oplus D$.

2.1. DEFINITION. Let A and B be C^*-subalgebras of a C^*-algebra C. Suppose $f: A \to D$, $g: B \to D$ are homomorphisms of C^*-algebras. We say that f is *γ-contained in g* if $G_f \overset{\gamma}{\subset} G_g$ (as sub-algebras of $C \oplus D$).

2.2. REMARK. Let A, B, f and g be as in 2.1. Then $A \overset{\gamma}{\subset} B$ and $f(A) \overset{\gamma}{\subset} g(B)$. Moreover for any $a \in A$ and $b \in B$, $\|f(a) - g(b)\| < 2\gamma\|a\| + \|a - b\|$.

2.3. LEMMA. *Let A and B be C^*-subalgebras of a C^*-algebra C with A nuclear. Suppose that $f: A \to D$ and $g: B \to D$ are $*$-homomorphism. If f is γ-contained in g, then $f \otimes \mathrm{id}: A \otimes E \to D \otimes E$ is 6γ-contained in $g \otimes \mathrm{id}: B \otimes E \to$ $\to D \otimes E$ for any nuclear C^*-algebra E.*

Proof. Note that $G_{f\otimes \mathrm{id}} = G_f \otimes E$ and $G_{g\otimes \mathrm{id}} = G_g \otimes E$. Then $G_{f\otimes \mathrm{id}} \overset{6\gamma}{\subset} G_{g\otimes \mathrm{id}}$ by 1.3.

Let A and B be C^*-subalgebras of a C^*-algebra C. Suppose $A \overset{\gamma}{\subset} B$ with $\gamma \leqslant 1/38$. Let $p \in M_n(A^+)$ be a projection. By 1.3, 1.4 and 1.10 ii) we can choose a projection $q \in M_n(B^+)$ such that $\|p - q\| < \dfrac{\alpha(12\gamma)}{2}$. Let τ be the map that sends $[p]$ to $[q]$. We also denote by τ the map $[u] \to [v]$ where $u \in U_n(A^+)$ and $v \in U_n(B^+)$ are such that $\|u - v\| < \alpha(12\gamma)$. In the following proposition we prove that τ is well defined and extends to a natural homomorphism of $\mathrm{K}_*(A)$ into $\mathrm{K}_*(B)$.

2.4. PROPOSITION. *Let A and B be C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$ with $\gamma \leqslant 1/38$ and A nuclear. Then*

i) *τ defines a homomorphism of $\mathrm{K}_*(A)$ into $\mathrm{K}^*(B)$;*

ii) *if $f: A \to D$ is γ-contained in $g: B \to D$, then the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{K}_*(A) & \xrightarrow{\ \tau\ } & \mathrm{K}_*(B) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{K}_*(A') & \xrightarrow{\ \tau\ } & \mathrm{K}_*(B') \end{array}$$

where A' and B' are C^*-subalgebras of the C^*-algebra D such that $f(A) \subset A'$, $g(B) \subset B'$ and $A' \overset{\gamma}{\subset} B'$ with A' nuclear.

Proof. We first prove the proposition for A^+ and B^+, noting that by 1.4 we have $A^+ \overset{2\gamma}{\subset} B^+$. The general case follows by applying *ii)* to A^+ and B^+ with $D = \mathbf{C}$.

Note that if $\gamma \leqslant 1/38$, then $\alpha(12\gamma) \leqslant 1/3$. To show that τ is well defined let $p \in M_n(A)$ and $q_1, q_2 \in M_n(B^+)$ be such that $\|p - q_i\| < \dfrac{\alpha(12\gamma)}{2}$, $i = 1, 2$. Then $\|q_1 - q_2\| < \alpha(12\gamma) < 1$ and by 1.11 i), $[q_1] = [q_2]$. Now suppose $p_1 = up_2u^*$ for some $u \in U_\infty(A)$. Let q_1, $q_2 \in M_\infty(B)$ be projections such that $\|p_i - q_i\| < \alpha(12\gamma)/2$, $i = 1,2$. By 1.10 i) there exists $v \in U_\infty(B^+)$ such that $\|u - v\| < \alpha(12\gamma)$. Then we have $\|q_1 - vq_2v^*\| < 3\alpha(12\gamma) < 1$ and by 1.11 ii) $[q_1] = [q_2]$. Hence τ is well defined. Suppose $p = p_1 \oplus p_2$ and q, q_1 and q_2 are projections $(\alpha(12\gamma)/2)$-close to p, p_1 and p_2 respectively. One has $\|q - (q_1 \oplus q_2)\| < \alpha(12\gamma) < 1$ so that $[q] = [q_1 \oplus q_2]$. This shows that τ is additive. Now by the universal property of the Grothendick group associated with a semigroup, τ extends to a homomorphism of $\mathrm{K}_0(A^+)$ into $\mathrm{K}_0(B^+)$.

Next we prove that the closeness map τ: $K_1(A) \to K_1(B)$ is a well-defined homomorphism. If $u \in U_\infty(A^+)$ and $v_1, v_2 \in U_\infty(B^+)$ are such that $\|u - v_i\| < \alpha(12\gamma)$, $i = 1,2$, then $\|v_1 - v_2\| < 2$ and by 1.11 ii) $[v_1] = [v_2]$. This shows that τ is well-defined on $U_\infty(A^+)$. Let $u, v \in U_\infty(A^+)$ and $u', v', w \in U_\infty(B^+)$ be such that $\|u - u'\|$, $\|v - v'\|$ and $\|uv - w\|$ are smaller than $\alpha(12\gamma)$. Then $\|w - u'v'\| < 2$ and by 1.11 ii) $[w] = [u'v']$, i.e. τ is a homomorphism. Finally we show that τ is well defined on the quotient $U_\infty(A^+)/U^0_\infty(A^+)$. Let $u \in U^0_\infty(A^+)$ and $v \in U_\infty(B^+)$ be such that $\|u - v\| < \alpha(12\gamma)$. As $U^0_\infty(B^+)$ is generated by unitaries near the identity we may assume that $\|u - 1\| < \alpha(12\gamma)$. Then $\|v - 1\| < 2\alpha(12\gamma) < 2$ and by 1.11 ii) $v \in U^0_\infty(B^+)$ and the proof is complete.

ii) Follows directly from 2.3 and the fact that τ is defined by closeness.

2.5. REMARK. We note that 2.4 i) will remain valid if we replace the hypothesis "$A \overset{\gamma}{\subset} B$, with $\gamma \leqslant 1/38$ and A nuclear" by: $M_n(A) \overset{\gamma}{\subset} M_n(B)$ for all n, with $\alpha(\gamma) \leqslant 1/3$.

2.6. REMARK. If A, B and D are C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$ and $B \overset{\gamma'}{\subset} D$ with $\gamma + \gamma' + \gamma\gamma' \leqslant 1/38$, then the following diagram is commutative:

$$\begin{array}{ccc} K^*(A) & \xrightarrow{\ \tau\ } & K^*(B) \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & K_*(D) & \end{array}$$

Note that $A \overset{\gamma}{\subset} B$ and $B \overset{\gamma'}{\subset} D$ implies $A \overset{\gamma+\gamma'+\gamma\gamma'}{\subset} D$.

2.7. REMARK. We note that by considering homotopy classes of projections (instead of unitary equivalent classes) in the definition of K_0, we can obtain a sharper constant γ ($\leqslant 1/20$).

2.8. PROPOSITION. *Let A and B be C^*-subalgebras of a C^*-algebra C. Suppose that $d(A, B) < \gamma \leqslant 1/100$ and A nuclear. Then τ is an isomorphism from* $K_*(A)$ *onto* $K_*(B)$.

Proof. Since $d(A, B) < \gamma$, we have $A \overset{\gamma}{\subset} B$ and $B \overset{\gamma}{\subset} A$. As $d(A, B) < 1/100$ by 2, Theorem 6.5] B is nuclear. Now the homomorphism $K_*(A) \to K_*(B)$ and $K_*(B) \to K_*(A)$ given by 2.4 are obviously the inverse of each other.

2.9. REMARK. In the proof of 2.4 the nuclearity of A is only used to ensure the relation $A \overset{\gamma}{\subset} B$ implies that $M_n(A) \overset{k\gamma}{\subset} M_n(B)$ for every positive integer n and a fixed number k. However, this holds for a larger class of C^*-algebras. For instance if for any representation π of A, $\pi(A)''$ is properly infinite, then $A \overset{\gamma}{\subset} B$ implies that $M_n(A) \overset{3/2\gamma}{\subset} M_n(B)$ for every positive integer n [2, Proposition 2.7]. Hence 2.4 (and 2.8) is true in this case.

In the proof of the following lemma we use the idea of [6, Lemma 1.2]. However this is a slightly different situation and our proof is also a bit different.

2.10. LEMMA. *Let A and B be C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$ with $\gamma \leqslant 1/6$ and A nuclear. Let J be a closed two sided ideal in B.*

i) *There exists a unique closed two sided ideal I in A characterized by the following property:*

There exists $\varphi : A \to D, \psi : B \to D$ such that φ is $(\gamma + \alpha(2\gamma)/2)$-contained in ψ with $I = \ker\varphi$ and $J = \ker\psi$.

ii) *Moreover $I \overset{2\gamma+\alpha(2\gamma)/2}{\subset} J$ and if $A_0 = \varphi(A)$ and $B_0 = \psi(B)$, then $A_0 \overset{\gamma+\alpha(2\gamma)/2}{\subset} B_0$.*

Proof. Let π be a representation of B on a Hilbert space H such that $\ker\pi = J$. By [3, 2.10.2] there exists a representation $\tilde{\pi}$ of the C^*-algebra generated by A and B on a Hilbert space K containing H such that $\tilde{\pi}(b)$ restricted to H is $\pi(b)$ for every $b \in B$. Let P be the projection of K onto H. Then $P \in \tilde{\pi}(B)'$, and since A is nuclear 1.5 gives $\tilde{\pi}(B)' \overset{2\gamma}{\subset} \tilde{\pi}(A)'$. Hence by 1.10 ii) there exists a projection $Q \in \tilde{\pi}(A)$ such that $\|P - Q\| < \alpha(2\gamma)/2$. Now J is the kernel of the map $\psi : b \to P\tilde{\pi}(b)$, $b \in B$. Let I be the kernel of the map $\varphi : a \to Q\tilde{\pi}(a)$, $a \in A$. Let $a \in A$, $\|a\| \leqslant 1$, choose $b \in B$ with $\|a - b\| < \gamma$. Then

$$\|(a - b, \varphi(a) - \psi(b))\| = \|(a - b, Q\tilde{\pi}(a) - P\tilde{\pi}(b))\| \leqslant$$

$$\leqslant \sup(\|a - b\|, \|a - b\| + \|P - Q\|) \leqslant \gamma + \frac{\alpha(2\gamma)}{2}$$

and it follows that φ is $(\gamma + \alpha(2\gamma)/2)$-contained in ψ. To show that I is unique, let I' be a closed ideal in A such that $I' = \ker\varphi'$ and $J = \ker\psi'$ where $\varphi' : A \to D'$ is $(\gamma + \alpha(2\gamma)/2)$-contained in $\psi' : B \to D'$. Let $x \in A$, $\|x\| \leqslant 1$. Choose $y \in B$ such that $\|x - y\| < \gamma + \alpha(2\gamma)/2$ and $\|\varphi'(x) - \psi'(y)\| < \gamma + \alpha(2\gamma)/2$. Then since $\ker\psi = \ker\psi'$, $\|\psi(y)\| = \|\psi'(y)\|$ and we have

$$|\,\|\varphi(x)\| - \|\varphi'(x)\|\,| \leqslant \|\varphi'(x) - \psi'(y)\| + \|\psi(y) - \varphi(x)\| \leqslant$$

$$\leqslant \|\varphi'(x) - \psi'(y)\| + \|x - y\| + \|P - Q\| <$$

$$< \gamma + \frac{\alpha(2\gamma)}{2} + \gamma + \frac{\alpha(2\gamma)}{2} + \frac{\alpha(2\gamma)}{2} = 2\gamma + \frac{3\alpha(2\gamma)}{2} < 1.$$

This implies that $\|\varphi \mid I'\| < 1$ and $\|\varphi' \mid I\| < 1$, i.e. $\varphi \mid I' = 0$ and $\varphi' \mid I = 0$ and consequently $I = I'$.

ii) $A_0 \overset{\gamma+\alpha(2\gamma)/2}{\subset} B_0$ because φ is $(\gamma + \alpha(2\gamma)/2)$-contained in ψ. Finally we show that $I \overset{2\gamma+\alpha(2\gamma)/2}{\subset} J$. Let $x \in I$, $\|x\| \leqslant 1$. Choose $y \in B$ such that $\|x - y\| < \gamma$. Then as $\varphi(x) = 0$ we have $\|\psi(y)\| < \gamma + \alpha(2\gamma)/2$. Therefore there exists $y' \in J$ such that $\|y - y'\| < \gamma + \alpha(2\gamma)/2$. Then $\|x - y'\| < 2\gamma + \alpha(2\gamma)/2$ which ends the proof.

2.11. REMARK. Let A and B be C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$ and A nuclear. Let $\tilde{\pi}, \varphi, \psi, P, Q, A_0$ and B_0 be as given in 2.10. Then we denote by φ^+ and ψ^+ the induced homomorphism on $M_n(A^+)$ and $M_n(B^+)$ by φ and ψ respectively i.e. $\varphi^+((a_{ij}), \lambda) = ((Q\tilde{\pi}(a_{ij})), \lambda)$ and $\psi^+((b_{ij}), \lambda) = ((P\tilde{\pi}(b_{ij})), \lambda)$, $i, j = 1,2, \ldots, n$, where λ is a complex $n \times n$ matrix. Then $M_n(I) = \ker \varphi^+$ and $M_n(J) = \ker \psi^+$. If $x \in M_n(I)$, $\|x\| \leqslant 1$, then by 1.4 we can choose $y \in M_n(B)$ such that $\|x - y\| < 6\gamma$. Note that

$$\|\psi^+(y)\| = \|\varphi^+(x) - \psi^+(y)\| \leqslant \|P - Q\| + \|x - y\| < \frac{\alpha(2\gamma)}{2} + 6\gamma.$$

Thus there exists $y' \in M_n(J) = \ker \psi^+$ such that $\|y - y'\| < \alpha(2\gamma)/2 + 6\gamma$. Then $\|x - y'\| < \alpha(2\gamma)/2 + 12\gamma$ and this shows that $M_n(I) \overset{12\gamma+\alpha(2\gamma)/2}{\subset} M_n(J)$ for every positive integer n. We also have $M_n(I^+) \overset{24\gamma+\alpha(2\gamma)}{\subset} M_n(J^+)$. Note that by using relation $I \overset{2\gamma+\alpha(2\gamma)/2}{\subset} J$ (2.10 ii) and 1.4 we get $M_n(I) \overset{12\gamma+3\alpha(2\gamma)}{\subset} M_n(J)$. We also note that φ^+ is $\alpha(2\gamma) + 12\gamma$-contained in ψ^+. We will use these observations and notations throughout the proof of the following theorem without further explanation.

2.12 THEOREM. *Let A and B be C^*-subalgebras of a C^*-algebra C such that $A \overset{\gamma}{\subset} B$ with $\gamma \leqslant 1/100$ and A nuclear. Let J be a closed two-sided ideal in B and I the corresponding ideal in A given by* 2.10. *Then there exists a homomorphism $\hat{\tau} : \mathrm{K}_*(A/I) \to$ $\to \mathrm{K}_*(B/J)$ making the following diagram commutative:*

$$\begin{array}{ccccccccccccc}
\mathrm{K}_1(I) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \xrightarrow{\pi_*} & \mathrm{K}_1(A/I) & \xrightarrow{\delta} & \mathrm{K}_0(I) & \xrightarrow{\iota_*} & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(A/I) & \xrightarrow{\delta} & \mathrm{K}_1(I) \\
\downarrow{\scriptstyle\tau} & \mathrm{I} & \downarrow{\scriptstyle\tau} & \mathrm{II} & \downarrow{\scriptstyle\hat{\tau}} & \mathrm{III} & \downarrow{\scriptstyle\tau} & \mathrm{IV} & \downarrow{\scriptstyle\tau} & \mathrm{V} & \downarrow{\scriptstyle\hat{\tau}} & \mathrm{VI} & \downarrow{\scriptstyle\tau} \\
\mathrm{K}_1(J) & \xrightarrow{j_*} & \mathrm{K}_1(B) & \xrightarrow{\pi_*} & \mathrm{K}_1(B/J) & \xrightarrow{\delta} & \mathrm{K}_0(J) & \xrightarrow{j_*} & \mathrm{K}_0(B) & \xrightarrow{\pi_*} & \mathrm{K}_0(B/J) & \xrightarrow{\delta} & \mathrm{K}_1(J).
\end{array}$$

Proof. First note that the maps induced by $a + I \to Q\tilde{\pi}(a)$ and $b + J \to$ $\to P\tilde{\pi}(b)$ on the K-groups together with the closeness homomorphism $\tau\colon K_*(A_0) \to$ $\to K_*(B_0)$ yield the desired homomorphism $\hat{\tau}\colon K_*(A/I) \to K_*(B/J)$, noting that by 2.10 ii) $A_0 \overset{\gamma+\alpha(2\gamma)/2}{\subset} B_0$. Next we show the commutativity of the diagrams. Commutativity of diagrams I, IV, II and V follows by applying 2.4 ii) to the diagrams

$$\begin{array}{ccc} A & \overset{\gamma}{\subset} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A_0 & \overset{\gamma+\frac{\alpha(2\gamma)}{2}}{\subset} & B_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I & \overset{2\gamma+\frac{\alpha(2\gamma)}{2}}{\subset} & J \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A & \overset{\gamma}{\subset} & B \end{array},$$

where i and j are the inclusion maps. We note that by 2.11 $M_n(I^+) \overset{24\gamma+\alpha(2\gamma)}{\subset} M_n(J^+)$ and the assumption $2\gamma + \alpha(2\gamma)/2 \leqslant 1/38$ is not needed here in order to apply 2.4.

To show that diagram III is commutative choose $a \in U_n(A^+/I)$ for some n. Let $u \in U^0_{2n}(A^+)$ be a preimage for $a \oplus a^*$. Hence $u = \begin{pmatrix} x_1 & y_1 \\ y_2 & x_2 \end{pmatrix}$ where $x_i \in M_n(A^+)$ and $y_i \in M_n(I)$, $i = 1, 2$. By 1.4, $M_n(A^+) \overset{12\gamma}{\subset} M_n(B^+)$ and by 2.11 $M_n(I) \overset{12\gamma+\alpha(2\gamma)/2}{\subset} M_n(J)$. So there are $x_i' \in M_n(B^+)$ and $y_i' \in M_n(J)$ $i = 1, 2$, such that $\|x_i - x_i'\| < 12\gamma$ and $\|y_i - y_i'\| < 12\gamma + \alpha(2\gamma)/2$. If $x = \begin{pmatrix} x_1' & y_1' \\ y_2' & x_2' \end{pmatrix}$, then $\|x - u\| < 24\gamma + \alpha(2\gamma)/2$. Let $x = v|x|$ be the polar-decomposition of x. Then by [1, Lemma 2.7] $\|u - v\| < \alpha(24\gamma + \alpha(2\gamma)/2)$. Moreover the image of v in $M_{2n}(B^+/J)$ is of the form $b \oplus b'$. Now by definition of the map δ we have $\delta([a]) = [p] - [I_n]$ and $\delta([b]) = [q] - [I_n]$ where $p = u(I_n \oplus 0)u^*$ and $q = v(I_n \oplus 0)v^*$. Since $\|p - q\| <$ $< 2\alpha(24\gamma + \alpha(2\gamma)/2) < 1 - \alpha(24\gamma + \alpha(2\gamma))/2$ we have $\tau([p]) = [q]$. Hence it only remains to show that $\hat{\tau}([a]) = [b]$. Note that if $v = \begin{pmatrix} x_1'' & y_1'' \\ y_2'' & x_2'' \end{pmatrix}$ with $x_i'' \in M_n(B^+)$ and $y_i'' \in M_n(J)$ $i = 1,2$, then from $\|x_1 - x_1'\| < 12\gamma$ and [1, Lemma 2.7] we get that $\|x_1 - x_1''\| < \alpha(12\gamma)$. Now by 2.11 φ^+ is $\alpha(2\gamma) + 12\gamma$-contained in ψ^+. Hence

$$\|\varphi^+(x_1) - \psi^+(x_1'')\| < \alpha(2\gamma) + 12\gamma + \alpha(12\gamma) < 2 - \alpha(12\gamma + 6\alpha(2\gamma))$$

and this shows that $\hat{\tau}([a]) = [b]$.

Finally we show that the diagram VI commutes. Let $p \in M_\infty(A^+/I)$ be a projection. Choose a preimage $\xi \in M_\infty(A^+)$ for p such that $0 \leqslant \xi \leqslant 1$. Let $2\xi - 1 =$ $= (a, \lambda)$ where $a \in M_\infty(A)$. Since $0 \leqslant \xi \leqslant 1$ we have $\|a\| \leqslant 2$. Hence by 1.4 we can choose a self-adjoint element $b \in M_\infty(B)$ such that $\|a - b\| < 12\gamma$. Let $\hat{\eta} = (b, \lambda)$. If

$\eta = (\hat{\eta} + 1)/2$, then we have

$$\|\varphi^+(\xi) - \psi^+(\eta)\| = \left\|\varphi^+\left(\frac{a}{2}, \frac{\lambda+1}{2}\right) - \psi^+\left(\frac{b}{2}, \frac{\lambda+1}{2}\right)\right\| =$$

$$= \left\|\left(Q\tilde{\pi}\left(\frac{a}{2}\right), \frac{\lambda+1}{2}\right) - \left(P\tilde{\pi}\left(\frac{b}{2}\right), \frac{\lambda+1}{2}\right)\right\| =$$

$$= \left\|Q\tilde{\pi}\left(\frac{a}{2}\right) - P\tilde{\pi}\left(\frac{b}{2}\right)\right\| \leqslant \frac{1}{2}\|a\|\cdot\|Q - P\| + \frac{1}{2}\|a - b\| < \frac{\alpha(2\gamma)}{2} + 6\gamma.$$

Since $\varphi^+(\xi)$ is a projection we have $\sigma(\psi^+(\eta)) \subset (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon)$ where $\varepsilon = \alpha(2\gamma)/2 + 6\gamma$. Let $\hat{q}$ be the spectral projection of $\psi^+(\eta)$ corresponding to the interval $(1-\varepsilon, 1+\varepsilon)$. Then $\hat{q} \in M_\infty(B_0^+)$ and $\|\hat{q} - \psi^+(\eta)\| < \varepsilon$. Also

$$\|\hat{q} - \varphi^+(\xi)\| \leqslant \|\hat{q} - \psi^+(\eta)\| + \|\psi^+(\eta) - \varphi^+(\xi)\| <$$

$$< \frac{\alpha(2\gamma)}{2} + 6\gamma + \frac{\alpha(2\gamma)}{2} + 6\gamma = \alpha(2\gamma) + 12\gamma.$$

Next choose a self-adjoint element $\nu \in M_\infty(B^+)$ such that $\psi^+(\nu) = \hat{q}$ and $\|\nu - \eta\| < \varepsilon$. Let q be the image of $\hat{q}$ in $M_\infty(B^+/J)$ under the inverse of the $*$-isomorphism $b + J \to P\tilde{\pi}(b)$, $b \in B$. Then by definition of the Bott-map δ we have $\delta([p]) = [e^{2\pi i\xi}]$ and $\delta([q]) = [e^{2\pi i\nu}]$. Also since

$$\|\psi^+(\nu) - \varphi^+(\xi)\| < \alpha(2\gamma) + 12\gamma < 1 - \frac{\alpha(12\gamma + 6\alpha(2\gamma))}{2}$$

we have $\hat{\tau}([p]) = [q]$. Now to end the proof we must show that $\tau([e^{2\pi i\xi}]) = [e^{2\pi i\nu}]$, i.e. we need to show that $\|e^{2\pi i\xi} - e^{2\pi i\nu}\| < 2 - \alpha(24\gamma + 6\alpha(2\gamma))$. To that end let $f(t) = \exp(2\pi it\xi)\exp(2\pi i(1-t)\nu)$, $0 \leqslant t \leqslant 1$ and note that $\|\xi - \nu\| < \alpha(2\gamma)/2 + 12\gamma$. Then

$$\|e^{2\pi i\xi} - e^{2\pi i\nu}\| = \left\|\int_0^1 f'(t)\,dt\right\| =$$

$$= \left\|\int_0^1 2\pi i \exp(2\pi it\xi)(\xi - \nu)\exp(2\pi i(1-t)\nu)\,dt\right\| \leqslant 2\pi\int_0^1 \|\xi - \nu\|\,dt <$$

$$< 2\pi\left(\frac{\alpha(2\gamma)}{2} + 12\gamma\right) < 2 - \alpha(24\gamma + 3\alpha(2\gamma))$$

and this ends the proof.

We conclude with proving that if A and B are unital C^*-algebras with the same identity and $d(A, B) < 1$, then their groups of unitaries are homotopically equivalent. In the case that A and B have different units we need $d(A,B) < \gamma$, with $\gamma + \alpha(\alpha(\delta)) \leqslant \leqslant 1$ (see Remark 2.15). We will denote by $U(A)$ and $U(B)$ the set of unitaries of A and B respectively. We need the following lemma.

2.13. LEMMA. *Let u and v be unitaries in a unital C^*-algebra C. If $\|u - 1\| < < \sqrt{2}$ and $\|v - 1\| \leqslant \sqrt{2}$, then $\|uv - 1\| < 2$.*

Proof. Let C be represented faithfully on a Hilbert space H. Since $\|u - 1\| < \sqrt{2}$ there exists an $\varepsilon > 0$ such that $\mathrm{Re}(u^*) \geqslant \varepsilon$. Also $\|v - 1\| \leqslant \sqrt{2}$ implies that $\mathrm{Re}(v) < 0$. Let $V(u^*)$ and $V(v)$ denote the numerical ranges of u^* and v respectively. Then

$$V(u^*) \subseteq \{z : \mathrm{Re}(z) \geqslant \varepsilon\} \quad \text{and} \quad V(v) \subseteq \{z : \mathrm{Re}(z) \geqslant 0\}.$$

Then $V(u^*) \cap (-V(v)) = \emptyset$. Therefore the set

$$\{\lambda\mu^{-1} : \lambda \in V(v),\ \mu \in V(u^*)\}$$

does not contain -1. Hence by [11, Theorem 1] $-1 \notin \sigma(uv)$ and it follows that $\|uv - 1\| < 2$.

2.14. THEOREM. *Let A and B be unital C^*-subalgebras of a C^*-algebra C having the same unit. If $d(A, B) < 1$, then $U(A)$ and $U(B)$ are homotopically equivalent.*

Proof. For each $y \in B_1$ let $O_y = \{x \mid x \in A_1, \|x - y\| < 1\}$. Then $\{O_y\}_{y \in B_1}$ is an open covering for A_1. Since A_1 is a metric space it is paracompact cf. [10] and we can choose a locally finite partition of unity $\{f_y\}_{y \in B_1}$ subordinate to the open covering $\{O_y\}_{y \in B_1}$. Now set $\hat{\varphi}(x) = \sum_{y \in B_1} f_y(x) y$ for each $x \in A_1$. Suppose $x \in A_1$ and $f_y(x) \neq 0$; then $\|x - y\| < 1$ and we get

$$\|\hat{\varphi}(x) - x\| \leqslant \sum_{y \in B_1} f_y(x)\|x - y\| < 1.$$

If $x \in A_1$ is a unitary, $\|\hat{\varphi}(x) - x\| < 1$ implies that $\hat{\varphi}(x)$ is invertible. Now $\varphi(x)$ the unitary part of $\hat{\varphi}(x)$ belongs to the C^*-algebra B and φ defines a continuous map from $U(A)$ into $U(B)$. Moreover by [1, Lemma 2.7] $\|\varphi(x) - x\| < \alpha(1) = \sqrt{2}$.

In the same way we construct a continuous function ψ from $U(A)$ into $U(B)$ satisfying $\|\psi(y) - y\| < \sqrt{2}$ for each $y \in U(B)$. Let $u = x^*\varphi(x)$ and $v = \varphi(x)^*\psi(\varphi(x))$. Then Lemma 2.13 implies that $-1 \notin \sigma(x^*\psi(\varphi(x)))$. Thus the log function is continuous on the $\sigma(x^*\psi(\varphi(x)))$ and $F(x, t) = x\exp(t \log x^*\psi(\varphi(x)))$ is a homotopy between $\psi \circ \varphi$ and $1_{U(A)}$, the identity map of $U(A)$. In the same manner $\varphi \circ \psi$ is homotopic to $1_{U(B)}$. Hence φ and ψ define the desired homotopy equivalence between $U(A)$ and $U(B)$.

2.15. Remark. We note that in 2.14 the assumption that A and B have the same identity can be dropped if $d(A, B) < \gamma$, with $\alpha(\alpha(\gamma)) + \gamma \leqslant 1$. To see this, one can use 1.11 to show that $\|1_A - 1_B\| < \alpha(\gamma)/2$. Let $u \in C^+$ be the unitary part of the invertible element $x = 1_A 1_B + (1 - 1_A)(1 - 1_B)$, where 1 is the identity of C^+. Then $1_A = u 1_B u^*$ and [1, Lemma 2.7] implies that $\|u - 1\| < \alpha(\alpha(\gamma))$. Now A and uBu^* have the same unit and $d(A, uBu^*) < \alpha(\alpha(\gamma)) + \gamma \leqslant 1$. Hence 2.14 can be applied.

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MAHMOOD KHOSHKAM
Department of Mathematics and Statistics,
Queen's University at Kingston,
Kingston, K7L 3N6,
Canada.

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