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HYPERCONTRACTIONS AND SUBNORMALITY

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0. INTRODUCTION

In this paper $\mathscr{H}$ will always refer to a separable Hilbert space over the complex numbers. $\mathscr{L}(\mathscr{H})$ will denote the bounded linear transformations of $\mathscr{H}$. The following theorem is well known ([4], [9]).

THEOREM A. *Let S denote the unilateral unweighted shift of multiplicity one and let $S^{*(\infty)}$ denote the direct sum of a countably infinite number of copies of S^*. Let $T \in \mathscr{L}(\mathscr{H})$. T has an extension to $S^{*(\infty)}$ if and only if $\|T\| \leqslant 1$ and $T^m \to 0$ strongly as $m \to \infty$.*

In [2] the following analog of Theorem A with S replaced by B, the Bergman shift, was proved.

THEOREM B. *Let $T \in \mathscr{L}(\mathscr{H})$. T has an extension to $B^{*(\infty)}$ if and only if $\|T\| \leqslant 1$, $1 - 2T^*T + T^{*2}T^2 \geqslant 0$, and $T^m \to 0$ strongly as $m \to \infty$.*

A comparison of Theorem A and Theorem B above seems to suggest that they are but the first two theorems in an infinite sequence of theorems. This sequence of theorems is described in Section 1 of this paper. Loosely speaking, for each positive integer $n \geqslant 1$ we introduce a class of operators T, which is defined in a simple way using a system of polynomial inequalities in T and T^* (and which we call the n-hypercontractions); for each n there is also a weighted shift S_n. Our theorem (Theorem 1.12) then states that $T \in \mathscr{L}(\mathscr{H})$ has extension to $S_n^{*(\infty)}$ if and only if T is an n-hypercontraction. When $n = 1$ the theorem is Theorem A above and when $n = 2$ it reduces to Theorem B above.

Our initial proof of Theorem 1.12 which appears in Section 1 is based on the techniques in [2]. In Section 2 we give a different proof based on the Rovnyak-de Branges construction ([7] and [4]). While considerably less amenable to generalization than the proof in Section 1, the proof in Section 2 has the advantage of being vastly more elementary.

In Section 3 we answer the question: What are the operators that are n-hypercontractive for every n? The answer (Theorem 3.1) is surprising not only because it is so simple (T is an n-hypercontraction for every n if and only if T is a contractive subnormal) but also because it makes us realize how little we understand the gestalt behind many of the intrinsic conditions for subnormality. Section 3 concludes with an attempt to deal with this unpleasantness by translating Theorem 3.1 into an approximation theorem on the bidisc.

1. n-CONTRACTIVE MODELS

In this section for n a positive integer, $\mathcal{M}_n$ will denote the Hilbert space of functions $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$, defined and analytic on the open unit disc $\mathbf{D}$, and satisfying $\|f\|_n^2 = \sum_{k=0}^{\infty} (w_{nk})^{-1} |\hat{f}(k)|^2 < \infty$. Here w_{nk} is defined by

$$(1.1) \qquad (1 - x)^{-n} = \sum_{k=0}^{\infty} w_{nk} x^k, \quad |x| < 1.$$

Equivalently, $\mathcal{M}_n$ can be defined as the Hilbert space with reproducing kernel $k_w(z) = k(w, z) = (1 - \overline{w}z)^{-n}$ (see [3]). Define an operator S_n on $\mathcal{M}_n$ by

$$S_n(f)(z) = zf(z) \quad f \in \mathcal{M}_n.$$

Notice that S_1 is the unilateral shift and S_2 is the Bergman shift and that, in general, S_n is a weighted shift. The following proposition gives yet another description of the space $\mathcal{M}_n$, now as an $H^2(\mu)$ space. As a consequence it is clear that S_n is subnormal. Other relevant information about S_n is also summarized in the proposition.

Define a sequence of nonnegative measures ν_n on [0,1] inductively as follows. Let ν_1 be the unit point mass concentrated at 1. Define ν_{n+1} in terms of ν_n by requiring

$$(1.2) \qquad \operatorname{spt} \nu_n \subseteq [0, 1] \quad \text{and} \quad \nu_{n+1}(E) = \int_E 2nr\, \nu_n([r, 1])\, dr$$

for all Borel sets $E \subseteq [0, 1]$. Define a measure μ_n on $\mathbf{D}^-$ by $d\mu_n(re^{i\theta}) = \frac{d\theta}{2\pi}\, d\nu_n(r)$. Let $H^2(\mu_n)$ denote the completion of the analytic polynomials in $L^2(\mu_n)$ and let M_n denote the operator multiplication by z on $H^2(\mu_n)$.

1.3. PROPOSITION. *The map I_n which sends $p(z) \in \mathcal{M}_n$ to $p(z) \in H^2(\mu_n)$ is a densely defined (on polynomials) Hilbert space isomorphism. S_n is unitarily equivalent to M_n. $\sigma(S_n) = \mathbf{D}^-$, $\sigma_e(S_n) = \partial\mathbf{D}$ (equivalently, $\lambda - S_n$ is Fredholm for every $\lambda \in \mathbf{D}$), and S_n is essentially unitary.*

Proof. We first show that I_n is isometric. The other facts will then follow immediately. We proceed by induction. That I_1 is isometric is of course quite well known. Assume I_n is isometric. Since the monomials are orthogonal in both $H^2(\mu_n)$ and $\mathcal{M}_n$ it is enough to show that

$$\|I_{n+1}(z^k)\|^2_{\mu_{n+1}} = \|z^k\|^2_{n+1}$$

for every $k \geqslant 0$. The critical fact that we shall use is that $(k+1)w_{n\,k+1} = nw_{n+1\,k}$ which is obtained by differentiating (1.1). We have

$$\|z^k\|^2_{n+1} = \frac{1}{w_{n+1\,k}} = \frac{n}{k+1}\cdot\frac{1}{w_{n\,k+1}} =$$

$$= \frac{n}{k+1}\|z^{k+1}\|^2_n = \frac{n}{k+1}\|I_n z^{k+1}\|^2_{\mu_n} =$$

$$= \frac{n}{k+1}\int_0^1 r^{2k+2}\mathrm{d}\nu_n(r) = \int_0^1\left(\int_0^r s^{2k+1}\,\mathrm{d}s\right)2n\mathrm{d}\nu_n(r) =$$

$$= \int_0^1 s^{2k}\,2ns\,\nu_n([s,1])\mathrm{d}s = \int_0^1 s^{2k}\mathrm{d}\,\nu_{n+1}(s) = \|I_{n+1}(z^k)\|^2_{\mu_{n+1}},$$

and consequently, I_n is isometric for every n. Trivially, I_n has dense range. Hence I_n implements an equivalence between S_n and M_z on $H^2(\mu_n)$. It is easy to verify that μ_n is mutually boundedly absolutely continuous with respect to area measure on compact subsets of $\mathbf{D}$ when $n > 1$. Since we also have $\mathrm{spt}(\mu_n) \subseteq \mathbf{D}^-$ for every n it follows that $\sigma(S_n) = \mathbf{D}^-$ and that $\sigma_e(S_n) = \partial\mathbf{D}$. Finally, since S_n is a cyclic subnormal the Berger-Shaw Theorem implies that S_n is essentially normal. Since $\sigma_e(S_n) = \partial\mathbf{D}$ it must be the case that S_n is essentially unitary. In the case of this simple weighted shift this can also be shown by direct computation. This concludes the proof of Proposition 1.3.

We now are ready to describe the operators which can be modeled as parts of operators of the form $\pi(S_n^*)$ where π is a representation on $\mathcal{L}(\mathcal{M}_n)$ with $\pi(1) = 1$. The remainder of this section relies heavily on the techniques of [2]. In particular recall that an hereditary polynomial is a polynomial $p(x, y)$ in two noncommuting variables x and y, of the form,

$$p(x, y) = \sum c_{ij} y^j x^i. \tag{1.4}$$

If p is the hereditary polynomial of (1.4) and $b \in B$ where B is any C^*-algebra with identity then $p(b)$ is defined by

$$p(b) = \sum c_{ij} b^{*j} b^i.$$

Thus, for example, if $n \geqslant 0$ is an integer and T is an operator,

$$(1 - yx)^n (T) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k.$$

We let $\mathscr{P}$ denote the set of hereditary polynomials. If B is a C^*-algebra with identity and $b \in B$ we let $H(b)$, the hereditary manifold of b, be defined by $H(b) = \{p(b) \mid p \in \mathscr{P}\}$.

For the convenience of the reader we state the results from [2] that we will require. The first of these is based on Theorem 1.5 in [2].

1.5. THEOREM. *Let B be a C^*-algebra with identity and let $b \in B$. A bounded operator T has the form $\pi(b) \mid \mathscr{N}$ where π is a representation of B ($\pi(1) = 1$) and $\mathscr{N}$ is invariant for $\pi(b)$ if and only if the map φ defined on $H(b)$ by the formula $\varphi(p(b)) = p(T)$ is in fact well defined and completely positive.*

Proposition 1.3 guarantees that $\mathscr{M}_n$ is a *regular* analytic model atom (this is language from [2]). Thus the following theorem follows from Theorem 2.3 in [2].

1.6. THEOREM. *Let $T \in \mathscr{L}(\mathscr{H})$ with $\sigma(T) \subseteq \mathbf{D}$. T has an extension to $S_n^{*(\infty)}$ if and only if $(1 - yx)^n (T) \geqslant 0$.*

The following result may be deduced directly from Proposition 1.3 or obtained as a corollary to Theorem 2.8 in [2].

1.7. THEOREM. *If $\pi : \mathscr{L}(\mathscr{M}_n) \to \mathscr{L}(\mathscr{M})$ is a representation with $\pi(1) = 1$ then $\pi(S_n)$ has the form $S_n^{(m)} \oplus U$ (either summand may be absent) where m is a possibly countable infinite positive integer and U is unitary.*

Our next result follows from Lemma 2.4 in [2].

1.8. PROPOSITION. *Let $p \in \mathscr{P}$. $p(S_n^*) \geqslant 0$ if and only if $p(\bar{z}, w)(1 - \bar{z}w)^{-n}$ is positive definite on $\mathbf{D} \times \mathbf{D}$.*

The final result we shall need has nothing to do with [2].

1.9. LEMMA. *Let $T \in \mathscr{L}(\mathscr{H})$ satisfy $(1 - yx)^k(T) \geqslant 0$ for all $k \leqslant n$. Then $(1 - yx)^k(zT) \geqslant 0$ for all $k \leqslant n$ whenever $|z| \leqslant 1$.*

Proof. Let $k \leqslant n$. Then

$$(1 - yx)^k(zT) = (1 - |z|^2yx)^k(T) = (1 - yx + (1 - |z|^2)yx)^k(T) =$$

$$= \left(\sum_{j=0}^{k} \binom{k}{j}(1-|z|^2)^{k-j}y^{k-j}(1-yx)^j x^{k-j}\right)(T) =$$

$$= \sum_{j=0}^{k} \binom{k}{j}(1-|z|^2)^{k-j}T^{*k-j}(1-yx)^j(T)T^{k-j}.$$

But by assumption this last sum is a linear combination with positive coefficients of positive operators and hence is positive. We conclude that $(1 - yx)^k(zT) \geqslant 0$ and this establishes the lemma.

1.10. THEOREM. *Let $T \in \mathscr{L}(\mathscr{H})$ with $\mathscr{H}$ separable. T has an extension to an operator of the form $S_n^{*(\infty)} \oplus U$ where U is unitary if and only if $(1 - yx)^k(T) \geqslant 0$ for all $k \leqslant n$.*

Proof. We first verify the easy half of the theorem. Let $T = (S_n^{*(\infty)} \oplus U)|\mathscr{M}$ where U is unitary and $\mathscr{M}$ is invariant for $S_n^{*(\infty)} \oplus U$. If P denotes projection onto $\mathscr{M}$ then,

$$(1 - yx)^k(T) = P((1 - yx)^k(S_n^{*(\infty)}) \oplus (1 - yx)^k(U))|\mathscr{M}.$$

Hence $(1 - yx)^k(T) \geqslant 0$ follows from $(1 - yx)^k(S_n^{*(\infty)}) \geqslant 0$ and $(1 - yx)^k(U) \geqslant 0$. But $(1 - yx)^k(U) = 0$ for any $k \geqslant 1$ and by Proposition 1.8, $(1 - yx)^k(S_n^{*(\infty)}) \geqslant 0$ if $k \leqslant n$.

Now assume that $(1 - yx)^k(T) \geqslant 0$ for $k \leqslant n$. It follows from Lemma 1.9 that

$$(1 - yx)^k(zT) \geqslant 0$$

whenever $k \leqslant n$ and $|z| < 1$. Since T is a contraction it is also the case that $\sigma(zT) \subseteq \subseteq \mathbf{D}$ whenever $|z| < 1$. It follows from Theorem 1.6 that zT has an extension to an operator of the form $S_n^{*(\infty)}$ whenever $|z| < 1$. Consequently by Theorem 1.5 the map,

$$p(S_n^*) \mapsto p(zT),$$

defined on the hereditary manifold of S_n^* is well defined and completely positive for every $|z| < 1$. It follows that the map

$$p(S_n^*) \mapsto p(T)$$

is well defined and completely positive. Thus, by Theorem 1.5, T has an extension to an operator of the form $\pi(S_n^*)$, where π is a representation on $\mathscr{L}(\mathscr{M}_n)$, and $\pi(1)=$

$=1$. By Theorem 1.7, $\pi(S_n^*)$ has the form $S_n^{*(k)} \oplus U$ where U is unitary. Since $S_n^{*(k)}$ has an extension to $S_n^{*(\infty)}$ this concludes the proof of Theorem 1.10.

Theorem 1.10 suggests the following definition.

1.11. DEFINITION. $T \in \mathscr{L}(\mathscr{H})$ is a *hypercontraction of order n* if $(1 - yx)^k(T) \geqslant 0$ whenever $1 \leqslant k \leqslant n$. The collection of hypercontractions of order n on a space $\mathscr{H}$ will be denoted $\mathscr{C}_n(\mathscr{H})$. If $T \in \mathscr{C}_n(\mathscr{H})$ we say T is *strong* if $T^m \to 0$ strongly as $m \to \infty$.

We remark that it is a consequence of Lemma 2.11 in the next section that $T \in \mathscr{L}(\mathscr{H})$ is a strong hypercontraction of order n if and only if $T^m \to 0$ strongly and $(1 - yx)^n(T) \geqslant 0$. The significance of strong hypercontractions is revealed in the following theorem whose proof is obtained in a simple way from Theorem 1.10 (using the fact that $S_n^{*m} \to 0$ strongly).

1.12. THEOREM. *Let $T \in \mathscr{L}(\mathscr{H})$ with $\mathscr{H}$ separable. T has an extension to $S_n^{*(\infty)}$ if and only if T is a strong hypercontraction of order n.*

2. THE EXTENSION THEOREM VIA THE ROVNYAK-de BRANGES CONSTRUCTION

In this section we show how Theorem 1.12 in Section 1 can be obtained by the Rovnyak-de Branges construction ([7], p. 34; [4]). While from the point of view of the *general* coanalytic model this construction is severely restricted, in the case of hypercontractions it has the advantage of displaying the extension concretely and in a geometrical form, without reference to complete positivity. The natural level of generality of the Rovnyak-de Branges construction will be discussed in a future paper.

To begin let us first recall how Rovnyak and de Branges proved the following theorem.

2.1. THEOREM. *Let $T \in \mathscr{L}(\mathscr{H})$. T has an extension to $S_1^{*(\infty)}$ iff $\|T\| \leqslant 1$ and $T^k \xrightarrow{s} 0$.*

Set $\mathscr{K} = \mathscr{H}^{(\infty)} =$ set of square summable sequences of vectors from $\mathscr{H}$. If $\{f_k\}_{k=0}^{\infty} \in \mathscr{K}$ define

$$\|\{f_k\}_{k=0}^{\infty}\|^2 = \sum_{k=0}^{\infty} \|f_k\|^2 . \tag{2.2}$$

Let $W: \mathscr{H} \to \mathscr{K}$ be defined by

$$W(f) = \{(1 - T^*T)^{1/2} T^k f\}_{k=0}^{\infty}, \tag{2.3}$$

and let $S_- : \mathscr{K} \to \mathscr{K}$ be defined by

$$S_-(\{f_k\}_{k=0}^{\infty}) = \{f_{k+1}\}_{k=0}^{\infty}. \tag{2.4}$$

It is not immediately apparent that (2.3) defines a map into $\mathscr{K}$. That W is in fact well defined and indeed is an isometry follows by noticing that

$$\sum_{k=0}^{m} \|(1 - T^*T)^{1/2} T^k f\|^2 = \sum_{k=0}^{m} (\|T^k f\|^2 - \|T^{k+1} f\|^2) = \|f\|^2 - \|T^{m+1} f\|^2.$$

Since $\|T^{m+1} f\| \to 0$ as $m \to \infty$, W is well defined and isometric. The proof of Theorem 2.1 is now complete since

$$S_- \cong S_1^{*(\infty)}, \tag{2.5}$$

and

$$S_- W = WT \tag{2.6}$$

together imply that T has an extension to an operator of the form $S_1^{*(\infty)}$.

In order to use the argument just carried out to prove Theorem 1.12 only cosmetic changes are required. Accordingly fix $T \in \mathscr{L}(\mathscr{H})$ with $(1 - yx)^n(T) \geqslant 0$ and $T^k \xrightarrow{s} 0$. Define $\mathscr{K}_n$ to be the Hilbert space of sequences $\{f_k\}_{k=0}^{\infty}$ of vectors from $\mathscr{H}$ with

$$\|\{f_k\}_{k=0}^{\infty}\|^2 = \sum_{k=0}^{\infty} w_{nk} \|f_k\|^2 < \infty. \tag{2.7}$$

Evidently, if $n = 1$ then (2.7) reduces to (2.2). Define $S_- \in \mathscr{L}(\mathscr{H})$ as in (2.4). It is left as an exercise to verify that the analog of (2.5) is now

$$S_- \cong S_n^{*(\infty)}. \tag{2.8}$$

Since $1 - T^*T = (1 - yx)^1(T)$ it is natural to replace (2.3) with

$$W_n(f) = \{((1 - yx)^n (T))^{1/2} T^k f\}_{k=0}^{\infty} \in \mathscr{K}_n. \tag{2.9}$$

If we suspend concern for the moment over whether (2.9) actually defines an operator it is nevertheless clear that a least formally (2.6) holds and consequently in light of (2.8), to establish Theorem 1.12, it is enough to verify that the map W_n of (2.9) is isometric. This we now do. The unpleasant part of the proof is in the following two lemmas.

2.10. LEMMA. *Let* w_{nk} *be defined as in* (1.1). *Let* $j_k = \left[\frac{k+1}{2}\right]$. *Then*

a) $\frac{w_{n+1\,k}}{j_k} \leqslant \frac{2}{n} w_{n\,k+1}$;

b) $\lim_{k\to\infty} \frac{w_{n\,k+1}}{w_{nk}} = 1$;

c) $\lim_{k\to\infty} \frac{w_{n\,k+1}}{w_{n\,k-j_k}} = 1$;

d) $w_{nk} - w_{n\,k-1} = w_{n-1\,k}$.

Proof. By differentiating $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$ $n-1$ times one sees that $w_{nk} = \frac{1}{(n-1)!} \prod_{l=1}^{n-1} (k+l)$. The assertions of Lemma 2.10 follow immediately.

2.11. LEMMA. *Suppose* $T \in \mathcal{L}(\mathcal{H})$ *satisfies* $(1-yx)^n(T) \geqslant 0$ *and* $T^k \to 0$ *strongly as* $k \to \infty$. *Then* $T \in \mathcal{C}_n(\mathcal{H})$ *(*i.e. $(1-yx)^m(T) \geqslant 0$ *whenever* $m \leqslant n$*). Furthermore,* $w_{mk} \langle T^{*k+1}(1-yx)^{m-1}(T)\, T^{k+1}f, f\rangle \to 0$ *as* $k \to \infty$ *for every* $m \leqslant n$ *and every* $f \in \mathcal{H}$.

Proof. For convenience set $R_m = (1-yx)^m(T)$. Since $R_n \geqslant 0$ is equivalent to $T^*R_{n-1}T \leqslant R_{n-1}$ we see that $T^{*k}R_{n-1}T^k \leqslant R_{n-1}$ for every k. Since $T^k \to 0$ strongly it follows that $0 \leqslant R_{n-1}$. Thus we see by induction that $0 \leqslant R_m$ for every $m \leqslant n$ and $T \in \mathcal{C}_n(\mathcal{H})$. The second assertion of the lemma is also shown by induction. It is apparent that when $m = 1$, $w_{mk}\langle T^{*k+1}R_{m-1}T^{k+1}f, f\rangle = \|T^{k+1}f\|^2 \to 0$. Assume that $w_{mk}\langle T^{k+1}R_{m-1}\, T^{k+1}f, f\rangle \to 0$ as $k \to \infty$ for some $m < n$ and some $f \in \mathcal{H}$. We want to show that $w_{m+1\,k}\langle T^{*k+1} R_n T^{k+1} f, f\rangle \to 0$ as $k \to \infty$. For each k let $j_k = \left[\frac{k+1}{2}\right]$. We have

$$w_{m+1\,k}\langle T^{*k+1} R_m T^{k+1} f, f\rangle \leqslant$$

$$\text{(i)} \qquad \leqslant \frac{2}{m} w_{m\,k+1} j_k \langle T^{*k+1} R_m\, T^{k+1} f, f\rangle =$$

$$= \frac{2}{m} w_{m\,k+1} \sum_{l=1}^{j_k} \langle T^{*k+1} R_m\, T^{k+1} f, f\rangle \leqslant$$

(ii) $$\leq \frac{2}{m} w_{m\,k+1} \sum_{l=1}^{j_k} \langle T^{*k+1-l} R_m T^{k+1-l} f, f\rangle =$$

(iii) $$= \frac{2}{m} w_{m\,k+1}(\langle T^{*k+1-j_k} R_{m-1} T^{k+1-j_k} f, f\rangle - \langle T^{*k+1} R_{m-1} T^{k+1} f, f\rangle) =$$

$$= \frac{2}{m}\left[\frac{w_{m\,k+1}}{w_{m\,k-j_k}}(w_{m\,k-j_k}\langle T^{*k+1-j_k} R_{m-1} T^{k+1-j_k} f, f\rangle) - \right.$$

$$\left. - \frac{w_{m\,k+1}}{w_{mk}}(w_{mk}\langle T^{*k+1} R_{m-1} T^{k+1} f, f\rangle)\right] \to$$

(iv) $$\to 0 \text{ as } k \to \infty.$$

In the estimate above (i) follows from Lemma 2.10 (a). To see (ii) note that since $m + 1 \leq n$ and T is n-hypercontractive (by the first part of Lemma 2.11), $0 \leq R_{m+1} = R_m - T^* R_m T$. Thus $T^* R_m T \leq R_m$ and by induction, $T^{*k+1-l} R_m T^{k+1-l} \leq T^{*k+1} R_m T^{k+1}$. (iii) follows since $R_m = R_{m-1} - T^* R_{m-1} T$ so that the preceding series telescopes. Finally, (iv) follows from Lemma 2.10(b) and (c) and the inductive assumption. This concludes the proof of Lemma 2.11.

We now verify that (2.9) defines an isometry and thereby establish Theorem 1.12. Accordingly fix $T \in \mathscr{L}(\mathscr{H})$ with $(1 - yx)^n (T) \geq 0$ and $T^k \to 0$ strongly. We proceed by induction on the following assertion:

$$\mathrm{P}(m)\begin{cases}\text{The map which sends } f \text{ in } \mathscr{H} \text{ to} \\ \{((1 - yx)^m (T))^{1/2} T^k f\}_{k=0}^{\infty} \text{ in } \mathscr{H}_m \\ \text{is a well defined isometry.}\end{cases}$$

Evidently, since Lemma 2.11 guarantees that T is a contraction, the original Rovnyak-de Branges argument yields P(1). Now suppose P($m - 1$) holds where $1 \leq m \leq n$. We show that P(m) holds. For convenience set $R_k = (1 - yx)^k(T)$. Then

$$\sum_{r=0}^{k} w_{mr} \| R_m^{1/2} T^r f\|^2 = \sum_{r=0}^{k} w_{mr}\langle T^{*r} R_m T^r f, f\rangle =$$

$$= \sum_{r=0}^{k} w_{mr}(\langle T^{*r} R_{m-1} T^r f, f\rangle - \langle T^{*r+1} R_{m-1} T^{r+1} f, f\rangle) =$$

$$\text{(i)} \qquad = w_{m0}\langle R_{m-1}f, f\rangle + \sum_{k=1}^{k} (w_{mr} - w_{m\,r-1})\langle T^{*r}R_{m-1}T^{r}f, f\rangle -$$

$$- w_{mk}\langle T^{*k+1}R_{m-1}T^{k+1}f, f\rangle =$$

$$\text{(ii)} \qquad = \sum_{r=0}^{k} w_{m-1\,r}\langle T^{*r}R_{m-1}T^{r}f,f\rangle - w_{mk}\langle T^{*k+1}R_{m-1}T^{k+1}f,f\rangle,$$

where (i) is summation by parts and (ii) follows from Lemma 2.10 (d). Now notice that

$$\sum_{r=0}^{k} w_{m-1r}\langle T^{*r}R_{m-1}T^{r}f,f\rangle \to \|f\|^2$$

as $k \to \infty$ by the induction assumption. Also observe that

$$w_{mk}\langle T^{*k+1}R_{m-1}T^{k+1}f,f\rangle \to 0$$

by Lemma 2.11. Consequently,

$$\sum_{r=0}^{k} w_{mr}\|(R_{m-1})T^{r}f\|^2 \to \|f\|^2$$

as $k \to \infty$ and P(m) holds. This completes the proof of Theorem 1.12 via the Rovnyak-de Branges construction.

3. HYPERCONTRACTIONS AND SUBNORMALITY

In this section we examine the following remarkable fact implicitly contained in the work of Mary Embry.

THEOREM 3.1. *T is an n-hypercontraction for every n if and only if $\|T\| \leqslant 1$ and T is subnormal.*

The proof of Theorem 3.1 is a simple matter based on Embry's Theorem and Sz.-Nagy's operator valued version of the Hausdorff moment theorem (also see [6]). For the convenience of the reader we state these two theorems.

THEOREM. (Sz.-Nagy [8]). *A sequence of operators M_n satisfies $\sum_{k=0}^{n}\binom{n}{k}(-1)^k M_k M_{n-k} \geqslant 0$ for every n if and only if there exists an operator measure A on $[0, 1]$ such that $M_n = \int_0^1 t^n \mathrm{d}A$.*

THEOREM. (Embry [5]). *An operator T is subnormal if and only if there exists an operator measure A concentrated on an interval* $[0, a]$ *(a may be taken to be $\|T\|$) such that* $T^{*n}T^n = \int_0^a t^{2n} \mathrm{d}A$ *for all* $n \geq 0$.

Proof of Theorem 3.1. Since T is n-hypercontractive for all n, Sz.-Nagy's Theorem implies there is a measure such that $T^{*n}T^n = \int_0^1 t^n \mathrm{d}A(t)$. A simple change of variable and an application of Embry's Theorem yield that T is a contractive subnormal. Conversely, if $T \in \mathscr{L}(\mathscr{H})$ is a contractive subnormal with normal extension $N \in \mathscr{L}(\mathscr{K})$ ($\|N\| \leq 1$), then $(1 - yx)^n(T) = P_{\mathscr{H}}(1 - yx)^n(N)|\mathscr{H}$. Since $(1 - \bar{z}z)^n \geq 0$ for $z \in \mathbf{D}^-$ the functional calculus implies that $(1 - yx)^n(N) \geq 0$. Hence $(1 - yx)^n(T) \geq 0$ and T is an n-hypercontraction for every n.

One of the motivating points of the remainder of this section is that the proof we have just given of Theorem 3.1 is not easily generalized either to the case of sub-Jordan operators (equivalently, sub-n-normal) or to the study of n-tuples of subnormal operators. Nor, upon examination, is the proof natural from the point of view of C^*-algebra. The fundamental problem here is that the proof of Embry's Theorem relies heavily on the proof of the Halmos-Bram condition; the reason that the remarkable trickery of the latter proof works is hard to fathom in simple intuitive terms. To get a handle on the non C^*-algebraic content of Theorem 3.1 let us recall the following C^*-algebraic triviality from [1].

THEOREM 3.2. *T is subnormal with $\|T\| \leq 1$ if and only if $p(T) \geq 0$ whenever $p \in \mathscr{P}$ and $p(z, \bar{z}) \geq 0$ for all $z \in \mathbf{D}^-$.*

It is apparent that Theorem 3.1 and Theorem 3.2 combine to give the following theorem.

THEOREM 3.3. *The following conditions on an operator T are equivalent:*

(1) *$p(T) \geq 0$ whenever $p \in \mathscr{P}$ and $p(z, \bar{z}) \geq 0$ on $\mathbf{D}$;*

(2) *$p(T) \geq 0$ whenever $p \in \mathscr{P}$ and $p(x, y) = (1 - yx)^n$.*

An examination of Theorem 3.3 seems to suggest that Theorem 3.1 is the operator theoretic reflection of a two variable approximation theorem. We now state this theorem and show how to pass between it and Theorem 3.1.

THEOREM 3.4. *Let A^2 denote the set of continuous functions on $\mathbf{D}^- \times \mathbf{D}^-$ that are analytic on $\mathbf{D} \times \mathbf{D}$ with the topology of uniform convergence. Let $\mathscr{M} = \{f \in A^2 \mid f(z, \bar{z}) \geq 0$ for all $z \in \mathbf{D}\}$ and let $\mathscr{C}$ be the convex hull of the set of $f \in A^2$ of* *the form* $f(z, w) = (1 - zw)^n p(z)\overline{p(\bar{w})}$ *where n is a nonnegative integer and p is a polynomial in one variable. Then* $\mathscr{C}^- = \mathscr{M}$.

We first show how Theorem 3.1 implies Theorem 3.4. It is clear that $\mathscr{C} \subseteq \mathscr{M}$. Hence $\mathscr{C}^- \subseteq \mathscr{M}$. To show the reverse inclusion we argue by contradiction. Let $h_0 \in \mathscr{M}/\mathscr{C}^-$. Using one of the many variants of the Hahn-Banach Theorem, there exists a linear functional Λ on A^2 with

$$\operatorname{Re} \Lambda(h_0) < 0 \leqslant \operatorname{Re} \Lambda(f)$$

for all $f \in \mathscr{C}$. Extending Λ to $C(\mathbf{D}^- \times \mathbf{D}^-)$ and representing the extension with a measure yield a measure ν with support in $\mathbf{D}^- \times \mathbf{D}^-$ and satisfying

$$\text{(3.5)} \qquad \operatorname{Re} \int h_0 \, d\nu < 0 \leqslant \operatorname{Re} \int f d\nu \quad \text{for all } f \in \mathscr{C}.$$

Let $\varphi \colon \mathbf{D}^- \times \mathbf{D}^- \to \mathbf{D}^- \times \mathbf{D}^-$ be defined by $\varphi(z, w) = (\bar{w}, \bar{z})$ and define a measure μ by the formula $\mu = (1/2)(\nu + \overline{\nu \circ \varphi})$. If $h \in \mathscr{M}$, then in particular $h(z, \bar{z})$ is real valued. It follows that $g = h - \overline{h \circ \varphi}$ is holomorphic on $\mathbf{D} \times \mathbf{D}$ and vanishes on $\{(z, \bar{z}) \mid z \in \mathbf{D}\}$. Hence $g \equiv 0$ on $\mathbf{D} \times \mathbf{D}$ and we conclude $h = \overline{h \circ \varphi}$ for all $h \in \mathscr{M}$. Consequently, if $h \in \mathscr{M}$, then

$$\operatorname{Re} \int h d\nu = \frac{1}{2}\left(\int h d\nu + \int \bar{h} d\bar{\nu}\right) = \frac{1}{2} \int h d\nu + \frac{1}{2} \int \overline{h \circ \varphi} \, d\overline{\nu \circ \varphi} = \int h d\mu.$$

Combining this last observation with (3.5) yields

$$\text{(3.6)} \qquad \int h_0 d\mu < 0 \leqslant \int f d\mu \quad \text{for all } f \in \mathscr{C}.$$

We now define a Hilbert space. Let P denote the polynomials in one complex variable and let $N \subseteq P$ denote those polynomials p with the property that $\int p(z) \overline{p(\bar{w})} d\mu = 0$. Cauchy's inequality and (3.6) guarantee that N is a subspace. Define an inner product on P/N by the formula

$$[p + N, q + N] = \int p(z) \, \overline{q(\bar{w})} d\mu.$$

That $[\cdot, \cdot]$ actually is an inner product is guaranteed by (3.6). Let $H^2(\mu)$ denote the completion of P/N with respect to $[\cdot, \cdot]$. It is apparent that the polynomials are dense in $H^2(\mu)$. Densely define an operator M on polynomials $p(z)$ in $H^2(\mu)$ by the formula

$$M(p(z)) = zp(z).$$

Fix a polynomial p. Then

$$\|p\|^2 - \|Mp\|^2 = [p, p] - [Mp, Mp] = \int p(z)\overline{p(\overline{w})}\mathrm{d}\mu - \int zp(z)\overline{wp(\overline{w})}\mathrm{d}\mu =$$

$$= \int (1 - zw)p(z)\overline{p(\overline{w})}\mathrm{d}\mu \geqslant 0,$$

by (3.6). It follows that M extends to a contraction on $H^2(\mu)$ which we will denote by S_μ. In fact, S_μ is an n-hypercontraction for every n since

$$[(1 - yx)^n(S_\mu)p, p] = \left[\left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} S_\mu^{*k} S_\mu^k\right) p, p\right] =$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} [S_\mu^k p, S_\mu^k p] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int z^k p(z)\overline{\overline{w}^k p(\overline{w})}\mathrm{d}\mu =$$

$$= \int \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k w^k\right) p(z)\overline{p(\overline{w})}\mathrm{d}\mu = \int (1 - zw)^n p(z)\overline{p(\overline{w})}\mathrm{d}\mu \geqslant 0$$

by (3.6), and the polynomials are dense in $H^2(\mu)$.

It follows by Theorem 3.1 that S_μ is a contractive subnormal. Also, since the polynomials are dense in $H^2(\mu)$ the vector 1 is cyclic for S_μ. By Bram's Theorem there exists a nonnegative measure σ with support in $\mathbf{D}^-$ such that

$$[p(S_\mu)1, q(S_\mu)1] = \int p\overline{q}\mathrm{d}\sigma$$

for all polynomials p and q. In particular, for n and m arbitrary nonnegative integers we have

$$\int z^m w^n \mathrm{d}\mu = \int z^m \overline{z}^n \mathrm{d}\sigma.$$

Taking linear combinations of this last fact and noting that the polynomials in two variables are uniformly dense in A^2 we conclude that

$$\int h(z, w)\mathrm{d}\mu = \int h(z, \overline{z})\mathrm{d}\sigma$$

for all $h \in A^2$. In particular since $h_0(z, \bar{z}) \geqslant 0$ and $\sigma \geqslant 0$ we conclude that

$$\int h_0 \mathrm{d}\mu \geqslant 0.$$

But this contradicts (3.6). Hence $\mathscr{M} \subseteq \mathscr{C}^-$ and the proof of Theorem 3.4 is complete.

We now show how Theorem 3.1 can be made to follow from Theorem 3.4. Fix $T \in \mathscr{L}(\mathscr{H})$ with T n-hypercontractive for every n. Assume that $0 \leqslant r \leqslant 1$. By Lemma 1.9, rT is n-hypercontractive for every n. In particular,

$$(1 - yx)^n(rT) \geqslant 0$$

for every n. Hence we also have,

$$p(rT)^*(1 - yx)^n(rT)p(rT) \geqslant 0, \tag{3.7}$$

for all $n \geqslant 0$ and every polynomial p. For $f(z, w) = \sum_{m,n=0}^{\infty} a_{mn}z^m w^n \in A^2$ and R and S operators with $\sigma(R)$, $\sigma(S) \subseteq \mathbf{D}$ *define* $f(R, S)$ by the formula,

$$f(R, S) = \sum a_{mn}S^n R^m.$$

It is apparent that if $p \in \mathscr{P}$ and $\tilde{p} \in A^2$ is defined by $\tilde{p}(z, w) = p(z, w)$ and R is as above, then $p(R) = \tilde{p}(R, R^*)$. It also can be easily verified that if R and S are as above and f_k tends uniformly to f in A^2, then $f_n(R, S)$ tends to $f(R, S)$ in operator norm.

The assertion (3.7) now translates into the assertion:

$$f(rT, rT^*) \geqslant 0 \quad \text{for all } f \in \mathscr{C}. \tag{3.8}$$

But Theorem 3.4 now allows us to conclude that in fact 3.8 holds for all $f \in \mathscr{M}$. In particular, $p(rT) \geqslant 0$ for all $p \in \mathscr{P}$ that satisfy $p(z, \bar{z}) \geqslant 0$ for all $z \in \mathbf{D}$. But $p(T) = \lim_{r \to 1} p(rT)$. Thus $p(T) \geqslant 0$ whenever $p \in \mathscr{P}$ and $p(z, \bar{z}) \geqslant 0$. By Theorem 3.2 we conclude that T is subnormal. This concludes the proof of Theorem 3.1 using Theorem 3.4.

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