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AVERAGES OF PROJECTIONS

C. K. FONG and G. J. MURPHY

INTRODUCTION

Let H be an infinite-dimensional Hilbert space, and $\mathscr{B}(H)$ the algebra of all (bounded linear) operators on H. It is well known that the closed convex hull of the projections in $\mathscr{B}(H)$ is the set of all positive contractions (i.e. positive operators of norm less than or equal to one). However there appears to be no characterization of the convex hull of the projections known. This latter set cannot be closed, for as we shall see below, it cannot contain any infinite-rank compact operator. This was proved originally in [2].

In this paper we give a number of conditions, either necessary or sufficient, for an operator A to be a convex combination, or even an average, of projections. Our main result is that if A is a positive noncompact contraction whose essential spectrum $\sigma_e(A)$ contains a positive number $\lambda < 1$, then A is an average of projections.

Interest in this circle of ideas was initiated by Fillmore in [2], where he showed that every operator is a linear combination of projections (for an easy proof of this result see [4]), and that every positive invertible operator is a linear combination of projections with the coefficients positive. We considerably extend this result, giving also an estimate on the size of the coefficients.

TERMINOLOGY

All operators will be assumed to act on Hilbert spaces and all Hilbert spaces will be assumed to be separable. We say an operator A is a *rational convex combination* of projections if $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ with λ_i non-negative rational numbers and P_i projections, and $\lambda_1 + \ldots + \lambda_n = 1$. We say A is an *average of projections* if $A = (P_1 + \ldots + P_n)/n$ where $P_1, \ldots, P_n$ are projections.

The following two easily-proved lemmas will be very useful.

LEMMA 1. *If $A_1, \ldots, A_n$ are averages of projections, so is $A_1 \oplus \ldots \oplus A_n$.*

Proof. This follows immediately from the observation that if an operator is an average of n projections then it is an average also of nk projections for $k = 1, 2, 3, \ldots$.

LEMMA 2. *An operator is an average of projections if and only if it is a rational convex combination of projections.*

Proof. Suppose $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ where $\lambda_1, \ldots, \lambda_n$ are non-negative rational numbers, and $P_1, \ldots, P_n$ are projections, and $\lambda_1 + \ldots + \lambda_n = 1$. We write $\lambda_i = r_i/s_i$ where r_i, s_i are non-negative integers. Let $s = s_1 \ldots s_n$. Then $A = \frac{1}{s} \sum_{i=1}^{n} \left(\frac{sr_i}{s_i} \right) P_i$, and the coefficients sr_i/s_i are integers. Thus, we have a sum $\sum_{i=1}^{n} \frac{sr_i}{s_i} P_i$ involving $\sum_{i=1}^{n} \frac{sr_i}{s_i} = s$ projections. Thus, A is an average of projections.

THEOREM 3. *A positive contraction A with finite spectrum $\sigma(A)$ is a convex combination of projections. Moreover $\sigma(A)$ consists of rational numbers if and only if A is an average of commuting projections.*

Proof. It follows from the Spectral Theorem that we can write $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ with $1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$, and with $P_1, \ldots, P_n$ mutually orthogonal projections. Let $\mu_i = \lambda_i - \lambda_{i+1}$ $(1 \leqslant i \leqslant n-1)$ and $\mu_n = \lambda_n$. Then $\mu_i \geqslant 0$ and $\lambda_i = \mu_i + \ldots + \mu_n$ $(1 \leqslant i \leqslant n)$, and $\mu_1 + \ldots + \mu_n = \lambda_1 \leqslant 1$. Put $Q_i = P_1 + P_2 + \ldots + P_i$ $(1 \leqslant i \leqslant n)$. Then

$$A = \lambda_1 P_1 + \ldots + \lambda_n P_n = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \mu_j \right) P_i = \sum_{i=1}^{n} \mu_i Q_i .$$

Thus, A is a convex combination of the projections Q_i.

Suppose now $\sigma(A)$ consists of rational numbers. Then the λ_i, and therefore, the μ_i above are rational. Hence, by Lemma 2, A is an average of projections, which we can take to be commuting since the Q_i are commuting.

Conversely, if $A = (P_1 + \ldots + P_n)/n$ is an average of commuting projections $P_1, \ldots, P_n$, then it is easy to see that $\sigma(A)$ consists of rational numbers by considering the commutative C^*-algebra generated by 1 and $P_1, \ldots, P_n$.

COROLLARY 4. *Let K denote the ideal of all compact operators on an infinite dimensional Hilbert space. Then the convex hull of the projections in K consists precisely of the finite-rank positive contractions.*

Proof. This follows directly from the above theorem and the following observation of Fillmore [2]: If $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ is compact with $\lambda_1, \ldots, \lambda_n > 0$, and $P_1, \ldots, P_n$ projections then $P_1, \ldots, P_n$ are finite-rank (for then $P_i \leqslant \lambda_i^{-1} A$ so P_i is compact).

It was shown in [1] that if A is a positive contraction unitarily equivalent to $1-A$ then A is an average of two projections. Our next result is related to this.

THEOREM 5. *Let $A_1, \ldots, A_n$ be commuting operators on a Hilbert space H such that $A_1 + \ldots + A_n = 1$. Then any positive contraction A unitarily equivalent to $A_1 \oplus \ldots \oplus A_n$ is an average of 2^{n-1} unitarily equivalent projections.*

Proof. Without loss of generality assume $A = A_1 \oplus \ldots \oplus A_n$. Put $H^{(n)} = H \oplus \ldots \oplus H$ (n times). An easy computation shows that the operator P on $H^{(n)}$ defined by the operator matrix

$$P = \begin{pmatrix} \sqrt{A_1A_1} & \sqrt{A_1A_2} & \cdots & \sqrt{A_1A_n} \\ \vdots & & & \vdots \\ \sqrt{A_nA_1} & & \cdots & \sqrt{A_nA_n} \end{pmatrix}$$

is a projection.

Let S be the set of all n-tuples $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ such that $\delta_1 = 1$, $\delta_i = \pm 1$ $(2 \leqslant i \leqslant n)$. For each $\delta \in S$ the operator $U_\delta = \delta_1 1_H \oplus \ldots \oplus \delta_n 1_H$ is a symmetry on $H^{(n)}$. Put $P_\delta = U_\delta P U_\delta$. Then P_δ is a projection unitarily equivalent to P.

We write the operator $Y = \sum_{\delta \in S} P_\delta$ in operator matrix form $Y = (Y_{ij})_{ij}$ where

$$Y_{ij} = \sum_{\delta \in S} (P_\delta)_{ij} = \sum_{\delta \in S} \delta_i \sqrt{A_iA_j}\delta_j .$$

Thus if $i \neq j$, then

$$Y_{ij} = \sum_{\substack{\delta \in S \\ \delta_i = 1}} \delta_i\delta_j\sqrt{A_iA_j} + \sum_{\substack{\delta \in S \\ \delta_i = -1}} \delta_i\delta_j\sqrt{A_iA_j} = 0.$$

Also, for $i = j$,

$$Y_{ii} = \sum_{\delta \in S} \sqrt{A_i^2} = 2^{n-1}A_i;$$

thus

$$Y = 2^{n-1}(A_1 \oplus \ldots \oplus A_n) = 2^{n-1}A,$$

and so $A = \dfrac{1}{2^{n-1}} \sum_{\delta \in S} P_\delta$ is an average of 2^{n-1} unitarily equivalent projections.

COROLLARY 6. *If A is a positive operator of trace 1 on a finite-dimensional Hilbert space of dimension n, then A is an average of 2^{n-1} rank 1 projections.*

Proof. Diagonalize A, and conclude that A is unitarily equivalent to $\lambda_1 1_{\mathbf{C}} \oplus \ldots \oplus \lambda_n 1_{\mathbf{C}}$ on $\mathbf{C}^n$ with $\lambda_1 + \ldots + \lambda_n = \mathrm{trace}(A) = 1$. By the above theorem, A is an average of 2^{n-1} unitarily equivalent projections P_i. Taking traces shows each projection P_i has trace 1 and therefore rank 1.

REMARK. The converse of the above corollary is also true — it is obvious that an average of rank one projections on a finite-dimensional space is a positive operator of trace 1.

We shall need the following result, which was proven in [4] with the α_i real only, but an inspection of the proof shows one can make the α_i rational.

THEOREM 7. (Fong [4]). *There exists a universal positive constant V_0 such that every Hermitian operator A on an infinite-dimensional Hilbert space can be written in the form $A = \alpha_1 P_1 + \ldots + \alpha_n P_n$ with $\alpha_1, \ldots, \alpha_n$ rational numbers, $P_1, \ldots, P_n$ projections and $|\alpha_1| + \ldots + |\alpha_n| \leqslant V_0 \|A\|$.*

We shall also use the following consequence of the Spectral Theorem for the proof of our next lemma — given a Hermitian operator A and $\varepsilon > 0$, there exists a Hermitian operator A_0 with finite rational spectrum such that $\|A - A_0\| < \varepsilon$.

LEMMA 8. *If A is a positive contraction such that A and $1 - A$ have closed range then A is an average of projections.*

Proof. Clearly $A = 0 \oplus B \oplus 1$ where B is a positive contraction with $0, 1 \notin \sigma(B)$. Thus, using Lemma 1, we may assume, without losing generality, that $0, 1 \notin \sigma(A)$.

Choose ε_1, ε_2 rational such that $0 < \varepsilon_1 < \varepsilon_2 < 1/2$ and $\|A\|, \|1 - A\| < 1 - \varepsilon_2$. Let V_0 be the universal constant of Theorem 7. Choose $\delta > 0$, $\delta < \varepsilon_1/V_0$ and $\delta < \varepsilon_2 - \varepsilon_1$. By the remark preceding this lemma we can choose A_0 Hermitian with finite rational spectrum and $\|A - A_0\| \leqslant \delta$. Thus

$$\|A_0\| \leqslant \|A - A_0\| + \|A\| \leqslant \delta + 1 - \varepsilon_2 \leqslant 1 - \varepsilon_1.$$

Likewise

$$\|1 - A_0\| \leqslant \|1 - A\| + \|A - A_0\| \leqslant 1 - \varepsilon_2 + \delta \leqslant 1 - \varepsilon_1.$$

Thus $\sigma(A_0) \subseteq [\varepsilon_1, 1 - \varepsilon_1]$. Also $\|A - A_0\| \leqslant \varepsilon_1/V_0$.

Now $(A_0 - \varepsilon_1)(1 - 2\varepsilon_1)^{-1}$ is a positive contraction with finite rational spectrum. Hence, by Theorem 3, it is an average of projections, and so $A_0 - \varepsilon_1 = \lambda_1 P_1 + \ldots + \lambda_m P_m$ with $\lambda_1, \ldots, \lambda_m$ nonnegative rational numbers, $P_1, \ldots, P_m$ projections, and $\lambda_1 + \ldots + \lambda_m \leqslant 1 - 2\varepsilon_1$. Moreover $A - A_0$ is Hermitian, so by Theorem 7, $A - A_0 = \alpha_1 Q_1 + \ldots + \alpha_n Q_n$ with $\alpha_1, \ldots, \alpha_n$ rationals, $Q_1, \ldots, Q_n$ projections, and $|\alpha_1| + \ldots + |\alpha_n| \leqslant \delta V_0 < \varepsilon_1$.

Thus

$$A = A_0 + A - A_0 = \sum_{i=1}^{m} \lambda_i P_i + \varepsilon_1 + \sum_{j=1}^{n} \alpha_j Q_j =$$

$$= \sum_{i=1}^{m} \lambda_i P_i + \sum_{\alpha_j \geqslant 0} \alpha_j Q_j + \sum_{\alpha_j < 0} (-\alpha_j)(1 - Q_j) + (\varepsilon_1 + \sum_{\alpha_j < 0} \alpha_j) 1.$$

Now $\varepsilon_1 + \sum_{\alpha_j<0} \alpha_j \geqslant 0$, since $\varepsilon_1 \geqslant V_0\delta \geqslant \sum_{j=1}^{n} |\alpha_j|$. Also,

$$\sum_{i=1}^{m} \lambda_i + \sum_{\alpha_j\geqslant 0} \alpha_j + \sum_{\alpha_j<0} -\alpha_j + (\varepsilon_1 + \sum_{\alpha_j<0} \alpha_j) = \sum_{i=1}^{m} \lambda_i + \sum_{\alpha_j\geqslant 0} \alpha_j + \varepsilon_1 \leqslant$$

$$\leqslant 1 - 2\varepsilon_1 + \varepsilon_1 + \varepsilon_1 = 1.$$

Thus A is a rational convex combination of projections, and so an average of projections.

LEMMA 9. *A noncompact positive operator A such that $\|A\| < 1$ is an average of projections.*

Proof. We have $1 \notin \sigma(A)$, so $1 - A$ has closed range. If A has closed range we can use Lemma 8 to conclude that A is an average of projections.

Therefore we can assume that the range of A is not closed. Choose $\delta > 0$ such that $\|A\| \leqslant 1 - 2\delta$, and put $B = (1 - \delta)^{-1}A$. Then $\|B\| < 1 - \delta$. Also, since B has nonclosed range, 0 is not an isolated point of $\sigma(B)$. Now, using the noncompactness of B, we can choose $\varepsilon > 0$ such that $\varepsilon \leqslant \delta$ and that the spectral projections $E[0, \varepsilon]$ and $E(\varepsilon, 1)$ of B are both of infinite rank.

Thus by a unitary transformation we may assume $H = K \oplus K$ for some infinite dimensional Hilbert space K, and $B = B_1 \oplus B_2$ where $B_1, B_2 \in \mathscr{B}(K)$ and $0 \leqslant B_1 \leqslant \varepsilon \leqslant B_2 \leqslant 1 - \delta$.

Put $W = \sqrt{2}(1_K \oplus B_2^{-1/2}(1_K - B_1)^{1/2})$. Then $W \in \mathscr{B}(H)$, W is invertible, and $\|W^{-1}\| \leqslant (1/\sqrt{2}) \max\{1, (1 - \delta)^{1/2}(1 - \varepsilon)^{-1/2}\} = 1/\sqrt{2}$.

We define the projections P_1, P_2 in $\mathscr{B}(H)$ by

$$P_1 = \begin{pmatrix} B_1 & \sqrt{B_1 - B_1^2} \\ \sqrt{B_1 - B_1^2} & 1 - B_1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} B_1 & -\sqrt{B_1 - B_1^2} \\ -\sqrt{B_1 - B_1^2} & 1 - B_1 \end{pmatrix}.$$

Then $P_1 + P_2 = 2\,(B_1 \oplus (1 - B_1))$, and so $(P_1 + P_2)^{1/2} = \sqrt{2}(B_1^{1/2} \oplus (1 - B_1)^{1/2}) = B^{1/2}W$. Hence $P_1 + P_2 = W^*BW$. Thus $B = C_1 + C_2$ where $C_j = W^{*-1}P_jW^{-1}$ are positive operators with closed range and $\|C_j\| \leqslant \|W^{-1}\|^2 \leqslant 1/2$.

Hence $2(1 - \delta)C_j$ satisfy the hypothesis of Lemma 8, implying they are averages of projections. But then $A = (1 - \delta)B = 1/2(2(1 - \delta)C_1 + 2(1 - \delta)C_2)$ is also an average of projections.

THEOREM 10. *Let A be a positive contraction whose essential spectrum $\sigma_e(A)$ intersects the open interval $(0, 1)$. Then A is an average of projections.*

Proof. Consider first the case where $\sigma_e(A)$ contains at least two points $\lambda_1 < \lambda_2$ of $(0, 1)$. Choose λ such that $\lambda_1 < \lambda < \lambda_2$. By the Spectral Theorem we can write

$A = A_1 \oplus A_2$ with $0 \leqslant A_1 \leqslant \lambda \leqslant A_2 \leqslant 1$ and $\lambda_j \in \sigma_e(A_j)$, $j = 1,2$. Now A_1 and $1 - A_2$ are noncompact operators with $\|A_1\|$, $\|1 - A_2\| < 1$. Hence by Lemma 9, A_1 and $1-A_2$ are averages of projections. Thus by Lemma 1, A is an average of projections.

Now consider the case where $\sigma_e(A)$ contains a unique point λ of (0, 1). Then there is an orthonormal sequence e_n of H, and a sequence λ_n of (0, 1) such that $Ae_n = \lambda_n e_n$ and $\lim_{n\to\infty} \lambda_n = \lambda$. Let K be the subspace of H spanned by the e_n (n even) and by the spectral subspace $E[0, \lambda/2]H$ of A. Let $K^\perp = H \ominus K$. Again, Lemma 9 can be applied to $A|K$ and $(1 - A)|K^\perp$, and again the conclusion that A is an average of projections follows using Lemma 1.

THEOREM 11. *A positive operator A is a positive combination of projections (*i.e. *$A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ with $\lambda_1, \ldots, \lambda_n \geqslant 0$, $P_1, \ldots, P_n$ projections) if and only if A is not an infinite-rank compact operator. Moreover, if A is a positive combination of projections and $\varepsilon > 0$ is given, we can choose an expression $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ with $\lambda_1 + \ldots + \lambda_n \leqslant \|A\| + \varepsilon$.*

Proof. If A is not an infinite-rank compact operator, neither is $B = (\|A\| + \varepsilon)^{-1}A$, and we have $\|B\| < 1$. If B is finite-rank, then B is a convex combination of projections by Theorem 3. If B is not compact, B is a convex combination of projections by Lemma 9. In either case $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ is a positive combination of projections with $\lambda_1 + \ldots + \lambda_n \leqslant \|A\| + \varepsilon$.

Conversely if A is a positive combination of projections, then we saw earlier (in the proof of Corollary 4) that A compact implies the projections, and hence A, are finite-rank.

THEOREM 12. *Let $\mathscr{C}$ be the Calkin algebra of an infinite-dimensional Hilbert space and $0 \leqslant a \leqslant 1$ in $\mathscr{C}$. Then a is an average of projections in $\mathscr{C}$.*

Proof. Choose an operator A in the coset a such that $0 \leqslant A \leqslant 1$. Thus $\sigma_e(A) = \sigma_{\mathscr{C}}(a)$, so if a is not a projection, $\sigma_e(A)$ contains a point of (0, 1). Hence by Theorem 10, A is an average of projections, which clearly implies a is an average of projections in $\mathscr{C}$.

CONCLUDING REMARKS. Fillmore [1] showed every *invertible* positive operator is a positive combination of projections, but gave no bound on the coefficients. Some details of the proof of Lemma 9 are modelled on the proof of a result (Theorem 2.6) in Fillmore-Williams [3].

Returning to the basic question raised in this paper — when is a positive contraction A a convex combination of projections — we have answered every case except where $\sigma_e(A) \subseteq \{0, 1\}$ and $\sigma(A)$ is infinite. But (assuming $\sigma(A)$ infinite) $\sigma_e(A) = \{0\}$ or $\{1\}$ implies A or $1 - A$ is compact, so A cannot be a convex combination of projections, by Theorem 11. Thus the remaining possibilities are $\sigma_e(A) = \{0,1\}$ and 0 or 1 (or both) are limit points of $\sigma(A)$. We shall show now that if not both 0, 1 are limit points, then A is not a convex combination of projections. (This leaves as the only open question the case where $\sigma_e(A) = \{0,1\}$ and 0, 1 are limit points of $\sigma(A)$). So suppose $\sigma_e(A) = \{0,1\}$, and exactly one of 0, 1 is a limit point of $\sigma(A)$. We may assume without loss of generality (by considering $1 - A$ if necessary) that 0 is a limit point, and 1 an isolated point of $\sigma(A)$. Write $A = K \oplus 1$ with K a compact operator of infinite rank. Now if $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ is a convex combination of projections, and $P = 0 \oplus 1$, then $K \oplus 0 = \sum_{j=1}^{n} \lambda_j (P_j - P)$. We show $P_j - P$ are projections by showing range$(P) \subseteq$ range(P_j) $(1 \leqslant j \leqslant n)$. For if $x \in$ range(P) and $\|x\| = 1$, then $\langle P_j x, x \rangle \leqslant 1$ and $1 = \langle x, x \rangle = \langle Ax, x \rangle = \sum_{j=1}^{n} \lambda_j \langle P_j x, x \rangle \leqslant \sum_{j=1}^{n} \lambda_j = 1$. Thus $\langle P_j x, x \rangle = 1$, hence $x \in$ range(P_j). Thus we have an infinite-rank compact operator $K \oplus 0$ written as a convex combination of projections, contradicting Theorem 11. Hence A is not a convex combination of projections.

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C. K. FONG and G. J. MURPHY
Department of Mathematics,
University of Toronto,
Toronto, M5S 1A1,
Ontario,
Canada.

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