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SOME APPLICATIONS OF A TECHNIQUE FOR CONSTRUCTING REFLEXIVE OPERATOR ALGEBRAS

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In [4] W. Arveson showed that if $\mathscr{A}$ is a nest algebra, then the distance from an arbitrary operator T in $L(H)$ to $\mathscr{A}$ is given by the formula

$$\mathrm{d}(T, \mathscr{A}) = \sup\{\|P^{\perp}TP\| : P \in \operatorname{lat}\mathscr{A}\}.$$

(E. C. Lance obtained this result independently [15].) This distance formula has proven to be very useful in investigating problems involving compact perturbations and similarity theory for nests [1, 2, 4, 11, 12, 15, 17]. Certain other reflexive algebras have been shown to possess the property that there exists a constant K such that

$$\mathrm{d}(T, \mathscr{A}) \leqslant K\sup\{\|P^{\perp}TP\| : P \in \operatorname{lat}\mathscr{A}\}$$

for all T in $L(H)$ [7, 8, 9, 10, 13]. Simple examples show that this constant need not be 1. K. Davidson asked in [10] whether every reflexive algebra has such a distance estimate for some constant K. (This question was also posed in [13, 15, 16].) In [5], W. Arveson conjectured that the answer to this question is no. In this paper we verify this conjecture by producing an explicit example of a reflexive algebra that does not satisfy a distance estimate for any K. We also give examples of points of discontinuity of the maps $\mathscr{L} \to \operatorname{alg}\mathscr{L}$ and $\mathscr{A} \to \operatorname{lat}\mathscr{A}$, where the topology for both lattices and algebras is that induced by the Hausdorff metric.

We would like to thank W. Arveson for several helpful conversations concerning this paper, and in particular for pointing out to us that if $\mathscr{S}$ is a reflexive subspace of $L(H)$, then

$$\begin{pmatrix} \mathbf{C} & \mathscr{S} \\ 0 & \mathbf{C} \end{pmatrix}$$

is a reflexive subalgebra of $L(H \oplus H)$. We use this observation to construct reflexive algebras with certain properties by first constructing reflexive subspaces with analogous properties. The research for this paper was done while both authors were visiting the University of California at Berkeley.

1. REFLEXIVE AND HYPERREFLEXIVE SUBSPACES OF $L(H)$

In [18] Loginov and Sulman introduced the following notion of a reflexive (linear) subspace of $L(H)$.

DEFINITION. A subspace $\mathscr{S}$ of $L(H)$ is *reflexive* if whenever $T \in L(H)$ satisfies the condition that $Tx \in [\mathscr{S}x]$ for all $x \in H$, then T is in $\mathscr{S}$. (Here $[\cdot]$ denotes normed closed linear span.)

It is easily verified that reflexive subspaces are weakly closed, and that a unital algebra $\mathscr{A}$ is reflexive as a subspace if and only if it is reflexive as an algebra (i.e. $\mathscr{A} = \operatorname{alg}\operatorname{lat}\mathscr{A}$, where $\operatorname{lat}\mathscr{A}$ denotes the lattice of invariant subspaces (or projections) for $\mathscr{A}$, and $\operatorname{alg}\operatorname{lat}\mathscr{A}$ is the algebra of bounded linear operators on H leaving every element of $\operatorname{lat}\mathscr{A}$ invariant).

PROPOSITION 1. *Let $\mathscr{S}$ be a subspace of $L(H)$, and let $\mathscr{A}$ be the (algebra) of all operators on $H \oplus H$ which admit a matrix representation*

$$\begin{pmatrix} \lambda I & S \\ 0 & \mu I \end{pmatrix}$$

for $\lambda, \mu \in \mathbf{C}$, $S \in \mathscr{S}$. Then $\mathscr{A}$ is reflexive if and only if $\mathscr{S}$ is reflexive. Moreover, a subspace of $H \oplus H$ is in lat $\mathscr{A}$ *if and only if it is of the form $F \oplus E$, where E and F are closed subspaces of H such that $\mathscr{S}E \subseteq F$.*

Proof. Let $K \in \operatorname{lat}\mathscr{A}$. Since $\mathscr{A}$ contains $\mathbf{C}I \oplus \mathbf{C}I$, *$K$ has the form $F \oplus E$*, and since $\mathscr{A}$ contains all operators

$$\tilde{S} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

for $S \in \mathscr{S}$, we have $\mathscr{S}E \subseteq F$. Conversely, if $\mathscr{S}E \subseteq F$ it is clear that $\mathscr{A}(F \oplus E) \subseteq$ $\subseteq (F \oplus E)$. The second assertion is thus verified.

Now suppose $\mathscr{A}$ is reflexive. If $S \in L(H)$ satisfies $Sx \in [\mathscr{S}x]$, $x \in H$, then $SE \subseteq F$ whenever $\mathscr{S}E \subseteq F$, so $\tilde{S}(F \oplus E) \subseteq F \oplus E$ whenever $\mathscr{S}F \subseteq E$, hence $\tilde{S} \in \mathscr{A}$, so $S \in \mathscr{S}$. Hence $\mathscr{S}$ is reflexive.

Finally, suppose $\mathscr{S}$ is reflexive and $A \in \operatorname{alg}\operatorname{lat}\mathscr{A}$. Since $\mathbf{C}x \oplus 0$ and $[\mathscr{S}x] \oplus \mathbf{C}x$ are in $\operatorname{lat}\mathscr{A}$ for all x in H, A is of the form

$$\begin{pmatrix} \lambda I & T \\ 0 & \mu I \end{pmatrix}$$

where $\lambda, \mu \in \mathbf{C}$ and $Tx \in [\mathscr{S}x]$ for all x in H. But $\mathscr{S}$ is reflexive hence $T \in \mathscr{S}$, so $A \in \mathscr{A}$. Hence $\mathscr{A}$ is reflexive. ▨

We will require generalizations of certain elements of the duality theory for unital algebras developed in [16] and [5]. Let $\mathscr{L}_*$ denote the space of trace class operators. Then $L(H)$ can be identified with $(\mathscr{L}_*)^*$ via the pairing $(f, T) = \mathrm{Tr}(Tf)$, $f \in \mathscr{L}_*$. The σ-weak (ultraweak) topology on $L(H)$ is just the weak-* topology under this identification. If $\mathscr{S}$ is a σ-weakly closed linear subspace of $L(H)$, the *pre-annihilator* of $\mathscr{S}$ is the set $\mathscr{S}_\perp = \{f \in \mathscr{L}_*: \mathrm{Tr}(Sf)=0,\ S \in \mathscr{S}\}$. (The notation annih$(\mathscr{S})$ was used for this set in [16].) The trace class norm will be denoted by $\|\cdot\|_1$. Lemma 2 in [16] states that a unital σ-weakly closed algebra $\mathscr{A}$ is reflexive if and only if $\mathscr{A}_\perp$ is the closed span of its rank-1 elements. (This fact, in somewhat modified form, had been independently observed earlier by E. Azoff [6], and also by K. Tsuji [19].) The same result is true for subspaces of $L(H)$, an observation that has also been made by E. Azoff and by J. Erdos. This yields in a sense a "machine" for constructing examples of reflexive subspaces, and via Proposition 1, reflexive algebras of a particularly special but potentially pathological form. We use this technique in the next section to construct an example of a reflexive algebra without a distance estimate.

LEMMA 2. *Let $\mathscr{S}$ be a σ-weakly closed subspace of $L(H)$. Then $\mathscr{S}$ is reflexive if and only if $\mathscr{S}_\perp$ is the $\|\cdot\|_1$-closed linear span of its rank-1 elements.*

Proof. For x, y in H, let $x \otimes y$ denote the rank-1 operator $u \to (u, x)y$. Since $\mathrm{Tr}(T(x \otimes y)) = (Ty, x)$, $x, y \in H$, $T \in L(H)$, $x \otimes y \in \mathscr{S}_\perp$ if and only if $x \in [\mathscr{S}y]^\perp$. Hence $\mathrm{Tr}(Tf) = 0$ for all rank-1 operators $f \in \mathscr{S}_\perp$ if and only if $Ty \in [\mathscr{S}y]$ for all $y \in H$. Thus $\mathscr{S}_\perp$ is the $\|\cdot\|_1$-closed linear span of its rank-1 elements if and only if $T \in \mathscr{S}$ whenever $Ty \in [\mathscr{S}y]$ for all $y \in H$ if and only if $\mathscr{S}$ is reflexive. ▨

A routine modification of the proof of Proposition 12 in [16] yields the following result.

PROPOSITION 3. *Let $\mathscr{S}$ be a reflexive subspace. Let $\mathscr{C}_1$ denote the $\|\cdot\|_1$-closed convex hull of the operators in the closed unit ball of $\mathscr{S}_\perp$ with rank $\leqslant 1$. Then there exists a constant K such that for all $T \in L(H)$*

$$(1) \qquad \mathrm{d}(T, \mathscr{S}) \leqslant K \sup\{\|Q^\perp TP\| : Q\mathscr{S}P = \mathscr{S}P,\ P, Q \text{ projections in } L(H)\}$$

if and only if $\mathscr{C}_1$ has nonempty relative interior in $\mathscr{S}_\perp$. In this case the smallest constant K for which this estimate holds is $1/R$, where R is the largest radius such that $\{f \in \mathscr{S}_\perp: \|f\|_1 \leqslant R\} \subseteq \mathscr{C}_1$.

DEFINITION. A norm closed subspace $\mathscr{S}$ of $L(H)$ is said to be *hyperreflexive* if (1) holds for some K. The smallest K for which (1) holds is called the *distance constant* for $\mathscr{S}$.

Note that hyperreflexive subspaces are reflexive. For if $Tx \in [\mathscr{S}x]$ for all x in H, then $Q^\perp TP = 0$ whenever $Q\mathscr{S}P = \mathscr{S}P$, and so by (1) we have $T \in \mathscr{S}$.

In [5], Arveson defines a norm closed unital subalgebra $\mathscr{A}$ of $L(H)$ to be hyperreflexive if it has the following property:

for every sequence $B_n \in L(H)$ satisfying

$$\sup_{P \in \operatorname{lat} \mathscr{A}} \|P^{\perp} B_n P\| \to 0 \text{ as } n \to \infty, \tag{2}$$

there is a sequence A_n in $\mathscr{A}$ such that

$$\|A_n - B_n\| \to 0 \quad \text{as } n \to \infty.$$

This condition is equivalent [5] to the existence of a constant K such that

$$\mathrm{d}(T, \mathscr{A}) \leqslant K \sup \{\|P^{\perp} T P\| : P \in \operatorname{lat} \mathscr{A}\}. \tag{3}$$

Moreover, if $\mathscr{A}$ is a unital algebra, then it is easily verified that (3) is equivalent to (1) (with $\mathscr{S} = \mathscr{A}$). Finally, it is also easily checked that $\mathscr{S}$ is hyperreflexive if and only if the analogue of (2) holds, with $\sup\{\|P^{\perp} B_n P\| : P \in \operatorname{lat} \mathscr{A}\}$ replaced by $\sup\{\|Q^{\perp} B_n P\| : Q\mathscr{S}P = \mathscr{S}P,\ P, Q \text{ projections in } L(H)\}$.

PROPOSITION 4. *Let $\mathscr{S}$ be a norm closed subspace of $L(H)$ and let* $\mathscr{A} = \begin{pmatrix} \mathbf{C} & \mathscr{S} \\ 0 & \mathbf{C} \end{pmatrix}$. *If $\mathscr{A}$ is hyperreflexive with distance constant K, then $\mathscr{S}$ is hyperreflexive with distance constant $\leqslant K$.*

Proof. Let $T \in L(H)$, and let

$$\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

Then $\mathrm{d}(T, \mathscr{S}) = \mathrm{d}(\tilde{T}, \mathscr{A}) \leqslant K \sup\{\|R^{\perp} \tilde{T} R\| : R \in \operatorname{lat} \mathscr{A}\}$. By Proposition 1, $R \in \operatorname{lat} \mathscr{A}$ if and only if there are projections P and Q in $L(H)$ with $R = Q \oplus P$ and $Q\mathscr{S}P = \mathscr{S}P$. Then $\|R^{\perp} \tilde{T} R\| = \|Q^{\perp} T P\|$, so $\sup\{\|R^{\perp} \tilde{T} R\| : R \in \operatorname{lat} \mathscr{A}\} = \sup\{\|Q^{\perp} T P\| : Q\mathscr{S}P = \mathscr{S}P\}$, and the result follows. ▨

We remark that in [5] W. Arveson has determined a stronger characterization of hyperreflexivity than used in this paper. While computations in this study do not require this, the result is quite intuitive and will be used in further work.

We will term a subspace lattice hyperreflexive if it is the invariant subspace (or projection) lattice of a unital hyperreflexive algebra.

Prior to [16], a duality approach to problems concerning distance estimates was used by E. C. Lance [15] for the class of nest algebras, and by E. Christensen [9] in the study of derivations on C^*-algebras. In particular, parts (5) and (6) of [9, Theorem 3.1] really contain the duality characterization of hyperreflexivity for the class of von Neumann algebras.

2. AN EXAMPLE

Let H_2 be a 2-dimensional Hilbert space with orthonormal basis $\{e_1, e_2\}$. Fix $0 < \varepsilon < 1/3$. Let $u_1 = e_1$, $u_2 = \frac{e_1 + \varepsilon e_2}{\sqrt{1+\varepsilon^2}}$, $g_i = u_i \otimes u_i$, $i = 1,2$. Let $\mathscr{E}$ be the linear span of $\{g_1, g_2\}$, and let $\mathscr{S} = \mathscr{E}^\perp = \{A \in L(H_2) : \mathrm{Tr}(Af) = 0,\ f \in \mathscr{E}\}$. Then $\mathscr{S}$ is a reflexive subspace of $L(H_2)$ with $\mathscr{S}_\perp = \mathscr{E}$. Since $\mathscr{S}$ is acting on a finite dimensional space, it is hyperreflexive. We will show that its distance constant is larger than $1/3\varepsilon$. The operators g_1 and $(1+\varepsilon^2)g_2$, and the space $\mathscr{E}$, have respective matrix representations

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon^2 \end{pmatrix}, \quad \left\{ \begin{pmatrix} \lambda + \mu & \mu\varepsilon \\ \mu\varepsilon & \mu\varepsilon^2 \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\}.$$

So $A = (a_{ij}) \in \mathscr{S}$ if and only if for every $\lambda, \mu \in \mathbf{C}$ we have $0 = (\lambda + \mu)a_{11} + \mu\varepsilon a_{21} + \mu\varepsilon a_{12} + \mu\varepsilon^2 a_{22}$, hence if and only if $a_{11} = 0$ and $a_{12} + a_{21} + \varepsilon a_{22} = 0$. Thus $\mathscr{S}$ has the form in the next lemma.

LEMMA 5. *Let* $0 < \varepsilon < 1/3$, *and let*

$$\mathscr{S}_\varepsilon = \left\{ \begin{pmatrix} 0 & \lambda \\ \mu & \dfrac{-(\lambda + \mu)}{\varepsilon} \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\}.$$

Then $\mathscr{S}_\varepsilon$ *is hyperreflexive with distance constant larger than* $1/3\varepsilon$.

Proof. Fix $\varepsilon > 0$, and let $\mathscr{S} = \mathscr{S}_\varepsilon$. Let $\mathscr{C}_1$ be the $\|\cdot\|_1$-closed convex hull of the operators in the unit ball of $\mathscr{S}_\perp$ with rank $\leqslant 1$, and let R be the largest radius with $\{f \in \mathscr{S}_\perp : \|f\|_1 \leqslant R\} \subseteq \mathscr{C}_1$. By Proposition 3 it suffices to show that $R < 3\varepsilon$. From the discussion above, we know that $\mathscr{S}_\perp = \mathrm{span}\{g_1, g_2\}$, $g_1 = e_1 \otimes e_1$, $g_2 = \frac{1}{1+\varepsilon^2}(e_1 + \varepsilon e_2) \otimes (e_1 + \varepsilon e_2)$. Now let $f = \alpha_1 g_1 + \alpha_2 g_2$ be an arbitrary operator in $\mathscr{S}_\perp$ of rank $\leqslant 1$. Then $fe_1 = \alpha_1 u_1 + \frac{\alpha_2}{\sqrt{1+\varepsilon^2}} u_2$, while $fe_2 = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \alpha_2 u_2$. But u_1 and u_2 are linearly independent, and rank $f \leqslant 1$, so either α_1 or α_2 is zero. So the only rank-1 operators in $\mathscr{E}$ are scalar multiples of g_1 or g_2. Let $f \in C_1$. Since $\mathscr{E}$ is finite dimensional, C_1 is just the convex hull of the operators in the unit ball of $\mathscr{E}$ with rank $\leqslant 1$, and so $f = \lambda_1 g_1 + \lambda_2 g_2$, where $\lambda_1, \lambda_2 \in \mathbf{C}$ with $|\lambda_1| + |\lambda_2| \leqslant 1$. Hence $fe_2 = \lambda_2 g_2 e_2 = \lambda_2 (e_2, u_2) u_2 = \lambda_2 \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} u_2$, and $\|fe_2\| \leqslant \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} < \varepsilon$. Now let $h = \frac{(1+\varepsilon^2)g_2 - g_1}{\|(1+\varepsilon^2)g_2 - g_1\|_1}$. Then $h \in \mathscr{S}_\perp$ and $\|Rh\|_1 = R$. By the definition of R,

$Rh \in \mathscr{C}_1$, so $\|Rhe_2\| < \varepsilon$. On the other hand,

$$\|(1+\varepsilon^2)g_2 - g_1\|_1 he_2 = (1+\varepsilon^2)g_2e_2 = (1+\varepsilon^2)\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}u_2 = (\sqrt{1+\varepsilon^2})\varepsilon u_2$$

and

$$\|(1+\varepsilon^2)g_2 - g_1\|_1 = \|\varepsilon(e_2\otimes e_1) + \varepsilon(e_1\otimes e_2) + \varepsilon^2(e_2\otimes e_2)\|_1 \leqslant 3\varepsilon.$$

Hence $\|he_2\| \geqslant \dfrac{(\sqrt{1+\varepsilon^2})\varepsilon}{3\varepsilon} > 1/3$. So $\varepsilon > \|Rhe_2\| = R\|he_2\| > R/3$. This shows $R < 3\varepsilon$, as desired. ▨

THEOREM 6. *There is a reflexive unital algebra acting on infinite dimensional separable Hilbert space which is not hyperreflexive.*

Proof. Let $\varepsilon_n = 1/n$, $n \geqslant 4$, let $\mathscr{S}_{\varepsilon_n}$ be the subspace defined above for $\varepsilon = \varepsilon_n$, and let $\mathscr{A}_n$ be the algebra described in the statement of Proposition 1 for $\mathscr{S} = \mathscr{S}_{\varepsilon_n}$. Then $\mathscr{A}_n$ is reflexive with distance constant $K_n > n/3$ by Propositions 1 and 4, and Lemma 5. Let H_n denote the Hilbert space on which $\mathscr{A}_n$ acts, and let $\mathscr{A} = \bigoplus_{n=4}^{\infty}\mathscr{A}_n$ acting on $H = \bigoplus_{n=4}^{\infty} H_n$. Then $\mathscr{A}$ is reflexive because each $\mathscr{A}_n$ is reflexive. If $T_n \in L(H_n)$ with $d(T_n, \mathscr{A}_n) > (n/3)\sup\{\|P^{\perp}T_nP\| : P \in \operatorname{lat}\mathscr{A}_n\}$, write $\hat{T}_n$ for the infinite direct sum $\bigoplus_{i=4}^{\infty} S_i$ where $S_n = T_n$ and $S_i = 0$, $i \neq n$. Every projection in $\operatorname{lat}\mathscr{A}$ has the form $P = \bigoplus_{i=4}^{\infty} P_i$, $P_i \in \operatorname{lat}\mathscr{A}_i$. It follows that $d(\hat{T}_n, \mathscr{A}) > n/3 \cdot \sup\{\|P^{\perp}\hat{T}_nP\| : P \in \operatorname{lat}\mathscr{A}\}$. So $\mathscr{A}$ has no finite distance constant. ▨

The above construction can be modified in several ways:

(i) If in place of the algebras $\mathscr{A}_n$ a direct sum of the subspaces $\mathscr{S}_{\varepsilon_n}$ is taken one obtains a reflexive selfadjoint nonhyperreflexive subspace of $L(H)$. (This is of course not an algebra.)

(ii) If in the beginning of this section we replaced g_1, g_2 with *any* pair of sufficiently close rank-1, norm-1 operators for which no nontrivial linear combination is rank-1 the balance of the proof will carry through nearly intact yielding a subspace $\mathscr{S}$ with large distance constant. The pair we choose seemed to best illustrate our techniques.

(iii) The subspace $\mathscr{S}_\varepsilon$ of Lemma 5 can be modified to yield an *algebra* acting on H_2 with distance constant greater than $1/3\varepsilon$. We thank K. Davidson for pointing this out to us. This can be obtained by replacing g_1, g_2 with $g_1 = e_2 \otimes e_1$, $g_2 = \left(\dfrac{e_2 - \varepsilon e_1}{\sqrt{1+\varepsilon^2}}\right) \otimes \left(\dfrac{e_1 + \varepsilon e_2}{\sqrt{1+\varepsilon^2}}\right)$.

The resultant subspace $\mathscr{S}_\varepsilon$ becomes

$$\mathscr{S}_\varepsilon = \left\{ \begin{pmatrix} \lambda & \dfrac{\mu - \lambda}{\varepsilon} \\ 0 & \mu \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\},$$

and this is indeed an algebra. A direct verification that $K > 1/3\varepsilon$ via $\operatorname{lat}(\mathscr{S}_\varepsilon)$ is now also possible. Thus if H is a Hilbert space of dimension greater than 1 there is no universal distance constant valid for all reflexive subalgebras of $L(H)$.

3. THE MAPS $\mathscr{L} \to \operatorname{alg} \mathscr{L}$ and $\mathscr{A} \to \operatorname{lat} \mathscr{A}$

The Hausdorff metric on the space of all bounded closed subsets of a given metric space is defined by

$$\mathrm{d}(E, F) = \max\{\sup_{x\in E} \inf_{y\in F} \mathrm{d}(x, y),\ \sup_{y\in F} \inf_{x\in E} \mathrm{d}(x, y)\}.$$

The distance between two norm closed subspaces of $L(H)$ is taken to be the Hausdorff distance between their unit balls, and the distance between two closed sets of projections is taken to be simply the Hausdorff distance.

In [14] R. V. Kadison and D. Kastler initiated perturbation theory for von Neumann algebras, and in particular proved that the type of a factor is invariant under small perturbations. Since then much work has been done concerning properties of algebras preserved under such small perturbations in the Hausdorff metric. Somewhat recently, as part of a perturbation theory for nests and nest algebras, Lance [15] showed that within these classes both maps $\mathscr{L} \to \operatorname{alg} \mathscr{L}$ and $\mathscr{A} \to \operatorname{lat} \mathscr{A}$ are uniformly continuous. The purpose of this section is to give some examples of the type of pathology one may encounter (even in finite dimensions) in such a theory for more general reflexive algebras.

We first wish to note a connection with hyperreflexivity. The idea in the next proposition has been used in the class of nest algebras [1, 11, 15] and in the class of von Neumann algebras [7, 8, 9].

PROPOSITION 7. *Let $\mathscr{A}$ and $\mathscr{B}$ be hyperreflexive algebras with distance constants $\leqslant K$ acting on a Hilbert space H. Then*

$$\mathrm{d}(\mathscr{A}, \mathscr{B}) \leqslant 4K\, \mathrm{d}(\operatorname{lat}\mathscr{A}, \operatorname{lat}\mathscr{B}).$$

Proof. Let $\alpha = \mathrm{d}(\operatorname{lat}\mathscr{A}, \operatorname{lat}\mathscr{B})$. Let $A \in \mathscr{A}$, $\|A\| \leqslant 1$, and let $P \in \operatorname{lat}\mathscr{B}$. If $Q \in \operatorname{lat} \mathscr{A}$ then $Q^\perp A Q = 0$ so we have $P^\perp A P = Q^\perp A(P - Q) + (Q - P)AP$. Hence

$\|P^\perp AP\| \leqslant 2\|P - Q\|$. Taking the infimum over $Q \in \operatorname{lat}\mathscr{A}$ shows that $\|P^\perp AP\| \leqslant \leqslant 2\alpha$, so $d(A, \mathscr{B}) \leqslant 2K\alpha$ by hyperreflexivity. It follows that the distance from A to the unit ball of $\mathscr{B}$ is $\leqslant 4K\alpha$. Now reverse the argument, obtaining $d(\mathscr{A}, \mathscr{B}) \leqslant \leqslant 4K\alpha$. ▨

COROLLARY 8. *If $\mathscr{R}$ is a class of hyperreflexive lattices acting on a Hilbert space H for which countable direct sums of lattices in $\mathscr{R}$ are also hyperreflexive, then the map $\mathscr{L} \to \operatorname{alg}\mathscr{L}$ is uniformly continuous on $\mathscr{R}$.*

Proof. An adaptation of the proof of Theorem 6 shows that the set of distance constants for the class must be bounded. Proposition 7 then implies uniform continuity. ▨

The above suggests that the examples of Section 2 may be used to reveal the discontinuity of the $\mathscr{L} \to \operatorname{alg}\mathscr{L}$ map.

EXAMPLE 9. ($\mathscr{L} \to \operatorname{alg}\mathscr{L}$). Let $\mathscr{S} = \left\{\begin{pmatrix} 0 & \lambda \\ \mu & \delta \end{pmatrix} : \mu, \lambda, \delta \in \mathbf{C}\right\}$ and let $\mathscr{S}_\varepsilon$ be the subspace of Lemma 5. The preannihilator of $\mathscr{S}$ is $\left\{\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} : \alpha \in \mathbf{C}\right\}$ so $\mathscr{S}$ is reflexive. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathscr{S}$. Then $d(A, \mathscr{S}_\varepsilon) > 1/4$, for otherwise by inspection there would exist scalars λ, μ with $|\lambda - 1| \leqslant 1/4$, $|\mu| \leqslant 1/4$, and $\left|\dfrac{\lambda + \mu}{\varepsilon}\right| \leqslant 1/4$. But then $|\lambda + \mu| \geqslant ||\lambda| - |\mu|| \geqslant 3/4 - 1/4 = 1/2$, so $1/4 \geqslant 1/2\varepsilon \geqslant 1/2$, a contradiction. Hence $d(\mathscr{S}, \mathscr{S}_\varepsilon) > 1/4$, $0 < \varepsilon < 1$.

Now let $\mathscr{A} = \begin{pmatrix} \mathbf{C} & \mathscr{S} \\ 0 & \mathbf{C} \end{pmatrix}$, $\mathscr{A}_\varepsilon = \begin{pmatrix} \mathbf{C} & \mathscr{S}_\varepsilon \\ 0 & \mathbf{C} \end{pmatrix}$, and let $\mathscr{L} = \operatorname{lat}\mathscr{A}$, $\mathscr{L}_\varepsilon = \operatorname{lat}\mathscr{A}_\varepsilon$. Then also $d(\mathscr{A}, \mathscr{A}_\varepsilon) > 1/4$. An elementary computation via Proposition 1 shows that

$$\mathscr{L} = \{F \oplus E : \text{either } E = 0 \text{ or } F = H_2\} \cup \{\mathbf{C}e_2 \oplus \mathbf{C}e_1\}$$

and

$$\mathscr{L}_\varepsilon = \mathscr{L} \cup \{\mathbf{C}(e_2 - \varepsilon e_1) \oplus \mathbf{C}(e_1 + \varepsilon e_2)\}.$$

If P is the projection onto $\mathbf{C}e_2 \oplus \mathbf{C}e_1$ and P_ε the projection onto $\mathbf{C}(e_2 - \varepsilon e_1) \oplus \oplus \mathbf{C}(e_1 + \varepsilon e_2)$ it is clear that $\|P - P_\varepsilon\| \to 0$ as $\varepsilon \to 0$. Hence $d(\mathscr{L}, \mathscr{L}_\varepsilon) \to 0$ as $\varepsilon \to 0$. This demonstrates the discontinuity of the $\mathscr{L} \to \operatorname{alg}\mathscr{L}$ map. ▨

LEMMA 10. *Let $T \in L(H)$. Then the subspace $\mathscr{S} = \{\lambda T : \lambda \in \mathbf{C}\}$ is reflexive.*

Proof. We must show that if $S \in L(H)$ and if for each $x \in H$ there is a $\lambda(x) \in \mathbf{C}$ with $Sx = \lambda(x)Tx$, then $\lambda(x)$ can be taken constant. We may assume $S \neq 0$. Choose $x \in H$ so that $Sx \neq 0$. Let $y \in H$, and let $\lambda = \lambda(x)$, $\alpha = \lambda(y)$, $\beta = \lambda(x + y)$. If Tx and Ty are linearly independent, then $\lambda Tx + \alpha Ty = Sx + Sy = S(x + y) =$

$= \beta Tx + \beta Ty$, so $\alpha = \beta = \lambda$. Next, if $Ty = 0$, then $Sy = 0$, so $Sy = 0 = \lambda Ty$. Finally, suppose $Ty = \mu\, Tx$, $\mu \neq 0$. Then $T(\mu x - y) = 0$, so $S(\mu x - y) = 0$ and so $\alpha Ty = \mu\lambda Tx = \lambda(\mu Tx) = \lambda Ty$, so again $\alpha = \lambda$. ▨

EXAMPLE 11. ($\mathscr{A} \to \operatorname{lat}\mathscr{A}$). Let H_2 be the two dimensional Hilbert space with orthonormal basis $\{e_1, e_2\}$, and let E_i denote the projection onto $\mathbf{C}e_i$, $i = 1,2$. Let $\varepsilon > 0$, and let $T = E_1$, $T_\varepsilon = E_1 + \varepsilon E_2$. Let $\mathscr{S} = \mathbf{C}T, \mathscr{S}_\varepsilon = \mathbf{C}T_\varepsilon$, and let

$$\mathscr{A} = \begin{pmatrix} \mathbf{C} & \mathscr{S} \\ 0 & \mathbf{C} \end{pmatrix}, \quad \mathscr{A}_\varepsilon = \begin{pmatrix} \mathbf{C} & \mathscr{S}_\varepsilon \\ 0 & \mathbf{C} \end{pmatrix}.$$

Then $\mathscr{A}$, $\mathscr{A}_\varepsilon$ are reflexive since $\mathscr{S}$, $\mathscr{S}_\varepsilon$ are reflexive, and $\mathrm{d}(\mathscr{A}, \mathscr{A}_\varepsilon) \to 0$ as $\varepsilon \to 0$ since $\mathrm{d}(\mathscr{S}, \mathscr{S}_\varepsilon) \to 0$.

Let $\mathscr{L} = \operatorname{lat}\mathscr{A}$, $\mathscr{L}_\varepsilon = \operatorname{lat}\mathscr{A}_\varepsilon$. We will show that $\mathrm{d}(\mathscr{L}, \mathscr{L}_\varepsilon) = 1$ for all $\varepsilon > 0$, thus demonstrating discontinuity of the $\mathscr{A} \to \operatorname{lat}\mathscr{A}$ map.

It is clear that $0 \oplus E_2 \in \mathscr{L}$. Let $R_1 = 0 \oplus E_2$, and let R_2 be any projection in $\mathscr{L}_\varepsilon$. Then by Proposition 1, R_2 has the form $Q \oplus P$ where $\mathscr{S}_\varepsilon PH \subseteq QH$. Now $\|0 \oplus E_2 - Q \oplus P\| = \max\{\|Q\|, \|E_2 - P\|\}$, so if $Q \neq 0$, then $\|R_1 - R_2\| \geqslant 1$. But if $Q = 0$, then $T_\varepsilon P = 0$ so $P = 0$, and so also $\|R_1 - R_2\| = 1$. Hence $\mathrm{d}(\mathscr{L}, \mathscr{L}_\varepsilon) = 1$. ▨

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