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ON THE GROUP OF EXTENSIONS RELATIVE TO A SEMIFINITE FACTOR

GEORGES SKANDALIS

0. INTRODUCTION

In G. G. Kasparov's work on extensions ([7], [8]) an important hypothesis is that the ideal has a countable approximate unit.

However, extensions of the form $A \to N/\mathscr{K}_N$, where N is a II_∞ factor and $\mathscr{K}_N$ is the ideal of compact operators of N, seem quite interesting (cf. [14], [6], [2], [11], [4]).

Such extensions give rise to a semi-group $\mathrm{Ext}_N(A)$. If A is separable and nuclear this semi-group is a group ([3]). If not, consider instead the group $\mathrm{Ext}_N(A)^{-1}$. A natural question is then: Does this functor have all the nice properties of the Kasparov $\mathrm{Ext}(A, B)$ functor? (For instance homotopy invariance, Bott periodicity, Thom isomorphism cf. [8]; see also [9] and [5].)

Let M be a II_1 factor such that $N = M \otimes \mathscr{L}(H)$ (where H is a separable Hilbert space). We prove that $\mathrm{Ext}_N(A)^{-1} = \mathrm{Ext}(A, M)^{-1}$ and hence has all these properties.

We conclude by some remarks on the KK functor based on the technique used in the proof of our main result.

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1. DEFINITIONS AND NOTATIONS

Let N be a countably decomposable II_∞ factor, and let $\mathscr{K}_N$ be the closed ideal of compact operators of N.

1.1. We are interested in extensions of the form $0 \to \mathscr{K}_N \to E \to A \to 0$. Such an extension of A by $\mathscr{K}_N$ is exactly equivalent to a homomorphism $A \xrightarrow{\varphi} N/\mathscr{K}_N$.

1.2. Such an extension is called trivial if there exists a splitting which is a homomorphism $E \leftarrow A$. This is equivalent to saying that the map $\varphi: A \to N/\mathscr{K}_N$ admits a lifting π which is a *-homomorphism $A \xrightarrow{\pi} N$.

1.3. The sum of extensions φ, $\varphi': A \to N/\mathscr{K}_N$ is the extension $\varphi \oplus \varphi': A \to \to M_2(N/\mathscr{K}_N) \cong N/\mathscr{K}_N$ (M_2 stands for 2 by 2 matrices).

1.4. The extensions φ and φ' are said to be unitarily equivalent (one writes $\varphi \underset{u}{\sim} \varphi'$) if there exists a unitary $U \in N$ such that $\forall a \in A$

$$\varphi'(a) = q(U)\varphi(a)q(U^*)$$

(where $q: N \to N/\mathscr{K}_N$ is the quotient map).

1.5. Define $\mathrm{Ext}_N(A)$ as the semi-group of extensions divided by the equivalence relations: $\varphi \sim \varphi'$ iff there exist trivial τ and τ' such that $\varphi \oplus \tau \underset{u}{\sim} \varphi' \oplus \tau'$.

The (stably) trivial extensions form the zero element of $\mathrm{Ext}_N(A)$.

1.6. Consider also $\mathrm{Ext}_N(A)^{-1}$ as the group of invertible elements of the semigroup $\mathrm{Ext}_N(A)$. If A is separable, the class of the extension φ is in $\mathrm{Ext}_N(A)^{-1}$ iff it admits a completely positive lifting. Hence ([3]) if A is nuclear and separable $\mathrm{Ext}_N(A) = \mathrm{Ext}_N(A)^{-1}$.

Let us also recall briefly the definition of $\mathrm{Ext}(A, B)$ (cf. [8], § 7, Definition 1).

1.7. Extensions are maps $A \xrightarrow{\varphi} \mathscr{L}(\mathscr{E})/\mathscr{K}(\mathscr{E})$ where $\mathscr{E}$ is a countably generated Hilbert B module ([7], Definitions 1,2,3,4). Addition of extensions is defined through the map

$$\mathscr{L}(\mathscr{E})/\mathscr{K}(\mathscr{E}) \oplus \mathscr{L}(\mathscr{E}')/\mathscr{K}(\mathscr{E}') \to \mathscr{L}(\mathscr{E} \oplus \mathscr{E}')/\mathscr{K}(\mathscr{E} \oplus \mathscr{E}').$$

1.8. The extension φ is said to be trivial if it factors through a *-homomorphism $\pi: A \to \mathscr{L}(\mathscr{E})$.

The extensions φ and φ' are unitarily equivalent (write $\varphi \underset{u}{\sim} \varphi'$) iff there exists a unitary $U \in \mathscr{L}(\mathscr{E}, \mathscr{E}')$ such that for all a in A, $\varphi'(a) = U\varphi(a)U^*$ (with obvious meaning).

1.9. $\mathrm{Ext}(A, B)$ is the semi-group of extensions up to the equivalence relation: $\varphi \sim \varphi'$ iff there exist trivial τ and τ' such that $\varphi \oplus \tau \underset{u}{\sim} \varphi' \oplus \tau'$.

1.10. We will in fact mostly consider $\mathrm{Ext}(A, B)^{-1}$, the group of invertible elements of $\mathrm{Ext}(A, B)$. If A is separable, the extension φ is in $\mathrm{Ext}(A, B)^{-1}$ iff it admits a completely positive lifting $A \to \mathscr{L}(\mathscr{E})$ ([7], Theorem 3). Hence if A is separable and nuclear $\mathrm{Ext}(A, B)^{-1} = \mathrm{Ext}(A, B)$ (cf. [8], § 7, Lemma 1.2). Moreover, if A is separable $\mathrm{Ext}(A, B)^{-1} = \mathrm{KK}^1(A, B)$ (cf. [8], § 5, Definition 1, and § 7 proof of Lemma 1, and Lemma 2, see also [13], Theorem 19).

We will use the rather obvious:

1.11. LEMMA. *Let $\mathscr{E}$ be a Hilbert B module where B is any (graded) C^*-algebra. Let $J \subset \mathscr{K}(\mathscr{E})$ be a C^*-subalgebra with countable approximate unit. Then the closure $\mathscr{E}'$ of $J\mathscr{E}$ in $\mathscr{E}$ is countably generated. Moreover $J \subset \mathscr{K}(\mathscr{E}')$ $(\subset \mathscr{K}(\mathscr{E}))$.*

Proof. Let h be a strictly positive element of J. As for all $x \in J$, x is a norm limit of elements of the form hy_n $(y_n \in J)$, $\mathscr{E}'$ is the closure of $h\mathscr{E}$ in $\mathscr{E}$.

Write then

$$h^{1/2} = \lim_n \sum_{k=1}^{m_n} \theta_{\xi_{n,k}, \eta_{n,k}}.$$

If $\xi \in \mathscr{E}$ write $h\xi = \lim \sum h^{1/2} \xi_{n,k} \langle \eta_{n,k}, \xi \rangle$. Hence the family $h^{1/2}\xi_{n,k}$ generates $\mathscr{E}'$. Moreover, for all $y \in J$, $hy = \lim \sum \theta_{h^{1/2}\xi_{n,k}, y^*\eta_{n,k}}$. Hence $hJ \subset \mathscr{K}(\mathscr{E}')$. Thus $J \subset \mathscr{K}(\mathscr{E}')$.

∎

2. THE MAIN RESULT

Let Q be a finite projection in N and let M be the II_1 factor $M = N_Q = QNQ$.

Let $\mathscr{E}_N$ be the Hilbert M module $\mathscr{E}_N = NQ$ (for $\xi \in \mathscr{E}_N$, $x \in M$, $\xi x \in \mathscr{E}_N$. For $\xi, \eta \in \mathscr{E}_N$ one writes $\langle \xi, \eta \rangle = \xi^* \eta \in M$.).

One has $\mathscr{K}(\mathscr{E}_N) = \mathscr{K}_N$ and $\mathscr{L}(\mathscr{E}_N) = N$.

Let Q_n be a family of pairwise orthogonal projections of N equivalent to Q

Let then $\mathscr{E} = \bigoplus_{n=1}^{\infty} Q_n NQ \subset \mathscr{E}_N$. One has $\mathscr{E} \cong \mathscr{H}_M$. (Note that even if $\sum Q_n = 1$, $\mathscr{E} \neq \mathscr{E}_N$.) We thus get the rather obvious:

2.1. PROPOSITION. a) *For any countably generated Hilbert M module $\mathscr{E}$, there exists a submodule $\mathscr{E}_1$ of $\mathscr{E}_N$ which is isomorphic to $\mathscr{E}$.*

b) *The map $\mathscr{K}(\mathscr{E}) \cong \mathscr{K}(\mathscr{E}_1) \to \mathscr{K}(\mathscr{E}_N)$ thus defined extends to a homomophism $\mathscr{L}(\mathscr{E}) \xrightarrow{\pi} \mathscr{L}(\mathscr{E}_N) = N$.*

c) *If $\mathscr{E}_1$ and $\mathscr{E}_2$ are submodules of $\mathscr{E}_N$ and $U_1 \in \mathscr{L}(\mathscr{E}, \mathscr{E}_1)$, $U_2 \in \mathscr{L}(\mathscr{E}, \mathscr{E}_2)$ are unitaries then there exists a unitary $V \in \mathscr{L}(\mathscr{E}_N \oplus \mathscr{E}_N)$ with $V(\mathscr{E}_1 \oplus 0) = 0 \oplus \mathscr{E}_2$ and $(V|(\mathscr{E}_1 \oplus 0))(U_1 \oplus 0) = 0 \oplus U_2$.*

Proof. a) follows from the above discussion and from the stabilization theorem ([7], Theorem 2).

b) The map $\mathscr{K}(\mathscr{E}) \to \mathscr{K}(\mathscr{E}_N) \subset N$ is a representation. Hence it extends to the multiplier algebra $\mathscr{M}(\mathscr{K}(\mathscr{E})) = \mathscr{L}(\mathscr{E})$ ([7], Theorem 1; [10], Proposition 3.12.3).

c) Let $W \in \mathscr{L}(\mathscr{E}_1 \oplus \mathscr{E}_2)$, $W = \begin{bmatrix} 0 & U_1 U_2^* \\ U_2 U_1^* & 0 \end{bmatrix}$.

Using b) there exists $\overline{W} = \pi(W) \in \mathscr{L}(\mathscr{E}_N \oplus \mathscr{E}_N)$ whose restriction to $\mathscr{E}_1 \oplus \mathscr{E}_2$ coincides with W. One has $\overline{W}^2 = \overline{W}\overline{W}^* = P$ is a projection of $\mathscr{L}(\mathscr{E}_N \oplus \mathscr{E}_N)$. Put then $V = (1 - P) + \overline{W}$. ▨

Let $\mathscr{E}$ be a countably generated Hilbert M module and $\pi: \mathscr{L}(\mathscr{E}) \to N$ the *-homomorphism described by the above proposition. As $\pi(\mathscr{K}(\mathscr{E})) \subset \mathscr{K}(\mathscr{E}_N) = \mathscr{K}_N$ we have the *-homomorphism

$$\pi_*: \mathscr{L}(\mathscr{E})/\mathscr{K}(\mathscr{E}) \to N/\mathscr{K}_N.$$

Using Proposition 2.1 c) we get then a well defined homomorphism W_A: Ext$(A, M) \to$ Ext$_N(A)$ which to the extension φ assigns $\pi_* \circ \varphi$.

Our main result is then:

2.2. THEOREM. *Assume A is separable.*

a) *The map W_A is surjective.*

b) *The induced map W_A*: Ext$(A, M)^{-1} \to$ Ext$_N(A)^{-1}$ *is a bijection.*

The following lemma implies Part a) of Theorem 2.2 and the surjectivity in Part b).

2.3. LEMMA. *Assume A is separable.*

Let $\varphi: A \to \mathscr{L}(\mathscr{E}_N)/\mathscr{K}(\mathscr{E}_N)$ be an extension. Then there exists a countably generated submodule $\mathscr{E}$ of $\mathscr{E}_N$ and an extension $\psi: A \to \mathscr{L}(\mathscr{E})/\mathscr{K}(\mathscr{E})$ such that $\varphi = \pi_* \circ \psi$.

Moreover if φ admits a complete positive lifting or is trivial, ψ may be chosen to have the same property.

Proof. Let A_0 be a separable subspace of N such that $q(\overline{A_0}) = \varphi(A)$. If φ admits the completely positive or *-homomorphism lifting φ_0 let $A_0 = \varphi_0(A)$.

Let h be an injective positive element of $\mathscr{K}_N$. Put $D = C^*(A_0, h)$. It is the C^*-subalgebra of N generated by A_0 and h.

Put $J = D \cap \mathscr{K}(\mathscr{E}_N)$, and let $\mathscr{E}$ be the closure of $J\mathscr{E}_N$. Then $J \subset \mathscr{K}(\mathscr{E})$ and $\mathscr{E}$ is countably generated (Lemma 1.11). As $h \in \mathscr{K}(\mathscr{E})$ the representation $\mathscr{K}(\mathscr{E}) \to N$ is non degenerate.

Hence if $\pi: \mathscr{L}(\mathscr{E}) \to N$ is the extension to the multiplier algebra $\pi(1) = 1$.

As $\mathscr{E} = J\mathscr{E} = J \otimes_J \mathscr{E}$, and as J is an ideal in D, D acts naturally in $\mathscr{E}$. Moreover, the composition $D \to \mathscr{L}(\mathscr{E}) \xrightarrow{\pi} N$ is the identity. Indeed it is the restriction to D of the unique extension of the inclusion $J \hookrightarrow N$ to the multiplier algebra $\mathscr{M}(J)$.

Call ψ the composition $A \to D/J \to \mathscr{L}(\mathscr{E})/\mathscr{K}(\mathscr{E})$ $(J \subset \mathscr{K}(\mathscr{E}))$. One has $\varphi = \pi_* \circ \psi$.

Note that ψ admits the lifting φ_0 which is completely positive or a *-homomorphism when desired. ▨

End of proof of Theorem 2.2. What remains to be proved is that if $\varphi \in \mathrm{Ext}(A,M)^{-1}$ satisfies $\pi_* \circ \varphi = 0$ in $\mathrm{Ext}_N(A)$, then $\varphi = 0$ in $\mathrm{Ext}(A, M)$.

If $\pi_* \circ \varphi = 0$ in $\mathrm{Ext}_N(A)$, there exists a trivial τ such that $\pi_* \circ \varphi \oplus \tau$ is trivial. But the trivial extension τ may be written as $\pi_* \circ \sigma$, where σ is a trivial extension (Lemma 2.3).

So that all which now remains to be proved, is that if $\varphi \in \mathrm{Ext}(A, M)^{-1}$ satisfies $\pi_* \circ \varphi$ is trivial, then φ is 0 in $\mathrm{Ext}(A, M)^{-1}$.

We may consider φ as a completely positive map $\varphi : A \to \mathscr{L}(\mathscr{E})$ where $\mathscr{E} \subset \mathscr{E}_N$ is a countably generated Hilbert C^*-module over M, and φ is a *-homomorphism modulo $\mathscr{K}(\mathscr{E})$.

As $\pi_* \circ \varphi$ is trivial, there exists $\tau : A \to N$ which is a *-homomorphism such that $\tau(a) - \varphi(a) \in \mathscr{K}_N \quad \forall a \in A$.

Let $h \in \mathscr{K}(\mathscr{E}) \subset \mathscr{K}_N$ be a strictly positive element of $\mathscr{K}(\mathscr{E})$. Put $D = C^*(\varphi(A), \tau(A), h)$ and $J = D \cap \mathscr{K}_N$. Let $\mathscr{E}'$ be the Hilbert M submodule of $\mathscr{E}_N$ generated by $J\mathscr{E}_N$. It contains $\mathscr{E}$ ($h \in J$) and is countably generated. Also $J \subset \mathscr{K}(\mathscr{E}')$ (Lemma 1.11).

Let $\mathscr{E}''$ be the Hilbert $M \otimes C([0, 1])$ module $\mathscr{E}'' = \{\xi : [0, 1] \to \mathscr{E}' \mid \xi(0) \in \mathscr{E}\}$.

Let $\psi : A \to \mathscr{L}(\mathscr{E}'')$ be the completely positive map given by $(\psi(a)\xi)(t) = \varphi(a)\xi(t)$. As $\mathscr{K}(\mathscr{E}) \otimes C([0, 1]) \subset \mathscr{K}(\mathscr{E}'')$ one has $\psi(a_1a_2) - \psi(a_1)\psi(a_2) \in \mathscr{K}(\mathscr{E}'')$. So that ψ defines a *-homomorphism $A \to \mathscr{L}(\mathscr{E}'')/\mathscr{K}(\mathscr{E}'')$ and hence an element of $\mathrm{Ext}(A, M \otimes C([0, 1]))^{-1}$.

By homotopy invariance of the bifunctor $\mathrm{Ext}(A, B)^{-1} = \mathrm{KK}^1(A, B)$ we deduce that the extension $A \xrightarrow{\varphi} \mathscr{L}(\mathscr{E})/\mathscr{K}(\mathscr{E})$ is equal in this group to the extension $A \xrightarrow{\varphi} \mathscr{L}(\mathscr{E}')/\mathscr{K}(\mathscr{E}')$. But this last extension is trivial. ▨

2.4. COROLLARY. a) *For separable A the functor* $\mathrm{Ext}_N(A)^{-1}$ *is equal to* $\mathrm{KK}^1(A, M)$ *In particular, it is homotopy invariant, periodic, invariant under tensor products with* $\mathscr{K}(H)$.

b) *When moreover A is nuclear,* $\mathrm{Ext}_N(A)$ *is equal to the group of absorbing extensions (those* φ *for which* $\varphi \oplus \tau \underset{u}{\sim} \varphi$ *for all trivial* τ*), up to unitary equivalence.*

Proof: a) is a direct consequence of Theorem 2.2 and of [8].

b) is a direct consequence of Lemma 2.3 and of [8], §7, Lemma 1. ▨

2.5. REMARKS. a) If A is abelian, or continuous trace or inductive limit of continuous trace, all non unital injective extensions are absorbing as proved in [6] and in [4]. One may wish to extend this result to more general cases.

However, our technique should fail, the corresponding result being wrong even in the case $A = \mathbf{C}$ for $\mathrm{Ext}(A, M)$. (One very easily constructs a finite projection in $\mathscr{L}(\mathscr{H}_M) \subset N$, which is not in $\mathscr{K}(\mathscr{H}_M)$.)

b) If A is in the category of C^*-algebras containing type I algebras and closed under direct limits, extensions, crossed products by $\mathbf{R}$ on $\mathbf{Z}$, the equality $\mathrm{Ext}_N(A) = \mathrm{Ext}(A, M)$ reads $\mathrm{Ext}_N(A) = \mathrm{Hom}(\mathrm{K}_1(A), \mathbf{R})$ using the "universal coefficient formula" of [12].

In the case where A is abelian the universal coefficient formula was proved in [2] by Cho who remarked that the techniques of [1] work in the II_∞ case.

c) In general, if the fundamental group of N is $\mathbf{R}_+^*$, then $\mathrm{Ext}_N(A)^{-1}$ has a natural real vector space structure. The dependence of the functor $\mathrm{Ext}_N(A)$ on N is not clear.

One would suspect that in the nuclear case there is no dependence at all.

2.6. Remark. It is not difficult to give the list of all countably generated Hilbert M modules up to isomorphism.

If $\mathscr{E}$ is a submodule of $\mathscr{E}_N$ and $\pi: \mathscr{L}(\mathscr{E}) \to N$ the corresponding homomorphism put $P_{\mathscr{E}} = \pi(1)$. (Then $P_{\mathscr{E}}\mathscr{E}_N$ is the weak closure of $\mathscr{E}$ in $\mathscr{E}_N$.) Put $D(\mathscr{E}) = \mathrm{Tr}(P_{\mathscr{E}})$ where Tr is the semifinite trace of N.

According to Proposition 2.1 c) this number $D(\mathscr{E}) \in [0, +\infty]$ only depends upon the isomorphism class of $\mathscr{E}$. One has:

a) If $D(\mathscr{E}) = 0$ then $\mathscr{E} = 0$.

b) If $\mathscr{E}$ is countably generated and $D(\mathscr{E}) = +\infty$ then $D(\mathscr{E}) \cong \mathscr{H}_M$.

c) There are exactly two isomorphism classes of countably generated $\mathscr{E}$ with $D(\mathscr{E}) = d$, $0 < d < \infty$:

$\mathscr{E}_d = P_d\mathscr{E}$ where $P_d \in N$ is a projection of trace d;

$\mathscr{E}'_d$ caracterized by the fact $1 \notin \mathscr{K}(\mathscr{E}'_d)$.

3. CONCLUDING REMARKS

Let us conclude these notes by making some remarks on the $\mathrm{KK}(A, B)$ bifunctor for A, B graded algebras, with A separable.

Our first remark will be that if A is separable the definition of $\mathrm{KK}(A, B)$ given in [13] (Definition 2.7) coincides with the original one given by G. G. Kasparov in [8] (§ 4, Definition 3). (These two definitions are a priori different if B has no countable approximate unit.)

Let us recall these definitions and add a third one:

$\mathbb{E}(A, B)$ (resp. $\mathbb{E}'(A, B)$, $\mathbb{E}''(A, B)$) is the set of pairs $(\mathscr{E}, F)$ where $\mathscr{E}$ is a graded Hilbert B module isomorphic to $\mathscr{H}_B$ (resp. countable generated, any graded Hilbert B module). A acts on $\mathscr{E}$ through a grading preserving homomorphism $A \to \mathscr{L}(\mathscr{E})$. $F \in \mathscr{L}(\mathscr{E})$ satisfies $a(F^2 - 1) \in \mathscr{K}(\mathscr{E})$, $[a, F] \in \mathscr{K}(\mathscr{E})$, $a(F - F^*) \in \mathscr{K}(\mathscr{E})$ for all a in A.

A homotopy is an element of $\mathscr{E}(A, B \otimes C([0, 1]))$ (resp. $\mathscr{E}'$, $\mathscr{E}''$).

The degenerate elements $\mathscr{D}(A, B)$ (resp. $\mathscr{D}'(A, B)$, $\mathscr{D}''(A, B)$) are those for which $a(F^2 - 1) = 0$, $[a, F] = 0$, $a(F - F^*) = 0$. $\mathrm{KK}(A, B)$ (resp. $\mathrm{KK}'(A, B)$, $\mathrm{KK}''(A, B)$) is the quotient of $\mathscr{E}(A, B)$ (resp. $\mathscr{E}'(A, B)$, $\mathscr{E}''(A, B)$) by the equivalence relation: $(\mathscr{E}_1, F_1) \sim (\mathscr{E}_2, F_2)$ iff $\exists (\mathscr{E}_1', F_1'), (\mathscr{E}_2', F_2')$ degenerate such that $(\mathscr{E}_1 \oplus \mathscr{E}_1', F_1 \oplus F_1')$ and $(\mathscr{E}_2 \oplus \mathscr{E}_2', F_2 \oplus F_2')$ are homotopic.

Thanks to the stabilization theorem ([7], Theorem 2) there is a well defined map $\mathrm{KK}'(A, B) \to \mathrm{KK}(A, B)$, which to $(\mathscr{E}, F)$ assigns $(\mathscr{E} \oplus \mathscr{H}_B, F \oplus 0)$ where A acts through the zero map in $\mathscr{H}_B$.

There is also a well defined map $\mathrm{KK}(A, B) \to \mathrm{KK}''(A, B)$ $((\mathscr{E}, F) \to (\mathscr{E}, F)!)$.

3.1 Proposition. *These two maps are isomorphisms.*

Proof. Surjectivity. It is enough to show that the two maps $\mathrm{KK}'(A, B) \to$ $\to \mathrm{KK}(A, B)$ and $\mathrm{KK}'(A, B) \to \mathrm{KK}''(A, B)$ are onto.

Take then $(\mathscr{E}, F) \in \mathscr{E}''(A, B)$. Set $D = C^*(A, F) \subset \mathscr{L}(\mathscr{E})$ and $J = D \cap \mathscr{K}(\mathscr{E})$. Let $\mathscr{E}_0$ be the closure of $J\mathscr{E}$ in $\mathscr{E}$. It is countably generated (Lemma 1.11).

Let $(\overline{\mathscr{E}}, \overline{F}) \in \mathscr{E}''(A, B \otimes C([0, 1]))$, where $\overline{\mathscr{E}} = \{\xi \in \mathscr{E} \otimes C([0, 1]) \mid \xi(0) \in \mathscr{E}_0\}$. The action of A is given by $(a\xi)(t) = a\xi(t)$, $(\overline{F}\xi)(t) = F\xi(t)$, $t \in [0, 1]$.

One thus gets a homotopy between $(\mathscr{E}, F)$ and $(\mathscr{E}_0, F)$.

To get the surjectivity for the map $\mathrm{KK}' \to \mathrm{KK}$, one just has to check that if $\mathscr{E} \cong \mathscr{H}_B$, $\overline{\mathscr{E}} \oplus \mathscr{H}_{B \otimes C([0,1])} \cong \mathscr{H}_{B \otimes C([0,1])}$. This is true ([7], Theorem 2) if B has a unit. If not, one considers $\mathscr{H}_B$ as a submodule of $\mathscr{H}_{\tilde{B}}$.

Injectivity. It is now enough to show that the map $\mathrm{KK}' \to \mathrm{KK}''$ is injective. This is done by applying the above argument to $B \otimes C([0, 1])$ instead of B. ▨

Finally let us make the following:

3.2. Remark. *Let A be separable and $x \in \mathrm{KK}(A, B)$.*

Then there exists a separable (graded) subalgebra $B_1 \subset B$ and an element $x_1 \in \mathrm{KK}(A, B_1)$ such that $x = i_*(x_1)$ where $i\colon B_1 \to B$ is the inclusion.

Proof. Take an element $(\mathscr{H}_B, F)$ in $\mathrm{KK}(A, B)$. Write $D = C^*(A, F)$ and $J = D \cap \mathscr{K}(\mathscr{H}_B) = D \cap (\mathscr{K} \hat{\otimes} B)$.

Each element of J is an infinite matrix with coefficients in B. Let then B_1 be the (graded-separable) subalgebra of B generated by all these coefficients. ▨

This remark allows results to be extended from the separable case in B to the general case. For instance, the result of [5] (Theorem 1), as well as the more general Theorem 2 of § 6 of [9], extend to the case of nonseparable B.

Let us give the argument for extending [5], Theorem 1.

3.3. Consequence. *Let $(B, \mathbf{R}, \alpha)$ be a C^*-dynamical system, A a separable C^*-algebra. Then* $\mathrm{KK}(A, B \times_\alpha \mathbf{R})$ *is isomorphic to* $\mathrm{KK}^1(A, B)$.

Proof. Let $t_\alpha \in KK^1(B, B \times_\alpha \mathbf{R})$ and $t_{\hat{\alpha}} \in KK^1(B \times_\alpha \mathbf{R}, B)$ be the corresponding Thom elements ([5], Proposition 1). Let $x \in KK(A, B)$. In view of [5], all we have to prove is $(x \otimes_B t_\alpha) \otimes_{B \times_\alpha \mathbf{R}} t_{\hat{\alpha}} = x$.

Then let B_1 be a separable subalgebra of B and $x_1 \in KK(A, B_1)$ such that $x = i_*(x_1)$. We may replace B_1 by the subalgebra of B generated by B_1 and its transforms by α_t $t \in \mathbf{R}$ and x_1 by its image in this algebra. Thus we may furthermore assume that B_1 is invariant by the action of $\mathbf{R}$. Call α^1 the restriction of α in B_1.

We have

$$x \otimes_B t_\alpha = (i_* x_1) \otimes_B t_\alpha = x_1 \otimes_{B_1} i_* t_{\alpha^1} = i_*(x_1 \otimes_{B_1} t_{\alpha^1}).$$

Finally

$$(x \otimes_B t_\alpha) \otimes_{B \times_\alpha \mathbf{R}} t_{\hat{\alpha}} = i_*[(x_1 \otimes_{B_1} t_{\alpha^1}) \otimes_{B_1 \times_{\alpha^1} \mathbf{R}} t_{\hat{\alpha}^1}) = x$$

by [5], Theorem 2. ▧

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GEORGES SKANDALIS
Department of Mathematics and Statistics,
Queen's University at Kingston,
Kingston, K7L 3N6,
Canada.

Permanent address:

Laboratoire de Mathématiques Fondamentales,
U.E.R. 48,
Université Pierre et Marie Curie,
4, place Jussieu, F-75230,
Paris, Cedex 05,
France.

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