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ON SOME CONTINUATION PROBLEMS WHICH ARE CLOSELY RELATED TO THE THEORY OF OPERATORS IN SPACES $\Pi_\varkappa$. IV: CONTINUOUS ANALOGUES OF ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE WITH RESPECT TO AN INDEFINITE WEIGHT AND RELATED CONTINUATION PROBLEMS FOR SOME CLASSES OF FUNCTIONS

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In this paper the classes $\mathfrak{P}_{\varkappa;a}$ and $\mathfrak{G}_{\varkappa;a}$, which were introduced in [1], [2], play a fundamental role. Recall that a complex valued function h on $(-2a, 2a)$ is called *Hermitian* if $h(-t) = \overline{h(t)}$ $(-2a < t < 2a)$, and that $\mathfrak{P}_{\varkappa;a}$ $(\mathfrak{G}_{\varkappa;a})$, $0 < a \leqslant \infty$, is the set of all continuous functions f (g) on $(-2a, 2a)$ which are Hermitian and for which the kernel

$$F_f(t, s) := f(t-s) \quad (G_g(t, s) := g(t-s) - g(t) - \overline{g(s)} + g(0), \text{ resp.})$$

$(0 \leqslant s, t < 2a)$ has $\varkappa$ negative squares. We put $\mathfrak{P}_\varkappa := \mathfrak{P}_{\varkappa;\infty}$, $\mathfrak{G}_\varkappa := \mathfrak{G}_{\varkappa;\infty}$.

It turns out that an arbitrary function $f \in \mathfrak{P}_{\varkappa;a}$ has a continuation in $\mathfrak{P}_\varkappa$ (see [1], [3]), that is there exists a continuous Hermitian function $\tilde{f}$ on the real axis such that $\tilde{f}(t) = f(t)$ $(-2a < t < 2a)$ and that the kernels F_f and $F_{\tilde{f}}$ have the same number $\varkappa$ of negative squares. This continuation can be uniquely determined or not, and the question arises to give criteria for either case. If $f \in \mathfrak{P}_{\varkappa;a}$ has a unique continuation $\tilde{f} \in \mathfrak{P}_\varkappa$ we shall say that the continuation problem for f is *determined*, otherwise it is called *indetermined*. In the second case the following problem naturally arises: How to describe the totality of all the continuations $\tilde{f} \in \mathfrak{P}_\varkappa$ of f? The descriptions given in this paper will always be of the following form: There exist four entire functions w_{jk}, $j, k = 1,2$, such that the equality

$$-\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}zt} f(t)\,\mathrm{d}t = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)} \quad (\operatorname{Im} z < -\gamma)$$

for some $\gamma \geqslant 0$ establishes a bijective correspondence between all continuations $\tilde{f} \in \mathfrak{P}_\varkappa$ of f and all $T \in \tilde{N}_0$. Here $\tilde{N}_0$ denotes the "Nevanlinna class" N_0, consisting of all functions T which are holomorphic in the open upper half plane $\mathbf{C}_+$, map this half plane into itself, and are extended to the lower half plane $\mathbf{C}_-$ by $T(z)=\overline{T(\bar{z})}$, augmented by ∞. The matrix function $W=(w_{jk})_1^2$ giving such a description will be called a *resolvent matrix* of the function $f \in \mathfrak{P}_{\varkappa;a}$. If it is normalized by $W(0)=I_2$ and satisfies some further conditions the resolvent matrix of $f \in \mathfrak{P}_{\varkappa;a}$ will be uniquely determined.

Similar questions and results arise for functions $g \in \mathfrak{G}_{\varkappa;a}$, and we shall use a corresponding terminology. If $\varkappa=0$, for $f \in \mathfrak{P}_{0;a}$ and $g \in \mathfrak{G}_{0;a}$ these questions were completely answered by M. G. Kreĭn ([4], [5], [6]). In their full generality for $\varkappa \geqslant 0$ they will be considered in Part V of this series, see [7]. In the present Part IV we consider functions $f \in \mathfrak{P}_{\varkappa;a}$ and $g \in \mathfrak{G}_{\varkappa;a}$ of a special form, namely we suppose that they have an accelerant. Recall that the Hermitian function g on $(-2a, 2a)$, $0<a<\infty$, is said to have an *accelerant* H if it admits a representation

$$g(t)=g(0)-\alpha|t|-\int_0^t (t-s)H(s)\,\mathrm{d}s \quad (-2a<t<2a) \tag{0.1}$$

with some $\alpha>0$ and a function $H \in L^1(-2a, 2a)$ if $a<\infty$ and $H \in L^1_{\mathrm{loc}}(-\infty, \infty)$ if $a=\infty$. Evidently, $g''=-H$ exists on $(-2a, 0) \cup (0, 2a)$ and we have $g'(0+)-g'(0-)=-2\alpha<0$. It turns out that an arbitrary Hermitian function g with accelerant belongs to classes $\mathfrak{G}_{\varkappa;a}$ and $\mathfrak{P}_{\varkappa';a}$. In this special case the solutions to the problems formulated above can be given in much more explicit form than for general functions of $\mathfrak{P}_{\varkappa;a}$ or $\mathfrak{G}_{\varkappa;a}$.

The basic tool in this paper is the theory of π-selfadjoint and π-Hermitian operators, and in particular the theory of entire operators in $\pi_\varkappa$-spaces (Pontrjagin spaces of index $\varkappa$), see [8], [9]. The π-scalar product arises e.g. as follows:

$$[\varphi, \psi] := \int_{-a}^{a} \varphi(t)\overline{\psi(t)}\,\mathrm{d}t + \int_{-a}^{a}\int_{-a}^{a} H(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t \tag{0.2}$$

for $\varphi, \psi \in L^2(-a, a)$. In the construction of the resolvent matrix for a given function $g \in \mathfrak{G}_{\varkappa;a}$ of the form (0.1) there appear in a natural way entire functions which can be considered as continuous analogues of orthogonal polynomials of first and second kind on the unit circle with respect to an indefinite weight function. We call these functions the *orthogonal functions of first and second kind* associated with the accelerant H. Their study, in its turn, led us to statements about the zeros of some classes of entire functions which can be considered as continuous analogues of the theorems of Hermite and Schur-Cohn.

Now we describe the contents of the paper. In § 1 we study the integral equation

$$\varphi(t)+\int_0^{2a} H(t-s)\,\varphi(s)\,\mathrm{d}s = u(t) \quad (0 \leqslant t \leqslant 2a) \tag{0.3}$$

for a given function $H \in L^1(-2a, 2a)$. If $u = 0$ the solutions of (0.3) form a D-chain, see Theorem 1.1. This fact can be considered to be known. However, we prove some statements about the zeros of the Fourier transform of the generating element of this D-chain which are, apparently, new. Further, the resolvent kernel of the inhomogeneous equation (0.2) is studied.

In § 2 some basic facts for Hermitian functions with an accelerant are proved. In particular we show that these functions belong to classes $\mathfrak{G}_{\varkappa;a}$ and $\mathfrak{P}_{\varkappa;a}$, and give criteria for the corresponding continuation problems to be determined or indetermined. The theory of entire operators in $\pi_\varkappa$-spaces, in particular the resolvent matrix of an entire operator is used in § 3 in order to describe all the continuations of $f \in \mathfrak{P}_{\varkappa;a}$ with accelerant in the indetermined case. In § 4, after recalling some general results of De Branges, we show that the resolvent matrix of a function $f \in \mathfrak{P}_{\varkappa;a}$ with accelerant satisfies a canonical differential system with a Hamiltonian which has a constant determinant.

In § 5 it is shown that the functions $f \in \mathfrak{P}_{\varkappa;a}$ with $f(0) > 0$ and $g \in \mathfrak{G}_{\varkappa;a}$ are connected by a simple transformation (comp. [2]). If f has an accelerant H_f, then also the corresponding function g has an accelerant H_g and these accelerants satisfy an integral equation. This fact and the results of § 3 give the possibility to find a resolvent matrix of $g \in \mathfrak{G}_{\varkappa;a}$ with an accelerant in the indetermined case.

The orthogonal functions $D(z)$ and $E(z)$ of first and second kind, respectively, associated with the accelerant H, are introduced in § 6. We prove the fundamental identity

$$D(z)E^*(z) + D^*(z)E(z) = 2,$$

and give characteristic properties of the orthogonal functions. With a given accelerant H we construct a dual accelerant H_d such that the orthogonal functions of first and second kind exchange their roles if we replace H by H_d. In case $\varkappa = 0$ orthogonal functions of first and second kind were first introduced in [10]; for $\varkappa > 0$ orthogonal polynomials on the unit circle with respect to an indefinite weight were introduced in [11].

In § 7 a continuous analogue of the theorem of Schur-Cohn is proved. We mention that in [12], [13] a similar result was proved for functions which are, apparently, more general than functions corresponding to an accelerant. However, they are less general than the functions in Theorem 7.1, where it is shown that Hermite's theorem generalizes to entire functions of arbitrary order if only their Weierstrass expansion has certain properties.

In § 8 we express the δ_0-resolvent matrix of the operator $A_0 = \frac{1}{i}\frac{d}{dt}$ in the $\pi_\varkappa$-space, corresponding to the π-scalar product (0.2), by the orthogonal functions of first and second kind and give in § 9 a description of all the continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of the function $g \in \mathfrak{G}_{\varkappa;a}$ from (0.1), if this continuation problem is indetermined. If $\varkappa = 0$ this description can be considered as the continuous analogue of the formulae of Artjomenko and Geronimus (see [14], [15], [16]), describing the solutions of the problem of Carathéodory. Thus, in § 5 and § 9 we find two resolvent matrices for a given function $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant in the indetermined case. It follows from a general result of [9] that these resolvent matrices (after a suitable normalization) must coincide. It would be interesting to prove this in a straightforward way.

In § 10 we show that the results of §§ 5, 9 can be used in order to find resolvent matrices for a function $f \in \mathfrak{P}_{\varkappa;a}$ with accelerant in the indetermined case. These resolvent matrices are expressed in terms of the orthogonal functions of H_f or H_g where f and g are related by the transformation of § 5. Thus, for a function $f \in \mathfrak{P}_{\varkappa;a}$ with accelerant, $f(0) > 0$ and infinitely many continuations in $\mathfrak{P}_\varkappa$, in § 3 and § 10 we give three forms of the resolvent matrix.

In § 11 we derive canonical systems which are satisfied by the resolvent matrix of $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant, and show that in case $\varkappa = 0$ the spectral measures of g coincide with the spectral measures of these canonical systems. Finally, in § 12 we consider the particular case of a real function $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant H. We give a description of all its real continuations in $\mathfrak{G}_\varkappa$ and show that in dependence of the smoothness properties of H the canonical system can be transformed into an equation of a vibrating string or a Sturm-Liouville equation. Thus, in fact, we give a solution of the inverse spectral problem for Sturm-Liouville equations. There, apparently, the descriptions of all the spectral measures τ such that $\operatorname{supp}\tau \cap (-\infty, 0)$ consists of a given number $\varkappa$ of points is new.

If $\varkappa = 0$, the considerations in this paper lead us close to the solution of inverse spectral problems for a canonical differential operator. These questions, however, will not be considered here.

Some results of this paper were announced in [17]. There also the solution of the continuation problem for general functions of $\mathfrak{P}_{\varkappa;a}$ and $\mathfrak{G}_{\varkappa;a}$ was formulated, which will be proved in Part V. In Part V we shall also give a complete solution of the continuation problem for helical arcs in Bólyai-Lobačevskiĭ spaces. These problems were considered in [18], [19].

Finally we mention that, apparently, most of the results of this paper generalize to the case of matrix functions (for $\varkappa = 0$ comp. [20], [21]).

§ 1. INTEGRAL OPERATORS WITH DISPLACEMENT KERNELS

1.1. Let $0 < a < \infty$ and $H \in L^1(-2a, 2a)$. With the function H we associate the operator $\mathbf{H}$ in $L^2(0, 2a)$ or in $C(0, 2a)$, given by

$$(\mathbf{H}\varphi)(t) := \int_0^{2a} H(t-s)\varphi(s)\,\mathrm{d}s \quad (0 \leqslant t \leqslant 2a)$$

where $\varphi \in L^2(0, 2a)$ or $\varphi \in C(0, 2a)$. Then $\mathbf{H}$ is a bounded operator in $C(0, 2a)$. Indeed,

$$\max_{0 \leqslant t \leqslant 2a} |(\mathbf{H}\varphi)(t)| \leqslant \max_{0 \leqslant t \leqslant 2a} \int_0^{2a} |H(t-s)|\,\mathrm{d}s\,\|\varphi\|_C$$

whence

$$\|\mathbf{H}\|_C \leqslant \max_{0 \leqslant t \leqslant a} \int_0^{2a} |H(t-s)|\,\mathrm{d}s \leqslant \int_{-2a}^{2a} |H(s)|\,\mathrm{d}s.$$

Approximating the function H on $[-2a, 2a]$ by polynomials in the L^1-norm it is easy to see that $\mathbf{H}$ is even compact in $C(0, 2a)$ (comp. [22, III, § 10.1]).

Now suppose additionally that H is Hermitian:

$$H(t) = \overline{H(-t)} \quad (-2a \leqslant t \leqslant 2a). ^{*)}$$

Then $\mathbf{H}$ is selfadjoint in $L^2(0, 2a)$, and from a general result of the first author (see [23]) we conclude that $\mathbf{H}$ is compact in $L^2(0, 2a)$,

$$\|\mathbf{H}\|_{L^2} \leqslant \max_{0 \leqslant t \leqslant 2a} \int_0^{2a} |H(t-s)|\,\mathrm{d}s \leqslant \int_{-2a}^{2a} |H(s)|\,\mathrm{d}s, \tag{1.1}$$

*) If the considered functions are only summable, equalities are always understood to hold almost everywhere. Some of the results of this section (in particular the statements (1°) and (2°)) are true also for non-Hermitian H.

the spectra of **H** in $C(0, 2a)$ and $L^2(0, 2a)$ and the corresponding eigenspaces to nonzero eigenvalues coincide. Theorem 1.1 below describes the structure of these eigenspaces.

In order to formulate it, if $\lambda = \bar{\lambda} \neq 0$ by $\varkappa_{\mathbf{H}-\lambda I}$ we denote the number of positive (negative) squares of the Hermitian, possibly degenerated scalar product $((\mathbf{H} - \lambda I)\varphi, \psi)_{L^2}$, $\varphi, \psi \in L^2(0, 2a)$, if $\lambda > 0$ ($\lambda < 0$, respectively). Evidently, $\varkappa_{\mathbf{H}-\lambda I}$ is the total multiplicity of eigenvalues $> \lambda$ ($< \lambda$, respectively) of **H**.

THEOREM 1.1.*) *If $H \in L^1(-2a, 2a)$ is Hermitian and $\lambda \in \sigma(\mathbf{H})$, $\lambda \neq 0$, then the corresponding eigenspace is spanned by absolutely continuous functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ with the properties*

$$\varphi_0(2a - t) = \overline{\varphi_0(t)}, \quad \varphi_{k+1}(t) = \frac{d\varphi_k(t)}{dt} \quad (0 \leqslant t \leqslant 2a),$$

(1.2)

$$\varphi_k(0) = \varphi_k(2a) = 0 \quad (k = 0, 1, \ldots, n-1), \quad \varphi_n(2a) \neq 0.$$

The real entire function Φ_0:

$$\Phi_0(z) := e^{iza} \int_0^{2a} \varphi_0(t) e^{-izt}\, dt \quad (z \in \mathbf{C}) \tag{1.3}$$

*has only a finite number of nonreal and nonsimple real zeros. If $z_1, \ldots, z_l$ are its zeros in $\mathbf{C}_+$ **) and $z_{l+1}, \ldots, z_m$ its nonsimple real zeros, their multiplicities $\varkappa(z_j)$, $j = 1, \ldots, m$, satisfy the estimation*

$$\sum_{j=1}^{l} \varkappa(z_j) + \sum_{j=l+1}^{m} \left[\frac{\varkappa(z_j)}{2}\right] \leqslant \varkappa_{\mathbf{H}-\lambda I}. \tag{1.4}$$ ***)

The sequence of functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ in Theorem 1.1 is called a D-*chain* of eigenvectors of **H** corresponding to the eigenvalue λ.

An analogue of the first statement of the theorem for the half axis was proved in [24]. For a finite interval this statement (even for non-Hermitian H) was proved in [25] and under an additional assumption of H in [26]. For the convenience of the reader we shall give a complete proof of Theorem 1.1 in the next section.

1.2. With the function $H \in L^1(-2a, 2a)$ an operator **H** can also be defined on other spaces. Let L be one of the spaces $L^1(0, 2a)$ or $C'(0, 2a)$, where $C'(0, 2a)$ is

*) Recently a generalization of Theorem 1.1 to the case of a generalized function H was announced by L. A. Sahnovič in his article: On the eigenspaces of an operator with difference kernel (Russian), *Izv. Vysš. Učebn. Zaved. Matematika*, **12**(1983), 75—77.

**) $\mathbf{C}_+(\mathbf{C}_-)$ denotes the open upper (lower) half plane.

***) Here, e.g., the first sum is zero if Φ_0 has only real zeros.

the Banach space of all absolutely continuous functions φ on $[0, 2a]$ with the norm

$$\|\varphi\|' := \max_{0 \leqslant t \leqslant 2a} |\varphi(t)| + \int_0^{2a} |\varphi'(t)| \, dt.$$

Then the following statement is known (comp. [22]):

(1°) *If* $H \in L^1(-2a, 2a)$, *the operator* $\mathbf{H}$ *maps* L *into itself. Its restriction* $\mathbf{H}|L$ *is compact in* L, *we have* $\sigma(\mathbf{H}|L) = \sigma(\mathbf{H})$ *and the eigenspaces of* $\mathbf{H}|L$ *and* $\mathbf{H}$, *corresponding to nonzero eigenvalues, coincide.*

(2°) *If* $H \in L^1(-2a, 2a)$, $\lambda \in \sigma(\mathbf{H})$, $\lambda \neq 0$, *the corresponding eigenspace is spanned by absolutely continuous functions* $\varphi_0, \varphi_1, \ldots, \varphi_n$ *with the properties*

$$\varphi_{k+1}(t) = \frac{d\varphi_k(t)}{dt} \quad (0 \leqslant t \leqslant 2a), \tag{1.5}$$

$$\begin{gathered} \varphi_k(0) = \varphi_k(2a) = 0 \quad (k = 0, 1, \ldots, n-1), \\ |\varphi_n(0)| + |\varphi_n(2a)| \neq 0. \end{gathered} \tag{1.6}$$

Proof. Without loss of generality we suppose $\lambda = -1$; the corresponding eigenspace of $\mathbf{H}$ ($\mathbf{H}^*$) will be denoted by S (S^* respectively). According to (1°) the elements of S and S^* are absolutely continuous functions. It is easy to check that the mapping $\varphi \to \varphi^*$: $\varphi^*(t) := \overline{\varphi(2a-t)}$ establishes a bijective correspondence between S and S^*.

Let p be a nonnegative integer such that

$$\int_0^{2a} \varphi(t) t^k \, dt = 0 \quad \text{for all } \varphi \in S, \ k = 0, 1, \ldots, p-1, \tag{1.7}$$

$\int_0^{2a} \varphi(t) t^p \, dt \neq 0$ for some $\varphi \in S$. Then, by the correspondence between S and S^*, p is also the smallest integer such that $\int_0^{2a} \varphi(t) t^p \, dt \neq 0$ for some $\varphi \in S^*$. There exists a nonzero function $\varphi \in S$, such that

$$\int_0^{2a} \varphi(t) t^k \, dt = 0, \quad k = 0, 1, \ldots, n+p-1. \tag{1.8}$$

Indeed, if $\chi_0, \chi_1, \ldots, \chi_n$ is a basis of S, we put $\varphi(t) = c_0\chi_0(t) + \ldots + c_n\chi_n(t)$. Then the first p equations in (1.8) are satisfied by (1.7), the last n equations in (1.8) can be satisfied by a suitable choice of $c_0, c_1, \ldots, c_n$.

We consider the function

$$\psi_0(t) := \int_0^t \frac{(t-s)^{n+p-1}}{(n+p-1)} \varphi(s)\,\mathrm{d}s.$$

Then $\psi_0(0) = \ldots = \psi_0^{(n+p-1)}(0) = \psi_0(2a) = \ldots = \psi_0^{(n+p-1)}(2a) = 0$, and, denoting

$$\psi_0(t) + \int_0^{2a} H(t-s)\psi_0(s)\,\mathrm{d}s =: f(t),$$

by differentiation it follows that

$$\psi_0'(t) + \int_0^{2a} H(t-s)\psi_0'(s)\,\mathrm{d}s = f'(t),$$

$$\vdots$$

$$\psi_0^{(n+p)}(t) + \int_0^{2a} H(t-s)\psi_0^{(n+p)}(s)\,\mathrm{d}s = f^{(n+p)}(t)$$

and $f^{(n+p)}(t) \equiv 0$ as $\psi_0^{(n+p)} = \varphi \in S$. Therefore

$$f(t) = \gamma_0 + \gamma_1 t + \ldots + \gamma_{n+p-1}t^{n+p-1} \quad \text{with } \gamma_0, \gamma_1, \ldots, \gamma_{n+p-1} \in \mathbf{C}.$$

As the functions $f, f', \ldots, f^{(n+p-1)}$ belong to the range of $I + \mathbf{H}$ they are orthogonal to S^*. Observing that p is the greatest integer such that $t^k \perp S^*$ for all $k = 0, 1, \ldots, p-1$, it follows $\gamma_p = \gamma_{p+1} = \ldots = \gamma_{n+p-1} = 0$ and $f^{(p)} = f^{(p+1)} = \ldots$ $\ldots = f^{(n+p)} = 0$. Thus the elements

$$\varphi_0 := \psi^{(p)}, \ldots, \varphi_n := \psi^{(n+p)}$$

have the properties (1.5) and (1.6). They are linearly independent. Indeed, the relation

$$\alpha_0\varphi_0 + \ldots + \alpha_n\varphi_n = 0$$

for some $\alpha_0, \ldots, \alpha_n \in \mathbf{C}$, $\sum |\alpha_j| \neq 0$, implies that the initial problem

$$\alpha_0\varphi_0 + \ldots + \alpha_n\varphi_0^{(n)} = 0, \quad \varphi_0(0) = \ldots = \varphi_0^{(n-1)}(0) = 0$$

has a nontrivial solution, which is impossible. Assuming $\varphi_n(0) = \varphi_n(2a) = 0$, it would follow that $\varphi_{n+1} := \varphi_n' \in S$. Then, in the same way as before, the elements $\varphi_0, \ldots, \varphi_{n+1}$ would be linearly independent, which is impossible as $\dim S = n + 1$. The statement (2°) is proved.

Proof of Theorem 1.1. The first part of the theorem follows from (1°) and (2°). We have only to observe that for an Hermitian function H the eigenspace S_λ of $\mathbf{H}$, corresponding to $\lambda \in \sigma(\mathbf{H})$, $\lambda \neq 0$, has the property that $\varphi \in S$ implies $\varphi^* \in S$, $\varphi^*(t) := \overline{\varphi(2a-t)}$ $(0 \leqslant t \leqslant 2a)$. Therefore, if for a moment the function φ_0 in (2°) is denoted by $\hat{\varphi}_0$, we can put

$$\varphi_0(t) := \hat{\varphi}_0(t) + \overline{\hat{\varphi}_0(2a-t)},$$

or, if this vanishes identically,

$$\varphi_0(t) = \mathrm{i}\hat{\varphi}_0(t).$$

In order to prove the statement about the zeros of $\Phi_0(z)$ in (1.3), we introduce the (degenerated) scalar product

$$[\varphi, \psi] := ((\mathbf{H} - \lambda I)\varphi, \psi)_{L^2} \quad (\varphi, \psi \in L^2(0, 2a)) \tag{1.9}$$

on $L^2(0, 2a)$. The eigenspace S_λ of $\mathbf{H}$ corresponding to the eigenvalue $\lambda \in \sigma(\mathbf{H})$ is the isotropic subspace with respect to the scalar product (1.9), and the factor space $L^2(0,2a)/S_\lambda =: L^2(\mathbf{H} - \lambda I)$ equipped with this scalar product is a $\pi_\varkappa$-space with $\varkappa = \varkappa_{\mathbf{H}-\lambda I}$ positive squares if $\lambda > 0$ and $\varkappa = \varkappa_{\mathbf{H}-\lambda I}$ negative squares if $\lambda < 0$.

Let A_0 be the following operator in $L^2(0, 2a)$: $\mathfrak{D}(A_0)$ is the set of all absolutely continuous functions $\varphi \in L^2(0, 2a)$ such that $\varphi' \in L^2(0, 2a)$ and $\varphi(0) = \varphi(2a) = 0$,

$$A_0\varphi := \frac{1}{\mathrm{i}}\varphi' \quad (\varphi \in \mathfrak{D}(A_0)).$$

It is easy to check that

$$[A_0\varphi, \psi] = [\varphi, A_0\psi] \quad (\varphi, \psi \in \mathfrak{D}(A_0)),$$

hence A_0 induces a closed π-Hermitian operator in $L^2(\mathbf{H} - \lambda I)$, which will also be denoted by A_0. As $\lambda \in \sigma(\mathbf{H})$ this operator is π-selfadjoint. Indeed, for arbitrary $u \in L^2(0, 2a)$ the equation $(A_0 - zI)\varphi = u$, considered in $L^2(\mathbf{H} - \lambda I)$, has a solution $\varphi \in \mathfrak{D}(A_0)$:

$$\varphi(t) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}zt}\left(\int_0^t \mathrm{e}^{-\mathrm{i}zs}u(s)\,\mathrm{d}s - \frac{\mathscr{F}(u;z)}{\mathscr{F}(\varphi_0;z)}\int_0^t \mathrm{e}^{-\mathrm{i}zs}\varphi_0(s)\,\mathrm{d}s\right),$$

$\left(\mathscr{F}(u;z) := \int_0^{2a} \mathrm{e}^{-\mathrm{i}zs}u(s)\,\mathrm{d}s\right)$ if and only if $\mathscr{F}(\varphi_0; z) \neq 0$. The function $\tilde{\varphi}_z$: $\tilde{\varphi}_z(t) := \int_0^t \mathrm{e}^{\mathrm{i}z(t-s)}\varphi_0(s)\,\mathrm{d}s$ $(0 \leqslant t \leqslant 2a)$ does not belong to S_λ. Indeed, otherwise we would have

$$\tilde{\varphi}_z(t) = \sum_{j=0}^{n} \alpha_j \varphi_0^{(j)}(t). \tag{1.10}$$

As $\tilde{\varphi}_z(0) = \varphi_0(0) = \ldots = \varphi_0^{(n-1)}(0) = 0$, $\varphi_0^{(n)}(0) \neq 0$, it follows that $\alpha_n = 0$. Thus, if $n = 0$, the statement is proved. Otherwise ($n > 0$), the relation (1.10) implies

$$\tilde{\varphi}_z'(t) = \sum_{j=0}^{n-1} \alpha_j \varphi_0^{(j+1)}(t). \tag{1.11}$$

As $\tilde{\varphi}_z'(t) = \varphi_0(t) + \mathrm{i}z\tilde{\varphi}_z(t)$, we have $\tilde{\varphi}_z'(0) = 0$ and (1.11) yields $\alpha_{n-1} = 0$, etc.

The singularities of the resolvent $(A_0 - zI)^{-1}$ in $L^2(\mathbf{H} - \lambda I)$ are poles; they coincide with the zeros of $\mathscr{F}(\varphi_0; z)$ or of $\Phi_0(z)$, including the orders. Moreover, if $z_0 \in \sigma(A_0)$ its geometric eigenspace is of dimension one, spanned by $\tilde{\varphi}_{z_0}$. Therefore the algebraic multiplicity of the eigenvalue $z_0 \in \sigma(A_0)$ is equal to the order of the pole of the resolvent of A_0 at z_0 and hence equal to the order of the zero z_0 of Φ_0. Now the estimation (1.4) follows from the fact that it holds for the multiplicities of the nonreal or nonsimple real eigenvalues z_j of the π-selfadjoint operator A_0, see [8]. The theorem is proved.

REMARK 1. If $z_0 \in \sigma(A_0)$ is of algebraic multiplicity $m > 1$, the elements $\tilde{\varphi}_{z_0;j}$:

$$\tilde{\varphi}_{z_0;j}(t) := \mathrm{i}^j (j!)^{-1} \int_0^t (t-s)^j \mathrm{e}^{\mathrm{i}z_0(t-s)} \varphi_0(s)\,\mathrm{d}s, \quad j = 0, 1, \ldots, m-1, \tag{1.12}$$

($\tilde{\varphi}_{z_0;0} = \tilde{\varphi}_{z_0}$) form a Jordan chain of A_0 corresponding to z_0. It can be seen as above that no linear combination of these elements belongs to S_λ.

REMARK 2. A π-selfadjoint operator with discrete spectrum has a complete system of eigen and associated vectors. Thus, under the conditions of Theorem 1.1, all the functions $\tilde{\varphi}_{z_0;j}$ in (1.12) with z_0 running through $\sigma(A_0) = \{z : \mathscr{F}(\varphi_0; z) = 0\}$ form a complete system in $L^2(\mathbf{H} - \lambda I)$ and, after adjoining a basis of S_λ, we get a complete system in $L^2(0, 2a)$.

REMARK 3. If, in particular, $\varkappa = 0$, the function $\mathscr{F}(\varphi_0; \cdot)$ in Theorem 1.1 has only real and simple zeros. This was formulated in [6] (however, there it was not indicated that for φ_0 the first element of a D-chain must be chosen).

1.3. Let again $H \in L^1(-2a, 2a)$ be Hermitian. Besides the operator $\mathbf{H}$, with H we define the operators $\mathbf{H}_r$, $0 < r \leqslant a$:

$$(\mathbf{H}_r\varphi)(t) = \int_0^{2r} H(t-s)\varphi(s)\,\mathrm{d}s \quad (0 \leqslant t \leqslant 2r)$$

in $L^2(0, 2r)$ or $C(0, 2r)$. The point $r \in (0, a]$ is called a *singular point* of H of order $\varkappa(r)$ if -1 is an eigenvalue of the operator $\mathbf{H}_r$ of multiplicity $\varkappa(r)$, that is, if the equation

$$\varphi(t) + \int_0^{2r} H(t-s)\varphi(s)\,ds = 0 \quad (0 \leqslant t \leqslant 2r)$$

has $\varkappa(r)$ linearly independent solutions φ (in $C(0, 2r)$ or in $L^2(0, 2r)$). In other words, r is a singular point of H if and only if the scalar product

$$[\varphi, \psi]_r := \int_0^{2r} \varphi(t)\overline{\psi(t)}\,dt + \int_0^{2r}\int_0^{2r} H(t-s)\varphi(s)\overline{\psi(t)}\,ds\,dt$$

on $L^2(0, 2r)$ degenerates. By (1.1), this scalar product is positive definite if r is sufficiently small: $\int_{-2r}^{2r} |H(s)|\,ds < 1$.

(3°) *The number $\varkappa_{\mathbf{H}+I}$ of negative squares of the scalar product $[\cdot,\cdot]_a$ coincides with the total order of the singular points $< a$ of H. In particular, the number of singular points of H is finite.*

Proof. (a) If $0 < r \leqslant a$ the operators $\tilde{\mathbf{H}}_r$ in $L^2(0, 2a)$:

$$(\tilde{\mathbf{H}}_r\varphi)(t) := \chi_{[0,2r]}(t)\int_0^{2r} H(t-s)\varphi(s)\,ds \quad (0 \leqslant t \leqslant 2a),$$

depend continuously on r (with respect to the operator norm). Indeed, we have for $0 < r' < r \leqslant a$:

$$\int_0^{2r}\left|\int_0^{2r} H(t-s)\varphi(s)\,ds - \chi_{[0,2r']}(t)\int_0^{2r'} H(t-s)\varphi(s)\,ds\right|^2 dt \leqslant$$

$$\leqslant \int_0^{2r}\left|\int_{2r'}^{2r} H(t-s)\varphi(s)\,ds\right|^2 dt + \int_{2r'}^{2r}\left|\int_0^{2r} H(t-s)\varphi(s)\,ds\right|^2 dt \leqslant$$

$$\leqslant \int_0^{2r}\left|\int_{2r'}^{2r} |H(t-s)|\,ds\right|\int_{2r'}^{2r} |H(t-s)|\,|\varphi(s)|^2\,ds\,dt + \int_{2r'}^{2r}\int_0^{2r} |H(t-s)|\,ds\,\cdot$$

$$\cdot\int_0^{2r} |H(t-s)|\,|\varphi(s)|^2\,ds\,dt \leqslant$$

$$\leqslant 2\int_0^{2a} |\varphi(s)|^2\,ds \int_{-2a}^{2a} |H(t)|\,dt \cdot \max_{0 \leqslant t \leqslant 2r} \int_{2r'}^{2r} |H(t-s)|\,ds,$$

and the statement follows as the last factor becomes arbitrarily small if r' is sufficiently close to r.

Evidently, $\mathbf{H}_r$ and $\tilde{\mathbf{H}}_r$ have the same eigenvalues and their eigenfunctions are in a bijective correspondence. Denoting the negative eigenvalues of these operators by $\lambda_{r,1} \leqslant \lambda_{r,2} \leqslant \ldots$, we have

$$\lambda_{r,j} = \min_{\dim L_j = j} \max_{x \in L_j} (\tilde{\mathbf{H}}_r x, x)_{L^2},$$

where L_j is an arbitrary (j-dimensional) subspace of $L^2(0, 2a)$. However, it is sufficient to form the minimum with respect to all those j-dimensional subspaces of $L^2(0, 2a)$, whose elements vanish outside $[0, 2r]$. Therefore, if r increases the set of these subspaces becomes larger, hence $\lambda_{r,j}$ is a non-increasing function of r. By the continuity of $\tilde{\mathbf{H}}_r$ with respect to r and well-known results of perturbation theory each of the eigenvalues $\lambda_{r,j}$, $j = 1, 2, \ldots$, is a continuous function of r. Therefore the number of negative squares of the scalar product $[\cdot,\cdot]_a$, that is the total number $\varkappa_{I+\mathbf{H}}$ of eigenvalues < -1 of $\mathbf{H}$, is not greater than the total order $\varkappa$ of the singular points $< a$ of H.

(b) Let r, $0 < r < a$, be a singular point of H and $\varphi_{r,0}, \ldots, \varphi_{r,n}$ be a D-chain of solutions of the equation $\varphi + \mathbf{H}_r \varphi = 0$. We extend $\varphi_{r,j}$ to $[0, 2a]$ as follows:

$$\tilde{\varphi}_{r,j}(t) := \begin{cases} \varphi_{r,j}(t) & 0 \leqslant t \leqslant 2r, \\ 0 & 2r < t \leqslant 2a. \end{cases}$$

Then the elements $\tilde{\varphi}_{r;j}$, $j = 0, 1, \ldots, n$, are linearly independent. They form a neutral subspace of $L^2(I + \mathbf{H}) : [\tilde{\varphi}_{r;j}, \tilde{\varphi}_{r;k}] = 0$, $j, k = 0, 1, \ldots, n$, and no element of this subspace is isotropic in $L^2(I + \mathbf{H})$. Otherwise, for some $\tilde{\varphi} = \sum_0^n \alpha_j \tilde{\varphi}_{r,j}$, $\tilde{\varphi} \neq 0$, we would have $(I + \mathbf{H})\tilde{\varphi} = 0$, which is impossible as $\tilde{\varphi}$ vanishes identically in a neighbourhood of $2a$.

If r', $0 < r' < a$, is another singular point of H with corresponding elements $\tilde{\varphi}_{r';j'}$, $j' = 0, \ldots, n'$, we have also $[\tilde{\varphi}_{r,j}, \tilde{\varphi}_{r',j'}] = 0$, $j = 0, \ldots, n$; $j' = 0, \ldots, n'$. Moreover, the elements $\tilde{\varphi}_{r',0}, \ldots, \tilde{\varphi}_{r',n'}$, are linearly independent from the elements $\varphi_{r,0}, \ldots, \varphi_{r,n}$. This follows from the fact that if, e.g., $r' < r$, the elements $\tilde{\varphi}_{r;j}$ do not vanish identically in a neighbourhood of 0 (see Theorem 1.1). Thus the total order $\varkappa$ of the singular points $< a$ of H gives rise to a neutral subspace of $L^2(I + \mathbf{H})$ of dimension $\varkappa$. Therefore $\varkappa \leqslant \varkappa_{I+\mathbf{H}}$, and the statement (3°) is proved.

1.4. Now we suppose that a is not a singular point of the Hermitian function $H \in L^1(-2a, 2a)$, that is $-1 \notin \sigma_p(\mathbf{H})$. Then for arbitrary $f \in L^2(0, 2a)$ or $C(0, 2a)$ the integral equation

$$\varphi(t) + \int_0^{2a} H(t - s)\varphi(s)\,\mathrm{d}s = f(t) \quad (0 \leqslant t \leqslant 2a) \tag{1.13}$$

has a unique solution $\varphi \in L^2(0, 2a)$ or $\in C(0, 2a)$, which can be written as

$$\varphi(t) = f(t) - \int_0^{2a} \Gamma_a(t, s) f(s)\,\mathrm{d}s \quad (0 \leqslant t \leqslant 2a).$$

Here Γ_a is the resolvent kernel of (1.13), that is the (unique) solution of the equation

$$\Gamma_a(t, s) + \int_0^{2a} H(t-u)\Gamma_a(u, s)\,\mathrm{d}u = H(t-s) \quad (0 \leqslant s, t \leqslant 2a), \tag{1.14}$$

or, equivalently,

$$\Gamma_a(t, s) + \int_0^{2a} \Gamma_a(t, u)H(u-s)\,\mathrm{d}u = H(t-s) \quad (0 \leqslant s, t \leqslant 2a). \tag{1.15}$$

If in (1.14) $s \in [0, 2a]$ is fixed the function $H(\cdot - s)$ on the right hand side belongs to $L^1(0, 2a)$. Moreover, $s \to H(\cdot - s)$ is a continuous mapping from $[0, 2a]$ into $L^1(0, 2a)$. Hence also the solution $\Gamma_a(\cdot, s)$ of (1.14) belongs to $L^1(0, 2a)$ for fixed $s \in [0, 2a]$, and $s \to \Gamma_a(\cdot, s)$ is a continuous mapping from $[0, 2a]$ into $L^1(0, 2a)$. It follows that if one argument in Γ_a is fixed it represents an element of $L^1(0, 2a)$ with respect to the other argument, depending continuously on the previously fixed argument.

PROPOSITION 1.1. *The kernel Γ_a has the following properties $(0 \leqslant s, t \leqslant 2a)$:*
(1) *$s \to \Gamma_a(\cdot, s)$ is a continuous mapping from $[0, 2a]$ into $L^1(0, 2a)$;*
(2) $\Gamma_a(t, s) = \overline{\Gamma_a(s, t)}$, $\Gamma_a(t, s) = \Gamma_a(2a - s, 2a - t)$;
(3) $\dfrac{\partial \Gamma_a(t, s)}{\partial t} + \dfrac{\partial \Gamma_a(t, s)}{\partial s} = \Gamma_a(t, 2a)\Gamma_a(2a, s) - \Gamma_a(t, 0)\Gamma_a(0, s)$.

In (3) the derivatives are to be understood as generalized functions, that is, for arbitrary φ, $\psi \in C^\infty(0, 2a)$ with compact support in $(0, 2a)$ we have

$$\begin{aligned} &-\int_0^{2a}\int_0^{2a} \Gamma_a(t, s)(\varphi'(s)\psi(t) + \varphi(s)\psi'(t))\,\mathrm{d}s\,\mathrm{d}t = \\ &= \int_0^{2a} \Gamma_a(t, 2a)\psi(t)\,\mathrm{d}t \int_0^{2a} \Gamma_a(2a, s)\varphi(s)\,\mathrm{d}s - \int_0^{2a} \Gamma_a(t, 0)\psi(t)\,\mathrm{d}t \int_0^{2a} \Gamma_a(0, s)\varphi(s)\,\mathrm{d}s. \end{aligned} \tag{1.16}$$

For real smooth kernels the relation (3) has first been obtained by Sobolev [27] (see also [28]).

Proof. (1) has been shown above. The first relation in (2) holds because H is Hermitian. Furthermore, from (1.14) we have

$$\Gamma_a(2a-s,\,2a-t)+\int_0^{2a} H(2a-s-u)\Gamma_a(u,\,2a-t)\,du = H(t-s),$$

which can be written as

$$\Gamma_a(2a-s,\,2a-t)+\int_0^{2a} H(v-s)\Gamma_a(2a-v,\,2a-t)\,dv = H(t-s).$$

Comparing this relation with (1.14) the second equality in (2) follows from the uniqueness of Γ_a.

In order to prove (3) we first suppose that H is absolutely continuous on $[-2a, 2a]$. Then, by (1.14) and (1.15), $\frac{\partial\Gamma_a(t,s)}{\partial t}$ and $\frac{\partial\Gamma_a(t,s)}{\partial s}$ exist a.e. and we have

$$\frac{\partial\Gamma_a(t,s)}{\partial t}+\int_0^{2a} H'(t-u)\Gamma_a(u,s)\,du = H'(t-s),$$

$$\frac{\partial\Gamma_a(t,s)}{\partial s}+\int_0^{2a} H'(t-u)\,\frac{\partial\Gamma_a(u,s)}{\partial s}\,du = -H'(t-s).$$

Integrating by parts, the first of these relation becomes

$$\frac{\partial\Gamma_a(t,s)}{\partial t}-H(t-2a)\Gamma_a(2a,s)+H(t)\Gamma_a(0,s)+\int_0^{2a} H(t-u)\,\frac{\partial\Gamma_a(u,s)}{\partial u}\,du = H'(t-s),$$

hence $\Delta(t,s) := \frac{\partial\Gamma_a(t,s)}{\partial t}+\frac{\partial\Gamma_a(t,s)}{\partial s}$ satisfies the equation

$$\Delta(t,s)+\int_0^{2a} H(t-u)\Delta(u,s)\,du = H(t-2a)\Gamma_a(2a,s)-H(t)\Gamma_a(0,s). \tag{1.17}$$

On the other hand, putting in (1.14) $s = 2a$ ($s = 0$) and multiplying it with $\Gamma_a(2a, s)$ ($\Gamma_a(0, s)$, resp.) we get

$$\begin{aligned}\Gamma_a(t,2a)\Gamma_a(2a,s)+\int_0^{2a} H(t-u)\Gamma_a(u,2a)\Gamma_a(2a,s)\,du &= H(t-2a)\Gamma_a(2a,s),\\ \Gamma_a(t,0)\Gamma_a(0,s)+\int_0^{2a} H(t-u)\Gamma_a(u,0)\Gamma_a(0,s)\,du &= H(t)\Gamma_a(0,s).\end{aligned} \tag{1.18}$$

If we form the difference of these relations and compare it with (1.17) it follows that

$$\Delta(t, s) = \Gamma_a(t, 2a)\Gamma_a(2a, s) - \Gamma_a(t, 0)\Gamma_a(0, s).$$

Now let H be an arbitrary Hermitian function of $L^1(-2a, 2a)$. We choose a sequence $(H^{(n)})$ of absolutely continuous Hermitian functions which converges to H in the norm of $L^1(-2a, 2a)$. Then the corresponding operators $\mathbf{H}^{(n)}$ converge to $\mathbf{H}$ in the operator norm of $C(0, 2a)$ and, consequently, we have also $(I + \mathbf{H}^{(n)})^{-1} \to \to (I + \mathbf{H})^{-1}$ $(n \to \infty)$ with respect to the operator norm of $C(0, 2a)$. Denoting the resolvent kernel of $\mathbf{H}^{(n)}$ by $\Gamma_a^{(n)}$, it follows for arbitrary $\varphi, \psi \in C(0, 2a)$

$$\int_0^{2a}\int_0^{2a} \Gamma_a^{(n)}(t, s)\varphi(s)\psi(t)\,ds\,dt \to \int_0^{2a}\int_0^{2a} \Gamma_a(t, s)\varphi(s)\psi(t)\,ds\,dt \quad (n \to \infty). \tag{1.19}$$

As we have shown above, the relation (1.16) holds if Γ_a is replaced by $\Gamma_a^{(n)}$. Letting $n \to \infty$ and observing (1.19) the statement follows. The Proposition 1.1 is proved.

The differential equation (3) in Proposition 1.1 gives the possibility to express $\Gamma_a(t, s)$ by means of its "boundary values" $\Gamma_a(t, 0)$ (and $\Gamma_a(0, t) = \overline{\Gamma_a(t, 0)}$). Indeed, denoting the right hand side in (3) by $g(t, s)$:

$$\begin{aligned} g(t, s) &:= \Gamma_a(0, 2a - t)\Gamma_a(2a - s, 0) - \Gamma_a(t, 0)\Gamma_a(0, s) = \\ &= \Gamma_a(2a - s, 0)\overline{\Gamma_a(2a - t, 0)} - \Gamma_a(t, 0)\overline{\Gamma_a(s, 0)} \end{aligned} \tag{1.20}$$

we have

$$\Gamma_a(t, s) = h(t - s) + \int_0^{\min(s, t)} g(t - r, s - r)\,dr \quad (0 \leqslant s, t \leqslant 2a),$$

with an arbitrary (summable) function h on $[-2a, 2a]$, which can be determined from the boundary condition. Thus we have proved the

COROLLARY 1.1. *The kernel $\Gamma_a(t, s)$ is given by*

$$\Gamma_a(t, s) = \Gamma_a(t-s, 0) + \overline{\Gamma_a(s - t, 0)} + \int_0^{\min(s, t)} g(t-r, s-r)\,dr \quad (0 \leqslant s, t \leqslant 2a), \tag{1.21}$$

with g from (1.20) *and $\Gamma_a(u, 0) = 0$ if $u < 0$.*

Thus the resolvent kernel $\Gamma_a(t, s)$ and also the function H are determined by the function $\Gamma_a(t, 0)$ $(0 \leqslant t \leqslant 2a)$. In § 3 we shall give necessary and sufficient conditions (by means of the zeros of the orthogonal functions) for a given function $\Gamma(t)$ $(0 \leqslant t \leqslant 2a)$ to coincide with $\Gamma_a(t, 0)$ for some $H \in L^1(-2a, 2a)$.

The representation (1.21) of the resolvent kernel $\Gamma_a(t, s)$ was proved (even for non-Hermitian functions H but in a more complicated way) in [29]; the discrete analogue had been considered in [30].

If we restrict the function H to $[-2r, 2r]$, $0 \leqslant r \leqslant a$, it generates an integral operator in $L^2(0, 2r)$ or in $C(0, 2r)$. Its resolvent kernel will be denoted by Γ_r.

In the following proposition by ρ_H we denote the set of all $r \in (0, a]$ which are not singular points of H.

PROPOSITION 1.2. *For arbitrary $\varphi, \psi \in C(0, 2a)$ and $r \in \rho_H$ the following relation holds:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{2r}\int_0^{2r}\Gamma_r(t, s)\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t &= -2\int_0^{2r}\Gamma_r(t, 2r)\psi(t)\,\mathrm{d}t\int_0^{2r}\Gamma_r(2r, s)\varphi(s)\,\mathrm{d}s + \\ &+\int_0^{2r}\Gamma_r(2r, s)\varphi(s)\,\mathrm{d}s\,\psi(2r) + \int_0^{2r}\Gamma_r(t, 2r)\psi(t)\,\mathrm{d}t\,\varphi(2r).\end{aligned} \tag{1.22}$$

If, additionally, the accelerant H is continuous on $[-2a, 2a]$, then $\Gamma_r(t, s)$ depends continuously on the three arguments r, s, t ($r \in \rho_H$, $0 \leqslant s, t \leqslant 2r$) and

$$\frac{\partial \Gamma_r(t, s)}{\partial r} = -2\Gamma_r(t, 2r)\Gamma_r(2r, s) \quad (0 \leqslant t, s \leqslant 2r). \tag{1.23}$$

We mention that the relation (1.23) for a continuous accelerant H is well known (see [49]). It also holds if we start not with a displacement kernel but with an arbitrary continuous kernel $H(t, s)$ $(0 \leqslant t, s \leqslant 2a)$.

Proof. If $H \in C(-2a, 2a)$ we consider the Fredholm determinant $\mathscr{D}_r(-1)$ for the interval $(0, 2r)$ $(0 < r \leqslant a)$ and its minor $\Delta_r(t, s; -1)$. Then $r \in \rho_H$ if and only if $\mathscr{D}_r(-1) \neq 0$, which is equivalent to the existence of the resolvent $\Gamma_r(t, s)$, and we have

$$\Gamma_r(t, s) = \frac{\Delta_r(t, s; -1)}{\mathscr{D}_r(-1)}.$$

The Fredholm formulae for $\mathscr{D}_r$ and Δ_r imply that these functions as well as $\Gamma_r(t, s)$ have a continuous derivative with respect to $r \in \rho_H$.

If in (1.14) we replace a by $r \in \rho_H$ and differentiate with respect to r then we obtain

$$\frac{\partial \Gamma_r(t, s)}{\partial r} + \int_0^{2r} H(t-u)\frac{\partial \Gamma_r(u, s)}{\partial r}\,\mathrm{d}u = -2H(t-2r)\Gamma_r(2r, s) \quad (0 \leqslant t, s \leqslant 2r).$$

Comparing this equality with (1.14) (with a replaced by r) the relation (1.23) follows.

In the general case $H \in L^1(-2a, 2a)$ for a given $\delta > 0$ we choose a Hermitian function $H^\delta \in C(-2a, 2a)$ such that

$$\int_{-2a}^{2a} |H(t) - H^\delta(t)|\, \mathrm{d}t < \delta.$$

Then for the corresponding operators $\mathbf{H}$ and $\mathbf{H}^\delta$ in $C(0, 2a)$ we have $\|\mathbf{H} - \mathbf{H}^\delta\|_C < \delta$. Therefore, if $r \in \rho_H$ there exists a neighbourhood $\mathfrak{U}$ of r belonging to $\rho_H \cap \rho_{H^\delta}$ for sufficiently small δ. As H^δ is continuous, the relation (1.23) and so the relation (1.22) with Γ replaced by the resolvent kernel Γ^δ of $\mathbf{H}^\delta$ hold. If we integrate this relation from r_0 to r ($r_0 < r$; $r_0 \in \mathfrak{U}$) we find

$$\int_0^{2r}\int_0^{2r} \Gamma_r^\delta(t, s)\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t - \int_0^{2r_0}\int_0^{2r_0} \Gamma_{r_0}^\delta(t, s)\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t =$$

$$(1.24)\qquad = -2\int_{r_0}^{r} \Big\{ \int_0^{2u} \Gamma_u^\delta(t, 2u)\psi(t)\,\mathrm{d}t \int_0^{2u} \Gamma_u^\delta(2u, s)\varphi(s)\,\mathrm{d}s +$$

$$+ \int_0^{2u} \Gamma_u^\delta(2u, s)\varphi(s)\,\mathrm{d}s\,\psi(2u) + \int_0^{2u} \Gamma_u^\delta(t, 2u)\psi(t)\,\mathrm{d}t\,\varphi(2u)\Big\}\,\mathrm{d}u.$$

As the operators $\mathbf{H}$ and $\mathbf{H}^\delta$ are arbitrarily close if $\delta \downarrow 0$ the functions $\Gamma_u(\cdot, 2u)$ and $\Gamma_u^\delta(\cdot, 2u)$ are arbitrarily close in the norm of $L^1(0, 2a)$, and the same holds for $\Gamma_u(2u, \cdot)$ and $\Gamma_u^\delta(2u, \cdot)$. Moreover, it is easy to see that for fixed $r \in \mathfrak{U}$

$$\lim_{\delta \downarrow 0} \int_0^{2r}\int_0^{2r} \Gamma_r^\delta(t, s)\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t = \int_0^{2r}\int_0^{2r} \Gamma_r(t, s)\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore we can pass in (1.24) to the limit $\delta \downarrow 0$, i.e. we can replace in (1.24) Γ_r^δ, $\Gamma_{r_0}^\delta$, Γ_u^δ by Γ_r, Γ_{r_0}, Γ_u, respectively, and obtain a relation equivalent to (1.22). The Proposition 1.2 is proved.

1.5. Let again $H \in L^1(-2a, 2a)$ be a Hermitian function. In the space $L^2(0, 2a)$ we introduce again the (possibly indefinite or degenerated) scalar product

$$(1.25)\qquad [\varphi, \psi] := \int_0^{2a} \varphi(t)\overline{\psi(t)}\,\mathrm{d}t + \int_0^{2a}\int_0^{2a} H(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t.$$

If $-1 \notin \sigma(\mathbf{H})$ then $L^2(0, 2a)$, equipped with this scalar product, is a $\pi_\varkappa$-space, $\varkappa =: \varkappa_{I+\mathbf{H}}$; if $-1 \in \sigma(\mathbf{H})$, this holds for the factor space $L^2(0, 2a)/\ker(I + \mathbf{H})$. In both cases this $\pi_\varkappa$-space will be denoted by $L^2(I + \mathbf{H})$. By A_0 we denote the following operator in $L^2(I + \mathbf{H})$ (comp. the proof of Theorem 1.1):

$\mathfrak{D}(A_0)$ is the set of all absolutely continuous functions $\varphi \in L^2(0, 2a)$ such that $\varphi' \in L^2(0, 2a)$ and $\varphi(0) = \varphi(2a) = 0$, and

$$A_0\varphi := \frac{1}{\mathrm{i}}\,\varphi' \quad (\varphi \in \mathfrak{D}(A_0)).$$

Then A_0 induces a closed π-Hermitian operator in $L^2(I + \mathbf{H})$, which will also be denoted by A_0.

(4°) *If* $-1 \in \sigma(\mathbf{H})$ *then the operator* A_0 *is* π*-selfadjoint. If* $-1 \notin \sigma(\mathbf{H})$ *then* A_0 *is closed,* π*-Hermitian with deficiency index* $(1\,;1)$ *and simple.*

Proof. The first statement was proved already in the course of the proof of Theorem 1.1. We only recall that for $z \in \rho(A_0)$ the resolvent $(A_0 - zI)^{-1}$ is given by the formula

$$((A_0 - zI)^{-1}u)(t) = \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}z(t-s)}u(s)\,\mathrm{d}s - \mathrm{i}\,\frac{\mathscr{F}(u; z)}{\mathscr{F}(\varphi_0; z)}\int_0^t \mathrm{e}^{\mathrm{i}z(t-s)}\varphi_0(s)\,\mathrm{d}s \quad (0 \leqslant t \leqslant 2a),$$

where $u \in L^2(I + \mathbf{H})$ and φ_0 is the first element of a D-chain of $\mathbf{H}$ corresponding to $\lambda = -1$.

Now suppose that $-1 \notin \sigma(\mathbf{H})$. By $e_{\mathrm{i}z}$ we denote the function $e_{\mathrm{i}z}(t) := \mathrm{e}^{\mathrm{i}zt}$ $(0 \leqslant t \leqslant 2a)$. Then the function

$$q_z := (I - \mathbf{\Gamma}_a)e_{\mathrm{i}z}$$

is π-orthogonal on the range of $A_0 - \bar{z}I$:

$$[(A_0 - \bar{z}I)\varphi, q_z] = ((A_0 - \bar{z}I)\varphi, e_{\mathrm{i}z}) = 0 \quad (\varphi \in \mathfrak{D}(A_0)).$$

Therefore A_0 has deficiency index $(1; 1)$. As the elements q_z $(z \neq \bar{z})$ form a total subset of $L^2(I + \mathbf{H})$ the π-Hermitian operator A_0 is simple.

1.6. Sometimes it is more convenient to use the space $L^2(-a, a)$ instead of $L^2(0, 2a)$, and to equip it with the (possibly indefinite or degenerated) scalar product

$$(1.26) \qquad [\varphi, \psi] := \int_{-a}^{a} \varphi(t)\overline{\psi(t)}\,\mathrm{d}t + \int_{-a}^{a}\int_{-a}^{a} H(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t.$$

The mapping $\varphi(s) \to \varphi(s + a)$ $(-a \leqslant s \leqslant a)$ establishes an isomorphism between $L^2(-a, a)$ and $L^2(0, 2a)$, which is also isometric with respect to the scalar products

(1.26) and (1.25) on $L^2(-a,a)$ and $L^2(0,2a)$, respectively. In particular, with the operator $\dot{\mathbf{H}}$ in $L^2(-a,a)$:

$$(\dot{\mathbf{H}}\varphi)(t) := \int_{-a}^{a} H(t-s)\varphi(s)\,\mathrm{d}s \quad (-a \leqslant t \leqslant a)$$

we have $\sigma(\mathbf{H}) = \sigma(\dot{\mathbf{H}})$, and the scalar product (1.26) has the same signature as that of (1.25). Thus its number of negative squares equals the total multiplicity of eigenvalues < -1 of $\dot{\mathbf{H}}$ and it degenerates on the subspace $\ker(I+\dot{\mathbf{H}})$ which is nontrivial if and only if $-1 \in \sigma(\dot{\mathbf{H}})$. If $-1 \notin \sigma(\dot{\mathbf{H}})$, the resolvent kernel of $\dot{\mathbf{H}}$ in $L^2(-a,a)$ will be denoted by $\dot{\Gamma}_a(t,s)$; it is connected with the resolvent kernel $\Gamma_a(t,s)$ of $\mathbf{H}$ in $L^2(0,2a)$ by the relation

$$\dot{\Gamma}_a(t,s) = \Gamma_a(t+a, s+a) \quad (-a \leqslant s, t \leqslant a).$$

The properties (2) and (3) in Proposition 1.1 look for $\dot{\Gamma}_a$ as follows $(-a \leqslant s, t \leqslant a)$:

$$\begin{gathered}\dot{\Gamma}_a(t,s) = \overline{\dot{\Gamma}_a(s,t)}, \quad \dot{\Gamma}_a(t,s) = \dot{\Gamma}_a(-s,-t),\\ \frac{\partial \dot{\Gamma}_a(t,s)}{\partial t} + \frac{\partial \dot{\Gamma}_a(t,s)}{\partial s} = \dot{\Gamma}_a(t,a)\dot{\Gamma}_a(a,s) - \dot{\Gamma}_a(t,-a)\dot{\Gamma}_a(-a,s).\end{gathered} \tag{1.27}$$

The factor space $L^2(-a,a)/\ker(I+\dot{\mathbf{H}})$ equipped with the scalar product (1.26) will be denoted by $L^2(I+\dot{\mathbf{H}})$. It is a $\pi_\varkappa$-space with

$$\varkappa = \varkappa_{I+\dot{\mathbf{H}}} = \varkappa_{I+\mathbf{H}}.$$

In $L^2(-a,a)$ we consider the operator $\dot{A}_0$: $\mathfrak{D}(\dot{A}_0)$ is the set of all absolutely continuous functions $\varphi \in L^2(-a,a)$ such that $\varphi' \in L^2(-a,a)$ and $\varphi(-a) = \varphi(a) = 0$,

$$\dot{A}_0\varphi := \frac{1}{\mathrm{i}}\varphi' \quad (\varphi \in \mathfrak{D}(\dot{A}_0)).$$

It has the property

$$[\dot{A}_0\varphi, \psi] = [\varphi, \dot{A}_0\psi] \quad (\varphi, \psi \in \mathfrak{D}(\dot{A}_0)),$$

hence it induces a closed π-Hermitian operator in $L^2(I+\dot{\mathbf{H}})$, which will again be denoted by $\dot{A}_0$.

The statement (4°) remains true if A_0 and $\mathbf{H}$ are replaced by $\dot{A}_0$ and $\dot{\mathbf{H}}$. If $-1 \notin \sigma(\dot{\mathbf{H}})$ the defect elements $\dot{q}_z$ of $\dot{A}_0$ are now given by the relation

$$\dot{q}_z = (I - \dot{\Gamma}_a)\dot{e}_{\mathrm{i}z} \tag{1.28}$$

$(\dot{e}_{\mathrm{i}z}(t) := \mathrm{e}^{\mathrm{i}zt}\ (-a \leqslant t \leqslant a))$.

§ 2. HERMITIAN FUNCTIONS WITH AN ACCELERANT

2.1. The Hermitian function g, defined on $(-2a, 2a)$, $0 < a \leqslant \infty$, is said to have an *accelerant* H if it admits a representation

$$(2.1) \qquad g(t) = g(0) - \alpha|t| - \int_0^t (t-s)H(s)\,\mathrm{d}s \quad (-2a < t < 2a)$$

with some $\alpha > 0$ and a function $H \in L^1(-2a, 2a)$ if $a < \infty$ and $H \in L^1_{\mathrm{loc}}(-\infty, \infty)$ if $a = \infty$. That is, g has an absolutely continuous first derivative on $(-2a, 0)$ and on $(0, 2a)$:

$$g'(t) = -\alpha \operatorname{sgn} t - \int_0^t H(s)\,\mathrm{d}s \quad (-2a \leqslant t \leqslant 2a,\ t \neq 0)$$

$g'(0+) = -\alpha < 0$, such that $H: H(t) := -g''(t)$ $(t \neq 0)$ is summable on $(-2a, 2a)$ if $a < \infty$ and locally summable on the real axis if $a = \infty$. The function g in (2.1) is Hermitian if and only if H is Hermitian and $g(0)$ is real. Without loss of generality we suppose in the following $\alpha = 1/2$, that is g is given by

$$(2.2) \qquad g(t) = g(0) - \frac{1}{2}|t| - \int_0^t (t-s)H(s)\,\mathrm{d}s \quad (-2a \leqslant t \leqslant 2a).$$

Recall (see [2, § 5], [1]) that for a nonnegative integer $\varkappa$ by $\mathfrak{G}_{\varkappa;a}$ ($\mathfrak{P}_{\varkappa;a}$) we denote the set of all continuous Hermitian functions g (f) on $[-2a, 2a]$ such that the kernel

$$G_g(t,s) := g(t-s) - g(t) - \overline{g(s)} + g(0) \quad (F_f(t,s) := f(t-s), \text{ resp.})$$

$(0 \leqslant t, s \leqslant 2a)$ has $\varkappa$ negative squares, and we write $\mathfrak{G}_{\varkappa;\infty} =: \mathfrak{G}_\varkappa$, $\mathfrak{P}_{\varkappa;\infty} =: \mathfrak{P}_\varkappa$. It was shown in [2] that for arbitrary $g \in \mathfrak{G}_\varkappa \cup \mathfrak{P}_\varkappa$ there exists a $\gamma > 0$ (depending on g) such that $g(t) = \mathrm{O}(\mathrm{e}^{\gamma t})$ if $t \to \infty$.

Let H be as above. If $a < \infty$ and $-1 \in \sigma(\mathbf{H})$, by φ_0 we denote the first element of a D-chain of $\mathbf{H}$ corresponding to the eigenvalue $\lambda = -1$, and we put

$$\psi_0(t) := \frac{1}{2}\varphi_0(t) + 2\int_t^{2a} H(t-s)\varphi_0(s)\,\mathrm{d}s \quad (0 \leqslant s \leqslant 2a).$$

THEOREM 2.1. *Let g on $(-2a, 2a)$, $0 < a < \infty$, be a Hermitian function with accelerant $H \in L^1(-2a, 2a)$, that is, g admits a representation* (2.2). *Then the following statements hold true:*

(1) *$g \in \mathfrak{G}_{\varkappa;a}$ with $\varkappa := \varkappa_{\mathbf{H}+I}$.*

(2) *g has at least one continuation $\tilde{g} \in \mathfrak{G}_\varkappa$.*

(3) *The continuation* $\tilde{g} \in \mathfrak{G}_\varkappa$ *in* (2) *is uniquely determined if and only if* $-1 \in \sigma(\mathbf{H})$. *In this case* $\tilde{g}$ *is given by the relation*

$$z^2 \int_0^\infty e^{-itz}\tilde{g}(t)\,dt + izg(0) = -\frac{\mathcal{F}(\psi_0; z)}{\mathcal{F}(\varphi_0; z)} \quad (\operatorname{Im} z < -\gamma) \tag{2.3}$$

for some $\gamma \leqslant 0$.

(4) *If* $-1 \notin \sigma(\mathbf{H})$ *there are infinitely many continuations* $\tilde{g} \in \mathfrak{G}_\varkappa$ *of* g *which have an accelerant.*

Proof. Forming the kernel $G_g(t, s) := g(t-s) - g(t) - \overline{g(s)} + g(0)$, integration by parts shows that the following relation holds true for arbitrary $\varphi \in C(0, 2a)$:

$$\int_0^{2a}\int_0^{2a} G_g(t, s)\varphi(s)\overline{\varphi(t)}\,ds\,dt = \int_0^{2a} |\Phi(s)|^2\,ds + \int_0^{2a}\int_0^{2a} H(t-s)\varphi(s)\overline{\varphi(t)}\,ds\,dt \tag{2.4}$$

with $\Phi(t) := \int_t^{2a} \varphi(s)\,ds$ $(0 \leqslant t \leqslant 2a)$. As the set of these functions Φ is dense in $L^2(0, 2a)$ this relation implies $g \in \mathfrak{G}_{\varkappa;a}$ with $\varkappa = \varkappa_{\mathbf{H}+I}$, and (1) is proved.

The statement (2), which holds for an arbitrary function $g \in \mathfrak{G}_{\varkappa;a}$ (see [17]), follows from the remark after Behauptung 4.4 in [31] and the fact that each π--Hermitian operator has at least one index-preserving extension, see [31, Satz 1.2]. It also follows from the fact that there is a bijective correspondence between all continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of a function $g \in \mathfrak{G}_{\varkappa;a}$ and all continuations $\widetilde{g''} \in \mathfrak{P}^d_{\varkappa;\infty}$ of the second derivative $g'' \in \mathfrak{P}^d_{\varkappa;a}$ (for the definition of $\mathfrak{P}^d_{\varkappa;a}$ see [32]), and [32, Proposition 1]. Moreover, the continuation $\widetilde{g''} \in \mathfrak{P}^d_{\varkappa;\infty}$ is uniquely determined if and only if the operator $A = \frac{1}{i}\frac{d}{dt}$ is π-selfadjoint in $\Pi_\varkappa(-g'')$ (this is not hard to see, comp. [32], [7]). On the other hand, $\Pi_\varkappa(-g'')$ coincides with the space $L^2(I + \dot{\mathbf{H}})$, and A is the operator $\dot{A}_0$ introduced in § 1.5. Thus the first statement in (3) follows from § 1.5, (4°). The second part of (3) will be proved in § 9.4, and a more explicite form of $\tilde{g}$ is given in Theorem 9.4. Finally, (4) follows from Theorem 9.2.

COROLLARY 2.1. *If* r, $0 < r \leqslant a$, *is sufficiently small, the restriction* $g_r := g_{[-2r, 2r]}$ *of* g *in* (2.2) *belongs to the class* $\mathfrak{G}_{0;r}$. *In particular, this holds if* $\max_{0 \leqslant t \leqslant 2r} \int_0^{2r} |H(t-s)|\,ds \leqslant 1$ *or if* $\int_{-2a}^{2a} |H(s)|\,ds \leqslant 1$.

PROPOSITION 2.1. *Let the Hermitian function* g *on* $(-2a, 2a)$ *be given by* (2.2). *If* $\dim\ker(I + \mathbf{H}) \geqslant 2$ *and the functions* $\varphi_0, \ldots, \varphi_n$ *form a* D*-chain of eigenvectors* *of* $\mathbf{H}$ *corresponding to* $\lambda = -1$, *then the linear span of* $\varphi_1, \ldots, \varphi_n$ *is neutral with respect to the scalar product*

$$[\varphi, \psi]' := \int_0^{2a}\int_0^{2a} g(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t. \tag{2.5}$$

Proof. If $\varphi, \psi \in C(0, 2a)$, $\Phi(t) := \int_t^{2a} \varphi(s)\,\mathrm{d}s$, $\Psi(t) := \int_t^{2a} \psi(s)\,\mathrm{d}s$, then

$$\int_0^{2a}\int_0^{2a} g(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t = g(0)\Phi(0)\overline{\Psi(0)} + \Phi(0)\int_0^{2a} g'(t)\overline{\Psi(t)}\,\mathrm{d}t + \overline{\Psi}(0)\int_0^{2a}\Phi(s)g'(s)\,\mathrm{d}s +$$

$$+ \int_0^{2a}\int_0^{2a} H(t-s)\Phi(s)\overline{\Psi(t)}\,\mathrm{d}s\,\mathrm{d}t + \int_0^{2a}\Phi(s)\overline{\Psi(t)}\,\mathrm{d}s\,\mathrm{d}t. \tag{2.6}$$

Now put $\varphi = \varphi_j$, $\psi = \varphi_k$, $j, k = 1, \ldots, n$. Then $\Phi(0) = \Phi(2a) = \Psi(0) = \Psi(2a) = 0$, $\Phi(t) + \int_0^{2a} H(t-s)\Phi(s)\,\mathrm{d}s = 0$ $(0 \leqslant s \leqslant 2a)$ and we find $\int_0^{2a}\int_0^{2a} g(t-s)\varphi_j(s)\overline{\varphi_k(t)}\,\mathrm{d}s\,\mathrm{d}t = 0$, which completes the proof.

REMARK. Evidently, Proposition 2.1 can also be formulated for the D-chain of the operator $\dot{\mathbf{H}}$. In this case in (2.5) the interval $[0, 2a]$ has to be replaced by $[-a, a]$. If the function g is real, the linear span of the elements $\varphi_1, \ldots, \varphi_n$ of a D-chain $\varphi_0, \varphi_1, \ldots, \varphi_n$ of $\dot{\mathbf{H}}$ corresponding to $\lambda = -1$ is also neutral with respect to the scalar product

$$[\varphi, \psi]'' := \int_{-a}^{a}\int_{-a}^{a} g(t+s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t.$$

2.2. Let $f \in \mathfrak{P}_{\varkappa;a}$. With f we associate the following $\pi_\varkappa$-space $\Pi_\varkappa(f)$. Consider the linear set $\mathscr{L}$ of all functions $\varphi \in C(0, 2a)$, which vanish in some neighbourhoods of 0 and of $2a$, and define on $\mathscr{L}$ the scalar product *)

$$[\varphi, \psi] := \int_0^{2a}\int_0^{2a} f(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t \quad (\varphi, \psi \in \mathscr{L}).$$

*) This scalar product is different from the scalar product $[.,.]$ in § 1. We hope that it is always clear from the context which scalar product we have in mind.

Then $\mathscr{L}$ can be canonically embedded into a $\pi_\varkappa$-space which we shall denote by $\Pi_\varkappa(f)$.

In the space $\Pi_\varkappa(f)$ we consider the operator A, which is the closure of $\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}$ defined on all functions $\varphi \in C^1(0, 2a) \cap \mathscr{L}$. It is not hard to see that A is a π-Hermitian operator with equal deficiency indices zero or one.

In the same way, starting with the linear set $\dot{\mathscr{L}}$ of all functions of $C(-a, a)$, which vanish in some neighbourhoods of $-a$ and a and defining a scalar product on $\dot{\mathscr{L}}$ by

$$[\varphi, \psi] := \int_{-a}^{a}\int_{-a}^{a} f(t-s)\varphi(s)\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t \quad (\varphi, \psi \in \dot{\mathscr{L}}),$$

we introduce the space $\dot{\Pi}_\varkappa(f)$. The operator $\dot{A}$ in $\dot{\Pi}_\varkappa(f)$ is defined in an analogous way as A in $\Pi_\varkappa(f)$. It is again a π-Hermitian operator with equal deficiency indices zero or one. Evidently the spaces $\Pi_\varkappa(f)$ and $\dot{\Pi}_\varkappa(f)$ as well as the operators A and $\dot{A}$ are isomorphic.

In the sequel we shall use the following fact (see [31, Behauptung 4.3]):

(1°) *The relation*

$$-\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{f}(t)\,\mathrm{d}t = [(\tilde{A} - zI)^{-1}u, u] \quad (\operatorname{Im} z < -\gamma)$$

with some $\gamma \geqslant 0$ *and* $u := 2\delta_0$ $(u := \delta_0)$ *establishes a bijective correspondence between all continuations* $\tilde{f} \in \mathfrak{P}_\varkappa$ *of* $f \in \mathfrak{P}_{\varkappa;a}$ *and all u-resolvents of* A $(\dot{A}$, *respectively).*

Recall that for an arbitrary π-selfadjoint extension $\tilde{A}$ of A the function

$$z \to [(\tilde{A} - zI)^{-1}u, u] \quad (z \in \rho(\tilde{A}))$$

is called a *u-resolvent* of A. If $\varphi \in C(0, 2a)$ $(\varphi \in C(-a, a))$ we define $\int_0^{2a} \varphi(t)\delta_0(t)\,\mathrm{d}t =$ $= (\varphi(0)/2)$ $\left(\int_{-a}^{a} \varphi(t)\delta_0(t)\,\mathrm{d}t = \varphi(0), \text{ respectively}\right)$. Sometimes it is more convenient to use the space $\dot{\Pi}_\varkappa(f)$ as the element $u = \delta_0 \in \dot{\Pi}_\varkappa(f)$ is real with respect to the involution $\varphi(t) \to \overline{\varphi(-t)}$.

Now let the function g be given by (2.2) and put $\varkappa = \varkappa_{\mathrm{H}+I}$, that is $g \in \mathfrak{G}_{\varkappa;a}$. Then g also belongs to some class $\mathfrak{P}_{\varkappa(g);a}$ with $\varkappa(g) = \varkappa$ or $\varkappa(g) = \varkappa + 1$. This is a general fact for functions $g \in \mathfrak{G}_{\varkappa;a}$ (see [2]); in the special case of a function g with accelerant considered here it is also an immediate consequence of the following statement.

(2°) *The space* $\Pi_{\varkappa(g)}(g)$ *is isomorphic to the space* $L^2(\mathfrak{G})$ *of all vectors* $(\Phi, \xi)^T$ $(\Phi \in L^2(0, 2a),\ \xi \in \mathbf{C})$ *with scalar product given by the Gram operator*

$$\mathfrak{G} := \begin{pmatrix} I+\mathbf{H} & g' \\ (\cdot, g') & g(0) \end{pmatrix} \quad in\ L^2(0, 2a) \oplus \mathbf{C}.$$

Consequently, $\varkappa(g) = \varkappa_{\mathfrak{G}}$.

Here, of course, $\varkappa_{\mathfrak{G}}$ denotes the total number of negative eigenvalues of $\mathfrak{G}$.

Proof. The relation (2.6) implies

$$\int_0^{2a}\int_0^{2a} g(t-s)\varphi(s)\overline{\varphi(t)}\,\mathrm{d}s\,\mathrm{d}t = \int_0^{2a} |\Phi(t)|^2\,\mathrm{d}t + \int_0^{2a}\int_0^{2a} H(t-s)\Phi(s)\overline{\Phi(t)}\,\mathrm{d}s\,\mathrm{d}t +$$

$$+ 2\mathrm{Re}\left(\int_0^{2a} \Phi(s)\overline{g'(s)}\,\mathrm{d}s\,\overline{\Phi(0)}\right) + g(0)|\Phi(0)|^2.$$

Therefore the mapping

$$\varphi \leftrightarrow (\Phi, \Phi(0))^T$$

is an isometry between the continuous functions φ of $\Pi_{\varkappa(g)}(g)$ and a dense subset of $L^2(\mathfrak{G})$. This isometry extends by continuity to all of $\Pi_{\varkappa(g)}(g)$, and the statement follows.

We mention that for an absolutely continuous function Φ on $[0, 2a]$ we have

(2.7) $$\varphi = -\Phi' + 2\delta_{2a}\Phi(2a) - 2\delta_0(\Phi(0) - \xi) \leftrightarrow (\Phi, \xi)^T.$$

(3°) *The operator* A *is* π*-selfadjoint in* $\Pi_{\varkappa(g)}(g)$ *if and only if* $0 \in \sigma(\mathfrak{G})$.

Proof. If $0 \in \sigma(\mathfrak{G})$ it is a normal eigenvalue of $\mathfrak{G}$. Hence there exists a non-zero element $(\Phi_0, \xi_0)^T \in L^2(0, 2a) \oplus \mathbf{C}$ such that

$$(I + \mathbf{H})\Phi_0 - \xi_0 g' = 0.$$

As g' is absolutely continuous on $[0, 2a]$, the statement (1°) of § 1 implies that Φ_0 is absolutely continuous. According to (2.7), the nonzero function

(2.8) $$\hat{\varphi}_0 = -\Phi_0' + 2\delta_{2a}\Phi_0(2a) - 2\delta_0(\Phi_0(0) - \xi_0)$$

is equivalent to the zero element in $\Pi_{\varkappa(g)}(g)$. It follows that

(2.9) $$\mathscr{F}(\hat{\varphi}_0; z) = -\int_0^{2a} \Phi_0'(t)\mathrm{e}^{-izt}\,\mathrm{d}t + \mathrm{e}^{-2iaz}\,\Phi_0(2a) - (\Phi_0(0) - \xi_0).$$

If $\mathscr{F}(\hat{\varphi}_0; z) \neq 0$, then for arbitrary $u \in L^2(0, 2a)$ the function φ:

(2.10) $$\varphi(t) := \mathrm{i}\, e^{izt} \left(\int_0^t e^{-izs} u(s)\, ds - \frac{\mathscr{F}(u; z)}{\mathscr{F}(\varphi_0; z)} \int_0^t e^{-izs} \hat{\varphi}_0(s)\, ds \right)$$

solves the boundary problem

$$\frac{1}{\mathrm{i}} \varphi' - z\varphi = u + z\hat{\varphi}_0 , \quad \varphi(0) = \varphi(2a) = 0,$$

that is, the equation $(A - zI)\varphi = u$ has a solution φ. It follows that A is π-selfadjoint in $\Pi_{\varkappa(g)}(g)$.

Let now $\mathfrak{G}$ be invertible. The relation

$$[(A - zI)u, q_{\bar{z}}] = 0 \quad (u \in \mathfrak{D}(A))$$

in $\Pi_{\varkappa(g)}(g)$ with some $q_{\bar{z}} \in \Pi_{\varkappa(g)}(g)$ takes the following form in $L^2(\mathfrak{G})$:

$$\left(\begin{pmatrix} -\frac{1}{\mathrm{i}} u(\cdot) - z \int^{2a} u(s)\, ds \\ \\ -z \int_0^{2a} u(s)\, ds \end{pmatrix}, \mathfrak{G} \begin{pmatrix} \Phi_{\bar{z}} \\ \xi_{\bar{z}} \end{pmatrix} \right) = 0 \quad (u \in \mathfrak{D}(A_0));$$

here $\begin{pmatrix} \Phi_{\bar{z}} \\ \xi_{\bar{z}} \end{pmatrix}$ is the vector in $L^2(\mathfrak{G})$ corresponding to q_z in $\Pi_{\varkappa(g)}(g)$. Now it is easy to check that this relation is satisfied with

(2.11) $$\begin{pmatrix} \Phi_{\bar{z}} \\ \xi_{\bar{z}} \end{pmatrix} = \mathfrak{G}^{-1} \begin{pmatrix} \mathrm{i}\bar{z}\, e_{\mathrm{i}\bar{z}} \\ 1 \end{pmatrix}.$$

As this is a non-zero vector in $L^2(\mathfrak{G})$, the operator A in $\Pi_{\varkappa(g)}(g)$ has defect one. The proposition (3°) is proved.

THEOREM 2.2. *Let g on $(-2a, 2a)$, $0 < a < \infty$, be a Hermitian function with accelerant $H \in L^1(-2a, 2a)$, that is, g admits a representation* (2.2). *Then the following statements hold true:*

(1) *$g \in \mathfrak{P}_{\varkappa;a}$ with $\varkappa = \varkappa_{\mathfrak{G}}$.*

(2) *g has at least one continuation $\tilde{g} \in \mathfrak{P}_\varkappa$.*

(3) *The continuation $\tilde{g} \in \mathfrak{P}_\varkappa$ in* (2) *is uniquely determined if and only if $0 \in \sigma(\mathfrak{G})$. In this case it is given by the formula*

(2.12) $$\int_0^\infty e^{-izt} \tilde{g}(t)\, dt = \frac{1}{\mathscr{F}(\varphi_0 , z)} \int_0^{2a} \overline{g(t)} \int_t^{2a} e^{iz(t-s)} \hat{\varphi}_0(s)\, ds\, dt \quad (\operatorname{Im} z < -\gamma)$$

with some $\gamma \geqslant 0$, where $\hat{\varphi}_0$ was defined in (2.8).

Proof. The statements (1), (2) and the first claim of (3) follow from (1°), (2°) and (3°) above. In order to prove (2.12) it is sufficient to show that the right hand side in (2.12) equals $i^{-1}[(A - zI)^{-1}2\delta_0, 2\delta_0]$, where the resolvent of A is given by (2.10). This will be left to the reader.

2.3. Let g be a Hermitian function with an accelerant as above. If $-1 \notin \sigma(\mathbf{H})$ we put

(2.13) $$\Delta := g(0) - ((I + \mathbf{H})^{-1}g', g') \quad (= g(0) - ((I - \mathbf{\Gamma})g', g')).$$

Then the inverse $\mathfrak{G}^{-1}$ is given by

$$\mathfrak{G}^{-1} = \begin{pmatrix} (I - \mathbf{H})^{-1} + \dfrac{h}{\Delta}(\cdot, h) & -\dfrac{1}{\Delta}h \\ \\ -\dfrac{1}{\Delta}(\cdot, h) & \dfrac{1}{\Delta} \end{pmatrix}$$

with $h := (I + \mathbf{H})^{-1}g$.

(4°) *If* $-1 \notin \sigma(\mathbf{H})$, *then we have*

$$\varkappa_{\mathfrak{G}} = \begin{cases} \varkappa_{\mathbf{H}+I} & \text{if } \Delta \geqslant 0, \\ \\ \varkappa_{\mathbf{H}+I} + 1 & \text{if } \Delta < 0, \end{cases}$$

and the inverse $\mathfrak{G}^{-1}$ *exists if and only if* $\Delta \neq 0$.

This statement follows immediately from the relation

$$\left(\mathfrak{G}\begin{pmatrix}\Phi\\ \xi\end{pmatrix}, \begin{pmatrix}\Psi\\ \eta\end{pmatrix}\right) = ((I + \mathbf{H})(\Phi + \xi(I + \mathbf{H})^{-1}g'), \ \Psi + \eta(I + \mathbf{H})^{-1}g') + \\ + (g(0) - ((I + \mathbf{H})^{-1}g', g'))\xi\bar{\eta}.$$

Combining the results of (4°) and Theorem 2.2, we get:

THEOREM 2.3. *Let g on $(-2a, 2a)$, $0 < a < \infty$, be a Hermitian function with an accelerant $H \in L^1(-2a, 2a)$, and suppose that $-1 \notin \sigma(\mathbf{H})$. With $\varkappa := \varkappa_{\mathbf{H}+I}$ the following statements hold true:*

(a) *If $\Delta > 0$ then $g \in \mathfrak{P}_{\varkappa;a}$ and there exist infinitely many continuations $\tilde{g} \in \mathfrak{P}_\varkappa$.*

(b) *If $\Delta = 0$ then $g \in \mathfrak{P}_{\varkappa;a}$ and it has a unique continuation $\tilde{g}_0 \in \mathfrak{P}_\varkappa$.*

(c) *If $\Delta < 0$ then $g \in \mathfrak{P}_{\varkappa+1;a}$ and there exist infinitely many continuations $\tilde{g} \in \mathfrak{P}_{\varkappa+1}$.*

REMARK 1. Let g be as in Theorem 2.3 and $-1 \in \sigma(\mathbf{H})$. Then we have the following alternative ($\varkappa := \varkappa_{\mathbf{H}+I}$): Either

(1) $\varkappa_{\mathfrak{G}} = \varkappa$. In this case $g \in \mathfrak{P}_{\varkappa;a} \cap \mathfrak{G}_{\varkappa;a}$, and the interlacing property of the eigenvalues of $I + \mathbf{H}$ and $\mathfrak{G}$ implies that $0 \in \sigma(I + \mathbf{H}) \cap \sigma(\mathfrak{G})$. Hence g has a unique continuation in $\mathfrak{G}_\varkappa$ and this is also its unique continuation in $\mathfrak{P}_\varkappa$; it is given by (2.3), or

(2) $\varkappa_{\mathfrak{G}} = \varkappa + 1$. In this case $g \in \mathfrak{G}_{\varkappa;a} \cap \mathfrak{P}_{\varkappa+1,a}$. It has a unique continuation in $\mathfrak{G}_\varkappa$ and a unique or infinitely many continuations in $\mathfrak{P}_{\varkappa+1}$.

It remains to give examples for these two possibilities. Let $H(s) := -e^{2i\pi s}$ $(-1 \leqslant s \leqslant 1)$. Then 0 is a simple eigenvalue of $I + \mathbf{H}$ $(\geqslant 0)$ and $0 \in \rho(\mathfrak{G})$, $\varkappa_{\mathfrak{G}} = 1$, hence the corresponding function g has infinitely many continuations in $\mathfrak{P}_{\varkappa+1}$. If $H(s) := -e^{2i\pi s} - e^{4i\pi s}$ $(-1 \leqslant s \leqslant 1)$, then 0 is an eigenvalue of multiplicity two of $I + \mathbf{H}$ $(\geqslant 0)$. From the interlacing property it follows that $0 \in \sigma(\mathfrak{G})$ and, finally, it is easy to see that $\varkappa_{\mathfrak{G}} = 1$. Thus the continuation of the corresponding function g in $\mathfrak{P}_{\varkappa+1}$ is uniquely determined.

REMARK 2. Theorem 2.3 and Remark 2.1 imply that a Hermitian function g on $(-2a, 2a)$ with accelerant $H \in L^1(-2a, 2a)$ belongs to $\mathfrak{P}_{0:a}$ and has infinitely many continuations in $\mathfrak{P}_0$ if and only if

$$I + \mathbf{H} > 0 \quad \text{and} \quad \Delta > 0.$$

In a more complicated way this result was proved by I. V. Mihailova and V. P. Potapov [33].

2.4. Let g be as above. We suppose in this n° that $0 \notin \sigma(\mathfrak{G})$. Then the operator A in $\Pi_{\varkappa(g)}(g)$ is not π-selfadjoint. By q_0 we denote the defect vector of $\mathfrak{R}(A)$, normalized according to the equation

$$\int_0^{2a} g(t-s)\, q_0(s)\, ds = 1 \quad (0 \leqslant t \leqslant 2a). \tag{2.14}$$

(5°) *If* $-1 \notin \sigma(\mathbf{H})$ *we have with* $h := (I + \mathbf{H})^{-1} g'$:

$$q_0 = (h' - 2\delta_{2a} h(2a) + 2\delta_0 (h(0) + 1))\Delta^{-1};$$

if $-1 \in \sigma(\mathbf{H})$ *we have with* $\Phi_0 \in \ker(I + \mathbf{H})$, $\Phi_0 \neq 0$:

$$q_0 = \left(-\int_0^{2a} \overline{g'(s)} \Phi_0(s)\, ds \right)^{-1} (\Phi_0' - 2\delta_{2a}\Phi_0(2a) + 2\delta_0 \Phi_0(0)).$$

It is easy to check that these functions q_0 satisfy the relation (2.14). The normalizing numbers in the nominator are $\neq 0$ as $0 \notin \sigma(\mathfrak{G})$. In the second case ($-1 \in \sigma(\mathbf{H})$) we also observe that for this reason 0 is a simple eigenvalue of $I + \mathbf{H}$, hence $\Phi_0(0) \neq 0$ and $\Phi_0(2a) \neq 0$.

(6°) *If* $\Delta \neq 0$ *then* $\Delta^{-1} = [q_0, q_0]$.

Proof. In the proof of (3°) it was shown that the image of q_0 in $L^2(\mathfrak{G})$ is $c\mathfrak{G}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with some $c \in \mathbf{C}$, see (2.11). As the normalization (2.14) implies

$[q_0, \mathbf{1}] = 2a$ and $\mathbf{1} \in \Pi_{\varkappa(g)}(g)$ corresponds to $(2a - t, 2a)^{\mathrm{T}} \in L^2(\mathfrak{G})$, it follows that

$$c\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2a-t\\2a\end{pmatrix}\right) = 2a, \quad \text{or } c = 1.$$

This gives finally

$$[q_0, q_0] = \left(\mathfrak{G}^{-1}\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = \Delta^{-1},$$

and (6°) is proved.

If $r \in (0, a]$ is not a singular point of H, the "determinant" Δ, corresponding to the restriction of H to $(-2r, 2r)$, will be denoted by Δ_r. The following statement gives some information about the function $r \to \Delta_r$ (comp. [33]):

(7°) $$\frac{\mathrm{d}\Delta_r}{\mathrm{d}r} = -2\left|\int_0^{2r} \Gamma_r(2r, s)\, g'(s)\, \mathrm{d}s - g'(2r)\right|^2.$$

Indeed,

$$\Delta_r = g(0) - \int_0^{2r} |g'(t)|^2\, \mathrm{d}t + \int_0^{2r}\int_0^{2r} \Gamma_r(t, s) g'(s)\overline{g'(t)}\, \mathrm{d}s\, \mathrm{d}t,$$

and, using (1.22), we get

$$\frac{\mathrm{d}\Delta_r}{\mathrm{d}r} = -2|g'(2r)|^2 + 2\int_0^{2r} \Gamma_r(2r, s) g'(s)\, \mathrm{d}s\, \overline{g'(2r)} +$$

$$+ 2\int_0^{2r} \Gamma_r(t, 2r) g'(t)\, \mathrm{d}t\, \overline{g'(2r)} -$$

$$- 2\int_0^{2r}\int_0^{2r} \Gamma_r(t, 2r)\Gamma_r(2r, s) g'(s)\overline{g'(t)}\, \mathrm{d}s\, \mathrm{d}t.$$

Remark. In the situation considered in Theorem 2.3 ($-1 \notin \sigma(\mathbf{H})$), the function $g \in \mathfrak{G}_{\varkappa; a}$ has infinitely many continuations in $\mathfrak{G}_\varkappa$, see Theorem 2.1. In case (a) ($\Delta > 0$) infinitely many of these continuations in $\mathfrak{G}_\varkappa$ belong to $\mathfrak{P}_\varkappa$ and infinitely many belong to $\mathfrak{P}_{\varkappa+1}$; their description will be given in [7]. In case (b) ($\Delta = 0$) all the continuations $\tilde{g}$ in $\mathfrak{G}_\varkappa$ with exception of $\tilde{g}_0\, (\in \mathfrak{P}_\varkappa)$ belong to $\mathfrak{P}_{\varkappa+1}$. Finally, in case (c) ($\Delta < 0$) all the continuations of $g \in \mathfrak{G}_{\varkappa; a}$ belong to $\mathfrak{P}_{\varkappa+1}$.

§3. SOLUTION OF THE CONTINUATION PROBLEM FOR $f \in \mathfrak{P}_{\varkappa;a}$ WITH ACCELERANT

3.1. Let f on $[-2a, 2a]$, $0 < a < \infty$, be a Hermitian function with an accelerant $H \in L^1(-2a, 2a)$:

$$f(t) = f(0) - \frac{1}{2}|t| - \int_0^t (t-s)H(s)\,\mathrm{d}s \quad (-2a \leqslant t \leqslant 2a) \tag{3.1}$$

and suppose $f \in \mathfrak{P}_{\varkappa;a}$. If f has a unique continuation $\tilde{f} \in \mathfrak{P}_\varkappa$ it is given by (2.12) (with g replaced by f). In this n° we suppose that f has more than one continuation $\tilde{f} \in \mathfrak{P}_\varkappa$ and we give a description of all these continuations. To this end we find the $2\delta_0$-resolvent matrix of the operator A in $\Pi_\varkappa(f)$ according to [9, (3.19)] and apply the statement (1°) of § 2.

Let $\mathfrak{G}$ be as in § 2, (2°) with g replaced by f. Then its inverse $\mathfrak{G}^{-1}$ exists (see Theorem 2.2, (3)) and the operator A in $\Pi_\varkappa(f)$ has defect one. Its defect vectors $(\Phi_z, \xi_z)^{\mathrm{T}}$ in the canonical image $L^2(\mathfrak{G})$ of $\Pi_\varkappa(f)$ are given by the relation

$$\begin{pmatrix} \Phi_z \\ \xi_z \end{pmatrix} = \mathfrak{G}^{-1} \begin{pmatrix} \mathrm{i}z\, e_{\mathrm{i}z} \\ 1 \end{pmatrix}.$$

In order to find $\mathscr{P}(z)x = \dfrac{[x, \varphi(\bar{z})]}{[u, \varphi(\bar{z})]}$ (see [9, (3.3)]) where $\varphi(z)$ denotes the defect vector of A in $\Pi_\varkappa(f)$ and $u = 2\delta_0$, we shall calculate these scalar products in the space $L^2(\mathfrak{G})$:

$$[u, \varphi(\bar{z})] = \left(\mathfrak{G}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathfrak{G}^{-1}\begin{pmatrix} \mathrm{i}\bar{z}\, e_{\mathrm{i}\bar{z}} \\ 1 \end{pmatrix}\right)_{L^2 \oplus \mathbf{C}} = 1,$$

$$[x, \varphi(\bar{z})] = \left(\mathfrak{G}\begin{pmatrix} \int_{\cdot}^{2a} x(s)\,\mathrm{d}s \\ \int_0^{2a} x(s)\,\mathrm{d}s \end{pmatrix}, \mathfrak{G}^{-1}\begin{pmatrix} \mathrm{i}\bar{z}\, e_{\mathrm{i}\bar{z}} \\ 1 \end{pmatrix}\right)_{L^2 \oplus \mathbf{C}} = \int_0^{2a} \mathrm{e}^{-\mathrm{i}zt} x(t)\,\mathrm{d}t,$$

hence

$$\mathscr{P}(z)x = \int_0^{2a} x(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t.$$

For arbitrary $f \in L^2(0, 2a)$ this is an entire function, hence A is an entire π-Hermitian operator in $\Pi_\varkappa(f)$ (see [9, § 6]).

In order to find $Q(z)$ we have to choose a π-selfadjoint extension of A in $\Pi_\varkappa(f)$. These extensions are given by a boundary condition $x(2a) = \beta x(0)$ with some β, $|\beta| = 1$. We choose the π-selfadjoint extension $\mathring{A}$ corresponding to $\beta = 1$. Its resolvent has the form

$$((\mathring{A} - zI)^{-1}x)(t) = \frac{\mathrm{i}}{1 - \mathrm{e}^{2\mathrm{i}az}} \left\{ \int_0^t x(s)\, \mathrm{e}^{\mathrm{i}z(t-s)}\, \mathrm{d}s + \mathrm{e}^{2\mathrm{i}az} \int_t^{2a} x(s)\, \mathrm{e}^{\mathrm{i}z(t-s)}\, \mathrm{d}s \right\}.$$

It follows that

$$((\mathring{A} - zI)^{-1}u)(t) = \begin{cases} \dfrac{\mathrm{i}}{1 - \mathrm{e}^{2\mathrm{i}az}}\, \mathrm{e}^{\mathrm{i}zt} & t > 0, \\[2ex] \dfrac{\mathrm{i}}{1 - \mathrm{e}^{2\mathrm{i}az}}\, \mathrm{e}^{2\mathrm{i}az} & t = 0, \end{cases}$$

$$(\mathring{A} - zI)^{-1}(x - (\mathscr{P}(z)x)u)(t) = -\mathrm{i} \int_t^{2a} x(s)\, \mathrm{e}^{\mathrm{i}z(t-s)}\, \mathrm{d}s$$

and, according to [9, (3.3)]

$$Q(z)x = -\mathrm{i} \int_0^{2a} \overline{f(s)} \int_{t=s}^{2a} x(t)\, \mathrm{e}^{-\mathrm{i}zt}\, \mathrm{d}t\, \mathrm{e}^{\mathrm{i}zs}\, \mathrm{d}s =$$

$$= -\mathrm{i} \int_0^{2a} \overline{f(s)} \int_{t=s}^{2a} x(t)\, \mathrm{d}t\, \mathrm{d}s - z \int_0^{2a} \overline{f(s)} \int_{\tau=t}^{2a} x(\tau)\, \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}z(s-t)}\, \mathrm{d}t\, \mathrm{d}s.$$

It remains to find $\mathscr{P}(0)^*$ and $Q(0)^*$, given by [9, (3.5)] for $z = 0$. We have

$$\mathscr{P}(0)^* \alpha = \alpha q_0 \quad (\alpha \in \mathbf{C}),$$

here $q_0 = \varphi(0)$ is a defect vector of $z=0$, that is, the element of $\Pi_\varkappa(f)$ corresponding to $\mathfrak{G}^{-1}\begin{pmatrix}0\\1\end{pmatrix}$. This element q_0 has been calculated already (see § 2, (5°)).

The element $\hat{q}_0 \in \Pi_\varkappa(f)$ which defines $Q(0)^*$ according to

$$Q(0)^*\alpha = \alpha\hat{q}_0 \quad (\alpha \in \mathbf{C})$$

can be found from the relation

$$[x, Q(0)^*\alpha] = (Q(0)x)\bar{\alpha} =$$

$$= -\mathrm{i}\bar{\alpha}\int_0^{2a}\overline{f(t)}\int_{s=t}^{2a} x(s)\,\mathrm{d}s\,\mathrm{d}t = \left(\mathfrak{G}\begin{pmatrix}\int_0^{2a} x(s)\,\mathrm{d}s \\ \int_0^{2a} x(s)\,\mathrm{d}s\end{pmatrix}, \ \alpha\mathfrak{G}^{-1}\begin{pmatrix}\mathrm{i}f \\ 0\end{pmatrix}\right)_{L^2\oplus\mathbf{C}}$$

hence it corresponds to $\mathfrak{G}^{-1}\begin{pmatrix}\mathrm{i}f \\ 0\end{pmatrix}$.

In order to give q_0 and $\hat{q}_0$ more explicitely we have to consider the two cases $-1 \notin \sigma(\mathbf{H})$ and $-1 \in \sigma(\mathbf{H})$ separately.

1) If $-1 \notin \sigma(\mathbf{H})$ then

(3.2)
$$q_0 = \Delta^{-1}(h' - 2\delta_{2a}h(2a) + 2\delta_0(h(0) + 1)) \text{ with}$$
$$h := (I + \mathbf{H})^{-1}f';$$

(3.3)
$$\hat{q}_0 = -c' + 2\delta_{2a}c(2a) - 2\delta_0(c(0) - \beta) \text{ with}$$
$$c := \mathrm{i}\left\{(I + \mathbf{H})^{-1}f + \frac{(f, h)}{\Delta}h\right\}, \quad \beta := -\mathrm{i}\Delta^{-1}(f, h).$$

Here we observe that according to statement (1°) of § 1 the function h is absolutely continuous.

2) If $-1 \in \sigma(\mathbf{H})$ then

$$q_0 = -(\Phi_0, f')^{-1}(\Phi_0' - 2\delta_{2a}\Phi_0(2a) + 2\delta_0\Phi_0(0))$$

with $\Phi_0 \in \ker(I + \mathbf{H})$, $\Phi_0 \neq 0$;

$$\hat{q}_0 = -\tilde{c}' + 2\delta_{2a}\tilde{c}(2a) - 2\delta_0(\tilde{c}(0) - \tilde{\beta})$$

with $\tilde{c} := \alpha\Phi_0 + \mathrm{i}(I + \mathbf{H})_0^{-1}\left(f - \dfrac{(f, \Phi_0)}{(f', \Phi_0)}f'\right)$,

$$\alpha := \mathrm{i}|(\Phi_0, f')|^{-2}\{(f, \Phi_0)f(0) + ((I + \mathbf{H})_0^{-1}(f(f', \Phi_0) - f'(f, \Phi_0)), f')\}$$

$$\tilde{\beta} := \mathrm{i}(f', \Phi_0)^{-1}(f, \Phi_0).$$

The expressions for q_0 follow immediately from § 2, (5°), and the expressions for $\hat{q}_0$ from the explicit form of $\mathfrak{G}^{-1}\begin{pmatrix} \mathrm{i}f \\ 0 \end{pmatrix}$ and (2.7).

Thus the resolvent matrix $W = (w_{jk})_1^2$ of [9, (3.16)], normalized according to $W(0) = I_2$ (that is in [9, (3.16)] we put $a = 0$), is given by the relations

$$
\begin{aligned}
w_{11}(z) &= 1 - \mathrm{i}z \int_0^{2a} \overline{f(t)} \int_{s=t}^{2a} q_0(s)\, \mathrm{e}^{-\mathrm{i}z(s-t)}\, \mathrm{d}s\, \mathrm{d}t, \\
w_{12}(z) &= -\mathrm{i}z \int_0^{2a} \overline{f(t)} \int_{s=t}^{2a} \hat{q}_0(s)\, \mathrm{e}^{-\mathrm{i}z(s-t)}\, \mathrm{d}s\, \mathrm{d}t, \\
w_{21}(z) &= -z \int_0^{2a} \mathrm{e}^{-\mathrm{i}zt} q_0(t)\, \mathrm{d}t, \\
w_{22}(z) &= 1 - z \int_0^{2a} \mathrm{e}^{-\mathrm{i}zt} \hat{q}_0(t)\, \mathrm{d}t,
\end{aligned} \tag{3.4}
$$

where q_0 and $\hat{q}_0$ are defined in 1) and 2) above.

Combining [9, Satz 3.9] and statement (1°) of § 2 we obtain the following result.

THEOREM 3.1. *If the function $f \in \mathfrak{P}_{\varkappa; a}$, $0 < a < \infty$, has an accelerant H and admits more than one continuation $\tilde{f} \in \mathfrak{P}_\varkappa$, then the relation*

$$
-\mathrm{i} \int_0^{\infty} \mathrm{e}^{-\mathrm{i}zt} \tilde{f}(t)\, \mathrm{d}t = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)} \quad (\operatorname{Im} z < -\gamma) \tag{3.5}
$$

for some $\gamma \geqslant 0$ establishes a bijective correspondence between all such continuations $\tilde{f} \in \mathfrak{P}_\varkappa$ and all $T \in \tilde{N}_0$. The matrix function $W(z) = (w_{jk}(z))_1^2$ is given by (3.4), $W(0) = I_2$.

REMARK 1. In the description (3.5) the number $\gamma \geqslant 0$ is fixed and independent of the parameter T. That is, for all the continuations $\tilde{f}$ of f the singularities of their Fourier transforms in $\mathbf{C}_-$ lie in a strip $\{z : -\gamma \leqslant \operatorname{Im} z \leqslant 0\}$. This is a consequence of the fact that for a densely defined π-Hermitian operator in a $\pi_\varkappa$-space the spectra of all its π-selfadjoint extensions lie in a strip around the real axis (see [34]).

This remark about γ concerns also the other descriptions of continuations $\tilde{f} \in \mathfrak{P}_\varkappa$ or $\tilde{g} \in \mathfrak{G}_\varkappa$ below. Of course, the numbers γ in different statements can be different.

3.2. The resolvent matrix W of Theorem 3.1 is not real, that is it does not have the property $W(\bar{z}) = \overline{W(z)}$. If, instead of $\Pi_\varkappa(f)$, we start with the space $\dot{\Pi}_\varkappa(f)$ (see § 2.2), then the operator $\dot{A}$ and the scale vector $u = \delta_0$ will be real with respect to the involution $\varphi(t) \to \overline{\varphi(-t)}$, and we can apply [9, Satz 3.10] which gives us a real resolvent matrix. We shall formulate the corresponding result in case $-1 \notin \sigma(\mathbf{H})$. The proof of it is similar to the proof of Theorem 3.1 and the details can be left to the reader.

The Gram matrix in $L^2(-a, a) \oplus \mathbf{C}$ is now given by

$$\dot{\mathfrak{G}} := \begin{pmatrix} I + \dot{\mathbf{H}} & f'_a \\ (\cdot, f'_a) & f(0) \end{pmatrix}, \quad f'_a(t) := f'(a+t) \quad (-a \leqslant t \leqslant a).$$

The operator $\dot{A}$ is defined as in § 2. We suppose again, that the continuation problem for $f \in \mathfrak{P}_{\varkappa;a}$ in (3.1) is not determined, that is, $\dot{\mathfrak{G}}^{-1}$ exists or, equivalently, the deficiency indices of $\dot{A}$ are equal to one.

It is easy to see that

$$\dot{\mathscr{P}}(z)x = \int_{-a}^{a} x(t)\, \mathrm{e}^{-\mathrm{i}zt}\, \mathrm{d}t.$$

Further, if we suppose that $(I+\dot{\mathbf{H}})^{-1}$ exists, the elements $\dot{q}_0, \hat{\dot{q}}_0 \in \dot{\Pi}_\varkappa(f)$ which span the ranges of $\dot{\mathscr{P}}(0)^*$ and $\dot{Q}(0)^*$:

$$\dot{\mathscr{P}}(0)^*\xi = \xi\dot{q}_0, \quad \dot{Q}(0)^*\xi = \xi\hat{\dot{q}}_0 \quad (\xi \in \mathbf{C}),$$

are given by

$$\dot{q}_0 = \frac{1}{\Delta}[\dot{h}' - 2\delta_a\dot{h}(a) + 2\delta_{-a}(\dot{h}(-a) + 1)]$$

$$\hat{\dot{q}}_0 = \dot{c}' - 2\delta_a\dot{c}(a) + 2\delta_{-a}(\dot{c}(-a) - \dot{\beta})$$

with

$$\dot{h} := (I + \dot{\mathbf{H}})^{-1}f'_a,$$

$$\dot{c} := \mathrm{i}(I + \dot{\mathbf{H}})^{-1}f + \mathrm{i}\dot{\beta}\,\dot{h}, \quad \dot{\beta} := \frac{\mathrm{i}}{\Delta}\int_{-a}^{0} f(t)\mathrm{d}t + \frac{\mathrm{i}}{\Delta}(f, \dot{h}).$$

The resolvent matrix $\dot{W} = (\dot{w}_{jk})$ of [9, Satz 3.10] is now given as follows:

$$
\begin{aligned}
\dot{w}_{11}(z) &= 1 - \mathrm{i}z \int_{-a}^{a} \overline{f(t)} \int_{s=t}^{a} \dot{q}_0(s)\, \mathrm{e}^{-\mathrm{i}z(s-t)}\, \mathrm{d}s\, \mathrm{d}t, \\
\dot{w}_{12}(z) &= -\mathrm{i}z \int_{-a}^{a} \overline{f(t)} \int_{s=t}^{a} \hat{q}_0(s)\, \mathrm{e}^{-\mathrm{i}z(s-t)}\, \mathrm{d}s\, \mathrm{d}t, \\
\dot{w}_{21}(z) &= -z \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}zt} \dot{q}_0(t)\, \mathrm{d}t, \\
\dot{w}_{22}(z) &= 1 - z \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}zt} \hat{q}_0(t)\, \mathrm{d}t.
\end{aligned} \tag{3.6}
$$

THEOREM 3.2. *If the function* $f \in \mathfrak{P}_{\varkappa;a}$, $0 < a < \infty$, *has an accelerant* H, $-1 \notin \sigma(\mathbf{H})$, *and* f *admits more than one continuation* $\tilde{f} \in \mathfrak{P}_\varkappa$, *then the relation*

$$
-\mathrm{i} \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}zt} \tilde{f}(t) \mathrm{d}t = \frac{\dot{w}_{11}(z)T(z) + \dot{w}_{12}(z)}{\dot{w}_{21}(z)T(z) + \dot{w}_{22}(z)} \quad (\operatorname{Im} z < -\gamma) \tag{3.7}
$$

for some $\gamma \geqslant 0$ *establishes a bijective correspondence between all such continuations* $\tilde{f} \in \mathfrak{P}_\varkappa$ *and all* $T \in \tilde{N}_0$. *The matrix function* $\dot{W}(z) = (\dot{w}_{jk}(z))_1^2$ *is given by* (3.6). *Its entries* $\dot{w}_{jk}$ *are real entire functions of exponential type a and of Cartwright class, and we have* $\dot{W}(0) = I_2$, $\det \dot{W}(z) = 1$ $(z \in \mathbf{C})$.

REMARK 1. According to the general results of [9] the resolvent matrix W in (3.7) has also the property that the kernel K_W:

$$
K_W(z, \zeta) := \frac{W(\zeta)^* J W(z) - J}{z - \bar{\zeta}}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

has $\varkappa$ negative squares. This remark concerns also other resolvent matrices in this paper.

REMARK 2. According to [9] the resolvent matrices $W(z)$ and $\dot{W}(z)$ are connected by a relation $\dot{W}(z) = W(z)\alpha(z)$ with some entire function $\alpha(z)$. Comparing the elements $w_{21}(z)$ and $\dot{w}_{21}(z)$ it follows easily that $\alpha(z) = \mathrm{e}^{\mathrm{i}az}$:

$$
\dot{W}(z) = W(z)\, \mathrm{e}^{\mathrm{i}az}. \tag{3.8}
$$

3.3. If $\varkappa=0$, for an arbitrary function $f \in \mathfrak{P}_{0;a}$ with more than one continuation $\tilde{f} \in \mathfrak{P}_0$ (even for an arbitrary resolvent matrix whose entries are entire functions) the resolvent matrix is determined by its elements of one line or one row, see [35]. In the general case $\varkappa \geqslant 0$ the functions w_{11} and w_{12} in Theorem 3.1 can be obtained from w_{21} and w_{22}, respectively, by means of the following functional Ω_f, which is associated with an arbitrary function $f \in \mathfrak{P}_{\varkappa;a}$.

Let φ be a function of bounded variation on $[-2a, 2a]$ and

$$\Phi(\lambda) := \int_{-2a}^{2a} e^{i\lambda t}\, d\varphi(t) \quad (\lambda \in \mathbf{C}). \tag{3.9}$$

We define the functional Ω_f on the set of all these functions Φ by the relation

$$\Omega_f(\Phi) := \int_{-2a}^{2a} f(t)\, d\varphi(t).$$

Then the following equalities hold:

$$\Omega_f^\lambda\left(\frac{w_{2j}(z) - w_{2j}(\lambda)}{z-\lambda}\right) = -w_{1j}(z), \quad j = 1, 2. \tag{3.10}$$

Here the λ at Ω_f indicates that the functional Ω_f acts with respect to λ. In the relations (3.10) w_{jk} can be replaced by $\dot{w}_{jk}$ of Theorem 3.2.

In order to prove (3.10) we observe the equality

$$\frac{ze^{-itz} - \lambda e^{-it\lambda}}{z-\lambda} = -iz\, e^{-itz}\int_0^t e^{-i\lambda s} e^{izs}\, ds + e^{-i\lambda t},$$

which gives e.g.

$$\Omega_f^\lambda\left(\frac{w_{21}(z) - w_{21}(\lambda)}{z-\lambda}\right) = \Omega_f^\lambda\left(\frac{-z\int_0^{2a} e^{-izt} q_0(t)\, dt + \lambda\int_0^{2a} e^{-i\lambda t} q_0(t)\, dt}{z-\lambda}\right) =$$

$$= \Omega_f^\lambda\left(\int_0^{2a} q_0(t)\left(ize^{-itz}\int_0^t e^{-i\lambda s} e^{izs}\, ds - e^{-i\lambda t}\right) dt\right) =$$

$$= -\int_0^{2a} \overline{f(t)}\, q_0(t)\, dt + iz\int_0^{2a}\int_0^t \overline{f(s)}\, e^{-iz(t-s)} q_0(t)\, ds\, dt.$$

As $\int_0^{2a} \overline{f(t)} q_0(t)\,dt = 1$ (see (2.14)), the right hand side of this relation coincides with $-w_{11}(z)$ from (3.4). The proof for $w_{12}(z)$ is similar.

If $\varkappa = 0$ the function $f \in \mathfrak{P}_{0;a}$ admits the representation

$$f(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\,d\sigma(\lambda) \quad (-2a \leqslant t \leqslant 2a) \tag{3.11}$$

with at least one bounded nondecreasing function σ on the real axis $\mathbf{R}$. Then the functional Ω_f can be expressed as follows:

$$\Omega_f(\Phi) = \int_{-\infty}^{\infty} \Phi(\lambda)\,d\sigma(\lambda).$$

This value does not depend on the special choice of the spectral measure σ in the representation (3.11). A similar but more complicated representation of Ω_f holds if $\varkappa > 0$. This will be considered in [7].

§ 4. CANONICAL DIFFERENTIAL SYSTEMS ASSOCIATED WITH THE RESOLVENT MATRIX OF $f \in \mathfrak{P}_{\varkappa;a}$ WITH ACCELEANT

4.1. Let $W = (w_{jk})_1^2$ be a 2×2-matrix function on $\mathbf{C}$ with the following properties:

1) w_{jk} are real entire functions, $j, k = 1, 2$.
2) $\det W(z) = 1 \quad (z \in \mathbf{C})$.
3) For arbitrary $\alpha \in \mathbf{R}$ we have
$$(w_{11}\cos\alpha + w_{12}\sin\alpha)(w_{21}\cos\alpha + w_{22}\sin\alpha)^{-1} \in N_0.$$
4) $W(0) = I_2$.

A Hermitian 2×2-matrix function $\mathscr{H}$ on some interval $\mathscr{I}$ with the property $\mathscr{H}(r) \geqslant 0$ $(r \in \mathscr{I})$ and which is summable on $\mathscr{I}$ is called a *Hamiltonian*. If, additionally, $\operatorname{tr}\mathscr{H}(r) = 1$ ($\det\mathscr{H}(r) = 1$) on $\mathscr{I}$, the Hamiltonian $\mathscr{H}$ is called *trace-normed* (*det-normed*, respectively) on $\mathscr{I}$. De Branges [36] has proved the following remarkable theorem.

The 2×2-matrix function W has the properties 1) —4) if and only if it admits a representation

$$W(z) = \overset{\curvearrowright}{\int_0^L} e^{-z\mathscr{H}_W(r)J}\,dr, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.1}$$

with a real trace-normed Hamiltonian $\mathscr{H}_W$ on $[0, L]$. The real trace-normed Hamiltonian in the representation (4.1) is uniquely determined.

That is, $W(z)$ is the solution at $r = L$ of the following canonical differential system on $[0, L]$:

$$\frac{\mathrm{d}W(r;z)}{\mathrm{d}r} J = zW(r;z)\mathscr{H}_W(r), \quad W(0;z) = I_2. \tag{4.2}$$

In this case the elements of W are real entire functions of exponential type a given by

$$a = \int_0^L \sqrt{\det \mathscr{H}_W(r)}\,\mathrm{d}r,$$

see [37], [36].

If, in particular, on some interval $[0, m]$ we have $\mathscr{H}_W(r) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $0 \leqslant r \leqslant m$, then the representation (4.1) becomes

$$W(z) = \theta_m(z) \overset{\curvearrowright}{\int_m^L} \mathrm{e}^{-z\mathscr{H}_W(r)J}\,\mathrm{d}r$$

with $\theta_m(z) := \begin{pmatrix} 1 & 0 \\ -mz & 1 \end{pmatrix}$.

In this Part IV we shall not use the theorem of De Branges explicitely. However, it enlightens the considerations in this § 4 and in § 11. Namely, we shall show by direct calculations, that the resolvent matrices W and $\dot{W}$ of Theorems 3.1 and 3.2 satisfy certain canonical differential equations and find the corresponding Hamiltonians.

If a canonical system

$$Jx'(s) = z\mathscr{H}(r)x(r) \quad (0 \leqslant r \leqslant L) \tag{4.3}$$

is given and the Hamiltonian $\mathscr{H}$ satisfies the condition $\det \mathscr{H}(r) > 0$ on some interval $\mathscr{I} \subset [0, L]$, then a new variable $\hat{r} = \varphi(r)$ can be introduced such that the system (4.3) becomes $J\hat{x}'(\hat{r}) = z\hat{\mathscr{H}}(\hat{r})\hat{x}(\hat{r})$ and that $\hat{\mathscr{H}}$ is det-normed on $\hat{\mathscr{I}} = \varphi(\mathscr{I})$.

Let $\mathscr{H}_\mathrm{d}$ be a real det-normed Hamiltonian on some interval $\mathscr{I}$. We introduce the complex Hamiltonian $\mathscr{H}_\mathrm{c}$:

$$\mathscr{H}_\mathrm{c} := \mathscr{H}_\mathrm{d} - \mathrm{i}J. \tag{4.4}$$

Then we have

$$\det \mathscr{H}_\mathrm{c}(r) = 0 \quad (r \in \mathscr{I}) \tag{4.5}$$

$$(\mathscr{H}_\mathrm{c}J)^2 = 2\,\mathrm{i}\mathscr{H}_\mathrm{c}J. \tag{4.6}$$

In particular, $\operatorname{rank} \mathscr{H}_\mathrm{c}(r) = 1 \quad (r \in \mathscr{I})$.

Conversely, given a complex Hamiltonian $\mathscr{H}_c = (h_{jk})_1^2$ on $\mathscr{I}$, $\mathscr{H}_c(r) \neq 0$ $(r \in \mathscr{I})$ with the property (4.6) and $\operatorname{Im} h_{21}(r) = 1$ $(r \in \mathscr{I})$. Then it is of the form (4.4) with a real det-normed Hamiltonian $\mathscr{H}_d$.

If $\mathscr{H}_d$ and $\mathscr{H}_c$, given on $\mathscr{I} = [r_0, r_1]$, are connected by (4.4) between the solutions $W_d(r; z)$ and $W_c(r; z)$ of the canonical differential systems

$$\frac{\mathrm{d}W_d(r;z)}{\mathrm{d}r} J = zW_d(r;z)\mathscr{H}_d(r), \quad \frac{\mathrm{d}W_c(r;z)}{\mathrm{d}r} J = zW_c(r;z)\mathscr{H}_c(r),$$

$W_d(r_0; z) = W_c(r_0; z)$, the following relation holds:

$$W_c(r;z) = \mathrm{e}^{-\mathrm{i}z(r-r_0)} W_d(r;\ z) \quad (r_0 \leqslant r \leqslant r_1).$$

Suppose we are given a real det-normed Hamiltonian $\mathscr{H}$ on some interval $\mathscr{I}$, such that the derivative $\mathscr{H}'$ exists and is continuous on $\mathscr{I}$. Then the canonical system

$$\text{(4.7)} \qquad \frac{\mathrm{d}W}{\mathrm{d}r} J = zW\mathscr{H} \quad \text{on } \mathscr{I}$$

can be transformed into a canonical system with a "potential":

$$\text{(4.8)} \qquad \frac{\mathrm{d}V}{\mathrm{d}s} J = zV + VP \quad \text{on } \mathscr{I}.$$

Here P, the potential, is a real continuous symmetric 2×2-matrix function on $\mathscr{I}$.

In order to see this we first observe that $\det\mathscr{H}(r)=1$ implies that $\mathscr{H}(r)J\mathscr{H}(r)=J$ or $\mathscr{H}(r)J = J\mathscr{H}(r)^{-1}$, which yields $\mathscr{H}(r)^{1/2}J = J\mathscr{H}(r)^{-1/2}$. Then the function V:

$$V(r;z) := W(r;z)\mathscr{H}(r)^{1/2}$$

satisfies the equation

$$\frac{\mathrm{d}V}{\mathrm{d}r} J = \frac{\mathrm{d}W}{\mathrm{d}r}\mathscr{H}^{1/2}J + W\frac{\mathrm{d}\mathscr{H}^{1/2}}{\mathrm{d}s} J =$$

$$= -zW\mathscr{H}J\mathscr{H}^{1/2}J + W\frac{\mathrm{d}\mathscr{H}^{1/2}}{\mathrm{d}r} J = zV + VP$$

with $P(r) := \mathscr{H}(r)^{-1/2}\dfrac{\mathrm{d}\mathscr{H}(r)^{1/2}}{\mathrm{d}r} J$. It is easy to see that $P(r)$ is real and symmetric and a continuous function of r.

Conversely, a canonical system of the form (4.8) with a real symmetric and continuous potential P can easily be transformed into a canonical system (4.7) with a real det-normed and continuously differentiable Hamiltonian $\mathscr{H}$. To this end we introduce the solution U_0 of the initial problem

$$\frac{dU_0}{ds}J = U_0 P, \quad U_0(r_0) = I_2 \quad \text{for some } r_0 \in \mathscr{I},$$

that is

$$U_0(r) = \overset{\curvearrowright}{\int_{r_0}^{r}} \exp(-P(s)J)\,ds.$$

As P is real and continuous, the 2×2-matrix function U_0 has a continuous derivative and its values $U_0(r)$ are real matrices. Moreover, $U_0(r)$ is J-unitary:

$$U_0(r)JU_0(r)^* = J \quad (r \in \mathscr{I}).$$

We write the solution $V(r; z)$ of (4.8) in the form $V(r; z) = W(r; z)U_0(r)$. Then the function W satisfies the canonical differential equation

$$\frac{dW}{ds}J = zW\mathscr{H} \quad \text{on } \mathscr{I}$$

with $\mathscr{H}(r) := U_0(r)U_0(r)^*$. Evidently, $\mathscr{H}$ is real, det-normed and $\mathscr{H}'$ is continuous.

4.2. Let now $f \in \mathfrak{P}_{\varkappa;a}$ be of the form (3.1). With f we consider its restrictions $f_r := f|_{[-2r, 2r]}$, $0 < r \leqslant a$, which belong to some class $\mathfrak{P}_{\varkappa(r);r}$, $0 \leqslant \varkappa(r) \leqslant \varkappa$. If f_r has more than one continuation in $\mathfrak{P}_{\varkappa(r)}$, by Δ_r, β_r, h_r, c_r we denote numbers or functions, given by (2.13), (3.2) and (3.3) with a replaced by r, that is we define:

$$h_r(t) := f'(t) - \int_0^{2r} \Gamma_r(t, s)f'(s)\,ds \quad (0 \leqslant t \leqslant 2r),$$

$$b_r(t) := f(t) - \int_0^{2r} \Gamma_r(t, s)f(s)\,ds \quad (0 \leqslant t \leqslant 2r),$$

$$\Delta_r := f(0) - (f_r', h_r),$$

$$\beta_r := -i\Delta_r^{-1}(f_r, h_r),$$

$$c_r(t) := i\{b_r(t) + \Delta_r^{-1}(f_r, h_r)h_r(t)\} \quad (0 \leqslant t \leqslant 2r),$$

$$j_r(t) := e^{-izt} - \int_0^{2r} \Gamma_r(t, s)\,e^{-izs}\,ds \quad (0 \leqslant t \leqslant 2r).$$

The resolvent matrix of f_r which appears in Theorem 3.1 will be denoted by $W(r;z)$.

THEOREM 4.1. *Let $f \in \mathfrak{P}_{\varkappa;a}$ be a Hermitian function on $[-2a, 2a]$ with an accelerant H. Suppose that for some interval $[r_0, r_1] \subset [0, a]$ we have*

1) $-1 \notin \sigma(\mathbf{H}_r)$ $(r_0 \leqslant r \leqslant r_1)$;

2) the restriction $f_r := f|_{[-2r,2r]}$, $f_r \in \mathfrak{P}_{\varkappa';r}$ has more than one continuation $\tilde{f}_r \in \mathfrak{P}_{\varkappa}$ $(r_0 \leqslant r \leqslant r_1)$.

Then the resolvent matrix $W(r;z)$ of f_r, given by Theorem 3.1, satisfies the canonical differential equation

$$\frac{\mathrm{d}W(r;z)}{\mathrm{d}r} J = zW(r;z)\mathscr{H}_{\mathrm{c}}(r) \quad (r_0 \leqslant r \leqslant r_1) \tag{4.9}$$

with the continuous Hamiltonian

$$\mathscr{H}_{\mathrm{c}}(r) = 2\begin{pmatrix} c_r(2r) \\ \dfrac{h_r(2r)}{\Delta_r} \end{pmatrix}\begin{pmatrix} \overline{c_r(2r)} & \dfrac{\overline{h_r(2r)}}{\Delta_r} \end{pmatrix}. \tag{4.10}$$

Proof. (I) In this part of the proof we establish some relations between the functions h_r, c_r, etc. We have for almost all $t \in [0, 2r]$:

$$h_r'(t) = -H(t) - \int_0^{2r} \frac{\partial \Gamma_r(t,s)}{\partial t} f'(s)\,\mathrm{d}s =$$

$$= -H(t) - \int_0^{2r}\Big(-\frac{\partial \Gamma_r(t,s)}{\partial s} + \Gamma_r(t,2r)\Gamma_r(2r,s) - \Gamma_r(t,0)\Gamma_r(0,s)\Big) f'(s)\,\mathrm{d}s =$$

$$= -H(t) + \Gamma_r(t,s)f'(s)\Big|_0^{2r} + \int_0^{2r} \Gamma_r(t,s)H(s)\,\mathrm{d}s -$$

$$- \Gamma_r(t,2r)\int_0^{2r}\Gamma_r(2r,s)f'(s)\,\mathrm{d}s + \Gamma_r(t,0)\int_0^{2r}\Gamma_r(0,s)f'(s)\,\mathrm{d}s =$$

$$= -\Gamma_r(t,0) + \Gamma_r(t,2r)h_r(2r) - \Gamma_r(t,0)h_r(0),$$

hence

$$(f_r', h_r) = \int_0^{2r} \Gamma_r(0,t)f(t)\,\mathrm{d}t + \overline{h_r(2r)}b_r(2r) - \overline{h_r(0)}b_r(0),$$

$$\Delta_r = b_r(0) - \overline{h_r(2r)}b_r(2r) + \overline{h_r(0)}b_r(0).$$

In the same way

$$b_r'(t) = h_r(t) + \Gamma_r(t, 2r)b_r(2r) - \Gamma_r(t, 0)b_r(0)$$

and we find

$$(h_r - \mathrm{i}zb_r, e_{\mathrm{i}\bar{z}}) = (h_r, e_{\mathrm{i}\bar{z}}) + b_r e_{-\mathrm{i}z}\Big|_0^{2r} - (b_r', e_{\mathrm{i}\bar{z}}) =$$

$$= b_r(2r)\overline{j_r(2r)} - b_r(0)\overline{j_r(0)},$$

$$1 + \mathrm{i}z(h_r, e_{\mathrm{i}\bar{z}}) = h_r(0)\overline{j_r(0)} - h_r(2r)\overline{j_r(2r)} + \overline{j_r(0)}.$$

Moreover,

$$(f_r, h_r) = (b_r, f_r') = b_r f_r\Big|_0^{2r} - (b_r', f_r) =$$

$$= |b_r(2r)|^2 - |b_r(0)|^2 - (h_r, f_r).$$

Finally, we need the relation

$$\overline{h_r(2r)} = -1 - h_r(0). \tag{4.11}$$

In order to see that (4.11) holds we write it as

$$\frac{1}{2} + \int_0^{2r} \overline{H(t)}\,\mathrm{d}t + \int_0^{2r} \overline{\Gamma_r(2r-t, 0)}\left(-\frac{1}{2} - \int_0^t \overline{H(s)}\,\mathrm{d}s\right)\mathrm{d}t =$$

$$= 1 - \frac{1}{2} - \int_0^{2r} \Gamma_r(0, t)\left(-\frac{1}{2} - \int_0^t H(s)\,\mathrm{d}s\right)\mathrm{d}t$$

and this relation can be verified easily.

(2) We need the following derivatives with respect to r:

$$\frac{\mathrm{d}\Delta_r}{\mathrm{d}r} = -2|h_r(2r)|^2 \qquad \text{(see § 2, (7°))}, \tag{4.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(h_r, f_r) = \frac{\mathrm{d}}{\mathrm{d}s}\left[(f_r', f_r) - \int_0^{2r}\int_0^{2r} \Gamma_r(t, s)f'(s)\overline{f(t)}\,\mathrm{d}s\,\mathrm{d}t\right] =$$

$$= 2f'(2r)\overline{f(2r)} - 2\int_0^{2r} \Gamma_r(2r, s)\overline{f(2r)}f'(s)\,\mathrm{d}s - 2\int_0^{2r} \Gamma_r(t, 2r)f'(2r)\overline{f(t)}\,\mathrm{d}t + \tag{4.13}$$

$$+ 2\int_0^{2r}\int_0^{2r} \Gamma_r(t, 2r)\Gamma_r(2r, s)f'(s)\overline{f(t)}\,\mathrm{d}s\,\mathrm{d}t = 2h_r(2r)\overline{b_r(2r)},$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(h_r, e_{\mathrm{i}\bar z}) = 2h_r(2r)\overline{j_r(2r)},$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(c_r, e_{\mathrm{i}\bar z}) = 2c_r(2r)(\overline{j_r(2r)} - (h_r, e_{\mathrm{i}\bar z})\Delta_r^{-1}\overline{h_r(2r)}),$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(c_r, f_r) = 2\,\mathrm{i}|c_r(2r)|^2.$$

(3) Integrating by parts, the functions w_{jk} of (3.4) can be written as follows:

$$w_{11}(r; z) = 1 + (\mathrm{i}z/\Delta_r)\int_{s=0}^{2r}\overline{f(s)}h_r(s)\,\mathrm{d}s + (z^2/\Delta_r)\int_{s=0}^{2r}\int_{t=0}^{s}\overline{f(t)}\,\mathrm{e}^{\mathrm{i}z(t-s)}\,\mathrm{d}t\,h_r(s)\,\mathrm{d}s,$$

$$w_{12}(r; z) = -\mathrm{i}z\int_{s=0}^{2r}\overline{f(s)}c_r(s)\,\mathrm{d}s - z^2\int_{s=0}^{2r}\int_{t=0}^{s}\overline{f(t)}\,\mathrm{e}^{\mathrm{i}z(t-s)}\,\mathrm{d}t\,c_r(s)\,\mathrm{d}s,$$

(4.14)

$$w_{21}(r; z) = -(z/\Delta_r) - (\mathrm{i}z^2/\Delta_r)\int_0^{2r}\mathrm{e}^{-\mathrm{i}zt}h_r(t)\,\mathrm{d}t,$$

$$w_{22}(r; z) = 1 - z\beta_r + \mathrm{i}z^2\int_0^{2r}\mathrm{e}^{-\mathrm{i}zt}c_r(t)\,\mathrm{d}t.$$

Now it is easy to verify (observe that Δ_r is real) that the corresponding elements on the second lines of (4.9) coincide, that is ($'$ denotes derivative with respect to r, $\mathscr{H}_c(r) = (h_{jk}(r))_1^2$):

$$w_{22}'(r; z) = z(w_{21}(r; z)h_{11}(r) + w_{22}(r; z)h_{21}(r)),$$

(4.15)

$$-w_{21}'(r; z) = z(w_{21}(r; z)h_{12}(r) + w_{22}(r; z)h_{22}(r)).$$

This will be left to the reader.

In order to show that the first lines in (4.9) also coincide, we use the functional Ω_f. It is not hard to see that the relation

$$\Omega_f^\lambda\left(\frac{w_{22}'(r; z)/z - w_{22}'(r; \lambda)/\lambda}{z - \lambda}\right) = -\frac{w_{12}'(r; z)}{z}$$

holds. Thus (4.15) and (3.10) imply that

$$\frac{w'_{12}(r;z)}{z} = w_{11}(r;z)h_{11}(r) + w_{12}(r;z)h_{21}(r).$$

Similarly the relation

$$-\frac{w'_{11}(r;z)}{z} = w_{11}(r;z)h_{12}(r) + w_{12}(r;z)h_{22}(r)$$

follows and the theorem is proved.

REMARK 1. The expression for $\mathscr{H}_c(r)$ in (4.10) was originally found from the relation

$$\mathscr{H}_c(r) = W'_1(r)J$$

$W'_1(r) := \lim_{z\to 0} (W(r;z) - I_2)z^{-1}$, which is an immediate consequence of (4.2).

REMARK 2. The condition 1) in Theorem 3.4 can be dropped. Then in the corresponding singular points of the interval $[r_0, r_1]$ the Hamiltonian can be found using the expressions of § 3.1, 2).

4.3. If in addition to the conditions of Theorem 4.1 we suppose that $\varkappa = 0$ there is a more complete result.

THEOREM 4.2. *Let the Hermitian function* $f \in \mathfrak{P}_{0;a}$ *be given by* (3.1) *and suppose that* f *has more than one continuation* $\tilde{f} \in \mathfrak{P}_0$. *Then the resolvent matrix* $W(r;z)$ *of the restriction* $f_r := f|_{[-2r,\,2r]}$, $0 < r \leqslant a$, *given by Theorem* 3.1, *is the solution of the initial problem*

$$\frac{\mathrm{d}W(r;z)}{\mathrm{d}r}J = zW(r;z)\mathscr{H}_c(r), \qquad W(0;z) = \begin{pmatrix} 1 & 0 \\ -z/f(0) & 1 \end{pmatrix}.$$

Here the complex Hamiltonian $\mathscr{H}_c$ *is given by* (4.10); *it is continuous and satisfies the relations* (4.5), (4.6).

Proof. The relations (4.14) imply that

$$\lim_{r\downarrow 0} W(r;z) = \begin{pmatrix} 1 & 0 \\ -z/f(0) & 1 \end{pmatrix}.$$

According to the remarks in n° 1 it is sufficient to show that $\operatorname{Im} h_{12}(r) = 1$ $(0 \leqslant r \leqslant a)$, that is

$$\frac{1}{\mathrm{i}\Delta_r}\left(h_r(2r)\overline{c_r(2r)} - \overline{h_r(2r)}c_r(2r)\right) = 1 \quad (0 \leqslant r \leqslant a). \tag{4.16}$$

In order to see this we prove the following:

LEMMA 4.1. *Under the conditions of Theorem* 4.2 *for* $0 \leqslant r \leqslant a$ *the following relation holds:*

$$\Delta_r = +2r|h_r(2r)|^2 - h_r(2r)\overline{b_r(2r)} - \overline{h_r(2r)}b_r(2r). \tag{4.17}$$

Proof. We first suppose that H is continuous. Then the following relations are easy to check (observe (1.22), Proposition 1.1 (3) and (4.11)):

$$\frac{\mathrm{d}h_r(2r)}{\mathrm{d}r} = -2\Gamma_r(2r, 0)h_r(0) - 2\Gamma_r(2r, 0) = 2\Gamma_r(2r, 0)\overline{h_r(2r)},$$

$$\frac{\mathrm{d}b_r(2r)}{\mathrm{d}r} = 2h_r(2r) - 2\Gamma_r(2r, 0)b_r(0).$$

Further, we have

$$\int_0^{2r} f(s-t)\Gamma_r(t, 0)\,\mathrm{d}t = f(s) - f(0) + \int_0^{2r} \overline{f(t)}\Gamma_r(t, 0)\,\mathrm{d}t -$$

$$-s\left(f'(s) - \int_0^{2r} f'(s-t)\Gamma_r(t, 0)\,\mathrm{d}t\right) \quad (0 \leqslant s \leqslant 2r).$$

Indeed, this relation holds for $s = 0$ and the derivatives of both sides coincide. Choosing in particular $s = 2r$, it follows that

$$b(2r) = \overline{b(0)} + 2rh(r).$$

With these relations it is easy to see that the derivative of the right hand side in (4.17) equals $-|h_r(2r)|^2$, which, on the other hand, is also the derivative of Δ_r (see § 2, (7°)). As $\Delta_0 = f(0)$, $h(0) = -1/2$ and $c(0) = f(0)$, the relation (4.17) follows.

If H is not continuous, we approximate it in the $L^1(-2a, 2a)$-norm by a sequence of continuous functions $H^{(n)}$, and the relation (4.17) remains true in the limit. The lemma is proved.

The relation (4.17) yields also the following identities for $0 \leqslant r \leqslant a$:

$$\begin{aligned} f(0) - (f'_r, h_r) &= 2r|h_r(2r)|^2 - h_r(2r)\overline{b_r(2r)} - \overline{h_r(2r)}b_r(2r), \\ r\Delta_r &= -\frac{1}{2}((f_r, h_r) + \overline{(f_r, h_r)}). \end{aligned} \tag{4.18}$$

The first relation follows immediately from (4.17) and the definition of Δ_r. In order to show (4.18), we observe (4.12) and (4.13). Then (4.17) implies

$$\Delta_r = -r\frac{\mathrm{d}\Delta_r}{\mathrm{d}r} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}[(f_r, h_r) + \overline{(f_r, h_r)}],$$

or

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\Delta_r) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}[(f_r, h_r) + \overline{(f_r, h_r)}],$$

which gives (4.18).

Now it is easy to prove (4.16). Indeed, we have from (4.18) and (4.17)

$$\frac{1}{\mathrm{i}\Delta_r}[h_r(2r)\overline{c_r(2r)} - \overline{h_r(2r)}c_r(2r)] =$$

$$= \frac{1}{\mathrm{i}\Delta_r}\left[h_r(2r)\left(-\mathrm{i}\overline{b_r(2r)} - \mathrm{i}\frac{\overline{(f_r, h_r)}}{\Delta_r}\overline{h_r(2r)}\right) - \right.$$

$$\left. - \mathrm{i}\overline{h_r(2r)}\left(b_r(2r) + \frac{(f_r, h_r)}{r}h_r(2r)\right)\right] =$$

$$= \frac{1}{\Delta_r}[2r|h_r(2r)|^2 - h_r(2r)\overline{b_r(2r)} - \overline{h_r(2r)}b_r(2r)] = 1.$$

Theorem 4.2 is proved.

REMARK. The function h_r and the number $((I + \mathbf{H}_r)^{-1}f_r', f_r')$ are independent of $f(0)$. Therefore, if we take the derivative of (4.18) with respect to $f(0)$ it follows that

$$r = -\mathrm{Re}\int_0^{2r} h_r(t)\,\mathrm{d}t.$$

Finally, we consider the canonical system which corresponds to the resolvent matrix $\dot{W}$ of Theorem 3.2. As $W(r, z)$ and $\dot{W}(r, z)$ are connected by the relation (3.6):

$$\dot{W}(r; z) = W(r; z)\,\mathrm{e}^{\mathrm{i}rz},$$

Theorem 4.2 and the considerations of $\mathscr{H}_{\mathrm{c}}$ and $\mathscr{H}_{\mathrm{d}}$ in n° 1 imply the following result.

THEOREM 4.3. *Let $f \in \mathfrak{P}_{0;a}$ be given by* (3.1) *and suppose that f has more than one continuation $\tilde{f} \in \mathfrak{P}_0$. Then the resolvent matrix $\dot{W}(r; z)$ of the restriction $f_r = f|_{[-2r,2r]}$, $0 < r \leqslant a$, given by Theorem* 3.2, *is the solution of the initial problem*

$$\frac{\mathrm{d}\dot{W}(r;z)}{\mathrm{d}r} J = z\dot{W}(r;z)\mathscr{H}_{\mathrm{d}}(r), \quad \dot{W}(0;z) = \begin{pmatrix} 1 & 0 \\ -z/f(0) & 1 \end{pmatrix}.$$

Here $\mathscr{H}_{\mathrm{d}}$ is the continuous real det-normed Hamiltonian

$$\mathscr{H}_{\mathrm{d}}(r) = \begin{pmatrix} 2|c_r(2r)|^2 & \frac{1}{\Delta_r}\left(\overline{c_r(2r)}h_r(2r) + c_r(2r)\overline{h_r(2r)}\right) \\ \frac{1}{\Delta_r}\left(\overline{c_r(2r)}h_r(2r) + c_r(2r)\overline{h_r(2r)}\right) & 2\frac{|h_r(2r)|^2}{\Delta_r^2} \end{pmatrix}.$$

REMARK. If the function f in Theorem 4.3 is real, then also h_r is real and c_r is purely imaginary. It follows that in this case we have

$$\mathscr{H}_{\mathrm{d}}(r) = \begin{pmatrix} 2|c_r(2r)|^2 & 0 \\ 0 & 2\frac{|h_r(2r)|^2}{\Delta_r^2} \end{pmatrix}.$$

Real Hermitian functions with an accelerant will be studied further in § 12.

§ 5. SOLUTION OF THE CONTINUATION PROBLEM FOR $g \in \mathfrak{G}_{\varkappa;a}$ WITH ACCELERANT

5.1. By $N_\varkappa$ we denote the class of all complex functions Q which are locally meromorphic on $\mathbf{C}_+ \cup \mathbf{C}_-$, have the property $Q(z) = \overline{Q(\bar{z})}$ $(z \in \mathfrak{D}_Q)$, and for which the kernel N_Q: $N_Q(z, \zeta) := (Q(z) - \overline{Q(\zeta)})(z - \bar{\zeta})^{-1}$ $(z, \zeta \in \mathfrak{D}_Q,\ z \neq \bar{\zeta})$ has $\varkappa$ negative squares (see [2], [38]). Here $\mathfrak{D}_Q$ denotes the domain of holomorphy of Q. A function Q which is defined and holomorphic on some open nonempty domain $\mathfrak{D} \subset \mathbf{C}_+$ and such that N_Q has $\varkappa$ negative squares on $\mathfrak{D}$ can always be extended to a function $\tilde{Q} \in N_\varkappa$. It was shown in [39] that the total order of all the poles of $Q \in N_\varkappa$ in $\mathbf{C}_+$ $(\mathbf{C}_-)$ is $\leqslant \varkappa$.

We need the following characterization of the one-sided Fourier transforms of functions from $\mathfrak{G}_\varkappa$, which completes [2, Satz 5.9].

PROPOSITION 5.1. *The equality*

$$Q(z) = -\mathrm{i}z^2 \int_0^\infty \mathrm{e}^{-\mathrm{i}tz} g(t)\,\mathrm{d}t \quad (\operatorname{Im} z < -\gamma) \tag{5.1}$$

for some $\gamma \geqslant 0$ establishes a bijective correspondence between all functions $g \in \mathfrak{G}_\varkappa$ with $g(0) = 0$ and all functions $Q \in N_\varkappa$ with the property

$$\lim_{y \downarrow -\infty} \frac{1}{y} Q(\mathrm{i}y) = 0. \tag{5.2}$$

Proof. From [2, Satz 5.9] it follows that for a given function $g \in \mathfrak{G}_\varkappa$, $g(0)=0$, the function Q in (5.1) belongs to $N_\varkappa$. Moreover, we have

$$-\frac{1}{\mathrm{i}y} Q(\mathrm{i}y) = -y \int_0^\infty \mathrm{e}^{ty} g(t)\,\mathrm{d}t \to g(0) = 0 \quad (y \downarrow -\infty).$$

Conversely, suppose we are given a function $Q \in N_\varkappa$ with the property (5.2). Then the function Q_1: $Q_1(z) = \frac{1}{z^2} Q(z)$ satisfies the conditions

$$\lim_{y \downarrow -\infty} Q_1(\mathrm{i}y) = 0, \quad \lim_{y \downarrow -\infty} y \operatorname{Im} Q_1(\mathrm{i}y) = 0.$$

Moreover, from the relations

$$\frac{Q_1(z) - \overline{Q_1(\zeta)}}{z - \bar\zeta} = -\frac{1}{z^2\bar\zeta} Q(z) + \frac{1}{z\bar\zeta}\,\frac{Q(z) - \overline{Q(\zeta)}}{z - \bar\zeta} - \frac{1}{z\bar\zeta^2}\overline{Q(\zeta)}$$

it follows that $Q_1 \in N_{\varkappa'}$ where $\varkappa - 1 \leqslant \varkappa' \leqslant \varkappa + 1$. Hence, by [2, Satz 5.3] we have

$$Q_1(z) = -\mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}tz} g_1(t)\,\mathrm{d}t \quad (\operatorname{Im} z < -\gamma)$$

for some function $g_1 \in \mathfrak{P}_{\varkappa'}$, which implies $g_1 \in \mathfrak{G}_{\varkappa''}$, $\varkappa'' \leqslant \varkappa$. Further,

$$g_1(0) = \lim_{y \downarrow -\infty} \left(-y \int_0^\infty \mathrm{e}^{ty} g_1(t)\,\mathrm{d}t \right) = \lim_{y \downarrow -\infty} (-\mathrm{i}y\, Q_1(\mathrm{i}y)) =$$

$$= \lim_{y \downarrow -\infty} \left(-\frac{1}{\mathrm{i}y} Q(\mathrm{i}y) \right) = 0,$$

and from the first part of the proof (or [2, Satz 5.9]) it follows that $\varkappa'' = \varkappa$.

REMARK. If $g \in \mathfrak{G}_\varkappa$ and $g(0) \neq 0$ the function $\hat{g} := g - g(0)$ has the properties $\hat{g} \in \mathfrak{G}_\varkappa$, $\hat{g}(0) = 0$, and we have

$$-iz^2 \int_0^\infty e^{-itz}\hat{g}(t)\,dt = -iz^2 \int_0^\infty e^{-itz} g(t)\,dt + zg(0).$$

Now suppose that we are given a Hermitian function g on $[-2a, 2a]$ with an accelerant $H = H_g \in L^1(-2a, 2a)$. Without loss of generality we suppose that $g(0)=0$:

$$g(t) = -\alpha|t| - \int_0^t (t-s)H_g(s)\,ds \quad (-2a \leqslant t \leqslant 2a) \tag{5.3}$$

with some $\alpha > 0$. According to Theorem 2.1 it belongs to some class $\mathfrak{G}_{\varkappa;a}$, and it admits at least one continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ to the whole real axis. We put

$$Q(z) := -iz^2 \int_0^\infty e^{-itz}\tilde{g}(t)\,dt.$$

Then we have $Q \in N_\varkappa$, $\lim\limits_{y \downarrow -\infty} \frac{1}{y} Q(iy) = 0$ and, as $g'(0)$ does not exist, it follows from [2, Lemma 5.10] that

$$\lim_{y \downarrow -\infty} |y\, Q(iy)| = \infty.$$

We consider the function R:

$$R(z) := Q(z)(1 - z\,Q(z))^{-1}. \tag{5.4}$$

According to [2, Satz 4.1] we have $R \in N_\varkappa$ and R admits a representation

$$R(z) = -i \int_0^\infty e^{-itz}\tilde{f}(t)\,dt \quad (\operatorname{Im} z < -\gamma)$$

with some function $\tilde{f} \in \mathfrak{P}_\varkappa$, $\tilde{f}(0) = 1$. Denote the restriction $\tilde{f}|_{[-2a,2a]}$ by f. In [2, § 5] it was shown that f is uniquely determined by g and conversely. This correspondence is given by the Volterra integral equation

$$-\int_0^t \frac{(t-s)^2}{2} f(s)\,ds + \int_0^t f(s)\, g(t-s)\,ds = \int_0^t g(s)\,ds \quad (0 \leqslant t \leqslant 2a). \tag{5.5}$$

We shall show that $f \in \mathfrak{P}_{\varkappa;a}$. Obviously, $f \in \mathfrak{P}_{\varkappa';a}$ with some $\varkappa'$, $0 \leqslant \varkappa' \leqslant \varkappa$. If $\varkappa = 0$ we obtain $(0 =)\varkappa = \varkappa'$. If $\varkappa > 0$, the proof of this equality $\varkappa' = \varkappa$ is more complicated.

We consider a continuation $\tilde{f}_1 \in \mathfrak{P}_{\varkappa'}$ and its Fourier transform

$$R_1(z) = -\,\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{f}_1(t)\mathrm{d}t \quad (\mathrm{Im}\, z < -\gamma).$$

Then $R_1 \in N_{\varkappa'}$ and

$$\lim_{y \downarrow -\infty} (-\,\mathrm{i}y\, R_1(\mathrm{i}y)) = \tilde{f}_1(0) = f(0) = 1.$$

Thus the function Q_1:

$$Q_1(z) := R_1(z)(1 + zR_1(z))^{-1}$$

has the property

(5.6) $$\lim_{y \downarrow -\infty} |yQ_1(\mathrm{i}y)| = \infty.$$

Observing that $R_1(z) = Q_1(z)(1 - zQ_1(z))^{-1}$, we obtain from [2, Satz 4.1] that $Q_1 \in N_{\varkappa'}$. Next we show that

(5.7) $$\lim_{y \downarrow -\infty} y^{-1}Q_1(\mathrm{i}y) = 0.$$

According to [2, Satz 1.4] the relation (5.7) is equivalent to

(5.8) $$\overline{\mathscr{R}(I - U_{Q_1})} = \Pi_\varkappa(Q_1).$$

Here we use the notation of [2, § 1], the operator U_{Q_1} was defined at the beginning of [2, § 1.5]. Observing (5.6) we can apply [2, Satz 4.3], hence (5.8) is equivalent to the fact that for the element $u \in \Pi_\varkappa(R_1)$ of the representation

$$R_1(z) = [(A_{R_1} - zI)^{-1}u, u], \quad [u, u] = 1$$

we have

$$u \notin \mathfrak{D}(A_{R_1}) \quad \text{or} \quad [A_{R_1}u, u] \neq 0.$$

As A_{R_1} coincides with the π-selfadjoint operator $A_{\tilde{f}_1}$ in $\Pi_\varkappa(\tilde{f}_1)$ (see [2, § 5.1]), $u = \delta_0$ and $f_1'(0)$ does not exist, the condition $u \notin \mathfrak{D}(A_{R_1})$ is satisfied according to [2, Satz 5.5] and (5.7) is proved.

Thus, according to Proposition 5.1, the function $Q_1 \in N_{\varkappa'}$ admits a representation

$$Q_1(z) = -\mathrm{i}z^2 \int_0^\infty \mathrm{e}^{-\mathrm{i}zt} \tilde{g}_1(t)\mathrm{d}t \quad (\operatorname{Im} z < -\gamma)$$

with some function $\tilde{g}_1 \in \mathfrak{G}_{\varkappa'}$. As $\tilde{f}$ and $\tilde{g}$ are related by an equation of the form (5.5) and $\tilde{f}_1$ is an extension of f, the function $\tilde{g}_1$ is an extension of g. Hence $\varkappa' \geqslant \varkappa$, and the equality $\varkappa' = \varkappa$ follows.

Thus, an essential part of the following theorem has been proved.

THEOREM 5.1. (1) *Suppose the Hermitian function* g *on* $[-2a, 2a]$ *is given by* (5.3) *with an accelerant* $H_g \in L^1(-2a, 2a)$ *and let* $\varkappa := \varkappa_{I+\mathrm{H}_g}$, *that is* $g \in \mathfrak{G}_{\varkappa;a}$. *Then the function* f *on* $[-2a, 2a]$, *which is defined by* (5.5), *belongs to the class* $\mathfrak{P}_{\varkappa;a}$, *it has an accelerant* $H_f \in L^1(-2a, 2a)$ *and admits a representation*

$$f(t) = 1 - \frac{1}{\alpha}|t| - \int_0^t (t-s)H_f(s)\mathrm{d}s \quad (-2a \leqslant t \leqslant 2a). \tag{5.9}$$

(2) *Suppose the Hermitian function* f *on* $[-2a, 2a]$ *is given by* (5.9) *with* $\alpha > 0$ *and an accelerant* $H_f \in L^1(-2a, 2a)$ *and let* $f \in \mathfrak{P}_{\varkappa;a}$. *Then the solution* g *of* (5.5) *admits a representation* (5.3) *with an accelerant* $H_g \in L^1(-2a, 2a)$ *and* $g \in \mathfrak{G}_{\varkappa;a}$.

(3) *The accelerants* H_g *and* H_f *in* (1) *or* (2) *determine each other uniquely by the relation*

$$\frac{1}{\alpha} + \int_0^t H_f(s)\mathrm{d}s + \int_0^t H_g(s)H_f(t-s)\mathrm{d}s - \alpha H_f(t) - \frac{1}{\alpha}H_g(t) = 0 \quad (0 \leqslant t \leqslant 2a);$$

H *and* H_g *have the same continuity or smoothness properties.*

(4) *If* f *and* g *are as in* (1) *or* (2), *the relation*

$$-\int_0^t \frac{(t-s)^2}{2}\tilde{f}(s)\mathrm{d}s + \int_0^t \tilde{f}(s)\tilde{g}(t-s)\mathrm{d}s = \int_0^t \tilde{g}(s)\mathrm{d}s \quad (0 \leqslant t < \infty) \tag{5.10}$$

establishes a bijective correspondence between all continuations $\tilde{} \in \mathfrak{G}_\varkappa$ *of* g *and all continuations* $\tilde{f} \in \mathfrak{P}_\varkappa$ *of* f.

Proof. Differentiating (5.5) on $(0, 2a)$ twice we get

$$-\int_0^t f(s)\mathrm{d}s - \alpha f(t) - \int_0^t H_g(t-s)f(s)\mathrm{d}s = g'(t) \quad (0 < t \leqslant 2a).$$

The solution f of this Volterra integral equation is absolutely continuous and, differentiating again, we find

$$-f(t) - \alpha f'(t) - \int_0^t H_g(s)f'(t-s)\mathrm{d}s = 0 \quad (0 < t \leqslant 2a).$$

As $f(0) = 1$, this relation implies $f'(0+) = -1/\alpha$ and f' is absolutely continuous on $[0, 2a]$ with a derivative $-H_f$, given by

$$-f'(t) + \alpha H_f(t) + \frac{1}{\alpha} H_g(t) + \int_0^t H_g(s)\, H_f(t-s)\mathrm{d}s = 0 \quad (0 < t \leqslant 2a).$$

With the remarks before Theorem 5.1 the statement (1) is proved.

If $f \in \mathfrak{P}_{\varkappa;a}$ is given as in (2), we choose a continuation $\tilde{f} \in \mathfrak{P}_\varkappa$ and the corresponding transform $\tilde{g}$, given by an equation of type (5.4) or by (5.10). The restriction $g = \tilde{g}|_{[-2a,\, 2a]}$ belongs to some class $\mathfrak{G}_{\varkappa';a}$, and as g and f are connected by the equality (5.5), it follows that g has an accelerant and that the representation (5.3) holds. Applying part (1) of the theorem, the statement (2) follows. (3) has been proved already, and (4) is an easy consequence of the above considerations.

5.2. Combining the results of § 3 and Theorem 5.1 we get immediately the following result:

THEOREM 5.2. *Suppose the function $g \in \mathfrak{G}_{\varkappa;a}$, $0 < a < \infty$, has a representation* (5.3) *with an accelerant $H_g \in L^1(-2a, 2a)$ and admits more than one continuation $\tilde{g} \in \mathfrak{G}_\varkappa$. Then the function $f \in \mathfrak{P}_{\varkappa;a}$, given by* (5.5), *admits more than one continuation $\tilde{f} \in \mathfrak{P}_\varkappa$. With the resolvent matrix $W = (w_{jk})_1^2$ of f in Theorem* 3.1, *the relation*

$$-\mathrm{i}z^2 \int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\mathrm{d}t = \frac{w_{11}(z)T(z) + w_{12}(z)}{(zw_{11}(z) + w_{21}(z))T(z) + zw_{12}(z) + w_{22}(z)} \quad (\operatorname{Im} z < -\gamma)$$

with some $\gamma \geqslant 0$ establishes a bijective correspondence between all continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of g and all $T \in \tilde{N}_0$.

This follows from the fact that the relation (5.10) between the continuations $\tilde{g}$ and $\tilde{f}$ can be written as

$$Q(z) = R(z)(zR(z) + 1)^{-1}$$

for the corresponding Fourier transforms. Therefore we find a resolvent matrix $W_g(z)$, giving a description of all continuations $\tilde{g} \in \mathfrak{G}_\varkappa$, from a resolvent matrix $W(z)$ of f by the relation

$$W_g(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} W(z). \tag{5.11}$$

REMARK 1. In Theorem 5.1 the resolvent matrix W from Theorem 3.1 can be replaced by the resolvent matrix $\dot{W}$ from Theorem 3.2.

REMARK 2. If we introduce $W_g(r; z)$ as the resolvent matrix of $g_r := g|_{[-2r, 2r]}$, where r is not a singular point of H, then $W_g(r; z)$ satisfies the same canonical differential equations as $W(r; z)$. If, in particular, $\varkappa = 0$, then the resolvent matrix $W_g(r; z)$ of the restriction g_r, $0 < r \leqslant a$, given by (5.11), is the solution of the initial problem

$$\frac{\mathrm{d}W_g(r; z)}{\mathrm{d}r} J = zW_g(r; z)\mathscr{H}_c(r), \qquad W_g(0; z) = I_2$$

with $\mathscr{H}_c$ from (4.10).

REMARK 3. Instead of (5.4) we can consider the more general transformation

$$R(z) = \omega_1 Q(z)(1 - z\omega_2 Q(z))^{-1}$$

with $\omega_1, \omega_2 > 0$. Then the above transformations hold between the functions $\frac{\omega_2}{\omega_1} f$ and $\omega_2 g$ instead of f, g, respectively.

§ 6. CONTINUOUS ANALOGUES OF ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

6.1. Let again $H \in L^1(-2a, 2a)$ be a Hermitian function. Suppose that $-1 \notin \sigma(\mathbf{H})$. Then the resolvent kernel $\Gamma_a(t; s)$ exists. We put in the following $\Gamma_a(t) := \Gamma_a(t, 0)$ $(0 \leqslant t \leqslant 2a)$.

The *orthogonal functions of first and second kind*, associated with H, are defined as follows:

$$D(a; z) := \mathrm{e}^{iaz}\left(1 - \int_0^{2a} \Gamma_a(s)\mathrm{e}^{-izs}\mathrm{d}s\right) \quad (z \in \mathbf{C}),$$

$$E(a; z) := \mathrm{e}^{iaz}\left(1 - \int_0^{2a} L_a(s)\mathrm{e}^{-izs}\mathrm{d}s\right) \quad (z \in \mathbf{C}),$$

respectively, with

$$L_a(s) := -2H(s) + \Gamma_a(s) + 2\int_0^s H(s-t)\Gamma_a(t)\,\mathrm{d}t \quad (0 \leqslant s \leqslant 2a). \tag{6.1}$$

Observing the relation (1.14), we have also

$$L_a(s) = -\Gamma_a(s) - 2\int_s^{2a} H(s-t)\,\Gamma_a(t)\,\mathrm{d}t \quad (0 \leqslant s \leqslant 2a). \tag{6.2}$$

The following lemma is well-known.

LEMMA 6.1. *If* $\chi \in L^1(0, 2a)$ *the function*

$$D_1(z) := 1 - \int_0^{2a} \chi(s)\,\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s$$

has the property

$$D_1(z) \to 1 \quad \textit{if} \quad |z| \to \infty, \quad \operatorname{Im} z \leqslant 0.$$

Indeed, if $z = x + \mathrm{i}y$ we have

$$\left|\int_0^{2a} \chi(s)\,\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s\right| \leqslant \int_0^{2a} |\chi(s)|\,\mathrm{e}^{ys}\,\mathrm{d}s$$

which becomes arbitrarily small for all $y \leqslant y_0$ with a suitable $y_0 < 0$. In the strip $0 \geqslant y \geqslant y_0$ we use the relation ($\tilde{\chi}(s) := \chi(s)$ if $s \in [0, 2a]$ and $= 0$ if $s \notin [0, 2a]$)

$$\left|\int_0^{2a} \chi(s)\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s\right| \leqslant \frac{1}{2}\int_{-\infty}^{\infty} \left|\tilde{\chi}(s) - \tilde{\chi}\left(s + \frac{\pi}{x}\right)\right| \mathrm{e}^{ys}\,\mathrm{d}s + $$
$$+ \frac{1}{2}\int_{-\infty}^{\infty} \left|\tilde{\chi}\left(s + \frac{\pi}{x}\right)\right| \mathrm{d}s\,|1 - \mathrm{e}^{y\pi/x}|,$$

where the terms on the right hand side become arbitrarily small if $|x| \to \infty$.

Lemma 6.1 implies that the functions $D(a;\,\cdot)$ and $E(a;\,\cdot)$ have only a finite number of zeros in the half plane $\operatorname{Im} z \leqslant 0$. This statement will be made more precise in n° 2 below.

Now let

$$E(z) := \mathrm{e}^{\mathrm{i}az}\left(1 - \int_0^{2a} L(s)\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s\right) \tag{6.3}$$

with some $L \in L^1(0, 2a)$, and choose $\gamma > 0$ such that $D(a; z)$ does not vanish if $\operatorname{Im} z \leqslant -\gamma$. Then a generalization of a theorem by N. Wiener implies for $\operatorname{Im} z \leqslant -\gamma$ the representation

$$(6.4) \qquad \left(1 - \int_0^{2a} L(s)\,\mathrm{e}^{-izs}\,\mathrm{d}s\right)\left(1 - \int_0^{2a} \Gamma_a(s)\,\mathrm{e}^{-izs}\,\mathrm{d}s\right)^{-1} = 1 + 2\int_0^{\infty} \tilde{H}_a(s)\,\mathrm{e}^{-izs}\,\mathrm{d}s$$

with some function $\tilde{H}_a$ on $[0, \infty]$ such that the function $e_{-\gamma}\tilde{H}_a$ belongs to $L^1(0, \infty)$. (Here $e_{-\gamma}$ denotes the function $t \to \mathrm{e}^{-\gamma t}$ $(0 \leqslant t < \infty)$.) The relation (6.4) can be written as

$$-\int_0^{2a} L(s)\,\mathrm{e}^{-isz}\,\mathrm{d}s = -\int_0^{2a} \Gamma_a(s)\,\mathrm{e}^{-isz}\,\mathrm{d}s + 2\int_0^{\infty} \tilde{H}_a(s)\,\mathrm{e}^{-izs}\,\mathrm{d}s -$$

$$-2\int_{t=0}^{2a} \mathrm{e}^{-itz} \int_{s=0}^{t} \Gamma_a(s)\tilde{H}_a(t-s)\,\mathrm{d}s\,\mathrm{d}t -$$

$$-2\int_{t=2a}^{\infty} \mathrm{e}^{-itz} \int_{s=0}^{2a} \Gamma_a(s)\tilde{H}_a(t-s)\,\mathrm{d}s\,\mathrm{d}t \quad (\operatorname{Im} z < -\gamma),$$

which is equivalent to

$$(6.5) \qquad \begin{aligned} L(t) &= -2\tilde{H}_a(t) + \Gamma_a(t) + 2\int_0^{t} \Gamma_a(s)\tilde{H}_a(t-s)\,\mathrm{d}s \quad (0 \leqslant t \leqslant 2a), \\ 0 &= -\tilde{H}_a(t) + \int_0^{2a} \Gamma_a(s)\,\tilde{H}_a(t-s)\,\mathrm{d}s \quad (2a < t < \infty). \end{aligned}$$

Now the following proposition can be proved easily. It explains the choice of L_a in (6.1).

PROPOSITION 6.1. *Let $E(z)$ be of the form* (6.3) *with $L \in L^1(0, 2a)$. Then L coincides with L_a from* (6.1) *(that is $E(z) \equiv E(a; z)$) if and only if the function $\tilde{H}_a$ in* (6.5) *is a continuation of H: $\tilde{H}_a(t) = H(t)$ $(0 \leqslant t \leqslant 2a)$.*

Proof. If $\tilde{H}_a(t) = H(t)$ $(0 \leqslant t \leqslant 2a)$, then from (6.5) and (6.1) it is evident that $L(t) = L_a(t)$ $(0 \leqslant t \leqslant 2a)$.

Conversely, given L_a by (6.2), and $L = L_a$, it follows that the function $\Delta(t) := \tilde{H}_a(t) - H(t)$ satisfies the Volterra integral equation

$$\Delta(t) = \int_0^t \Delta(t-u)\Gamma_a(u)\,du \quad (0 \leqslant t \leqslant 2a),$$

which has only the trivial solution $\Delta(t) = 0$ $(0 \leqslant t \leqslant 2a)$. The proposition is proved.

COROLLARY 6.1. *We have*

$$\frac{E(a;z)}{D(a;z)} = 1 + 2\int_0^\infty \tilde{H}_a(t)\,e^{-izt}\,dt \quad (\operatorname{Im} z \leqslant -\gamma)$$

with some function $\tilde{H}_a$ *on* $[0, \infty)$ *such that*

$$e_{-\gamma}\tilde{H}_a \in L^1(0,\infty) \quad \text{and} \quad \tilde{H}_a(t) = H(t) \quad (0 \leqslant t \leqslant 2a).$$

The following relation between $D(a; z)$ and $E(a; z)$ is of fundamental importance.

THEOREM 6.1. *The orthogonal functions of first and second kind satisfy the relation*

$$D(a;z)E^*(a;z) + D^*(a;z)E(a;z) = 2 \quad (z \in \mathbf{C}). \tag{6.6}$$

(If F is an entire function then we define $F^*(z) := \overline{F(\bar{z})}$ $(z \in \mathbf{C})$, $F_R := (F - F^*)/2$, $F_I := (F - F^*)/2i$.)

Proof. The relation (6.6) can be written as

$$-\int_0^{2a} L_a(s)\,e^{izs}\,ds - \int_0^{2a} \Gamma_a(t)\,e^{-izt}\,dt - \int_0^{2a} L_a(s)\,e^{-izs}\,ds -$$
$$-\int_0^{2a} \Gamma_a(t)\,e^{izt}\,dt + \int_0^{2a}\int_0^{2a} \overline{L_a(s)}\Gamma_a(t)\,e^{z(s-t)}\,ds\,dt +$$
$$+\int_0^{2a}\int_0^{2a} \Gamma_a\overline{(s)}L_a(t)\,e^{iz(s-t)}\,ds\,dt = 0$$

which is equivalent to

$$-L_a(u) - \Gamma_a(u) + \int_0^{2a-u} \overline{L_a(s)}\Gamma_a(s+u)\,ds + \int_0^{2a-u} \overline{\Gamma_a(s)}L_a(s+u)\,ds = 0 \tag{6.7}$$

($0 \leqslant u \leqslant 2a$). By the definition of L_a, the left hand side is equal to

$$(6.8)\quad \begin{aligned} &2\int_u^{2a} H(u-v)\Gamma_a(v)\,dv - 2\int_0^{2a-u} \overline{\Gamma_a(v)}\Gamma(u+v)\,dv - \\ &-2\int_{s=0}^{2a-u}\int_{t=s}^{2a} \overline{H(s-t)}\Gamma_a(t)\,dt\,\Gamma_a(s+u)\,ds - 2\int_{s=0}^{2a-u}\int_{t=s+u}^{2a} \overline{\Gamma_a(s)}H(s+u-t)\Gamma_a(t)\,dt\,ds. \end{aligned}$$

The first two terms give

$$2\int_u^{2a} \Gamma_a(v)(-\overline{\Gamma_a(t-u)} + \overline{H(t-u)})\,dt = 2\int_{t=u}^{2a} \Gamma_a(t)\int_{s=0}^{2a} \overline{H(t-s-u)}\,\overline{\Gamma_a(s)}\,ds\,dt,$$

and the last two terms of (6.8) can easily be transformed into the same expression with opposite sign. Thus (6.7) and therefore (6.6) are proved.

COROLLARY 6.2. *If $D(a; z_0) = 0$ then $D(a; \bar{z}_0) \neq 0$, that is, $D(a; \cdot)$ has no real zeros and no complex conjugate pairs of zeros. In the same way, $E(a; \cdot)$ has no real zeros and no complex conjugate pairs of zeros.*

In the next section we shall show that this property is characteristic for orthogonal functions.

6.2. THEOREM 6.2. *Let $\Gamma \in L^1(0, 2a)$ be such that the entire function*

$$(6.9)\qquad D(z) := e^{iaz}\left(1 - \int_0^{2a} \Gamma(t)e^{-izt}\,dt\right)$$

has no real zeros and no complex conjugate pairs of zeros. Then the equation

$$(6.10)\qquad \Gamma(t) + \int_0^{2a} H(t-s)\,\Gamma(s)\,ds = H(t) \quad (0 \leqslant t \leqslant 2a)$$

has a unique solution $H \in L^1(0, 2a)$. If we extend H to $[-2a, 2a]$ by $H(t) := \overline{H(-t)}$ ($-2a \leqslant t \leqslant 0$), then $-1 \notin \sigma(\mathbf{H})$ and $D(z)$ is the orthogonal function of first kind associated with H. The total number of zeros of $D(z)$ in $\mathbf{C}_-$ is equal to $\varkappa_{I+\mathbf{H}}$ (the total number of negative eigenvalues of $I + \mathbf{H}$).

Proof. (1). In this part of the proof we suppose additionally that $D(z)$ does not have any zeros in the closed lower half plane $\operatorname{Im} z \leqslant 0$. The function $(D(x)D^*(x))^{-1}$ belongs to the Wiener ring $\mathfrak{R}$, therefore it admits a representation

$$(D(x)\,D^*(x))^{-1} = 1 + \int_{-\infty}^{\infty} H(t)\,e^{-itx}\,dt \quad (x \in \mathbf{R})$$

with some Hermitian function $H \in L^1(\mathbf{R})$. This relation can be written as

$$
\begin{aligned}
0 = &- \int_{-2a}^{0} \overline{\Gamma(-t)} \mathrm{e}^{-itx}\,\mathrm{d}t - \int_{0}^{2a} \Gamma(t)\mathrm{e}^{-itx}\,\mathrm{d}t + \int_{-\infty}^{\infty} H(t)\mathrm{e}^{-itx}\,\mathrm{d}t + \\
&+ \int_{0}^{2a} \overline{\Gamma(t)}\mathrm{e}^{itx}\,\mathrm{d}x \int_{0}^{2a} \Gamma(s)\mathrm{e}^{isx}\,\mathrm{d}s - \int_{0}^{2a} \overline{\Gamma(t)}\mathrm{e}^{itx}\,\mathrm{d}t \int_{-\infty}^{\infty} H(s)\mathrm{e}^{-ixs}\,\mathrm{d}s - \\
&- \int_{0}^{2a} \Gamma(s)\mathrm{e}^{-ixs}\,\mathrm{d}s \int_{-\infty}^{\infty} H(t)\mathrm{e}^{ixt}\,\mathrm{d}t + \\
&+ \int_{0}^{2a} \overline{\Gamma(t)}\mathrm{e}^{itx}\,\mathrm{d}t \int_{0}^{2a} \Gamma(s)\mathrm{e}^{-isx}\,\mathrm{d}s \int_{-\infty}^{\infty} H(u)\mathrm{e}^{-ixu}\,\mathrm{d}u \quad (x \in \mathbf{R}).
\end{aligned} \tag{6.11}
$$

In the following it is convenient to extend the function Γ to the whole axis putting $\Gamma(t) = 0$ $(t \notin [0, 2a])$. Then (6.11) is equivalent to

$$
\begin{aligned}
0 = &-\overline{\Gamma(-t)} - \Gamma(t) + H(t) + \int_{0}^{2a-t} \overline{\Gamma(s)}\Gamma(s+t)\,\mathrm{d}s\,\chi_{[0,2a]}(t) + \\
&+ \int_{-t}^{2a} \overline{\Gamma(s)}\Gamma(s+t)\,\mathrm{d}s\,\chi_{[-2a,0]}(t) - \int_{0}^{2a} \overline{\Gamma(s)}H(s+t)\,\mathrm{d}s - \\
&- \int_{0}^{2a} \Gamma(s)H(t-s)\,\mathrm{d}s + \int_{0}^{2a}\int_{0}^{2a} H(t-u+s)\overline{\Gamma(s)}\Gamma(u)\,\mathrm{d}s\,\mathrm{d}u \quad (t \in \mathbf{R}),
\end{aligned} \tag{6.12}
$$

where χ_Δ denotes the characteristic function of the interval Δ. If we define the function v:

$$
v(t) := -\Gamma(t) + H(t) - \int_{s=0}^{2a} \Gamma(s)H(t-s)\,\mathrm{d}s \quad (0 \leqslant t < \infty), \tag{6.13}
$$

then (6.12) implies

$$
0 = v(t) - \int_{0}^{2a} \overline{\Gamma(s)}v(t+s)\,\mathrm{d}s \quad (0 \leqslant t < \infty),
$$

or

$$
0 = v(t) - \int_{0}^{\infty} v(s)\overline{\Gamma(s-t)}\,\mathrm{d}s \quad (0 \leqslant t < \infty). \tag{6.14}
$$

This Wiener-Hopf equation with kernel $k(t-s) := \overline{\Gamma(s-t)}$ has only the obvious solution: $v(t)=0$ $(0 \leqslant t < \infty)$. Indeed, the symbol of this equation is given by

$$1-\int_{-\infty}^{\infty} e^{izt}k(t)\,dt = 1-\int_{-\infty}^{\infty} e^{izt}\overline{\Gamma(-t)}\,dt = D^*(-z)e^{-iaz}.$$

This function is bounded on the closed lower half plane and does not vanish there. Hence its index (that is the increase of the argument if z runs through the real axis, divided by 2π) is zero. Now the statement follows from [24, Theorem 1]. Thus we have shown that the relation

$$\Gamma(t)+\int_0^{2a}\Gamma(s)H(t-s)\,ds = H(t) \quad (0 \leqslant t \leqslant 2a) \tag{6.15}$$

holds.

Next we prove that $-1 \notin \sigma(\mathbf{H})$. Assume to the contrary that there exists a non-trivial solution φ of the homogeneous equation

$$\varphi(t)+\int_0^{2a} H(t-s)\varphi(s)\,ds = 0 \quad (0 \leqslant t \leqslant 2a).$$

Then, according to Theorem 1.1, φ can be chosen such that $\varphi(0) \neq 0$, that is $\int_0^{2a} H(-s)\varphi(s)\,ds \neq 0$. On the other hand, φ must be orthogonal to the right hand side of (6.15):

$$\int_0^{2a}\overline{H(s)}\varphi(s)\,ds = 0.$$

As H is Hermitian this is a contradiction.

Now the resolvent kernel $\Gamma_a(t,s)$ of $\mathbf{H}$ is given by Corollary 1.1:

$$\begin{aligned}\Gamma_a(t,s) &= \Gamma(t-s)+\overline{\Gamma(s-t)}+\\ &+\int_0^{\min(s,t)}\left(\overline{\Gamma(2a-t+r)}\Gamma(2a-s+r)-\Gamma(t-r)\overline{\Gamma(s-r)}\right)dr \quad (0\leqslant s,t\leqslant 2a).\end{aligned} \tag{6.16}$$

In particular, $\Gamma_a(t,0)=\Gamma(t)$ $(0 \leqslant t \leqslant 2a)$, thus $D(z)$ in (6.9) is the orthogonal function of first kind associated with H.

(2) Now let $D(z)$ have zeros $z_1, \dots, z_k$ of multiplicities $\varkappa_1, \dots, \varkappa_k$, respectively, in $\mathbf{C}_-$. Then the function $D_0(z) := D(z) \prod_{j=1}^{k} \left(\frac{z-\bar{z}_j}{z-z_j}\right)^{\varkappa_j}$ admits also a representation of the form (6.9) with some Γ_0 instead of Γ (see [24, Lemma 4.1]). As $D_0(z)$ does not vanish in the closed lower half plane, by the first part of this proof it is the orthogonal function of first kind, corresponding to some Hermitian function $H_0 \in L^1(-2a, 2a)$. The corresponding orthogonal function of second kind is given by

(6.17) $$E_0(z) = \mathrm{e}^{\mathrm{i}az}\left(1 - \int_0^{2a} L_0(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t\right)$$

with some $L_0 \in L^1(0, 2a)$ and we have

(6.18) $$D_0(z)E_0^*(z) + D_0^*(z)E_0(z) = 2.$$

By $R(z)$ we denote the sum of the principal parts of $E_0(z)/D_0(z)$, corresponding to the poles $\bar{z}_1, \dots, \bar{z}_k$, and define a function $E(z)$:

$$E(z) := D(z)(E_0(z)/D_0(z) - R(z)) + D(z)R^*(z).$$

Then $E(z)$ is an entire function, which belongs to the Wiener ring $\mathfrak{R}_+$. Moreover, observing that in (6.17) we integrate only from zero to $2a$ and that $D(z)/D_0(z)$ is a rational function it follows that

$$\overline{\lim_{y\uparrow\infty}} \frac{\ln|\mathrm{e}^{-ay}E(\mathrm{i}y)|}{y} = a.$$

Therefore also $E(z)$ admits a representation

$$E(z) = \mathrm{e}^{\mathrm{i}az}\left(1 - \int_0^{2a} L(t)\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t\right)$$

with some $L \in L^1(0, 2a)$. If we chose $\gamma \geqslant 0$ such that $\operatorname{Im} z_j > -\gamma$, $j = 1, 2, \dots k$, then from the generalization of Wiener's theorem used already above it follows that

(6.19) $$\frac{E(z)}{D(z)} = 1 + 2\int_0^{\infty} H(t)\,\mathrm{e}^{-\mathrm{i}tz}\,\mathrm{d}t \quad (\operatorname{Im} z \leqslant -\gamma),$$

where H is such that the function $e_{-\gamma}H$ belongs to $L^1(0, \infty)$. Moreover, (6.18) implies that

(6.20) $$D(z)E^*(z) + D^*(z)E(z) = 2.$$

Now we shall establish the relation (6.10) between the given function Γ and H in (6.19). To this end we first observe that (6.19) is equivalent to

$$L(s) = -2H(s) + \Gamma(s) + 2\int_0^s \Gamma(u)H(s-u)\,du \quad (0 \leqslant s \leqslant 2a), \tag{6.21}$$

$$0 = -H(s) + \int_0^{2a} \Gamma(u)H(s-u)\,du \quad (2a < s < \infty). \tag{6.22}$$

If we extend L to the real axis putting $L(t) = 0$ ($t \notin [0, 2\pi]$), then (6.20) gives

$$-\Gamma(t) - L(t) - \overline{\Gamma(-t)} - \overline{L(-t)} +$$
$$+\int_0^{2a} \Gamma(u+t)\overline{L(u)}\,du + \int_0^{\infty} \overline{\Gamma(u-t)}L(u)\,du = 0 \quad (t \in \mathbf{R}).$$

Thus from (6.21) we obtain for $t \in \mathbf{R}$:

$$\begin{aligned} &2H(t) - 2\chi_{[0,\infty)}(t)\Big(\Gamma(t) + \int_0^t H(t-u)\Gamma(u)\,du\Big) - \\ &-2\chi_{(-\infty,0]}(t)\Big(\overline{\Gamma(-t)} + \int_0^{-t} \overline{H(-t-u)\Gamma(u)}\,du\Big) + \\ &+\int_0^{\infty} \Gamma(u+t)\Big(-2\overline{H(u)} + \overline{\Gamma(u)} + 2\int_0^u \overline{\Gamma(s)}H(u-s)\,ds\Big)\,du + \\ &+\int_0^{\infty} \overline{\Gamma(u-t)}\Big(-2H(u) + \Gamma(u) + 2\int_0^u \Gamma(s)H(u-s)\,ds\Big)\,du = 0. \end{aligned} \tag{6.23}$$

If H is extended to the negative axis by $H(t) = \overline{H(-t)}$ $(-\infty < t < 0)$ and we restrict t in (6.23) to be nonnegative, it follows that

$$H(t) - \Gamma(t) - \int_0^{2a} \Gamma(u)H(t-u)\,du + \int_0^{2a-t} \Gamma(u+t)\overline{\Gamma(u)}\,du - \int_0^{\infty} \overline{\Gamma(u-t)}H(u)\,du +$$
$$+\int_0^{2a}\int_0^{2a} \overline{\Gamma(s)}\Gamma(u)H(s+t-u)\,ds\,du = 0 \quad (t \geqslant 0).$$

This equation coincides with (6.12) if $t \geqslant 0$, therefore it can be transformed into (6.14) with v given by (6.13). As we know from (6.22) that $v(t) = 0$ if $t > 2a$, it follows

$$0 = v(t) - \int_t^{2a} v(s)\overline{\Gamma(s-t)}\, ds \quad (0 \leqslant t \leqslant 2a).$$

This Volterra integral equation has only the trivial solution, that is $v(t) = 0$ $(0 \leqslant t \leqslant 2a)$. Therefore the relation (6.10) holds. Now it follows as in Part (1) of the proof that $-1 \notin \sigma(\mathbf{H})$ and that $D(z)$ is the orthogonal function of first kind associated with H.

The statement about the zeros of $D(z)$ will be proved in § 7.2.

COROLLARY 6.3. *Suppose the entire function $D(z)$ has the same properties as in Theorem 6.2. Then the equation*

$$D(z)E^*(z) + D^*(z)E(z) = 2 \tag{6.24}$$

has a solution $E(z)$ which is of the form $E(z) = e^{iaz}\left(1 - \int_0^{2a} L(t)\, e^{-izt}\, dt\right)$ with some function $L \in L^1(0, 2a)$. Moreover, this solution is uniquely determined in the class of all entire functions which are bounded in the closed lower half plane and after division by e^{+iaz} tend to 1 if z goes to infinity along the negative imaginary axis. The functions $D(z)$ and $E(z)$ are the orthogonal functions of first and second kind, respectively, corresponding to some Hermitian function $H \in L^1(-2a, 2a)$, and we have

$$\frac{E(z)}{D(z)} = 1 + 2\int_0^{\infty} \tilde{H}(t)\, e^{-izt}\, dt \quad (\operatorname{Im} z \leqslant -\gamma) \tag{6.25}$$

for some $\gamma \geqslant 0$, where $\tilde{H}$ is such that $e_{-\gamma}\tilde{H} \in L^1(0, \infty)$ and $\tilde{H}(t) = H(t)$ if $-2a \leqslant t \leqslant 2a$.

It remains to prove the uniqueness statement, the other parts of the Corollary 6.3 have been established during the proof of Theorem 6.2. If there are two solutions $E_1(z)$, $E_2(z)$ of (6.24) with the indicated properties, then $E_3(z) := E_1(z) - E_2(z)$ satisfies the equation

$$\frac{E_3(z)}{D(z)} = -\frac{E_3^*(z)}{D^*(z)} \quad (=: F(z)). \tag{6.26}$$

After division by e^{iaz} it is bounded on the closed lower half plane and tends to zero if z tends to infinity along the negative imaginary axis. As $e^{iaz}D(z)^{-1}$ is bounded outside of a compact part of the closed lower half plane, the left expression in (6.26) is bounded there. Now it follows from (6.26) that $F(z)$ is a bounded entire function, hence $F(z) = \text{const.}$ and this constant must be zero as $\lim_{y\uparrow\infty} F(-\mathrm{i}y) = 0$. Therefore $E_3(z) = 0$ and the uniqueness is proved.

COROLLARY 6.4. *Let $D(a; z)$ and $E(a; z)$ be the orthogonal functions of first and second kind corresponding to some Hermitian function $H \in L^1(-2a, 2a)$. Then there exists a Hermitian function $H_\mathrm{d} \in L^1(-2a, 2a)$, such that $-1 \notin \sigma(\mathbf{H}_\mathrm{d})$ and $E(a; z)$ $(D(a; z))$ is its orthogonal function of first (second, respectively) kind. It is given by the integral equation*

$$H_\mathrm{d}(t) + H(t) + 2\int_0^t H_\mathrm{d}(s)H(t-s)\mathrm{d}s = 0 \quad (0 \leqslant t \leqslant 2a). \tag{6.27}$$

Indeed, if $\tilde{H}$ is defined by (6.25), then we have

$$\frac{D(a; z)}{E(a; z)} = \left(1 + 2\int_0^\infty \tilde{H}(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t\right)^{-1} = 1 + 2\int_0^\infty \tilde{H}_\mathrm{d}(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t$$

$(\operatorname{Im} z \leqslant -\gamma_\mathrm{d})$ with some function $\tilde{H}_\mathrm{d}$ on $[0, \infty)$ such that $e_{-\gamma_\mathrm{d}}\tilde{H}_\mathrm{d} \in L^1(0, \infty)$; here γ_d is such that all the zeros of $E(a; \cdot)$ are in the half plane $\operatorname{Im} z > -\gamma_\mathrm{d}$. The function H_d is the restriction of $\tilde{H}_\mathrm{d}$ to $[0, 2a]$, extended by the relation $H_\mathrm{d}(-t) = \overline{H_\mathrm{d}(t)}$ to $[-2a, 2a]$.

The connection between H and H_d can also be expressed as follows:

$$1 + 2\int_0^{2a} H(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t + o(\mathrm{e}^{-2\mathrm{i}az}) =$$

$$= \left[1 + 2\int_0^{2a} H_\mathrm{d}(t)\,\mathrm{e}^{-\mathrm{i}zt}\mathrm{d}t + o(\mathrm{e}^{-2\mathrm{i}az})\right]^{-1} \quad (\operatorname{Im} z \downarrow -\infty).$$

The function H_d will be called the *dual function of H*. Evidently $(H_\mathrm{d})_\mathrm{d} = H$. It is easy to see that the function L_a can be obtained from the resolvent kernel of $\mathbf{H}_\mathrm{d}$ in the same way as Γ_a was defined by means of the resolvent kernel of $\mathbf{H}$.

REMARK 1. Observe that the Hermitian function $H \in L^1(-2a, 2a)$, $-1 \notin \sigma(\mathbf{H})$, is completely determined by only one of the orthogonal functions D or E. In the positive definite case ($\varkappa = 0$) the following simple relation holds true:

$$\frac{1}{|D(x)|^2} = 1 + \int_{-\infty}^{\infty} \tilde{H}(t)\,\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t \quad (x \in \mathbf{R});$$

here $\tilde{H} \in L^1(\mathbf{R})$ is again a continuation of H.

6.3. Let again $H \in L^1(-2a, 2a)$ be a Hermitian function. We associate with it the dual function H_d, defined on $[0,2a]$ as the solution of the integral equation (6.27) and extended to $[-2a, 2a]$ as a Hermitian function. We shall establish a simple connection between the scalar product $[\cdot,\cdot]$ on $L^2(0, 2a)$ given by (1.25) and the scalar product $[\cdot,\cdot]_d$ defined in the same way with H replaced by H_d.

If $\chi \in L^2(0, 2a)$ we define $\hat{\chi} \in L^2(0, 2a)$:

$$\hat{\chi}(t) := \chi(t) + 2\int_0^t H(t-s)\chi(s)\,ds \quad (0 \leqslant t \leqslant 2a),$$

and $\check{\chi} \in L^2(0,2a)$:

$$\check{\chi}(t) := \chi(t) + 2\int_0^t H_d(t-s)\chi(s)\,ds \quad (0 \leqslant t \leqslant 2a).$$

These relations we shall formally write as

$$\hat{\chi} = \chi + 2H*\chi, \quad \check{\chi} = \chi + 2H_d*\chi, \tag{6.28}$$

that is, in these formulae H and H_d are considered to be zero outside of $[0,2a]$. With this agreement (6.27) takes the form

$$H_d + H + 2(H_d*H) = 0, \tag{6.29}$$

and the scalar product (1.25) can be written as

$$[x, y] = (x, y) + (H*x, y) + (x, H*y),$$

where $(\cdot,\cdot)$ denotes the scalar product of $L^2(0, 2a)$.

(1°) *The mapping $\chi \to \hat{\chi}$ establishes an isomorphism between $L^2(I + \mathbf{H})$ and $L^2(I + \mathbf{H}_d)$; its inverse is given by $\chi \to \check{\chi}$.*

Proof. We have

$$(\hat{\chi})^{\vee} = \hat{\chi} + 2H_d*\hat{\chi} = \chi + 2H*\chi + 2H_d*\chi + 2H*\chi) =$$
$$= \chi + (2H + 2H_d + 4H*H_d)*\chi = \chi$$

and

$$[\chi, \chi] - [\hat{\chi}, \hat{\chi}]_d = (\chi, \chi) + (H*\chi, \chi) + (\chi, H*\chi) -$$
$$-(\hat{\chi}, \hat{\chi}) - (H_d*\hat{\chi}, \hat{\chi}) - (\hat{\chi}, H_d*\hat{\chi}) = 0;$$

here the last equality is easy to check using (6.28) and (6.29).

As an immediate consequence of (1°) we have:

Proposition 6.2. *Let $H \in L^1(-2a, 2a)$ be a Hermitian function. Then*

(1) *The relations $-1 \in \sigma(\mathbf{H})$ and $-1 \in \sigma(\mathbf{H}_d)$ are equivalent. If they hold the mapping $\varphi \to \hat{\varphi}$, given by (6.28), establishes a bijection between the* D*-chains of* $\mathbf{H}$ *and* $\mathbf{H}_d$ *for the eigenvalue* -1.

(2) *The* $\pi_\varkappa$*-spaces* $L^2(I+\mathbf{H})$ *and* $L^2(I+\mathbf{H}_d)$ *are isomorphic.*

6.4. By C_0 we denote the set of all functions φ which are defined and continuous on $[-2a, 0)$ and $(0,2a]$ and such that $\varphi(0\pm)$ exist. On the linear set of all entire function Φ:

$$\Phi(z) = \int_{-2a}^{2a} e^{izt}\varphi(t)\,dt \quad (z\in \mathbf{C})$$

with $\varphi \in C_0$ we consider the following linear functional Ω_H:

$$\Omega_H(\Phi) = \frac{1}{i}\cdot\left(\varphi(0+) + \varphi(0-) + 2\int_{-2a}^{2a} H(t)\varphi(t)\,dt\right). \tag{6.30}$$

Now let $\chi \in L^2(0, 2a)$ be given, and

$$X(z) := \int_0^{2a} e^{-itz}\chi(t)\,dt \quad (z\in\mathbf{C})$$

be its Fourier transform.

(2°) *If* $\chi \in L^2(0,2a)$ *the following relation holds:*

$$\Omega_H^\lambda\left(\frac{e^{2iaz}X(z) - e^{2ia\lambda}X(\lambda)}{z-\lambda}\right) = \hat{X}(z)\,e^{2iaz}. \tag{6.31}$$

Here the λ at Ω_H indicates that the functional Ω_H acts with respect to the variable λ, and $\hat{X}$ is the Fourier transform of $\hat{\chi} = \chi + 2H*\chi$.

Proof. We have

$$\frac{e^{2iaz}X(z) - e^{2ia\lambda}X(\lambda)}{z-\lambda} = \int_0^{2a}\frac{e^{iz(2a-t)} - e^{i\lambda(2a-t)}}{z-\lambda}\chi(t)\,dt =$$

$$= i\int_0^{2a} e^{iz(2a-t)}\int_0^{2a-t} e^{i(\lambda-z)s}\,ds\,\chi(t)\,dt =$$

$$= i\int_{s=0}^{2a} e^{i\lambda s}\int_{t=0}^{2a-s} e^{-iz(t+s)}\chi(t)\,dt\,ds\cdot e^{2iaz},$$

and the definition (6.30) of the functional Ω_H gives

$$\Omega_H^\lambda\left(\frac{e^{2iaz}X(z)-e^{2ia\lambda}X(\lambda)}{z-\lambda}\right)=$$

$$=e^{2iaz}\left(\int_0^{2a}\chi(t)\,e^{-izt}\,dt+2\int_{t=0}^{2a}H(t)\int_{s=0}^{2a-t}e^{-iz(s+t)}\chi(s)\,ds\,dt\right)=$$

$$=e^{2iaz}\int_0^{2a}e^{-izt}\left(\chi(t)+2\int_{s=0}^{t}H(t-s)\chi(s)\,ds\right)dt=e^{2iaz}\hat{X}(z).$$

The proposition (2°) implies that the mapping $\chi\to\hat{\chi}$ of $L^2(I+\mathbf{H})$ into $L^2(I+\mathbf{H}_d)$ can also be realized by means of the functional Ω_H: If $\chi\in L^2(0,2a)$ is given then the left hand side of (6.31) is e^{2iaz} times the Fourier transform of $\hat{\chi}$.

In (2°) the functions $X(z)$ are generated by the powers $(e^{-iz})^t$ with $t\in[0,2a]$. A corresponding statement for the interval $[-a,a]$ is as follows.

(3°) *Let* $X_1(z)=\alpha\,e^{iaz}+\int_{-a}^{a}e^{-isz}\chi(s)\,ds$ *with some* $\alpha\in\mathbf{C}$, $\chi\in L^2(-a,a)$.
Then the following relation holds:

$$\Omega_H^\lambda\left(\frac{X_1(z)-X_1(\lambda)}{z-\lambda}\right)=\alpha\,e^{iaz}+\int_{-a}^{a}e^{-isz}\tilde{\chi}(s)\,ds$$

with

$$\tilde{\chi}(s):=\begin{cases}\chi(s)+2\int_{-a}^{s}H(s-u)\chi(u)\,du+2\alpha H(s+a), & -a\leqslant s<0,\\[2mm] \chi(s)-2\int_{s}^{a}H(s-u)\chi(u)\,du, & 0<s\leqslant a.\end{cases}$$

Proof. We have

$$\frac{X_1(z)-X_1(\lambda)}{z-\lambda}=\alpha i e^{iaz}\int_0^a e^{is(\lambda-z)}\,ds+i\int_{-a}^{a}\chi(s)\,e^{-isz}\int_0^s e^{-it(\lambda-z)}\,dt\,ds=$$

$$=\alpha i\int_{-a}^{0}e^{-it\lambda}e^{iz(a+t)}\,dt-i\int_{t=0}^{a}e^{-it\lambda}\int_{s=t}^{a}e^{-iz(s-t)}\chi(s)\,ds\,dt+$$

$$+\,\mathrm{i}\int\limits_{t=-a}^{0} \mathrm{e}^{-\mathrm{i}t\lambda}\int\limits_{s=-a}^{t} \mathrm{e}^{-\mathrm{i}z(s-t)}\chi(s)\,\mathrm{d}s\,\mathrm{d}t=\int\limits_{-a}^{a}\mathrm{e}^{-\mathrm{i}t\lambda}\chi_1(t)\,\mathrm{d}t,$$

$$\chi_1(t):=\begin{cases}\alpha\mathrm{i}\mathrm{e}^{\mathrm{i}z(a+t)}+\mathrm{i}\int\limits_{s=-a}^{t}\mathrm{e}^{-\mathrm{i}z(s-t)}\chi(s)\,\mathrm{d}s & -a\leqslant t<0,\\ -\,\mathrm{i}\int\limits_{s=t}^{a}\mathrm{e}^{-\mathrm{i}z(s-t)}\chi(s)\,\mathrm{d}s & 0\leqslant t\leqslant a.\end{cases}$$

It follows

$$\Omega_H^\lambda\left(\frac{X_1(z)-X_1(\lambda)}{z-\lambda}\right)=\alpha\mathrm{e}^{\mathrm{i}az}+\int\limits_{s=-a}^{0}\mathrm{e}^{-\mathrm{i}sz}\chi(s)\,\mathrm{d}s-\int\limits_{s=0}^{a}\mathrm{e}^{-\mathrm{i}sz}\chi(s)\,\mathrm{d}s+$$

$$+\,2\int\limits_{t=-a}^{0}\overline{H(t)}\left(\alpha\mathrm{e}^{\mathrm{i}z(a+t)}+\int\limits_{s=-a}^{t}\mathrm{e}^{-\mathrm{i}z(s-t)}\chi(s)\,\mathrm{d}s\right)\mathrm{d}t-2\int\limits_{t=0}^{a}\overline{H(t)}\int\limits_{s=t}^{a}\mathrm{e}^{-\mathrm{i}z(s-t)}\chi(s)\,\mathrm{d}s\,\mathrm{d}t=$$

$$=\alpha\mathrm{e}^{\mathrm{i}az}+\int\limits_{-a}^{0}\mathrm{e}^{-\mathrm{i}sz}\left(\chi(s)+2\alpha H(a+s)+2\int\limits_{-a}^{s}H(s-u)\chi(u)\,\mathrm{d}u\right)+$$

$$+\int\limits_{0}^{a}\mathrm{e}^{-\mathrm{i}sz}\left(-\chi(s)-2\int\limits_{s}^{a}H(s-u)\chi(u)\,\mathrm{d}u\right)\mathrm{d}s=\alpha\mathrm{e}^{\mathrm{i}az}+\int\limits_{-a}^{a}\mathrm{e}^{\mathrm{i}sz}\tilde{\chi}(s)\,\mathrm{d}s.$$

The following relations between the orthogonal functions of first and second kind are an immediate consequence of (3°). Here we write $D(z)$ instead of $D(a;z)$, etc.

(4°)

$$\Omega_H^\lambda\left(\frac{D(z)-D(\lambda)}{z-\lambda}\right)=E(z),\quad \Omega_H^\lambda\left(\frac{D^*(z)-D^*(\lambda)}{z-\lambda}\right)=-\,E^*(z),$$

$$\Omega_{H_\mathrm{d}}^\lambda\left(\frac{E(z)-E(\lambda)}{z-\lambda}\right)=D(z),\quad \Omega_{H_\mathrm{d}}^\lambda\left(\frac{E^*(z)-E^*(\lambda)}{z-\lambda}\right)=-D^*(z).$$

Proof. We have

$$D(z)=\mathrm{e}^{\mathrm{i}az}-\int\limits_{-a}^{a}\Gamma_a(s+a)\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s,\quad E(z)=\mathrm{e}^{\mathrm{i}az}-\int\limits_{-a}^{a}L_a(a+s)\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s,$$

therefore the first relation in (4°) follows if we show that $\chi(s)=-\Gamma_a(s+a)$, $\alpha=1$ implies $\tilde{\chi}(s)=-L_a(a+s)$; here χ and $\tilde{\chi}$ are connected as in (3°). This can be easily verified by means of (6.1) and (6.2).

Between $D(z)$ and $D^*(z)$ the relation $D^*(z) = \bar{D}(-z)$ holds with $\bar{D}(z) = e^{iaz} - \int_{-a}^{a} \overline{\Gamma_a(s+a)}\, e^{-izs} ds$. If in the first part of this proof we replace $\Gamma_a(s+a)$ and $L_a(s+a)$ by their complex conjugates and λ by $-\lambda$ (which implies that H has to be replaced by $\bar{H}$), it follows that

$$\Omega_H^\lambda\left(\frac{\bar{D}(-z) - \bar{D}(-\lambda)}{-z+\lambda}\right) = \bar{E}(-z).$$

This proves the second relation in (4°). The last two relations are now an immediate consequence of Corollary 6.4.

Finally, the first relations in (4°) and (6.6) imply

$$\begin{aligned}\Omega_H^\lambda\left(\frac{D(\lambda)D^*(\zeta) - D(\zeta)D^*(\lambda)}{\lambda - \zeta}\right) &= \Omega_H^\lambda\left(\frac{D(\lambda) - D(\zeta)}{\lambda - \zeta} D^*(\zeta) + \right.\\ &\left. + \frac{D^*(\zeta) - D^*(\lambda)}{\lambda - \zeta} D(\zeta)\right) = E(\zeta)D^*(\zeta) + E^*(\zeta)D(\zeta) = 2.\end{aligned} \tag{6.32}$$

7. A GENERALIZATION OF HERMITE'S THEOREM AND A CONTINUOUS ANALOGUE OF THE SCHUR-COHN THEOREM

7.1. In [38] the following statement was proved:

(1°) *If $F \in N_\varkappa$ then for arbitrary $c \in \mathbf{C}_+$ the function $F - c$ has in $\mathbf{C}_-$ zeros of total order $\varkappa$.*

Recall that for a kernel K by $\varkappa(K)$ we denote the number of its negative squares. If $\mathfrak{C} = (c_{jk})$ is an infinite matrix, then $\varkappa(\mathfrak{C})$ is the maximal number of negative eigenvalues of all the matrices $(c_{jk})_{j,k=0}^N$, $N = 1, 2, \ldots$.

PROPOSITION 7.1. *If $K(z, \zeta)$ is holomorphic in z and ζ on the closed unit disc, $K(z, \zeta) = \overline{K(\zeta, z)}$,*

$$K(z, \zeta) = \sum_{j,k=0}^{\infty} c_{jk} z^j \bar{\zeta}^k \quad \textit{and} \quad \mathfrak{C} := (c_{jk}),$$

then $\varkappa(K) = \varkappa(\mathfrak{C})$.

Proof. We observe the relation

$$c_{jk} = \frac{1}{4\pi^2} \oint_{|z|=1} \oint_{|\zeta|=1} K(z, \zeta) \frac{1}{z^{j+1}} \frac{1}{\bar{\zeta}^{k+1}} dz\, d\bar{\zeta}, \quad j, k = 0, 1, \ldots$$

and approximate the integrals by Riemann sums. This implies for arbitrary $N = 1,2, \ldots$

$$(c_{jk})_0^N = \lim_{M\to\infty} \frac{1}{4\pi^2}\left(\sum_{l,m=1}^{M} K(z_l, z_m)\frac{1}{z_l^{j+1}}\frac{1}{\bar z_m^{k+1}}\,\Delta z_l \Delta \bar z_m\right)_{j,k=0}^{N} =$$

$$= \lim_{M\to\infty} \frac{1}{4\pi^2}\, \mathfrak{Z}^*(k_{ml})_1^M \mathfrak{Z}$$

with $k_{ml} := K(z_l, z_m)$ and

$$\mathfrak{Z} := \begin{pmatrix} \dfrac{\Delta z_1}{z_1} & \ldots & \dfrac{\Delta z_1}{z_1^{N+1}} \\ \vdots & & \vdots \\ \dfrac{\Delta z_M}{z_M^1} & \ldots & \dfrac{\Delta z_M}{z_M^{N+1}} \end{pmatrix}.$$

As the matrix $\mathfrak{Z}^*(k_{ml})_1^M\mathfrak{Z}$ has at most $\varkappa(K)$ negative eigenvalues, we get $\varkappa(\mathfrak{C}) \leqslant \varkappa(K)$. The converse inequality follows from the relation

$$\left(\sum_{j,k=0}^{\infty} c_{jk} z_l^j \bar z_m^k\right)_{l,m=1}^{M} = \mathfrak{Z}_1^*(\hat c_{kj})_0^\infty \mathfrak{Z}_1 = \lim_{N\to\infty} \mathfrak{Z}_1^{(N)*}(\hat c_{kj})_0^\infty \mathfrak{Z}_1^{(N)}$$

with $\hat c_{kj} := c_{jk}$,

$$\mathfrak{Z}_1 := \begin{pmatrix} z_1^0 & z_2^0 & \ldots & z_M^0 \\ z_1^1 & z_2^1 & \ldots & z_M^1 \\ \vdots & \vdots & & \vdots \end{pmatrix} \quad \text{and} \quad \mathfrak{Z}_1^{(N)} := \begin{pmatrix} z_1^0 & z_2^0 & \ldots & z_M^0 \\ \vdots & \vdots & & \vdots \\ z_1^N & z_2^N & \ldots & z_M^N \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

In the following the multiplicity of a zero z_0 of an analytic function G will be denoted by $\varkappa_G(z_0)$, and we put $\varkappa_G(z) = 0$ if $G(z) \neq 0$. If $D(z)\,(\not\equiv 0)$ is an entire function, we consider the Hermite kernel K_D:

$$K_D(z, \zeta) := \frac{D(z)D^*(\bar\zeta) - D(\bar\zeta)D^*(z)}{\mathrm{i}(z - \bar\zeta)} \quad (z \neq \bar\zeta). \tag{7.1}$$

Evidently, K_D is determined by its values in an arbitrary neighbourhood of some point $z_0 \in \mathbf{C}$.

PROPOSITION 7.2. *If $D(z)$ is an entire function and $\varkappa(K_D)$ is finite we have*

$$\sum_{z\in\mathbf{C}_-} (\varkappa_D(z) - \varkappa_D(\bar z))^+ = \varkappa(K_D).^{*)}$$

*) If a is real we put $a^+ = \max(a, 0)$.

Proof. It is easy to verify that the relation

$$K_D(z, \zeta) = 2\, D_R(z)\overline{D_R(\zeta)}\, \frac{D_J(z)/D_R(z) - \overline{D_J(\zeta)/D_R(\zeta)}}{z - \bar{\zeta}} \tag{7.2}$$

holds. Therefore the hypothesis of the proposition implies $D_J/D_R \in N_\varkappa$. From the theorem quoted above it follows that the function

$$\frac{D(z)}{D_R(z)} = \mathrm{i}\, \frac{D_J(z) - \mathrm{i} D_R(z)}{D_R(z)} = \mathrm{i}\left(\frac{D_J(z)}{D_R(z)} - \mathrm{i}\right)$$

has $\varkappa$ zeros in $\mathbf{C}_-$. If $D(z_0) = 0$, $z_0 \in \mathbf{C}_-$, then z_0 is a zero of $\dfrac{D(z)}{D_R(z)}$ of multiplicity

$$\varkappa_D(z_0) - \min(\varkappa_D(z_0), \varkappa_D(\bar{z}_0)),$$

which is equal to $\varkappa_D(z_0) - \varkappa_D(\bar{z}_0)$ or zero, and the statement follows.

Propositions 7.1 and 7.2 imply immediately the following generalization of Hermite's theorem about the number of zeros of a polynomial in a half plane to entire functions. If D is an entire function by $\mathfrak{C}_D$ we denote the matrix (c_{jk}) with

$$K_D(z, \zeta) = \sum_{j,k=0}^{\infty} c_{jk} z^j \bar{\zeta}^k.$$

THEOREM 7.1. *If D is an entire function and $\varkappa(K_D)$ is finite then*

$$\sum_{z \in \mathbf{C}_-} (\varkappa_D(z) - \varkappa_D(\bar{z}))^+ = \varkappa(\mathfrak{C}_D).$$

Using a result of M. G. Kreĭn about entire functions of the Hermite-Biehler class a complete description of all the entire functions D with $\varkappa(K_D) < \infty$ can be given.

THEOREM 7.2. *Let $D(z)$ be an entire function. The kernel K_D has a finite number $\varkappa$ of negative squares if and only if $D(z)$ admits a representation*

$$D(z) = E(z)\,\mathrm{e}^{\mathrm{i}(az+b)} \prod_{j=1}^{\varkappa} (z - \beta_j) \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) \mathrm{e}^{\operatorname{Re} P_k(z/\alpha_k)} \tag{7.3}$$

with an arbitrary real entire function $E(z)$, $a \geqslant 0$, $b \in \mathbf{R}$, $\beta_j \in \mathbf{C}_-$ $(j = 1, \ldots, \varkappa)$, $\alpha_k \in \mathbf{C}_+$ $(k = 1,2,\ldots)$ and polynomials

$$P_k(z) := z + \frac{z^2}{2} + \ldots + \frac{z^{p_k}}{p_k}, \quad \operatorname{Re} P_k(z) = \frac{1}{2}(P_k(z) + P_k^*(z));$$

the integers p_k have to be chosen such that the second product in (7.3) converges (if it is infinite).

In the representation (7.3) we can always choose $p_k = k$. If $\rho := \inf\{\lambda : \sum_k |\alpha_k|^{-\lambda} < \infty\} < \infty$ we can put $p_k = [\rho]$.

Proof. With $l := (1 + iz)(1 - iz)^{-1}$, $\lambda := (1 + i\zeta)(1 - i\zeta)^{-1}$ and $X(l) := D(z)/D^*(z)$ it follows that

$$K_D(z, \zeta) = D^*(z)\frac{X(l)X^*(\bar{\lambda}) - 1}{2(l\bar{\lambda} - 1)} D(\bar{\zeta})(l + 1)(\bar{\lambda} + 1).$$

If $z, \zeta \in \mathbf{C}_+$ we have $|l|, |\lambda| < 1$. Therefore the kernel K_D has $\varkappa$ negative squares if and only if the function $X(l)\,(|l| < 1)$ belongs to the generalized Schur class $S_\varkappa$, introduced in [40]. It follows from [40, Theorem 3.2] that $X(l)$ has $\varkappa$ poles in the open unit disc. Thus the function $D(z)/D^*(z)$ has $\varkappa'$ $(\leqslant \varkappa)$ mutually different poles $\bar{\beta}_1, \ldots, \bar{\beta}_{\varkappa'}$ in $\mathbf{C}_+$ of total multiplicity $\varkappa$:

$$\sum_{j=1}^{\varkappa'} \varkappa_j = \varkappa \quad \text{with } \varkappa_j := \varkappa_{D^*}(\bar{\beta}_j) - \varkappa_D(\bar{\beta}_j) \ (> 0).$$

The factor $\prod_{j=1}^{\varkappa'} (z - \bar{\beta}_j)^{\varkappa_D(\bar{\beta}_j)}$ is common to $D(z)$ and $D^*(z)$, hence $\prod_{j=1}^{\varkappa'} ((z - \beta_j)(z - \bar{\beta}_j))^{\varkappa_D(\bar{\beta}_j)}$ is a real factor of $D(z)$. As this factor can be taken to $E(z)$ we can suppose that $D(\bar{\beta}_j) \neq 0$, $\varkappa_{D^*}(\bar{\beta}_j) = \varkappa_D(\beta_j) = \varkappa_j$.

Putting $P(z) = \prod_{j=1}^{\varkappa} (z - \beta_j)^{\varkappa_j}$, it follows that the function

$$D_1(z) = D(z)/P(z)$$

has no zeros in the lower half plane. Moreover,

$$K_D(z, \zeta) = D_1(z)K_P(z, \zeta)D_1^*(\bar{\zeta}) + P^*(z)K_{D_1}(z, \zeta)P(\bar{\zeta}).$$

The first term on the right hand side is a nonpositive kernel with $\varkappa$ negative squares, the second term has an at most finite number of negative squares. As $D_1(z)$ has no zeros in the lower half plane, by the considerations at the beginning of this proof (applied to D_1 instead of D) it follows that the kernel K_{D_1} is positive definite. According to [41] (see also [42, Theorem VII.6]) the function $D_1(z)$ admits a representation

$$D_1(z) = E(z)e^{i(az+b)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) e^{\operatorname{Re} P_k(z/\alpha_k)}$$

where $E(z)$, a, b, etc. have the properties mentioned in the theorem. With a suitable enumeration of the zeros of D in the lower half plane (according to their multiplicities) the representation (7.3) follows. The converse statement is now easy to see, and the theorem is proved.

7.2. Now let an entire function of the form

$$D(z) := e^{iaz}\left(1 - \int_0^{2a} \Gamma(t)\, e^{-itz}\, dt\right) \tag{7.4}$$

with some $\Gamma \in L^1(0, 2a)$ be given. It is not hard to check that the kernel $K_D(z, \zeta)$ admits the representation (comp. [29], in the discrete case see also [30])

$$K_D(z, \zeta) = e^{ia(z-\bar{\zeta})} \int_0^{2a}\int_0^{2a} e^{-izt}\, e^{i\bar{\zeta}s}(\delta_0(t-s) - \gamma(t, s))\, ds\, dt \tag{7.5}$$

with

$$\gamma(t, s) := \Gamma(t-s) + \overline{\Gamma(s-t)} + \int_0^{\min(s,t)} (\overline{\Gamma(2a-t+r)}\Gamma(2a-s+r) - \Gamma(t-r)\overline{\Gamma(s-r)})\, dr; \tag{7.6}$$

here we put again $\Gamma(t) := 0$ if $t \notin [0, 2a]$.

In the following the integral operator in $L^2(0, 2a)$ with the kernel $\gamma(t, s)$ will be denoted by $\mathbf{\Gamma}$. Recall that $\varkappa_{I-\mathbf{\Gamma}}$ denotes the total number of negative eigenvalues of the operator $I - \mathbf{\Gamma}$.

LEMMA 7.1. *If D is given by* (7.4) *we have*

$$\varkappa(K_D) = \varkappa_{I-\mathbf{\Gamma}}.$$

Proof. If, for short, $\varkappa(K_D) = \varkappa$, there exist n different points $z_1, z_2, \ldots, z_n$ $(n \geqslant \varkappa)$ such that the $n \times n$-matrix $\mathfrak{K} := (K_D(z_j, z_k))_1^n$ has $\varkappa$ negative eigenvalues. That is we can find $\varkappa$ linearly independent n-vectors $\vec{\xi}_1, \ldots, \vec{\xi}_\varkappa$ such that

$$(\mathfrak{K}\xi, \vec{\xi}) < 0$$

for each nonzero vector $\vec{\xi}$ of the linear span of the $\vec{\xi}_1, \ldots, \vec{\xi}_\varkappa$. If $\vec{\xi}_\rho = (\xi_{\rho 1}, \ldots, \xi_{\rho n})^{\mathrm{T}}$, $\rho = 1, \ldots, \varkappa$, we define the functions

$$\varphi_\rho(t) := \sum_{k=1}^{n} e^{-iz_k t}\, \bar{\xi}_{\rho k}, \quad \rho = 1, \ldots, \varkappa \ (0 \leqslant t \leqslant 2a). \tag{7.7}$$

Then for $\varphi(t) := \sum_{\rho=1}^{\varkappa} \bar{\gamma}_\rho \varphi_\rho(t)$, $\gamma_\rho \in \mathbf{C}$, it follows

(7.8) $$((I - \Gamma)\varphi, \varphi)_{L^2(0,2a)} = \sum_{k,l=1}^{n} e^{-ia(z_k - \bar{z}_l)} K_D(z_k, z_l) \sum_\sigma \gamma_\sigma \bar{\xi}_{\sigma l} \sum_\rho \bar{\gamma}_\rho \bar{\xi}_{\rho k}$$

and this is negative for each non-zero function φ in the linear span of $\varphi_1, \ldots, \varphi_\varkappa$. Therefore

$$\varkappa_{I-\Gamma} \geqslant \varkappa = \varkappa(K_D).$$

Let now $I - \Gamma$ have $\varkappa'$ negative eigenvalues. As the linear span of the functions e_{iz} $(z \in \mathbf{C})$ is dense in $L^2(0, 2a)$, there exist $\varkappa'$ functions φ_ρ, $\rho = 1, \ldots, \varkappa'$, of the form given in (7.7) such that

$$((I - \Gamma)\varphi, \varphi)_{L^2(0,2a)} < 0$$

for all non-zero functions φ from the linear span of $\varphi_1, \ldots, \varphi_{\varkappa'}$. This, by (7.8), implies that the matrix $(K_D(z_k, z_l))_1^{\varkappa'}$ has at least $\varkappa'$ negative eigenvalues, that is

$$\varkappa(K_D) \geqslant \varkappa' = \varkappa_{I-\Gamma},$$

and the lemma is proved.

Combining the results of Proposition 7.2 and Lemma 7.1 we have the following.

THEOREM 7.3. (Continuous analogue of the Schur-Cohn theorem*)). *If $D(z)$ is given by (7.4) and Γ denotes the integral operator with the kernel $\gamma(t, s)$ from (7.6) we have*

$$\sum_{z \in \mathbf{C}_-} (\varkappa_D(z) - \varkappa_D(\bar{z}))^+ = \varkappa_{I-\Gamma}.$$

That is, the total number of zeros of $D(z)$ in $\mathbf{C}_-$, which do not correspond to complex conjugate pairs of zeros, is equal to the number of negative eigenvalues of $I - \Gamma$. We mention that it was shown in [44] that the total number of real and of complex conjugate pairs of zeros of $D(z)$ coincides with the dimension of the kernel of $I - \Gamma$.

If the function $D(z)$, given by (7.4), does not have any real zeros or complex conjugate pairs of zeros, according to the statements of Theorem 7.2 which are already proved, there exists a Hermitian function $H \in L^1(-2a, 2a)$ such that $\gamma(t, s)$ in (7.6) is the resolvent kernel of $\mathbf{H}$ (observe that the right hand sides of (7.6) and (6.16) coincide):

$$I - \Gamma = (I + \mathbf{H})^{-1}.$$

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*) For the classical theorem of Schur-Cohn see [43].

Moreover, $D(z)$ is the orthogonal function of first kind associated with H. If we observe that $\varkappa_{I-\Gamma} = \varkappa_{I+\mathbf{H}}$, Theorem 7.1 implies that $D(z)$ has $\varkappa_{I+\mathbf{H}}$ zeros in $\mathbf{C}_-$. Thus also the last statement of Theorem 6.2 is proved.

Furthermore, if we start again from a Hermitian function $H \in L^1(-2a, 2a)$, the formulae (7.5) and (7.1) give a relation between (the Fourier transform of) the resolvent kernel $\Gamma(s, t)$ of $\mathbf{H}$ and the orthogonal functions of first kind of H:

$$\frac{D(z)D^*(\bar{\zeta}) - D(\bar{\zeta})D^*(z)}{i(z-\bar{\zeta})} = \tag{7.9}$$

$$= e^{ia(z-\bar{\zeta})} \int_0^{2a} e^{-izt} \left(e^{i\bar{\zeta}t} - \int_0^{2a} \Gamma(t, s)\, e^{i\bar{\zeta}s}\, ds \right) dt.$$

§ 8. THE OPERATOR $\dot{A}_0 = \frac{1}{i}\frac{d}{dt}$ IN $L^2(I+\dot{\mathbf{H}})$

8.1. Let $H \in L^1(-2a, 2a)$ be a Hermitian function. In this section we suppose that $-1 \notin \sigma(\dot{\mathbf{H}})$. According to the remarks at the end of § 1.6, the operator $\dot{A}_0$ in $L^2(I+\dot{\mathbf{H}})$ is closed, π-Hermitian with deficiency index (1; 1) and simple. Recall that $L^2(I+\dot{\mathbf{H}})$ is the space $L^2(-a, a)$, equipped with the scalar product (1.25):

$$[\varphi, \psi] = \int_{-a}^{a} \varphi(t)\overline{\psi(t)}\, dt + \int_{-a}^{a}\int_{-a}^{a} H(t-s)\varphi(s)\overline{\psi(t)}\, ds\, dt \qquad (\varphi, \psi \in L^2(-a, a)).$$

In this n° we shall give a description of all the π-selfadjoint extensions of $\dot{A}_0$ in $L^2(I+\dot{\mathbf{H}})$ along the well-known procedure, which can easily be generalized to $\pi_\varkappa$-spaces.

If $z \notin \sigma_p(\dot{A}_0)$, $z \neq \bar{z}$, by $\dot{q}_z$ we denote again a nonzero defect vector: $\dot{q}_z \perp \mathfrak{R}(\dot{A}_0 - \bar{z}I)$, see (1.28). Choose $z_0 \neq \bar{z}_0$, $z_0 \notin \sigma_p(\dot{A}_0)$ such that $[\dot{q}_{z_0}, \dot{q}_{z_0}] > 0$. According to [34] there exists an $h \geqslant 0$ such that this condition is always satisfied if $|\operatorname{Im} z_0| > h$. For arbitrary γ, $|\gamma| = 1$, we put

$$\psi_\gamma := \gamma\dot{q}_{\bar{z}_0} - \dot{q}_{z_0}$$

and define an extension $\dot{A}_0^\gamma$ of $\dot{A}_0$ in $L^2(I+\dot{\mathbf{H}})$ on the linear span of $\mathfrak{D}(\dot{A}_0)$ and ψ_γ:

$$\dot{A}_0^\gamma\psi_\gamma := \gamma\bar{z}_0\dot{q}_{\bar{z}_0} - z_0\dot{q}_{z_0}. \tag{8.1}$$

Then $\dot{A}_0^\gamma$ is π-selfadjoint. We mention that for arbitrary $\gamma \in \mathbf{C}$ the relation

$$\operatorname{Im}[\dot{A}_0^\gamma\psi_\gamma, \psi_\gamma] = \operatorname{Im} z_0(1-|\gamma|^2)[\dot{q}_{z_0}, \dot{q}_{z_0}]$$

holds true. An explicit description of the operator $\dot{A}_0^\gamma$ is as follows.

(1°) *The domain* $\mathfrak{D}(\dot{A}_0^\gamma)$ *consists of all absolutely continuous functions* $\varphi \in L^2(-a, a)$ *such that* $\varphi' \in L^2(-a, a)$ *and*

$$\varphi(a)(\gamma D^*(\bar{z}_0) - D^*(z_0)) - \varphi(-a)(\gamma D(\bar{z}_0) - D(z_0)) = 0, \tag{8.2}$$

the operator $\dot{A}_0^\gamma$ *is given by the relation*

$$(\dot{A}_0^\gamma\varphi)(t) = \frac{1}{i}\varphi'(t) + \frac{1}{i}\varphi(a)\left(\dot{\Gamma}_a(t, -a)\,\frac{D^*(z_0) - \gamma D^*(\bar{z}_0)}{D(z_0) - \gamma D(\bar{z}_0)} - \dot{\Gamma}_a(t; a)\right), \tag{8.3}$$

($\varphi \in \mathfrak{D}(\dot{A}_0^\gamma)$*).*

Here $D(z) = D(a; z)$ is the orthogonal function of first kind associated with H.

Proof. We first observe that the orthogonal function $D(z)$ can be written as

$$D(z) = e^{iaz} - \int_0^{2a} \Gamma_a(s, 0)\, e^{-iz(s-a)}\, ds = e^{iaz} - \int_{-a}^{a} \dot{\Gamma}_a(a, t)\, e^{izt}\, dt.$$

The defect vector $\dot{q}_{z_0}$, given by (1.28), is absolutely continuous (observe (1.27)):

$$\dot{q}'_{z_0}(t) = iz_0\dot{q}_{z_0}(t) + \dot{\Gamma}(t, a)\, D(z_0) - \dot{\Gamma}(t, -a)D^*(z_0) \tag{8.4}$$

and we have

$$\dot{q}_{z_0}(a) = D(z_0), \quad \dot{q}_{z_0}(-a) = D^*(z_0).$$

Therefore ψ_γ is absolutely continuous, $\psi'_\gamma \in L^2(-a, a)$ and

$$\psi_\gamma(a) = \gamma D(\bar{z}_0) - D(z_0), \quad \psi_\gamma(-a) = \gamma D^*(\bar{z}_0) - D^*(z_0).$$

It follows that also $\varphi \in \mathfrak{D}(\dot{A}_0^\gamma)$, which is by definition of the form $\varphi = \varphi_0 + \xi\psi_\gamma$ ($\varphi_0 \in \mathfrak{D}(\dot{A}_0)$, $\xi \in \mathbf{C}$), is absolutely continuous and satisfies the boundary condition

$$\varphi(a)\psi_\gamma(-a) - \varphi(-a)\psi_\gamma(a) = 0,$$

which is equivalent to (8.2).

Conversely, let φ be an absolutely continuous function, $\varphi' \in L^2(-a, a)$, which satisfies (8.2). Observe that $|\psi_\gamma(a)| + |\psi_\gamma(-a)| \neq 0$, otherwise $\psi_\gamma \in \mathfrak{D}(A_0)$, which is impossible as no linear combination of the defect vectors $\dot{q}_{z_0}$ and $\dot{q}_{\bar{z}_0}$ belongs to $\mathfrak{D}(\dot{A}_0)$. Assume e.g. $\psi_\gamma(a) \neq 0$. Then

$$\varphi = \left(\varphi - \frac{\varphi(a)}{\psi_\gamma(a)}\psi_\gamma\right) + \frac{\varphi(a)}{\psi_\gamma(a)}\psi_\gamma,$$

and the first term on the right hand side belongs to $\mathfrak{D}(\dot{A}_0)$.

In order to prove (8.3) we observe that (8.4) and (8.1) imply

$$\psi_\gamma' = \mathrm{i}\dot{A}_0^\gamma\psi_\gamma + \dot{\Gamma}(t, a)(\gamma D(\bar{z}_0) - D(z_0)) - \dot{\Gamma}(t, -a)(D^*(\bar{z}_0) - D^*(z_0)).$$

Consequently,

$$\dot{A}_0^\gamma(\varphi_0 + \xi\psi_\gamma) = \frac{1}{\mathrm{i}}(\varphi_0' + \xi\psi_\gamma') + \frac{1}{\mathrm{i}}\,\xi(\dot{\Gamma}(t, -a)(\gamma D^*(\bar{z}_0) - D^*(z_0)) -$$

$$- \dot{\Gamma}(t, a)(\gamma D(\bar{z}_0) - D(z_0))),$$

and, as

$$\xi = \varphi(a)\psi_\gamma(a)^{-1} = \varphi(a)(\gamma D(\bar{z}_0) - D(z_0))^{-1},$$

the relation (8.3) follows. The statement (1°) is proved.

Now it is easy to find the resolvent of $\dot{A}_0^\gamma$:

$$((\dot{A}_0^\gamma - zI)^{-1}u)(t) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}zt}\int_{-a}^{t}\mathrm{e}^{-\mathrm{i}zs}u(s)\,\mathrm{d}s +$$

8.5)

$$+ \mathrm{i}\frac{\dot{\mathscr{F}}(u; z)\,\mathrm{e}^{\mathrm{i}zt}}{D^*(z) - \Delta(z_0)D(z)}\left(\int_{-a}^{t}(\dot{\Gamma}(s, a) - \Delta(z_0)\dot{\Gamma}(s, -a))\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s + \Delta(z_0)\,\mathrm{e}^{\mathrm{i}za}\right)$$

with

$$\dot{\mathscr{F}}(u; z) := \int_{-a}^{a}\mathrm{e}^{-\mathrm{i}zt}u(t)\,\mathrm{d}t, \quad \Delta(z_0) := (D^*(z_0) - \gamma D^*(\bar{z}_0))(D(z_0) - \gamma D(\bar{z}_0))^{-1}.$$

8.2. Evidently, the spectrum of $\dot{A}_0^\gamma$ is the set of zeros of the entire function $D^*(z) - \Delta(z_0)D(z)$, which gives some information about these zeros. The formulation of this statement will be left to the reader. We shall show, however, how these connections between $\sigma(\dot{A}_0^\gamma)$ and the zeros of $D^*(z) - \Delta(z_0)D(z)$ can be used in order to show that $D(z)$ has $\varkappa$ zeros in the lower half plane.

To this end we first mention that the operators $\dot{A}_0^\gamma$ can be defined for arbitrary γ such that $|\gamma| \leqslant 1$, and that in this way we get all the maximal π-dissipative *) extensions of A_0. The relation (8.3) can be written as follows:

$$(\dot{A}_0^\gamma\varphi)(t) = \frac{1}{\mathrm{i}}\,\varphi'(t) + \frac{1}{\mathrm{i}}\,\varphi(-a)\left(\dot{\Gamma}(t, -a) - \frac{D(z_0) - \gamma D(\bar{z}_0)}{D^*(z_0) - \gamma D^*(\bar{z}_0)}\dot{\Gamma}(t, a)\right).$$

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*) An operator A in the $\pi_\varkappa$-space $\Pi_\varkappa$ is called π-dissipative if $\operatorname{Im}[A\varphi, \varphi] \geqslant 0$ ($\varphi \in \mathfrak{D}(A)$) and maximal π-dissipative if it does not have any proper π-dissipative extension, see [8].

Now we consider the following operator $\dot{B}$: $\mathfrak{D}(\dot{B})$ is the set of all absolutely continuous functions φ such that $\varphi' \in L^2(-a, a)$, $\varphi(a) = 0$, and

(8.6) $$(\dot{B}\varphi)(t) := \frac{1}{i}\varphi'(t) + \frac{1}{i}\varphi(-a)\dot{\Gamma}(t, -a).$$

Evidently,

(8.7) $$\dot{B} = \dot{A}_0^{\gamma_0} \quad \text{with } \gamma_0 := D(z_0)/D(\bar{z}_0).$$

This connection, however, we shall use only later. First we prove in a straightforward way the following proposition.

(2°) *The operator $\dot{B}$ in $L^2(I + \dot{\mathbf{H}})$ is maximal π-dissipative,*

(8.8) $$\operatorname{Im}[\dot{B}\varphi, \varphi] = |\varphi(-a)|^2 \quad (\varphi \in \mathfrak{D}(\dot{B})).$$

The linear span of the algebraic eigenspaces of $\dot{B}$, corresponding to eigenvalues in the lower half plane $\mathbf{C}_-$, is a $\varkappa$-dimensional negative subspace. All the eigenvalues of $\dot{B}$ are geometrically simple.

Proof. If we observe (8.6) it follows that

$$\operatorname{Im}[\dot{B}\varphi, \varphi] = -\operatorname{Re}\{((I + \dot{\mathbf{H}})\varphi', \varphi) + \varphi(-a)((I + \dot{\mathbf{H}})\dot{\Gamma}(\cdot, -a), \varphi)\} =$$
$$= -\frac{1}{2}\left\{\int_{-a}^{a} \varphi'\bar{\varphi}\,dt + \int_{-a}^{a} \bar{\varphi}'\varphi\,dt + \int_{-a}^{a}\int_{-a}^{a} H(t-s)\varphi'(s)\overline{\varphi(t)}\,ds\,dt + \right.$$
$$+ \int_{-a}^{a}\int_{-a}^{a} H(s-t)\overline{\varphi'(s)}\varphi(t)\,ds\,dt + \varphi(-a)\int_{-a}^{a} H(t+a)\overline{\varphi(t)}\,dt +$$
$$\left. + \overline{\varphi(-a)}\int_{-a}^{a} \overline{H(t+a)}\varphi(t)\,dt\right\},$$

and integration by parts gives (8.8). The relation (8.8) implies that $\dot{B}$ does not have any real eigenvalues. Indeed, $\dot{B}\varphi_0 = \lambda_0\varphi_0$, $\lambda_0 = \bar{\lambda}_0$, implies $\varphi_0(-a) = 0$, and the boundary problem $\frac{1}{i}\frac{d\varphi_0}{dt} = \lambda_0\varphi_0$, $\varphi_0(-a) = \varphi_0(a) = 0$ has only the obvious solution $\varphi_0 = 0$.

By (8.8), $\dot{B}$ is dissipative. If $\Delta(z) \neq 0$ its resolvent exists and is given by the formula (comp. (8.5))

(8.9) $$((\dot{B} - zI)^{-1}u)(t) = i\,e^{izt}\int_{-a}^{t} e^{-izs}u(s)\,ds +$$
$$+ i\,D(z)^{-1}\mathscr{F}(u; z)\,e^{izt}\left(\int_{-a}^{t} \dot{\Gamma}(s, -a)\,e^{izs}\,ds - e^{iza}\right).$$

According to [8] the maximal π-dissipative operator $\dot{B}$ in the $\pi_\varkappa$-space $L^2(I+\dot{\mathbf{H}})$ has a $\varkappa$-dimensional nonpositive invariant subspace $\mathscr{L}$ such that $\operatorname{Im}\sigma(\dot{B}|\mathscr{L}) \leqslant 0$ and all the algebraic eigenspaces corresponding to eigenvalues in $\mathbf{C}_-$ belong to $\mathscr{L}$. By the above remark we have even $\sigma(\dot{B}|\mathscr{L}) \subset \mathbf{C}_-$. The last statement in (2°) follows from the fact that the initial problem $\frac{1}{i}\frac{d\varphi}{dt} = \lambda\varphi$, $\varphi(a) = 0$, has a unique (up to constant multiples) solution. The proposition (2°) is proved.

As the eigenvalues of $\dot{B}$ are geometrically simple, their algebraic multiplicities coincide with the orders of the corresponding poles of the resolvent. Thus, from (2°) and (8.9) we have the

COROLLARY 8.1. *The orthogonal function $D(z)$ has zeros of total multiplicity $\varkappa$ in $\mathbf{C}_-$.*

This fact was stated in Theorem 6.2 and proved by a different method (using a generalization of the Schur-Cohn theorem) in § 7.2. The idea of the proof in this n° (considering the spectrum of the maximal π-dissipative operator $\dot{B}$) was also applied by L. A. Sahnovič [12], who proved the statement of Corollary 8.1 for the case of a generalized function H.

As we have mentioned already, the operators $\dot{A}_0^\gamma$, $|\gamma| \leqslant 1$, are the maximal π-dissipative extensions of A_0 in $L^2(I+\dot{\mathbf{H}})$. Here we have $|\gamma| < 1$ if and only if $\dot{A}_0^\gamma$ is not π-selfadjoint. Thus (8.7) and (2°) imply immediately the following statement.

(3°) *There exists an $h \geqslant 0$ such that*

$$|D(z)| < |D(\bar{z})| \quad \textit{if } \operatorname{Im} z > h.$$

In the positive definite case ($\varkappa = 0$) this property (with $h = 0$) plays a crucial role in the theory of De Branges [36].

8.3. Now we find the δ_0-resolvent matrix of the operator $\dot{A}_0 = \frac{1}{i}\frac{d}{dt}$ in $L^2(I+\dot{\mathbf{H}})$. To this end we apply [9, Satz 3.10]. This, however, is not so straightforward, as δ_0 is only a generalized scale vector. Therefore we shall choose a δ_0-sequence (u_n) of functions $u_n \in L^2(I+\dot{\mathbf{H}})$, which are symmetric with respect to zero, and shall calculate

$$\lim_{n\to\infty} W_{u_n}(z) =: W_{\delta_0}(z); \tag{8.10}$$

here W_{u_n} denotes the u_n-resolvent matrix of $\dot{A}_0$. It turns out that the limit in (8.10) exists for all $z \in \mathbf{C}$. *)

First of all we observe that with the defect vector $\dot{q}_z$ from (1.28) we have for arbitrary $f \in L^2(-a, a)$

$$[f, \dot{q}_{\bar{z}}] = ((I + \dot{\mathbf{H}})f, (I - \dot{\Gamma})e_z) = \int_{-a}^{a} f(t)\,\mathrm{e}^{-\mathrm{i}zt}\mathrm{d}t.$$

It follows (see [9, § 3.1]) that

$$\mathscr{P}(z)f := \lim_{n\to\infty} \frac{[f, \dot{q}_{\bar{z}}]}{[u_n, \dot{q}_{\bar{z}}]} = \int_{-a}^{a} f(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t. \; ^{**)}$$

Recall that $\mathscr{P}(z)$ is considered as a mapping from $L^2(I + \dot{\mathbf{H}})$ into $\mathbf{C}$. The adjoint mapping $\mathscr{P}(z)^*$ is given by

$$\mathscr{P}(z)^*\alpha = \alpha \dot{q}_{\bar{z}} \quad (\alpha \in \mathbf{C}),$$

and we get

$$\mathscr{P}(z)\mathscr{P}(\bar{\zeta})^* = \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}zt}\left(\mathrm{e}^{\mathrm{i}\zeta t} - \int_{-a}^{a} \dot{\Gamma}(t, s)\,\mathrm{e}^{\mathrm{i}\zeta s}\,\mathrm{d}s\right)\mathrm{d}t.$$

On the other hand, the term on the right hand side can be expressed by means of the orthogonal functions of first kind and it follows

$$\mathscr{P}(z)\mathscr{P}(\bar{\zeta})^* = \frac{D(z)\,D^*(\zeta) - D(\zeta)D^*(z)}{\mathrm{i}(z - \zeta)}.$$

For later use we formulate the following consequence of this identity:

(4°) *If D is the orthogonal function of first kind, the kernel K_D (see (7.1)) has infinitely many positive squares.*

Indeed, we have $K_D(z, \zeta) = \mathscr{P}(z)\mathscr{P}(\zeta)^*$ and the elements $\mathscr{P}(\zeta)^*\alpha$ ($\alpha \in \mathbf{C}$, $\zeta \neq \bar{\zeta}$) span the whole space $L^2(I + \dot{\mathbf{H}})$, which is infinite-dimensional.

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*) In fact, the generalized scale vector δ_0 belongs to some space with negative norm. Here, however, we shall avoid the theory of space triplets and use the simple limit procedure (8.10).

**) The functionals $\mathscr{P}(z)$, $Q(z)$ in this and the following sections correspond to the operator $\dot{A}_0$ in $L^2(I + \dot{\mathbf{H}})$. They are different from $\mathscr{P}(z)$ and $Q(z)$ in § 3, corresponding to A in $\Pi_\varkappa(f)$. We hope that no confusion arises.

Next we calculate $Q(z)$ from [9, (3.4)]:

$$Q(z)f := \lim_{n\to\infty} [(\dot{A}_0^\gamma - zI)^{-1}(f - \mathscr{P}(z)f)u_n), u_n].$$

Here $\dot{A}_0^\gamma$ is an arbitrary π-selfadjoint extension of A_0. It follows that

$$\lim_{n\to\infty}(\dot{A}_0^\gamma - zI)^{-1}(f - (\mathscr{P}(z)f)\,u_n) = \begin{cases} \mathrm{i}e^{\mathrm{i}zt}\displaystyle\int_{-a}^{t} e^{-\mathrm{i}zs}f(s)\,\mathrm{d}s & t < 0, \\ -\mathrm{i}e^{\mathrm{i}zt}\displaystyle\int_{t}^{a} e^{-\mathrm{i}zs}f(s)\,\mathrm{d}s & t > 0, \end{cases}$$

and

$$Q(z)f = \frac{\mathrm{i}}{2}\left\{\int_{-a}^{0} e^{-\mathrm{i}zs}f(s)\,\mathrm{d}s - \int_{0}^{a} e^{-\mathrm{i}zs}f(s)\,\mathrm{d}s\right\} + \tag{8.11}$$
$$+\,\mathrm{i}\int_{-a}^{0} H(-s)\,e^{\mathrm{i}zs}\int_{-a}^{s} e^{-\mathrm{i}zt}f(t)\,\mathrm{d}t\,\mathrm{d}s - \mathrm{i}\int_{0}^{a} H(-s)\,e^{\mathrm{i}zs}\int_{s}^{a} e^{-\mathrm{i}zt}f(t)\mathrm{d}t\,\mathrm{d}s.\ ^{*)}$$

On the other hand we have

$$\frac{\mathscr{P}(\lambda) - \mathscr{P}(z)}{\lambda - z} f = \mathrm{i}\int_{t=-a}^{a} e^{-\mathrm{i}\lambda t}\left\{\chi_{[-a,0]}(t)\int_{s=-a}^{t} - \chi_{[0,a]}(t)\int_{s=t}^{a}\right\} e^{\mathrm{i}z(t-s)}f(s)\,\mathrm{d}s\,\mathrm{d}t$$

and the definition of the functional Ω_H implies

$$\Omega_H^\lambda\left(\frac{\mathscr{P}(\lambda) - \mathscr{P}(z)}{\lambda - z} f\right) = \frac{2}{\mathrm{i}} Q(z)f. \tag{8.12}$$

---

*) We mention that $Q(0)f$ can also be written as follows:

$$Q(0)f = \mathrm{i}\int_{-a}^{a} f(t)\overline{g'(t)}\,\mathrm{d}t$$

where g is given by (2.2). It follows that

$$Q(0)^* = -\,\mathrm{i}(I - \dot{\Gamma})\,g', \quad Q(0)Q(0)^* = ((I - \dot{\Gamma})\,g', g').$$

Now the following relations are easy to check:

$$1 + (z - \zeta)Q(z)\mathscr{P}(\bar{\zeta})^* = \frac{1}{2}(E(z)D^*(\zeta) + E^*(z)D(\zeta)), \tag{8.13}$$

$$1 - (z - \zeta)\mathscr{P}(z)Q(\bar{\zeta})^* = \frac{1}{2}(D(z)E^*(\zeta) + D^*(z)E(\zeta)), \tag{8.14}$$

$$(z - \zeta)Q(z)Q(\bar{\zeta})^* = \frac{i}{4}(-E(z)E^*(\zeta) + E^*(z)E(\zeta)). \tag{8.15}$$

Indeed, we have from (8.12), (6.32) that

$$1+(z - \zeta)Q(z)\mathscr{P}(\bar{\zeta})^* = 1 + (z - \zeta)\frac{i}{2}\Omega_H^\gamma\left(\frac{\mathscr{P}(\lambda)\mathscr{P}(\bar{\zeta})^* - \mathscr{P}(z)\mathscr{P}(\bar{\zeta})^*}{\lambda - z}\right) = 1 +$$

$$+ (z-\zeta)\frac{i}{2}\Omega_H^\lambda\left(\frac{1}{\lambda - z}\left[\frac{D(\lambda)D^*(\zeta) - D(\zeta)D^*(\lambda)}{i(\lambda - \zeta)} - \frac{D(z)D^*(\zeta) - D(\zeta)D^*(z)}{i(z - \zeta)}\right]\right) =$$

$$= 1 + \frac{1}{2}\Omega_H^\lambda\left(\left(\frac{1}{\lambda - z} - \frac{1}{\lambda - \zeta}\right)(D(\lambda)D^*(\zeta) - D(\zeta)D^*(\lambda)) -\right.$$

$$\left. - \frac{1}{\lambda - z}(D(z)D^*(\zeta) - D(\zeta)D^*(z))\right) =$$

$$= 1 + \frac{1}{2}\Omega_H^\lambda\left(\frac{D(\lambda) - D(z)}{\lambda - z}D^*(\zeta) - \frac{D^*(\lambda) - D^*(z)}{\lambda - z}D(\zeta)\right) -$$

$$- \frac{1}{2}\Omega_H^\lambda\left(\frac{D(\lambda)D^*(\zeta) - D(\zeta)D^*(\lambda)}{\lambda - \zeta}\right) = \frac{1}{2}(E(z)D^*(\zeta) - D(\zeta)E^*(z)).$$

Formula (8.14) is an immediate consequence of (8.13), and the proof of (8.15) is similar.

Now [9, Satz 3.10] implies that the δ_0-resolvent matrix is given by the formula

$$W_{\delta_0}(z) = \begin{pmatrix} 1 + zQ(z)\mathscr{P}(0)^* & zQ(z)Q(0)^* \\ -z\mathscr{P}(z)\mathscr{P}(0)^* & 1 - z\mathscr{P}(z)Q(0)^* \end{pmatrix}.$$

Thus we have proved the following relation:

$$W_{\delta_0}(z) = \begin{pmatrix} -E^*(z)/2 & E(z)/2 \\ iD^*(z) & iD(z) \end{pmatrix}\begin{pmatrix} -D(0) & -iE(0)/2 \\ D^*(0) & -iE^*(0)/2 \end{pmatrix}. \tag{8.16}$$

In § 9 we shall need a description of all the δ_0-resolvents $[(\tilde{A} - zI)^{-1}\delta_0, \delta_0]$ of $\dot{A}_0$:

$$[(\tilde{A} - zI)^{-1}\delta_0, \delta_0] := \lim_{n\to\infty} [(\tilde{A} - zI)^{-1} u_n, u_n], \tag{8.17}$$

where $\tilde{A}$ is an arbitrary π-selfadjoint extension of $\dot{A}_0$ in some $\pi_\varkappa$-space $\tilde{\Pi}_\varkappa \supset L^2(I + \dot{\mathbf{H}})$. According to the definition of the u_n-resolvent matrix this description is given by the fractional linear transformation generated by $W_{\delta_0}(z)$, if the parameter T runs through the class $\tilde{N}_0$:

$$[(\tilde{A} - zI)^{-1}\delta_0, \delta_0] = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)} \quad (\operatorname{Im} z < -\gamma) \tag{8.18}$$

with $(w_{jk}(z))_1^2 = W_{\delta_0}(z)$ from (8.16). This δ_0-resolvent matrix is real and normalized such that

$$\det W_{\delta_0}(z) = 1, \quad W_{\delta_0}(0) = I.$$

The matrix

$$V_0 := \begin{pmatrix} -D(0) & -\mathrm{i}E(0)/2 \\ D^*(0) & -\mathrm{i}E^*(0)/2 \end{pmatrix} \begin{pmatrix} -1 & -\mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}$$

satisfies the relation $V_0 J V_0^* = 2J$. Therefore the fractional linear transformation generated by V_0 maps the class $\tilde{N}_0$ onto itself. If $W_{\delta_0}(z)$ in (8.16) is multiplied from the right by V_0 we get the matrix

$$W(z) = \begin{pmatrix} E_{\mathrm{R}}(z) & E_{\mathrm{J}}(z) \\ -2D_{\mathrm{J}}(z) & 2D_{\mathrm{R}}(z) \end{pmatrix}. \tag{8.19}$$

Its entries are real entire functions and it has the property $\det W(z) = 1$. Thus we have proved the following theorem.

THEOREM 8.1. *Suppose* $-1 \notin \sigma(\mathbf{H})$. *Then the matrices* $W = (w_{jk})_1^2$, *given by* (8.16) *and* (8.19), *are* δ_0*-resolvent matrices of* $\dot{A}_0$, *that is, for both matrices* W *the relation* (8.18) *with some* $\gamma \geqslant 0$ *establishes a bijective correspondence between the set* $\tilde{N}_0$ $(T \in \tilde{N}_0)$ *and the set of all* δ_0*-resolvents of the operator* $\dot{A}_0$.

REMARK 1. In the δ_0-resolvent matrices W from (8.16) or (8.19) the elements of the first line can be obtained from the elements of the second line by means of the functional Ω_H. This follows immediately from § 6, (4°). E.g., for the resolvent matrix W of (8.19) we have

$$\Omega_H^\lambda\left(\frac{-D_J(z)+D_J(\lambda)}{z-\lambda}\right) = \mathrm{i}\, E_R(z),$$

$$\Omega_H^\lambda\left(\frac{D_R(z)-D_R(\lambda)}{z-\lambda}\right) = \mathrm{i}\, E_J(z).$$

REMARK 2. Suppose that $\varkappa = 0$. Then for the entries $w_{jk}(z)$ of the resolvent matrix $W_{\delta_0}(z)$ in (8.16) the following analogues of the formulae of Christoffel-Darboux hold true:

$$w_{11}(z) = 1 + \frac{\mathrm{i}z}{2}\int_0^a (E(r;z)D^*(r;0) - E^*(r;z)D(r;0))\,\mathrm{d}r,$$

$$w_{12}(z) = \frac{z}{4}\int_0^a (E(r;z)E^*(r;0) + E^*(r;z)E(r;0))\,\mathrm{d}r,$$

$$w_{21}(z) = -z\int_0^a (D(r;z)D^*(r;0) + D^*(r;z)D(r;0))\,\mathrm{d}r,$$

$$w_{22}(z) = 1 + \frac{\mathrm{i}z}{2}\int_0^a (D(r;z)E^*(r;0) - D^*(r;z)D(r;0))\,\mathrm{d}r.$$

They will be proved in § 11.3.

§ 9. CONSTRUCTION OF THE RESOLVENT MATRIX OF $g \in \mathfrak{G}_{\varkappa;a}$ WITH ACCELERANT BY MEANS OF ORTHOGONAL FUNCTIONS

9.1. In the following theorem we give a description of the set of all continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of the function $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant in the indetermined case. A resolvent matrix for this problem was obtained already in § 5. Here, however, we find a resolvent matrix whose entries are in a simple way connected with the orthogonal functions of first and second kind.

THEOREM 9.1. *Let $g \in \mathfrak{G}_{\varkappa;a}$ be of the form* (2.2) *and suppose that* $-1 \notin \sigma(\mathrm{H})$. *Then the relation*

$$(1.9)\quad -\mathrm{i}z^2\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = -z\,g(0) + \frac{1}{2}\,\frac{E_R(z)T(z)+E_J(z)}{-D_J(z)T(z)+D_R(z)} \qquad (\operatorname{Im} z < -\gamma)$$

establishes a bijective correspondence between all $T \in \tilde{N}_0$ *and all continuations* $\tilde{g} \in \mathfrak{G}_\varkappa$ *of* g.

Proof. Without loss of generality we can suppose that $g(0) = 0$. The proof is divided into two steps.

(1) Let $\tilde{g} \in \mathfrak{G}_\varkappa$ be a continuation of g. We consider the linear space $\mathscr{L}$ of all functions $\hat{\varphi} \in C_0^\infty(\mathbf{R})$ with the property $\int_{-\infty}^{\infty} \hat{\varphi}(t)\,\mathrm{d}t = 0$, equipped with the scalar product

$$[\hat{\varphi}, \hat{\psi}]' := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}(t-s)\hat{\varphi}(s)\overline{\hat{\psi}(t)}\,\mathrm{d}s\,\mathrm{d}t \tag{9.2}$$

($\hat{\varphi}, \hat{\psi} \in C_0^\infty(\mathbf{R})$). It can be canonically embedded into a $\pi_{\varkappa'}$-space $\hat{\mathscr{L}}$. As on the right hand side of (9.2) the kernel $\tilde{g}(t-s)$ can be replaced by $G_{\tilde{g}}(t, s)$, we have $\varkappa' \leqslant \varkappa$. In order to prove the converse inequality with a function $\hat{\varphi} \in \mathscr{L}$ we associate the function φ:

$$\varphi(t) := \int_{-\infty}^{t} \hat{\varphi}(s)\,\mathrm{d}s.$$

If $\hat{\varphi}, \hat{\psi} \in \mathscr{L}$, such that their supports are contained in $(-a, a)$, it follows that

$$\begin{aligned}[\hat{\varphi}, \hat{\psi}]' &= -\int_{-a}^{a}\int_{-a}^{a} g''(t-s)\,\varphi(s)\,\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t = \\ &= \int_{-a}^{a} \varphi(t)\overline{\psi(t)}\,\mathrm{d}t + \int_{-a}^{a}\int_{-a}^{a} H(t-s)\,\varphi(s)\,\overline{\psi(t)}\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{9.3}$$

According to Theorem 2.1, (1) we have $\varkappa' \geqslant \varkappa$, hence $\varkappa' = \varkappa$ follows.

By $\hat{\mathscr{L}}_a$ we denote the $\pi_\varkappa$-space generated by all $\hat{\varphi} \in \mathscr{L}$ with support contained in $(-a, a)$. The relation (9.3) implies that the mapping

$$\hat{\varphi} \to \varphi \quad (\hat{\varphi} \in \mathscr{L}, \operatorname{supp} \hat{\varphi} \subset (-a, a))$$

establishes an isomorphism between the $\pi_\varkappa$-spaces $\hat{\mathscr{L}}_a$ and $L^2(I + \dot{\mathrm{H}})$. In $\hat{\mathscr{L}}\,(\hat{\mathscr{L}}_a)$ we consider the operator $\tilde{\hat{A}}_0$ $(\hat{A}_0)$, which is the closure of $\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}$, defined on all $\hat{\varphi} \in \mathscr{L}$

($\hat{\varphi} \in \mathscr{L}$ with $\operatorname{supp}\hat{\varphi} \subset (-a, a)$, respectively). It is easy to see that the isomorphism $\hat{\varphi} \to \varphi$ associates with $\hat{A}_0$ the operator $\dot{A}_0$ in $L^2(I + \dot{\mathbf{H}})$. The operator $\tilde{\hat{A}}_0$ is a π-self-adjoint extension of $\hat{A}_0$. Hence $[(\tilde{\hat{A}}_0 - zI)^{-1}\hat{\delta}_0, \hat{\delta}_0]$ is a δ_0-resolvent of the operator $\dot{A}_0$ in $L^2(I + \dot{\mathbf{H}})$. We shall find this δ_0-resolvent explicitely. To this end we observe that for arbitrary $\hat{\varphi} \in \mathscr{L}$ we have

$$((\hat{A} - zI)^{-1}\hat{\varphi})(t) = -\mathrm{i}\int_t^\infty \mathrm{e}^{\mathrm{i}z(t-s)}\hat{\varphi}(s)\,\mathrm{d}s \quad (\operatorname{Im} z < -\gamma),$$

hence

$$((\hat{A} - zI)^{-1}\hat{\delta}_0)(t) = z\mathrm{e}^{\mathrm{i}zt}\chi_{(-\infty,0]}(t) + \mathrm{i}\delta_0(t) \quad (\operatorname{Im} z < -\gamma).$$

Here we have used the relation $\hat{\delta}_0 = \delta_0'$ and the fact that δ_0 is always considered as a δ_0-sequence of even real nonnegative smooth functions. This implies in particular that $\int_{-a}^{a} g'(s)\,\delta_0(s)\,\mathrm{d}s = 0$, and we get finally

$$[(\tilde{\hat{A}}_0 - zI)^{-1}\hat{\delta}_0, \hat{\delta}_0)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t - s)\,((\tilde{\hat{A}}_0 - zI)^{-1}\hat{\delta}_0)(s)\hat{\delta}_0(t)\,\mathrm{d}s\,\mathrm{d}t =$$

$$= -\int_{-\infty}^{\infty} \delta_0(t) \int_{-\infty}^{\infty} \tilde{g}'(t - s)[\mathrm{i}\delta_0(s) + z\mathrm{e}^{\mathrm{i}zs}\chi_{(-\infty,0]}(s)]\,\mathrm{d}s\,\mathrm{d}t =$$

$$= -\mathrm{i}z^2\int_0^\infty \tilde{g}(s)\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s.$$

Thus, according to Theorem 8.1, for an arbitrary extension $\tilde{g} \in \mathfrak{G}_\varkappa$ of g there exists a $T \in \tilde{N}_0$ such that (9.1) holds.

(2) It remains to show that for arbitrary $T \in \tilde{N}_0$ there exists an extension $\tilde{g} \in \mathfrak{G}_\varkappa$ such that (9.1) holds. We first observe that the right hand side in (9.1) is a function of class $N_{\varkappa'}$ for some $\varkappa'$, $0 < \varkappa' < \varkappa$. This follows from the fact that it is a limit of a sequence $[R_z u_n, u_n]$, $n = 1,2,\ldots$, and that each function of this sequence belongs to some class $N_{\varkappa_n}$, $0 \leqslant \varkappa_n \leqslant \varkappa$. (It will follow from this proof that we always have $\varkappa' = \varkappa$).

If $T(z) = -\mathrm{i}$ $(\operatorname{Im} z < 0)$, the right hand side in (9.1) becomes $\dfrac{1}{2\mathrm{i}}\dfrac{E(z)}{D(z)}$, which according to Corollary 6.3 admits the representation

$$\frac{1}{2\mathrm{i}}\frac{E(z)}{D(z)} = \frac{1}{2\mathrm{i}}\left(1 + 2\int_0^\infty \tilde{H}_a(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t\right) \quad (\operatorname{Im} z < -\gamma)$$

with some locally summable function $\tilde{H}_a$ which is an extension of H and such that the integral exists. Then the function $\tilde{g}_0$:

$$\tilde{g}_0(t) = -\frac{1}{2}|t| - \int_0^t (t-s)\tilde{H}_a(s)\,ds \quad (t \in \mathbf{R})$$

is an extension of g and we have

$$\frac{1}{2i}\frac{E(z)}{D(z)} = -iz^2\int_0^\infty \tilde{g}_0(t)e^{-izt}\,dt \quad (\operatorname{Im} z < -\gamma).$$

From Proposition 5.1 it follows that $\tilde{g}_0 \in \mathfrak{G}_{\varkappa'}$, where $\varkappa'$ ($\leqslant \varkappa$) is such that $E/2iD \in N_{\varkappa'}$. On the other hand, $\tilde{g}_0$ is an extension of g, hence $\varkappa' = \varkappa$.

Now let $T \in \tilde{N}_0$ be arbitrary. Then we have

$$\left|\frac{E_R(z)T(z)+E_J(z)}{-D_J(z)T(z)+D_R(z)} + \frac{E_R(z)}{D_J(z)}\right| = \frac{1}{|D_J(z)|^2\left|T(z)-\dfrac{D_R(z)}{D_J(z)}\right|}. \tag{9.5}$$

The definition of $D(z)$ implies

$$D_J(iy) = \frac{1}{2i}(e^{-ay}-e^{+ay}) - \frac{1}{2i}\int_{-a}^{a}(\Gamma(s)e^{ys} - \overline{\Gamma(s)}e^{-ys})\,ds,$$

therefore $e^{ay}|D_J(iy)| \to \frac{1}{2}$ $(y \downarrow -\infty)$. Hence there exist $C > 0$, $y_0 < 0$ such that

$$|D_J(iy)|^{-2} \leqslant Ce^{2ay} \quad \text{for all } y \leqslant y_0. \tag{9.6}$$

The function $T - D_R/D_J$ does not vanish identically in $\mathbf{C}_-$; otherwise we would have $T = D_R/D_J$, which is impossible because of proposition (4°) in § 8.3 and (7.2). Furthermore, we have $-D_R/D_J \in N_\varkappa$ (see § 7.1), hence $-(T - D_R/D_J)^{-1} \in N_{\varkappa''}$ with some $\varkappa''$: $0 \leqslant \varkappa'' \leqslant \varkappa$. Therefore, if $y \downarrow -\infty$, then $|T(iy) - D_R(iy)/D_J(iy)|^{-1}$ is of polynomial increase (see [2]). Together with (9.5) and (9.6) this implies

$$\frac{1}{2}\frac{E_R(z)T(z)+E_J(z)}{-D_J(z)T(z)+D_R(z)} + \frac{E_R(z)}{2D_J(z)} = O(e^{2ay}p(|y|)) \tag{9.7}$$

$(z = iy,\ y \downarrow -\infty)$ with some polynomial p. If we put in this relation $T(z) = -i$ $(\operatorname{Im} z < 0)$, then, as was shown above, the first expression on the left hand side is of the form

$$-iz^2\int_0^\infty e^{-izt}\tilde{g}_0(t)\,dt \tag{9.8}$$

with some extension $\tilde{g}_0 \in \mathfrak{G}_\varkappa$ of g. It follows from (5.2) that

$$\lim_{y\downarrow -\infty} \frac{E_R(y)}{2yD_J(iy)} = 0.$$

By Proposition 5.1, also $-\dfrac{E_R(z)}{2D_J(z)}$ admits a representation (9.8) with $\tilde{g}_0$ replaced by $\tilde{g}$, $\tilde{g} \in \mathfrak{G}_{\varkappa''}$, $\varkappa'' \leqslant \varkappa$, which is by (9.7) also a continuation of g. Hence we have again $\varkappa'' = \varkappa$. If we apply the same argument to $\dfrac{1}{2}\,\dfrac{E_R(z)T(z)+E_J(z)}{-D_J(z)T(z)+D_R(z)}$, starting from $-\dfrac{E_R(z)}{2D_J(z)}$, we find that for arbitrary $T \in \tilde{N}_0$ the relation (9.1) holds with some $\tilde{g} \in \mathfrak{G}_\varkappa$ which extends g. The theorem is proved.

REMARK 1. The operator $\dot{A}_0$ in $L^2(I+\dot{\mathbf{H}})$ is π-selfadjoint if and only if g has a unique continuation $\tilde{g} \in \mathfrak{G}_\varkappa$. Indeed, if $\dot{A}_0$ is not π-selfadjoint its δ_0-resolvents generate infinitely many continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of g. Conversely, if g has infinitely many continuations $\tilde{g} \in \mathfrak{G}_\varkappa$, according to Part (1) of the proof each $\tilde{g}$ generates a δ_0-resolvent of $\dot{A}_0$, hence $\dot{A}_0$ is not π-selfadjoint.

REMARK 2. According to Theorem 8.1, the resolvent matrix

$$\begin{pmatrix} E_R(z) & E_J(z) \\ -2D_J(z) & 2D_R(z) \end{pmatrix}$$

in Theorem 9.1 can be replaced by the matrix $W_{\delta_0}(z)$ in (8.16). This amounts just to another "parametrization" of the set of all continuations $\tilde{g}$.

9.2. The continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ of g in Theorem 9.1 does, in general, not have an accelerant. That is, the second derivative $g''(t)$ exists on $[2a, \infty)$ only in the sense of generalized functions. The following theorem gives a sufficient condition that the extension $\tilde{g} \in \mathfrak{G}_\varkappa$ of g in Theorem 9.1 has an accelerant.

THEOREM 9.2. *Let $g \in \mathfrak{G}_{\varkappa;a}$ be as in Theorem 9.1. Then the extension $\tilde{g} \in \mathfrak{G}_\varkappa$ in (9.1) has an accelerant $\tilde{H} \in L^1_{\mathrm{loc}}$ with the property*

$$\int_0^\infty e^{-\beta t}\,|\tilde{H}(t)|\,dt < \infty \tag{9.9}$$

for some $\beta > 0$ if and only if $T \in N_0$ admits a representation

$$T(z) = -i + \int_0^\infty e^{-izt}\,\tau(t)\,dt \quad (\operatorname{Im} z < -\alpha) \tag{9.10}$$

with some function τ *on* $[0,\ \infty)$ *such that*

$$\int_0^\infty e^{-\alpha t}|\tau(t)|\,dt < \infty \quad \textit{for some } \alpha > 0.$$

If $\varkappa = 0$ we can put $\beta = \alpha = 0$, that is in this case $\tilde{g}$ has an accelerant $\tilde{H} \in L^1(-\infty,\infty)$ if T admits a representation (9.10) with $\tau \in L^1(0,\infty)$,

Proof. We suppose again that $g(0) = 0$. From (9.1) it follows that for a function T of the form (9.10) we have

$$-iz^2\int_0^\infty e^{-izt}\tilde{g}(t)\,dt = \frac{1}{2}\,\frac{-iE(z) + E_R(z)\int_0^\infty e^{-izt}\tau(t)\,dt}{D(z) - D_J(z)\int_0^\infty e^{-izt}\tau(t)\,dt} =$$

$$= \frac{1}{2}\,\frac{-i + \int_0^\infty \Gamma^{(1)}(t)e^{-izt}\,dt}{1 + \int_0^\infty \Gamma^{(2)}(t)e^{-izt}\,dt}$$

with functions $\Gamma^{(1)}$, $\Gamma^{(2)}$ on $[0,\infty)$ such that $\int_0^\infty e^{-\alpha t}|\Gamma^{(j)}(t)|\,dt < \infty$. According to a generalization of a result of N. Wiener (see [45]) this can be written as

$$\frac{1}{i}\left(\frac{1}{2} + \int_0^\infty \tilde{H}(t)e^{-izt}\,dt\right)$$

with a function $\tilde{H}$ having the property (9.9) if only $\beta > \max(\alpha, \gamma)$.

In order to prove the converse we first observe that the relation (9.1) can be written as:

$$T(z) = \frac{2D_R(z)Q_{\tilde{g}}(z) - E_J(z)}{2D_J(z)Q_{\tilde{g}}(z) + E_R(z)} \quad (\operatorname{Im} z < -\gamma),$$

$Q_{\tilde{g}}(z) := -iz^2\int_0^\infty e^{-izt}\tilde{g}(t)\,dt$. If $\tilde{g}$ has an accelerant $\tilde{H}$, then it follows that

$$Q_{\tilde{g}}(z) = \frac{i}{2} + i\int_0^\infty e^{-izt}\tilde{H}(t)\,dt$$

and we get the relation

$$T(z) = \frac{i + \int_0^\infty \tilde{\Gamma}^{(1)}(t)e^{-izt}\,dt}{1 + \int_0^\infty \tilde{\Gamma}^{(2)}(t)e^{-izt}\,dt}$$

with functions $\tilde{\Gamma}^{(1)}$, $\tilde{\Gamma}^{(2)}$ on $[0, \infty)$ such that

$$\int_0^\infty e^{-\beta t}|\tilde{\Gamma}^{(j)}(t)|\,dt < \infty.$$

Now the proof can be finished as in the first part. The theorem is proved.

9.3. The statements about the continuations of $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant H can be formulated for H as follows.

THEOREM 9.3. *Let $H \in L^1(-2a, 2a)$ be given. If $-1 \notin \sigma(\mathbf{H})$ the formula*

$$(9.11)\quad \frac{1}{i}\left(1+2\int_0^{2a} H(t)\,e^{-izt}\,dt\right) + e^{-2iaz}\Phi(z) = \frac{E_R(z)T(z) + E_J(z)}{-D_J(z)T(z) + D_R(z)} \quad (\operatorname{Im} z < -\gamma)$$

establishes a bijective correspondence between all $T \in \tilde{N}_0$ and all functions Φ with the properties:

(1) *For some $\gamma' > 0$ the function Φ is holomorphic in the half plane* $\operatorname{Im} z \leqslant -\gamma'$.

(2) $|\Phi(iy)| = O(|y|)$ *$(y \downarrow -\infty)$.*

(3) *The expression on the left hand side in* (9.11) *belongs to $N_\varkappa$.*

If $-1 \in \sigma(\mathbf{H})$ there exists a unique function Φ with the properties (1), (2), (3). *With φ_0 and ψ_0 from* § 2.1 *it is given by the relation*

$$\frac{1}{i}\left(1 + 2\int_0^{2a} H(t)e^{-izt}\,dt\right) + e^{-2iaz}\Phi(z) = 2i\,\frac{\mathcal{F}(\psi_0; z)}{\mathcal{F}(\varphi_0; z)}.$$

Proof. Let $-1 \notin \sigma(\mathbf{H})$. The expression on the left hand side of (9.1) can be written as

$$\frac{1}{2i}\left(1 + 2\int_0^{2a} H(t)e^{-izt}\,dt\right) + \frac{1}{2}\,e^{-2iaz}\Phi(z),$$

with

$$\Phi(z) := -az + \frac{i}{2} - z\int_0^{2a}(2a-s)\,H(s)\,ds + i\int_0^{2a} H(s)ds - iz^2\int_0^{\infty} e^{-izt}\tilde{g}(t+2a)\,dt.$$

Evidently, Φ has the properties (1), (2) and (3).

Conversely, given Φ with these properties and denoting the left hand side of (9.11) for a moment by $Q(z)$, it follows that $\frac{1}{y}Q(iy) \to 0$ if $y \downarrow -\infty$, hence by Proposition 5.1 we have

$$Q(z) = -iz^2\int_0^{\infty} e^{-izt}\tilde{g}(t)\,dt$$

with some function $\tilde{g} \in \mathfrak{G}_\varkappa$. Using the property (2) of Φ we find

$$iz^2\int_0^{\infty} e^{-izt}\tilde{g}(t)\,dt + \frac{1}{i} + \frac{2}{i}\int_0^{2a} H(t)e^{-izt}\,dt = O(|e^{-2iaz}z|).$$

This implies $\tilde{g}(t) = -\frac{1}{2}\cdot t - \int_0^t (t-s)H(s)\,ds$ $(0 \leqslant t \leqslant 2a)$. Hence $\tilde{g} \in \mathfrak{G}_\varkappa$ is an extension of g, and according to Theorem 9.1 the equality (9.11) holds with some $T \in \tilde{N}_0$

If $-1 \in \sigma(\mathbf{H})$ the statement follows in a similar way from Theorem 2.1, (3).

REMARK. The condition (2) can be replaced by the condition

$$|\Phi(iy)| = O(|y|^n) \quad (y \downarrow -\infty)$$

for an arbitrary positive integer n.

9.4. Now we prove the statement (3) of Theorem 2.1. Combining proposition (4°) of § 1.5 and the Remark after Theorem 9.1 we see that the following statements are equivalent:

(1) g has a unique continuation $\tilde{g} \in \mathfrak{G}_\varkappa$.

(2) $-1 \in \sigma(\mathbf{H})$.

(3) The operator $\dot{A}_0$ is π-selfadjoint in $L^2(I + \dot{\mathbf{H}})$.

If g has a unique continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ according to part (1) of the proof of Theorem 9.1 it is given by the equality

$$-iz^2\int_0^{\infty} e^{-izt}\tilde{g}(t)\,dt = [(\dot{A}_0 - zI)^{-1}\delta_0, \delta_0] \quad (\operatorname{Im} z < -\gamma),$$

where the expression on the right hand side is defined as $\lim_{n\to\infty}[(\dot{A}_0 - zI)^{-1}u_n, u_n]$ with a δ_0-sequence (u_n) of real even nonnegative smooth functions. An easy calculation gives

$$[(\dot{A}_0 - zI)^{-1}\delta_0, \delta_0] = \mathrm{i}\dot{\mathscr{F}}(\varphi_0; z)^{-1}\Big\{\dot{\mathscr{F}}(\varphi_0; z)\Big(\frac{1}{2} + \int_{-a}^{0} H(t)\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t\Big) - $$

$$- \int_{-a}^{0} \mathrm{e}^{-\mathrm{i}zs}\dot{\varphi}_0(s)\,\mathrm{d}s - \int_{-a}^{a} \overline{H(t)} \int_{-a}^{t} \mathrm{e}^{\mathrm{i}z(t-s)}\dot{\varphi}_0(s)\,\mathrm{d}s\,\mathrm{d}t\Big\},$$

$\dot{\mathscr{F}}(\varphi_0; z) := \int_{-a}^{a} \dot{\varphi}_0(s)\,\mathrm{e}^{-\mathrm{i}zs}\,\mathrm{d}s$. Here $\dot{\varphi}_0$ is again the first element of a D-chain of $\dot{\mathbf{H}}$ for the eigenvalue $\lambda = -1$. The expression in the brackets $\{\cdot\}$ can be written as

$$\dot{\mathscr{F}}(\dot{\psi}_0; z) = \int_{-a}^{a} \dot{\psi}_0(t)\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t, \quad \dot{\psi}_0(t) := \frac{1}{2}\dot{\varphi}_0(t) + \int_{t}^{a} H(t-s)\,\dot{\varphi}_0(s)\,\mathrm{d}s$$

$(-a \leqslant t \leqslant a)$. That is we have

$$-\mathrm{i}z^2\int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = \mathrm{i}\,\frac{\dot{\mathscr{F}}(\dot{\psi}_0; z)}{\dot{\mathscr{F}}(\varphi_0; z)} \quad (\operatorname{Im} z < -\gamma), \tag{9.12}$$

and the statement (3) of Theorem 2.1 is proved.

REMARK. The Fourier transformations in (9.12) are connected by the relation

$$\Omega_H^{\lambda}\Big(\frac{\dot{\mathscr{F}}(\varphi_0; z) - \dot{\mathscr{F}}(\varphi_0; \lambda)}{z-\lambda}\Big) = -\dot{\mathscr{F}}(\dot{\psi}_0; z).$$

This follows from statement (3°) in § 6.

9.5. If $g \in \mathfrak{G}_{\varkappa;a}$, given by (2.2), has a unique continuation $\tilde{g} \in \mathfrak{G}_{\varkappa}$, for arbitrary $b > a$ the restriction $g|_{[-2b, 2b]}$ does not have an accelerant. Indeed, assume to the contrary that for some $b > a$ we have

$$\tilde{g}(t) = g(0) - \frac{1}{2}|t| - \int_{0}^{t}(t-s)\tilde{H}(s)\,\mathrm{d}s \quad (-2b \leqslant t \leqslant 2b)$$

with a Hermitian function $\tilde{H} \in L^1(-2b, 2b)$. Then for arbitrary $c \in [a, b]$ the restriction $g|_{[-2c, 2c]}$ has the unique continuation $\tilde{g}$ in $\mathfrak{G}_{\varkappa}$, and the interval $[a, b]$ consists of singular points of $\tilde{H}$, which is impossible (see statement (3°) of § 1). A more explicit form of the continuation $\tilde{g}$ is obtained in the following theorem.

THEOREM 9.4. *Suppose that $g \in \mathfrak{G}_{\varkappa;a}$ in (2.2) with accelerant H has a unique continuation $\tilde{g} \in \mathfrak{G}_{\varkappa}$, that is $-1 \in \sigma(\mathbf{H})$, and let φ_0, n be the same as in Theorem* 1.1

with $\lambda = -1$. *Then this continuation* $\tilde{g}$ *is determined by the relation*

$$\tilde{g}''(t) = -\frac{1}{2}\delta_0(t) - \tilde{H}(t) - \sum_{j=1}^{\infty} c_j \delta_{2aj}(t) \quad (t > 0),$$

where $c_j := \left(\frac{\varphi_0^{(n)}(2a)}{\varphi_0^{(n)}(0)}\right)^j$, $\tilde{H}(t) := H(t)$ *if* $0 \leqslant t \leqslant 2a$, *and on* $(2aj, 2a(j+1))$ *the function* $\tilde{H}$ *is defined recursively by the Volterra integral equations*

$$\varphi_0^{(n)}(0)\,\tilde{H}(t) + \int_{2aj}^{t} \tilde{H}(v)\varphi_0^{(n+1)}(t-v)\,\mathrm{d}v =$$

$$= \varphi_0^{(n)}(2a)\,\tilde{H}(t-2a) - c_j\varphi_0^{(n+1)}(t-2aj) - \int_{v=t-2aj}^{2aj} \tilde{H}(v)\varphi_0^{(n+1)}(t-v)\,\mathrm{d}v$$

$(2aj \leqslant t \leqslant 2a(j+1),\ j = 1,2,\ldots)$.

Proof. According to Theorem 1.1 we have

$$\varphi_0^{(j)}(0) = \varphi_0^{(j)}(2a) = 0 \quad (0 \leqslant j \leqslant n-1),$$

$$|\varphi_0^{(n)}(0)| = |\varphi_0^{(n)}(2a)| > 0$$

and $\varphi_0^{(n)}$ is absolutely continuous. It is easy to see that the function ψ_0 has similar properties. Integration by parts gives the relation

$$\text{(9.13)} \qquad \frac{\mathscr{F}(\psi_0; z)}{\mathscr{F}(\varphi_0; z)} = \frac{\psi_0^{(n)}(0) + \int_0^{2a} \psi_0^{(n+1)}(t)\,\mathrm{e}^{-izt}\,\mathrm{d}t - \mathrm{e}^{-2aiz}\psi_0^{(n)}(2a)}{\varphi_0^{(n)}(0) + \int_0^{2a} \varphi_0^{(n+1)}(t)\,\mathrm{e}^{-izt}\,\mathrm{d}t - \mathrm{e}^{-2aiz}\varphi_0^{(n)}(2a)}.$$

If we consider the ring of summable functions on the group $\mathbf{R} \times \mathbf{N}$ and observe that $\mathscr{F}(\varphi_0; \cdot)$ has only a finite number of zeros in the lower half plane it follows from [45] that the right-hand side in (9.13) can be written as

$$\text{(9.14)} \qquad -\int_0^{\infty} \tilde{H}(t)\,\mathrm{e}^{-izt}\,\mathrm{d}t - \sum_{0}^{\infty} c_j\,\mathrm{e}^{-2iazj}.$$

Comparing (9.13) and (9.14) we obtain the following relations (we write $\varphi := \varphi_0^{(n)}$, $\psi := \psi_0^{(n)}$):

$$\psi(0) = -c_0\varphi(0),$$

$$-\psi(2a) = c_0\varphi(2a) - c_1\varphi(0),$$

$$0 = -c_j\varphi(0) + c_{j-1}\varphi(2a), \quad j = 2,3, \ldots,$$

$$(9.15) \qquad \psi'(t) = -c_0\varphi'(t) - \varphi(0)\tilde{H}(t) - \int_0^t \varphi'(u)\tilde{H}(t-u)\,\mathrm{d}u, \quad 0 \leqslant t \leqslant 2a$$

$$(9.16) \qquad \begin{aligned} 0 &= \tilde{H}(t-2a)\varphi(2a) - \tilde{H}(t)\varphi(0) - \\ &-c_j\varphi'(t-2aj) - \int_{u=0}^{2a} \tilde{H}(t-u)\varphi_0'(u)\,\mathrm{d}u, \quad 2aj \leqslant t \leqslant 2a(j+1), \end{aligned}$$

$j = 1,2, \ldots$. The first relations imply

$$c_0 = \frac{1}{2}, \quad c_1 = \frac{\frac{1}{2}\varphi(2a) + \psi(2a)}{\varphi(0)} = \frac{\varphi(2a)}{\varphi(0)}, \quad c_j = \frac{\varphi(2a)}{\varphi(0)} c_{j-1},$$

hence $c_j = \left(\dfrac{\varphi(2a)}{\varphi(0)}\right)^j$, $|c_j| = 1$, $j = 1,2, \ldots$ and the sum in (9.14) converges if $\operatorname{Im} z < 0$. Further, the relation (9.15) can be written as

$$(9.17) \qquad \psi'(t) = -\frac{1}{2}\varphi'(t) - \varphi(0)\tilde{H}(t) - \int_0^t \varphi'(t-s)\tilde{H}(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 2a.$$

On the other hand, the definition of ψ_0 implies

$$\psi_0(t) = -\frac{1}{2}\varphi_0(t) - \int_0^t H(s)\varphi_0(t-s)\,\mathrm{d}s,$$

which gives

$$\psi(t) = -\frac{1}{2}\varphi(t) - \int_0^t H(s)\varphi(t-s)\,\mathrm{d}s$$

or

$$(9.18) \qquad \psi'(t) = -\frac{1}{2}\varphi'(t) - H(t)\varphi(0) - \int_0^t \varphi'(t-s)H(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 2a.$$

As this Volterra integral equation has a unique solution, comparison of (9.17) and (9.18) yields $\tilde{H}(t) = H(t)$ $(0 \leqslant t \leqslant 2a)$.

Finally, the equation (9.16) can be written as

$$\tilde{H}(t) + \frac{1}{\varphi(0)} \int_{2aj}^{t} \tilde{H}(v)\varphi'(t-v)\,\mathrm{d}v =$$

(9.19)

$$= \frac{\varphi(2a)}{\varphi(0)} \tilde{H}(t-2a) - \frac{c_j}{\varphi(0)} \varphi'(t-2aj) - \int_{t-2a}^{2aj} \tilde{H}(v) \frac{\varphi'(t-v)}{\varphi(0)} \mathrm{d}v,$$

$2aj \leqslant t \leqslant 2a(j+1)$. If we put $H_j(t) := \tilde{H}(2aj+t)$ $(0 \leqslant t \leqslant 2a)$, $j = 0,1,2,\ldots$, the equations (9.19) become

$$H_j(t) + \frac{1}{\varphi(0)} \int_{0}^{t} H_j(v)\varphi'(s-v)\,\mathrm{d}v =$$

$$= \frac{\varphi(2a)}{\varphi(0)} H_{j-1}(s) - \frac{c_j}{\varphi(0)} \varphi'(s) - \int_{s}^{2a} H_{j-1}(v) \frac{\varphi'(t-v)}{\varphi(0)} \mathrm{d}v, \quad j = 1,2,\ldots.$$

Denoting the L^1-norm of H_j by $|H_j|_1$, it follows that

$$|H_j|_1 \leqslant \gamma_0 |H_{j-1}|_1 + \gamma_1, \quad j = 1,2,\ldots,$$

and $|H_j|_1 \leqslant C^j$ with some $C > 0$, $j = 1,2,\ldots$. This implies that the integral in (9.14) converges if $\operatorname{Im} z < -\gamma$ for some $\gamma \geqslant 0$:

$$\int_{0}^{\infty} |\tilde{H}(t)|\,|\mathrm{e}^{-\mathrm{i}zt}|\,\mathrm{d}t \leqslant \sum_{j=0}^{\infty} |H_j|_1 \mathrm{e}^{(\operatorname{Im} z)2aj} \leqslant \sum_{j=0}^{\infty} C^j \mathrm{e}^{(\operatorname{Im} z)2aj}.$$

Thus we have shown that

$$\frac{\mathscr{F}(\psi_0; z)}{\mathscr{F}(\varphi_0; z)} = -\int_{0}^{\infty} \tilde{H}(t)\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}t - \sum_{0}^{\infty} c_j \mathrm{e}^{-2\mathrm{i}azj} \qquad (\operatorname{Im} z < -\gamma)$$

for some $\gamma > 0$, where $\tilde{H}$ and c_j are as in the theorem. On the other hand, from (9.12) we have

$$\frac{\mathscr{F}(\psi_0; z)}{\mathscr{F}(\varphi_0; z)} = -\frac{1}{2} + \int_{0+}^{\infty} \mathrm{e}^{-\mathrm{i}zt}\tilde{g}''(t)\,\mathrm{d}t,$$

and the statement follows.

Remark. Evidently, the continuation $\tilde{g}$ has the property that $|g'(2aj+) - g'(2aj-)| = 1$, $j = 1,2,\ldots$.

§ 10. REPRESENTATION OF THE RESOLVENT MATRIX OF $f \in \mathfrak{P}_{\varkappa;a}$ WITH ACCELERANT BY MEANS OF ORTHOGONAL FUNCTIONS

10.1. Suppose we are given a function $f \in \mathfrak{P}_{\varkappa;a}$, $0 < a < \infty$, with the following properties:

(i) f has an accelerant H_f and admits a representation

$$f(t) = 1 - 2|t| - \int_0^t (t-s)H_f(s)\,\mathrm{d}s \qquad (|t| \leqslant 2a);$$

(ii) f has more than one continuation $\tilde{f} \in \mathfrak{P}_\varkappa$.

Then the transformation described in Theorem 5.1, (2) gives us a function $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant H_g and the property $-1 \notin \sigma(\mathbf{H}_g)$. Combining the results of § 5.1 and Theorem 9.1 we find:

THEOREM 10.1. *Let $f \in \mathfrak{P}_{\varkappa;a}$ have the properties* (i), (ii). *Then the formula*

$$-\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{f}(t)\,\mathrm{d}t = \frac{1}{2}\,\frac{E_\mathrm{J}(z)T(z) - E_\mathrm{R}(z)}{(D_\mathrm{R}(z) - zE_\mathrm{J}(z))T(z) + D_\mathrm{J}(z) + zE_\mathrm{R}(z)}$$

establishes a bijective correspondence between all $T \in \tilde{N}_0$ and all continuations $\tilde{f} \in \mathfrak{P}_\varkappa$ of f. Here D, E are the orthogonal functions of first and second, respectively, kind, associated with the accelerant H_g of g, given by Theorem 5.1, (3).

REMARK 1. If we observe Remark 3 after Theorem 5.2 it is easy to see that a similar result holds if the conditions $f(0) = 1$, $f'(0+) = -2$ are replaced by $f(0) > 0$, $f'(0+) < 0$.

REMARK 2. As the normed resolvent matrix of a function $f \in \mathfrak{P}_{\varkappa;a}$ is uniquely determined, the 2×2 matrix function

$$\begin{pmatrix} \frac{1}{2}E_\mathrm{J}(z) & -\frac{1}{2}E_\mathrm{R}(z) \\ D_\mathrm{R}(z) - zE_\mathrm{J}(z) & D_\mathrm{J}(z) + zE_\mathrm{R}(z) \end{pmatrix} \begin{pmatrix} \frac{1}{2}E_\mathrm{J}(0) & -\frac{1}{2}E_\mathrm{R}(0) \\ D_\mathrm{R}(0) & D_\mathrm{J}(0) \end{pmatrix}^{-1}$$

coincides with the matrix function $\dot{W}(z)$ of Theorem 3.2. A direct proof of this fact is left to the reader.

10.2. In n° 1 the resolvent matrix of a function $f \in \mathfrak{P}_{\varkappa;a}$ with accelerant was expressed by means of the orthogonal functions, corresponding to the accelerant H_g of a transformation g of f. Moreover, we had to suppose that $f(0) > 0$. In this n° we give another expression for the resolvent matrix of $f \in \mathfrak{P}_{\varkappa;a}$ without this restriction, but we suppose that $-1 \notin \sigma(\mathbf{H}_f)$. Then the orthogonal functions $D(z)$ and $E(z)$, corresponding to the accelerant H_f of f, exist and the resolvent matrix will be expressed in terms of these orthogonal functions.

Let $f \in \mathfrak{P}_{\varkappa;a}$ have the following properties:

(i) f has an accelerant H_f and admits a representation

$$f(t) = f(0) - \frac{1}{2}|t| - \int_0^t (t-s)H_f(s)\,ds \quad (|t| \leqslant 2a).$$

(ii) $-1 \notin \sigma(\mathbf{H}_f)$.

(iii) f has more than one continuation $\tilde{f} \in \mathfrak{P}_\varkappa$.

Then $f \in \mathfrak{G}_{\varkappa';a}$ with $\varkappa' = \varkappa$ or $\varkappa - 1$, and f has infinitely many continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ because of condition (ii), see § 2. A description of all these continuations is according to Theorem 9.1 given by the relation

$$\text{(10.1)} \quad \begin{aligned} -\mathrm{i}\int_0^\infty e^{-\mathrm{i}zt}\tilde{g}(t)\,dt &= -\frac{1}{z}f(0) + \frac{1}{2z^2}\frac{E_\mathrm{J}(z)T(z) - E_\mathrm{R}(z)}{D_\mathrm{R}(z)T(z) + D_\mathrm{J}(z)} = \\ &= \frac{\hat{w}_{11}(z)T(z) + \hat{w}_{12}(z)}{\hat{w}_{21}(z)T(z) + \hat{w}_{22}(z)} \quad (\operatorname{Im} z < -\gamma), \end{aligned}$$

with some $\gamma \geqslant 0$ and $T \in \tilde{N}_0$. Here $D(z)$, $E(z)$ are the orthogonal functions corresponding to the accelerant H_f of f, and $\hat{W}$ is the matrix function

$$\hat{W}(z) = (\hat{w}_{jk}(z))_1^2 = \begin{pmatrix} \frac{1}{2z^2} & -\frac{1}{z}f(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_\mathrm{J}(z) & -E_\mathrm{R}(z) \\ D_\mathrm{R}(z) & D_\mathrm{J}(z). \end{pmatrix}.$$

On the other hand, it will be shown in [7] (comp. [31]) that this set of functions $\tilde{g}$ is also given by the relation

$$\text{(10.2)} \quad -\mathrm{i}\int_0^\infty e^{-\mathrm{i}zt}g(t)\,dt = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)} \quad (\operatorname{Im} z < \gamma)$$

where $W = (w_{jk})_1^2$ is the resolvent matrix of $f \in \mathfrak{P}_{\varkappa;a}$ given by (3.4) and T_1 is an arbitrary function of the form

$$T_1(z) = T_0(z) + \frac{1}{z}\gamma_0 \quad (T_0 \in \tilde{N}_0)$$

with

$$\text{(10.3)} \quad \gamma_0 := \frac{|[u,\, q_0]|^2}{[q_0,\, q_0]}.$$

Here $u = 2\delta_0$, and q_0 is the defect vector of A in $\Pi_\varkappa(f)$ corresponding to $z = 0$, see (3.2). Thus the description (10.2) can be written as follows:

$$\text{(10.4)} \quad -\mathrm{i}\int_0^\infty e^{-\mathrm{i}zt}\tilde{g}(t)\,dt = \frac{\tilde{w}_{11}(z)T_0(z) + \tilde{w}_{12}(z)}{\tilde{w}_{21}(z)T_0(z) + \tilde{w}_{22}(z)} \quad (\operatorname{Im} z < -\gamma)$$

with the matrix function

$$\tilde{W}(z) \equiv (\tilde{w}_{jk}(z))_1^2 = W(z)\begin{pmatrix} 1 & \gamma_0/z \\ 0 & 1 \end{pmatrix}.$$

As the left hand sides of (10.1) and (10.4) define the same set of functions if T or T_0 runs through $\tilde{N}_0$, it follows that the relation

$$\tilde{W}(z) = \alpha(z)\hat{W}(z)V_1 \tag{10.5}$$

must hold with some locally holomorphic function $\alpha(z)$ in $\mathbf{C}_+ \cup \mathbf{C}_-$ and a J-unitary matrix $V_1 : V_1^* J V_1 = J$. We shall show that in this relation $\alpha(z)$, V_1 and γ_0 are determined from the condition that the entries of $W(z)$ are entire functions, $\det W(z) \equiv 1$ and $W(0) = I_2$.

In order to do this we introduce the matrix

$$\mathring{W}(z) := \begin{pmatrix} E_J(z) & -E_R(z) \\ D_R(z) & D_J(z) \end{pmatrix} \cdot \begin{pmatrix} E_J(0) & -E_R(0) \\ D_R(0) & D_J(0) \end{pmatrix}^{-1}. \tag{10.6}$$

Then (10.5) becomes

$$W(z) = \alpha(z)\begin{pmatrix} \dfrac{1}{2z^2} & -\dfrac{1}{2}f(0) \\ 0 & 1 \end{pmatrix} \mathring{W}(z) V \begin{pmatrix} 1 & \dfrac{-\gamma_0}{z} \\ 0 & 1 \end{pmatrix} \tag{10.7}$$

with

$$V = (v_{jk})_1^2 = \begin{pmatrix} E_J(0) & -E_R(0) \\ D_R(0) & D_J(0) \end{pmatrix} V_1.$$

The condition $\det W(z) = 1$ implies $\alpha(z)^2 = 2z^2$ or $\alpha(z) = \sqrt{2}\,z$. From (10.7) we have

$$w_{11}(z) = \frac{1}{\sqrt{2}\,z}(\mathring{w}_{11}(z)v_{11} + \mathring{w}_{12}(z)v_{21}) - \sqrt{2}\,f(0)(\mathring{w}_{21}(z)v_{11} + \mathring{w}_{22}(z)v_{21}), \tag{10.8}$$

hence $\mathring{w}_{11}(0) = 1$ yields $v_{11} = 0$, and, putting $z = 0$ in (10.8), we get

$$1 = \left(\frac{1}{\sqrt{2}}\mathring{w}_{12}'(0) - \sqrt{2}\,f(0)\right)v_{21}, \quad \text{or} \quad v_{21} = \frac{-1}{2(f(0) - \mathring{w}_{12}'(0)/2)}.$$

Further,

$$w_{22}(z) = z\sqrt{2}\left[\mathring{w}_{21}(z)v_{12} + \mathring{w}_{22}(z)\left(-\frac{\gamma_0}{z}v_{21} + v_{22}\right)\right]$$

which gives for $z = 0$ that $1 = -\sqrt{2}\,\gamma_0 v_{21}$, hence

$$\gamma_0 = f(0) - \frac{\mathring{w}_{21}'(0)}{2}. \tag{10.9}$$

Comparing in the relation

$$w_{12}(z) = \frac{1}{\sqrt{2}\,z}\left[\overset{\circ}{w}_{11}(z)v_{12} + \overset{\circ}{w}_{12}(z)\left(-\frac{\gamma_0}{z}v_{21} + v_{22}\right)\right] -$$

(10.10)

$$-\sqrt{2}\,f(0)\left[\overset{\circ}{w}_{21}(z)v_{12} + \overset{\circ}{w}_{22}(z)\left(-\frac{\gamma_0}{z}v_{21} + v_{22}\right)\right]$$

the terms with z^{-1} we find

$$0 = \frac{1}{\sqrt{2}}\,[v_{12} - \overset{\circ}{w}{}'_{12}(0)\gamma_0 v_{21}] + \sqrt{2}\,f(0)\gamma_0 v_{21},$$

hence

$$v_{12} = \sqrt{2}\left(f(0) - \frac{\overset{\circ}{w}{}'_{12}(0)}{2}\right).$$

Finally, comparing the terms with z^0 in (10.10) we get

$$v_{22} = \frac{1}{\sqrt{2}}\,\overset{\circ}{w}{}'_{11}(0) + \frac{1}{4\sqrt{2}}\,\frac{\overset{\circ}{w}{}''_{12}(0) - 4f(0)\overset{\circ}{w}{}'_{22}(0)}{f(0) - \dfrac{\overset{\circ}{w}{}'_{12}(0)}{2}}.$$

As $[u, q_0] = 1$ (see (2.14)), we find from (10.3), (10.9) and § 2, (6°):

$$\Delta = f(0) - \frac{\overset{\circ}{w}{}'_{12}(0)}{2}.$$

Thus the elements of V and γ_0 can be written by means of Δ:

$$\gamma_0 = \Delta, \quad v_{11} = 0, \quad v_{12} = \sqrt{2}\,\Delta, \quad v_{21} = -(\sqrt{2}\,\Delta)^{-1},$$

$$v_{22} = (\sqrt{2})^{-1}\overset{\circ}{w}{}'_{11}(0) + (4\sqrt{2})^{-1}(\overset{\circ}{w}{}''_{12}(0) - 4f(0)\overset{\circ}{w}{}'_{22}(0)).$$

In order to express these numbers by the orthogonal functions we observe the relations

$$\overset{\circ}{w}_{11}(z) = E_J(z)D_J(0) + E_R(z)D_R(0),$$

$$\overset{\circ}{w}_{12}(z) = E_J(z)E_R(0) - E_R(z)E_J(0),$$

$$\overset{\circ}{w}_{22}(z) = D_R(z)E_R(0) + D_J(z)E_J(0).$$

It follows that

$$\overset{\circ}{w}{}'_{11}(0) = E'_J(0)D_J(0) + E'_R(0)D_R(0),$$

$$\overset{\circ}{w}{}''_{12}(0) = E''_J(0)E_R(0) - E''_R(0)E_J(0),$$

$$\overset{\circ}{w}{}'_{22}(0) = D'_R(0)E_R(0) + D'_J(0)E_J(0),$$

$$\Delta = f(0) - \frac{1}{2}(E'_J(0)E_R(0) - E'_R(0)E_J(0)). \tag{10.11}$$

With the functionals $\mathscr{P}(z)$, $Q(z)$, corresponding to the operator A_0 in $L^2(I \dotplus \mathbf{H})$, we can write (see § 8.3):

$$\overset{\circ}{w}_{11}(z) = 1 + zQ(z)\mathscr{P}(0)^*,$$

$$\overset{\circ}{w}_{12}(z) = 2zQ(z)Q(0)^*,$$

$$\overset{\circ}{w}_{22}(z) = 1 - z\mathscr{P}(z)Q(0)^*,$$

hence

$$\overset{\circ}{w}{}'_{11}(0) = Q(0)\mathscr{P}(0)^*, \quad \overset{\circ}{w}{}''_{12}(0) = 4Q'(0)Q(0)^*,$$

$$\overset{\circ}{w}{}'_{22}(0) = -\mathscr{P}(0)Q(0)^*, \quad \Delta = f(0) - Q(0)Q(0)^*.$$

Thus we have proved the following

THEOREM 10.2. *Suppose that the function $f \in \mathfrak{P}_{\varkappa;a}$, $0 < a < \infty$, has the properties* (i—iii). *Then the resolvent matrix $W(z)$ of f in* (3.5) *is given by the relation*

$$W(z) = \begin{pmatrix} 1/z & -f(0) \\ 0 & z \end{pmatrix} \overset{\circ}{U}(z) \begin{pmatrix} 0 & \Delta \\ -1/\Delta & 1/z + \delta \end{pmatrix}$$

with

$$\overset{\circ}{U}(z) := U(z)U(0)^{-1}, \quad U(z) := \begin{pmatrix} E_J(z)/2 & -E_R(z)/2 \\ D_R(z) & D_J(z) \end{pmatrix},$$

$$\delta := \overset{\circ}{u}{}'_{11}(0) + \frac{1}{2\Delta}(\overset{\circ}{u}{}''_{12}(0) - 2f(0)\overset{\circ}{u}{}'_{22}(0))$$

and Δ given by (10.11), *where D, E are the orthogonal functions corresponding to the accelerant H_f.*

In the relation (3.5) the matrix function $W(z)$ can be replaced by

$$W_1(z) := \begin{pmatrix} \dfrac{1}{z} & -f(0) \\ 0 & z \end{pmatrix} \overset{\circ}{U}(z) \begin{pmatrix} 1 & 0 \\ \dfrac{1}{2\Delta z} & 1 \end{pmatrix}.$$

In order to see this we have only to observe the relation

$$\begin{pmatrix} 0 & \Delta \\ -\frac{1}{\Delta} & \frac{1}{z}+\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{z\Delta} & 1 \end{pmatrix} \begin{pmatrix} 0 & \Delta \\ -\frac{1}{\Delta} & \delta \end{pmatrix}$$

and the fact that the second factor on the right hand side is J-unitary. The matrix function $W_1(z)$ is in general not normalized: $W_1(0) \neq I_2$.

§ 11. THE CANONICAL DIFFERENTIAL SYSTEM FOR THE RESOLVENT MATRIX OF $g \in \mathfrak{G}_{\varkappa;a}$ WITH ACCELERANT

11.1. In this n° we suppose that the Hermitian accelerant H on $[-2a, 2a]$, $0 < a < \infty$, is continuous. Recall that $r \in (0, a)$ is a singular point of H if $-1 \in \sigma(\mathbf{H}_r)$. We consider an interval $[b_0, b_1] \subset (0, a]$ which does not contain any singular point of H. In particular the interval $[0, b_1]$ has this property if b_1 is sufficiently small. As H is continuous, also the kernel $\Gamma_r(t, s)$ $(0 \leqslant s, t \leqslant 2r)$ is continuous if r is not a singular point of H, and for these r the function A:

$$A(r) := 2\,\Gamma_r(2r, 0) = 2\Gamma_r(2r) \tag{11.1}$$

is well defined. The relation (1.23) implies

$$\frac{\partial \Gamma_r(t, 0)}{\partial r} = -\,A(r)\Gamma_r(t, 2r) \quad (0 \leqslant t \leqslant 2r).$$

Therefore the derivative of

$$D(r; z) = \mathrm{e}^{irz}\left(1 - \int_0^{2r} \Gamma_r(s)\,\mathrm{e}^{-izs}\mathrm{d}s\right), \quad \Gamma_r(s) := \Gamma_r(s, 0),$$

with respect to r exists and equals

$$\frac{\partial D(r; z)}{\partial r} = \mathrm{i}zD(r; z) + \mathrm{e}^{irz}\left(-2\Gamma_r(2r)\,\mathrm{e}^{-2irz} - \int_0^{2r} \frac{\partial \Gamma_r(s)}{\partial r}\,\mathrm{e}^{-izs}\mathrm{d}s\right) =$$

$$= \mathrm{i}zD(r; z) - A(r)\,\mathrm{e}^{-irz}\left(1 - \int_0^{2r} \overline{\Gamma_r(u)}\,\mathrm{e}^{izu}\mathrm{d}u\right)$$

(observe Proposition 1.1, (2)). This gives

$$\frac{\partial D(r; z)}{\partial r} = \mathrm{i}zD(r; z) - A(r)D^*(r; z). \tag{11.2}$$

In the same way it follows that

$$\frac{\partial E(r; z)}{\partial r} = \mathrm{i}zE(r; z) + A(r)E^*(r; z). \tag{11.3}$$

Now we consider the matrix function

$$V(r; z) := \begin{pmatrix} E_{\mathrm{R}}(r; z) & E_{\mathrm{J}}(r; z) \\ -D_{\mathrm{J}}(r; z) & D_{\mathrm{R}}(r; z) \end{pmatrix} \tag{11.4}$$

which for $r = a$ appears in (9.1). Evidently

$$\lim_{r \downarrow 0} V(r; z) = I_2 \quad (z \in \mathbf{C}). \tag{11.5}$$

Introducing the real continuous functions $\alpha(r)$, $\beta(r)$:

$$A(r) =: \alpha(r) + i\beta(r) \tag{11.6}$$

and the matrix function

$$P(r) := \begin{pmatrix} \beta(r) & -\alpha(r) \\ -\alpha(r) & -\beta(r) \end{pmatrix} \quad (b_0 \leqslant r \leqslant b_1), \tag{11.7}$$

the relations (11.2) and (11.3) imply:

THEOREM 11.1. *Let H be a continuous Hermitian function on $[-2a, 2a]$, $0 < a < \infty$, and suppose that the interval $[b_0, b_1] \subset [0, a]$ does not contain any singular point of H. Then the matrix function $V(\cdot\,; z)$ from* (11.4) *satisfies the canonical differential system*

$$\frac{\mathrm{d}V(r; z)}{\mathrm{d}r} J = zV(r; z) + V(r; z)P(r) \quad (b_0 \leqslant r \leqslant b_1) \tag{11.8}$$

with the continuous real 2×2-matrix function P given by (11.7).

COROLLARY 11.1. *Under the conditions of Theorem* 11.1 *we have*

$$V(b_1; z) = V(b_0; z)V_{b_0}(b_1; z),$$

where $V_{b_0}(r; z)$ is the solution of the initial problem

$$\begin{gathered} \frac{\mathrm{d}V_{b_0}(r; z)}{\mathrm{d}r} J = zV_{b_0}(r; z) + V_{b_0}(r; z)P(r) \quad (b_0 \leqslant r \leqslant b_1), \\ V_{b_0}(b_0; z) = I_2 . \end{gathered} \tag{11.9}$$

If, in particular, $b_0 = 0$, then the matrix function $V(r; z)$, $0 \leqslant r \leqslant b_1$, is uniquely determined by (11.9).

REMARK 1. We mention that, conversely, for an arbitrary continuous function $A(\cdot)$ on $[0, a]$ there exists a continuous Hermitian function H on $[-2a, 2a]$, $I + \mathbf{H} > 0$, such that (11.1) holds: $A(r) = 2\Gamma_r(2r, 0)$, where Γ_r is the resolvent kernel corresponding to $\mathbf{H}$ (see [10]).

REMARK 2. If H is real we have $D^*(r; 0) = D(r; 0)$ and the relation (11.2) implies

$$\frac{\mathrm{d}D(r; 0)}{\mathrm{d}r} = -A(r)D(r; 0) \quad (b_0 \leqslant r \leqslant b_1).$$

If, in particular, the interval [0, a] does not contain any singular point of H (that is $\varkappa = 0$ in Theorem 2.1, (1)), it follows that $D(r; 0) = \exp\left(-\int_0^r A(s)\,\mathrm{d}s\right)$ or

$$\ln\left(1 + \int_0^{2r} \Gamma_r(s, 0)\,\mathrm{d}s\right) = -\int_0^{2r} \Gamma_{t/2}(t, 0)\,\mathrm{d}t. \quad (0 \leqslant r \leqslant a).$$

According to the remarks at the end of § 4.1, if V satisfies the canonical system (11.8), and U_0 is the solution of the initial problem $U_0'J = U_0 P$, $U_0(r_0) = I_2$ for some $r_0 \in (b_0, b_1)$, then the function W: $W(r; z) = V(r; z)U_0(r)^{-1}$ satisfies the canonical system

$$W'J = zW\mathscr{H} \tag{11.10}$$

with $\mathscr{H} := U_0 U_0^*$. As $U_0(r)$ is iJ-unitary, in the relation (9.1) the matrix

$$V(a; z) = \begin{pmatrix} E_{\mathrm{R}}(z) & E_{\mathrm{J}}(z) \\ -D_{\mathrm{J}}(z) & D_{\mathrm{R}}(z) \end{pmatrix}$$

can be replaced by $W(a; z)$.

We mention that higher smoothness properties of H imply also higher smoothness properties of the function A in (11.1) and hence of α and β. In § 12 we shall need the following:

LEMMA 11.1. *If, additionally to the above conditions, the accelerant H is absolutely continuous (has a continuous derivative) on (b_0, b_1) then also A is absolutely continuous (has a continuous derivative) on (b_0, b_1).*

Proof. The relation

$$\Gamma_r(2r, 0) + \int_0^{2r} H(t - u)\Gamma_r(u, 0)\,\mathrm{d}u = H(2r) \quad (b_0 < r < b_1)$$

implies

$$\frac{1}{2}\frac{\mathrm{d}A(r)}{\mathrm{d}r} + H(t - 2r)A(r) + \int_0^{2a} H(t - u)\frac{\partial \Gamma_r(u, 0)}{\partial r}\,\mathrm{d}u = H'(2r),$$

and the statement follows easily.

11.2. If the accelerant H is not continuous the function $A(r)$ does in general not exist. In this case with the resolvent matrix W_{δ_0} in (8.16), which gives also a description of all the continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of g in (2.2) (see Remark 2 after Theorem 9.1), can be associated a canonical differential system with a real det-normed Hamiltonian (which, in general, cannot be written with a potential).

In this n° by $W(r; z)$ we denote the resolvent matrix given by (8.16):

$$(11.11)\quad W(r; z) := \begin{pmatrix} -E^*(r; z)/2 & E(r; z)/2 \\ \mathrm{i}D^*(r; z) & \mathrm{i}D(r; z) \end{pmatrix} \begin{pmatrix} -D(r; 0) & -\mathrm{i}E(r; 0)/2 \\ D^*(r; 0) & -\mathrm{i}E^*(r; 0)/2 \end{pmatrix}.$$

Evidently, $\lim_{r \downarrow 0} W(r; z) = I_2$.

THEOREM 11.2. *Let $H \in L^1(-2a, 2a)$ be a Hermitian function, $0 < a < \infty$, and suppose that the interval $[b_0, b_1] \subset [0, a]$ does not contain any singular point of H. Then the matrix function $W(\cdot\,; z)$ from (11.11) satisfies the canonical differential system*

$$(11.12)\qquad \frac{\mathrm{d}W(r; z)}{\mathrm{d}r} J = zW(r; z)\mathscr{H}(r) \quad (b_0 \leqslant r \leqslant b_1)$$

with the continuous real det-normed Hamiltonian $\mathscr{H}(r) = \hat{U}_0(r)\hat{U}_0(r)^*$, where

$$\hat{U}_0(r) := \begin{pmatrix} -E^*(r; 0)/2 & E(r; 0)/2 \\ \mathrm{i}D^*(r; 0) & \mathrm{i}D(r; 0) \end{pmatrix}.$$

Proof. First we suppose that H is continuous. Then we can differentiate the elements of $W(r; z) = (w_{jk}(r; z))_1^2$ with respect to r, using the relations (11.2), (11.3). It follows that

$$\frac{\mathrm{d}w_{11}(r; z)}{\mathrm{d}r} = \frac{1}{2}(-\mathrm{i}zE^*(r; z) + \overline{A(r)}E(r; z))D(r; 0) +$$

$$+ \frac{1}{2}E^*(r; z)(-A(r)D^*(r, 0)) +$$

$$(11.13)\qquad + \frac{1}{2}(\mathrm{i}zE(r; z) + A(r)E^*(r; z))D^*(0; z) - \frac{1}{2}E(r; z)\overline{A(r)}D(r; 0) =$$

$$= \frac{\mathrm{i}z}{2}(E(r; z)D^*(r; 0) - E^*(r; z)D(r; 0)),$$

and, in the same way,

$$\frac{\mathrm{d}w_{12}(r; z)}{\mathrm{d}r} = \frac{z}{4}(E(r; z)E^*(r; 0) + E^*(r; z)E(r; 0)),$$

$$(11.14)\qquad \frac{\mathrm{d}w_{21}(r; z)}{\mathrm{d}r} = -z(D(r; z)D^*(r; 0) + D^*(r; z)D(r; 0)),$$

$$\frac{\mathrm{d}w_{22}(r; z)}{\mathrm{d}r} = \frac{\mathrm{i}z}{2}(E(r; z)D^*(r; 0) - E^*(r; z)D(r; 0)).$$

These relations can be written as

$$\frac{\mathrm{d}W(r;z)}{\mathrm{d}r}J = zW(r;z)\begin{pmatrix} -D(r;0) & -\mathrm{i}E(r;0)/2 \\ D^*(r;0) & -\mathrm{i}E^*(r;0)/2 \end{pmatrix}^{-1}\begin{pmatrix} -E(r;0)/2 & D(r;0) \\ E^*(r;0)/2 & -\mathrm{i}D(r;0) \end{pmatrix}$$

and the product of the last two matrices is $\mathscr{H}(r)$.

If $H \in L^1(-2a, 2a)$ is general, we choose a sequence of continuous Hermitian functions $H^{(n)}$ on $[-2a, 2a]$ which converge to H in the L^1-norm. Denoting the orthogonal functions, corresponding to $H^{(n)}$, by D_n and E_n, it follows easily that $D_n(r; z) \to D(r; z)$ and $E_n(r; z) \to E(r; z)$ $(n \to \infty)$, hence (with evident notation)

$$W_n(r; z) \to W(r; z), \quad \mathscr{H}_n(r) \to \mathscr{H}(r) \quad (n \to \infty).$$

Finally, it is easy to check that $\det \mathscr{H}(r) = 1$. The theorem is proved.

REMARK. If $[0, a]$ does not contain any singular point of H we can put $b_0 = 0$, $b_1 = a$, and the relations (11.13), (11.14) are equivalent to the analogues of the Christoffel-Darboux formulae in Remark 2 after Theorem 8.1.

PROPOSITION 11.1. *Suppose that the conditions of Theorem* 11.2 *are satisfied for each interval* $[b_0, b_1]$, $b_0 < b_1 < r_1$ *with some* $r_1 \leqslant a$. *Then* r_1 *is a singular point of* H *if and only if*

$$\int_{b_0}^{r_1} \operatorname{tr} \mathscr{H}(r)\,\mathrm{d}r = \infty. \tag{11.15}$$

Proof. If $\int_{b_0}^{r_1} \operatorname{tr} \mathscr{H}(r)\,\mathrm{d}r < \infty$, the matrix

$$W(r_1; z) = \lim_{r \uparrow r_1} W(r; z)$$

exists and is nondegenerated. Remark 2 after Theorem 5.2 implies that the restriction $g_{r_1} = g|_{[-2r_1, 2r_1]}$ has more than one continuation in the corresponding class $\mathfrak{G}_{\varkappa'}$. Therefore r_1 is not a singular point of H. Conversely, if r_1 is not a singular point of H then the matrix $\mathscr{H}(r_1)$ exists and $\int_{b_0}^{r_1} \operatorname{tr} \mathscr{H}(r)\,\mathrm{d}r < \infty$. The proposition is proved.

Proposition 11.1 implies immediately a criterion for the uniqueness of the continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ of a given function $g \in \mathfrak{G}_{\varkappa;a}$ with accelerant. Indeed, this continuation $\tilde{g}$ is uniquely determined if and only if the Hamiltonian $\mathscr{H}$ associated with the accelerant H of g satisfies

$$\int_{b_0}^{a} \operatorname{tr} \mathscr{H}(r)\,\mathrm{d}r = \infty$$

if b_0 is sufficiently close to a, $b_0 < a$.

We mention that this criterion remains true for the more general situation of a function $g \in \mathfrak{G}_{\varkappa;a}$ with the property that each restriction $g|_{[-2b,2b]}$, $0 < b < a$, has an accelerant. This will follow from the more general considerations in [7].

11.3. Recall that an arbitrary function $g \in \mathfrak{G}_{0;a}$ admits at least one representation

$$(11.16)\qquad g(t) = g(0) + \mathrm{i}\gamma_0 t + \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\lambda t} - 1 - \frac{\mathrm{i}\lambda t}{1+\lambda^2}\right) \frac{\mathrm{d}\tau(\lambda)}{\lambda^2} \qquad (|t| \leqslant 2a)$$

with some $\gamma_0 \in \mathbf{R}$ and a nonnegative measure τ on $\mathbf{R}$ such that $\int_{-\infty}^{\infty} (1+\lambda^2)^{-1} \mathrm{d}\tau(\lambda) < \infty$.

It is easy to see that this representation is equivalent to the following representation of the kernel $G_g(t, s)$:

$$(11.17)\qquad G_g(t, s) = \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\lambda t} - 1)(\mathrm{e}^{-\mathrm{i}\lambda s} - 1)\lambda^{-2} \mathrm{d}\tau(\lambda) \qquad (|s|, |t| \leqslant 2a).$$

The measure τ in (11.16) or (11.17) is called a *spectral measure* of $g \in \mathfrak{G}_{0;a}$. It is easy to see that the right-hand side in (11.16) defines a continuation $\tilde{g} \in \mathfrak{G}_0$ of g:

$$(11.18)\qquad \tilde{g}(t) = g(0) + \mathrm{i}\gamma_0 t + \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\lambda t} - 1 - \frac{\mathrm{i}\lambda t}{1+\lambda^2}\right) \frac{\mathrm{d}\tau(\lambda)}{\lambda^2} \qquad (t \in \mathbf{R}).$$

Conversely, each continuation $\tilde{g} \in \mathfrak{G}_0$ admits a representation (11.18) and thus gives rise to a spectral measure τ of g. This correspondence between the continuations $\tilde{g} \in \mathfrak{G}_0$ of $g \in \mathfrak{G}_{0;a}$ and the spectral measures τ of g is bijective. An easy calculation gives

$$-\mathrm{i}z^2 \int_0^{\infty} \mathrm{e}^{-\mathrm{i}zt} \tilde{g}(t)\,\mathrm{d}t = -g(0)z - \gamma_0 + \int_{-\infty}^{\infty} \frac{1+\lambda z}{(\lambda - z)(1+\lambda^2)} \mathrm{d}\tau(\lambda) \qquad (z \in \mathbf{C}_-),$$

that is, the spectral measure of $\tilde{g}$ is the spectral measure of the Fourier transform

$$z \to -\mathrm{i}z^2 \int_0^{\infty} \mathrm{e}^{-\mathrm{i}zt} \tilde{g}(t)\,\mathrm{d}t \qquad (z \in \mathbf{C}_-),$$

which belongs to N_0.

Now let $H \in L^1(-2a, 2a)$ be a Hermitian function such that $I + \mathbf{H} > 0$. With the canonical system (11.12) we consider the canonical system

$$J\frac{\mathrm{d}y}{\mathrm{d}r} = z\mathscr{H}_1(r)y \quad (0 \leqslant r \leqslant a). \tag{11.19}$$

Here $\mathscr{H}_1$ is the Hamiltonian

$$\mathscr{H}_1(r) := J_1\mathscr{H}(r)^* J_1 , \quad J_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $W_1(r; z)$ denote the solution of the initial problem,

$$\frac{\mathrm{d}W_1(r;z)}{\mathrm{d}r} = z\mathscr{H}_1(r)W_1(r;z) \quad (0 \leqslant r \leqslant a),\ W_1(0;z) = I_2 , \tag{11.20}$$

and let $w_1(r, z)$ be the first column of $W_1(r; z)$, that is, $w_1(\cdot\,; z)$ satisfies the differential equation (11.19) and the initial condition $w_1(0; z)^{\mathrm{T}} = (1, 0)$. The Fourier transformation corresponding to the differential system (11.19) and its solution $w_1(\cdot\,; z)$ is defined as follows (see [46, § 14]):

$$\hat{\varphi}(\lambda) := \int_0^a \varphi(r)^{\mathrm{T}}\mathscr{H}_1(r)w_1(r;\lambda)\,\mathrm{d}r \quad (\lambda \in \mathbf{R}). \tag{11.21}$$

Here φ is an arbitrary bounded measurable 2-vector function on $[0, a]$ which vanishes in a neighbourhood of $r = a$. A measure τ on $\mathbf{R}$ is called a *spectral measure* of (11.19) corresponding to the Fourier transformation (11.21) if for each such function τ the Parseval-Plancherel relation

$$\int_{-\infty}^{\infty} |\hat{\varphi}(\lambda)|^2\,\mathrm{d}\tau(\lambda) = \int_0^a \varphi(r)^*\mathscr{H}_1(r)\varphi(r)\,\mathrm{d}r$$

holds (observe that $\mathscr{H}_1(r) > 0$ for all $r \in [0, a]$).

With the function H we associate the function g_1:

$$g_1(t) := -|t| - 2\int_0^t (t-s)H(s)\,\mathrm{d}s \quad (|t| \leqslant 2a). \tag{11.22}$$

That is we have $g_1 = 2g$, where g is given by (2.2) with $g(0) = 0$.

THEOREM 11.3. *Let H be a continuous Hermitian function on $[-2a, 2a]$ such that $I + \mathbf{H} > 0$. The set of all spectral measures of the canonical system* (11.19), *corresponding to the Fourier transformation* (11.21), *coincides with the set of all spectral measures of the function g_1 in* (11.22).

Proof. Let $W_1(a;z) =: (w^1_{jk}(z))^2_1$ $(z \in \mathbf{C})$. Then for arbitrary $T \in \tilde{N}_0$ the function

$$z \to \frac{w^1_{11}(z)T(z) + w^1_{12}(z)}{w^1_{21}(z)T(z) + w^1_{22}(z)} \quad (z \neq \bar{z}) \tag{11.23}$$

belongs to N_0, and the set of all spectral measures of the canonical system (11.19) and Fourier transformation (11.21) coincides with the set of the spectral measures of all the functions (11.23) $(T \in \tilde{N}_0)$, see [46, § 14].

Consider the matrix function

$$W(r;z) := J_1 W_1(r;z)^{\mathrm{T}} J_1 \quad (0 \leqslant r \leqslant a;\ z \in \mathbf{C}).$$

Then we have

$$W(a;z) = \begin{pmatrix} w^1_{22}(z) & w^1_{12}(z) \\ w^1_{21}(z) & w^1_{11}(z) \end{pmatrix}$$

and

$$\frac{\mathrm{d}W(r;z)}{\mathrm{d}r} J = zW(r;z)\mathscr{H}(r) \quad (0 \leqslant r \leqslant a),$$

that is, $W(\cdot, z)$ is the solution of (11.12) on $[0,a]$, $W(0;z) = I_2$. On the other hand, according to Remark 2 after Theorem 9.1 we get a description of all extensions $\tilde{g}_1 \in \mathfrak{G}_0$ of g_1 by the formula

$$-\mathrm{i}z^2 \int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}_1(t)\,\mathrm{d}t = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)} \quad (z \in \mathbf{C}_-) \tag{11.24}$$

$(w_{jk}(z))^2_1 := W(a;z)$, if T runs through $\tilde{N}_0$. The remarks at the beginning of this n° imply that the spectral measures of the functions on the right hand side of (11.24) for $T \in \tilde{N}_0$ coincide with the spectral measures of g_1. The theorem is proved.

We mention that by means of the results of De Branges it can be shown that in case $\varkappa = 0$ also for an arbitrary function $g \in \mathfrak{G}_{0;a}$, $0 < a < \infty$, the set of all spectral measures of g coincides with the set of all spectral measures of a canonical system.

§ 12. REAL ACCELERANTS, ASSOCIATED STRINGS AND STURM-LIOUVILLE EQUATIONS

12.1. In this section we assume that the Hermitian accelerant $H \in L^1(2a, 2a)$ is real: $H(t) = \overline{H(t)} = H(-t)$ $(-2a \leqslant t \leqslant 2a)$. Then also the resolvent kernel $\Gamma_a(t,s)$ is real if it exists (that is if $-1 \notin \sigma(\mathbf{H})$), hence $\Gamma_a(t,s) = \Gamma_a(s,t)$ $(0 \leqslant s, t \leqslant 2a)$ and also $\Gamma_a(s)$ and $L_a(s)$ are real. The definition of the orthogonal functions

$D(a, z) =: D(z)$ and $E(a, z) =: E(z)$ implies

$$D(-z)=D^*(z), \quad E(-z)=E^*(z) \quad (z \in \mathbf{C})$$

and

$$D_{\mathrm{R}}(-z) = D_{\mathrm{R}}(z), \quad D_{\mathrm{J}}(-z) = -D_{\mathrm{J}}(z),$$
$$E_{\mathrm{R}}(-z) = E_{\mathrm{R}}(z), \quad E_{\mathrm{J}}(-z) = -E_{\mathrm{J}}(z).$$

Hence there exist real entire functions d_R , d_J , e_R and e_J such that we have

$$D_{\mathrm{R}}(z) = d_R(z^2), \quad D_{\mathrm{J}}(z) = z^{-1}d_J(z^2),$$
$$E_{\mathrm{R}}(z) = e_R(z^2), \quad E_{\mathrm{J}}(z) = z\, e_J(z^2). \tag{12.1}$$

Evidently, $d_J(0) = 0$.

The function $\tilde{g} \in \mathfrak{G}_\varkappa$ is real if and only if the function

$$Q_{\tilde{g}}(z) := -\mathrm{i}z^2 \int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t \quad (\operatorname{Im} z < -\gamma)$$

has the property

$$\overline{Q_{\tilde{g}}(z)} = -Q_{\tilde{g}}(-\bar{z}) \quad (\operatorname{Im} z < -\gamma). \tag{12.2}$$

If $g \in \mathfrak{G}_{\varkappa;a}$ has an accelerant H, it is real if and only if H is real.

We consider now the the real Hermitian function $g \in \mathfrak{G}_{\varkappa;a}$:

$$g(t) := -\frac{1}{2}|t| - \int_0^t (t-s)H(s)\,\mathrm{d}s \quad (-2a \leqslant t \leqslant 2a) \tag{12.3}$$

with accelerant $H \in L^1(-2a, 2a)$. Suppose that $-1 \notin \sigma(\mathbf{H})$, that is g has infinitely many continuations $\tilde{g} \in \mathfrak{G}_\varkappa$. Such a continuation $\tilde{g}$ is real if and only if (12.2) holds. On the other hand we have from (7.17) and (12.1)

$$Q_{\tilde{g}}(z) = \frac{1}{2}\,\frac{e_R(z^2)T(z) + ze_J(z^2)}{-\dfrac{1}{z}\,d_J(z^2)T(z) + d_R(z^2)} \quad (\operatorname{Im} z < -\gamma), \tag{12.4}$$

and it follows easily that the relation (12.2) holds if and only if $\overline{T(z)} = -T(-\bar{z})$ $(z \in \mathbf{C}_-)$. As $T \in \tilde{N}_0$, according to [47, Theorem D I.5.4] the function T admits a representation

$$T(z) = zS(z^2) \tag{12.5}$$

with some function $S \in \tilde{S}$.*) Replacing $T(z)$ in (12.4) by $S(z)$ from (12.5) we have proved the first part of the following.

*) Recall that the function S belongs to $\tilde{S}$ if the functions S and $S_1 : S_1(z) = zS(z)$ belong to $\tilde{N}_0$. Other characterizations of the class $\tilde{S}\setminus\{\infty\}$ are given in [47, § 5]; in [2] this class is denoted by N_0^+.

THEOREM 12.1. *Let the function $H \in L^1(-2a, 2a)$ be real and Hermitian and $g \in \mathfrak{G}_{\varkappa;a}$ be given by* (12.3). *If $-1 \notin \sigma(\mathbf{H})$ then the relation*

(12.6)
$$-\mathrm{i}z\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = \frac{1}{2}\,\frac{e_R(z^2)S(z^2)+e_J(z^2)}{-d_J(z^2)S(z^2)+d_R(z^2)} \quad (\operatorname{Im} z \leqslant -\gamma)$$

establishes a bijective correspondence between all real continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of g and all $S \in \tilde{S}$. If $-1 \in \sigma(\mathbf{H})$ then the unique continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ which is given by (2.3) *is real.*

In order to see that the last statement holds we observe that the functions φ_0 and ψ_0 in § 2.1 can be chosen real. Hence their Fourier transforms have the properties $\mathscr{F}(\varphi_0; z) = \overline{\mathscr{F}(\varphi_0; \bar{z})}$, $\mathscr{F}(\psi_0; z) = \overline{\mathscr{F}(\psi_0; -\bar{z})}$, and it follows easily from (2.3) that $\tilde{g}$ is real.

If in particular $\varkappa = 0$, a real function $\tilde{g} \in \mathfrak{G}_0$ admits an integral representation (see, e.g., [6])

$$\tilde{g}(t) = \int_0^\infty \frac{\cos\sqrt{\lambda}\,t - 1}{\lambda}\,\mathrm{d}\tau_{\tilde{g}}(\lambda) \quad (t \in \mathbf{R})$$

with a unique nonnegative σ-finite measure $\tau_{\tilde{g}}$ on $[0, \infty)$ such that

$$\int_0^\infty (1+\lambda)^{-1}\,\mathrm{d}\tau_{\tilde{g}}(\lambda) < \infty.$$

The measure $\tau_{\tilde{g}}$ is called the *reduced spectral measure* of $\tilde{g}$. If $g \in \mathfrak{G}_{0;a}$ is real and $\tilde{g} \in \mathfrak{G}_0$ is an arbitrary real continuation of g then each $\tau_{\tilde{g}}$ is called a *reduced spectral measure* of g. That is, a reduced spectral measure τ of $g \in \mathfrak{G}_{0;a}$ is characterized by the property that the relation

(12.7)
$$g(t) = \int_0^\infty \frac{\cos\sqrt{\lambda}\,t - 1}{\lambda}\,\mathrm{d}\tau(\lambda) \quad (-2a \leqslant t \leqslant 2a)$$

holds. As in this section we shall deal only with reduced spectral measures of $g \in \mathfrak{G}_{0;a}$, the word 'reduced' will often be omitted.

If the conditions in the first part of Theorem 12.1 are satisfied and, additionally, $\varkappa = 0$, it follows from (12.6) that the relation

$$\int_0^\infty \frac{\mathrm{d}\tau(\lambda)}{\lambda - z} = \frac{1}{2}\,\frac{e_R(z)S(z)+e_J(z)}{-d_J(z)S(z)+d_R(z)} \quad (z \notin [0, \infty))$$

establishes a bijection between all spectral measures τ of g and all $S \in \tilde{S}$.

12.2. If $H \in L^1(-2a, 2a)$ is real and the interval $(b_0, b_1) \subset [0, a]$ does not contain any critical point of H, the (real det-normed) Hamiltonian $\mathscr{H}$ of the canonical system in Theorem 11.2 is of diagonal form:

$$\mathscr{H}(r) = \begin{pmatrix} \frac{1}{2} E(r;0)^2 & 0 \\ 0 & 2D(r;0)^2 \end{pmatrix} =: \begin{pmatrix} h_1(r) & 0 \\ 0 & h_2(r) \end{pmatrix} \quad (b_0 < r < b_1),$$

$h_j(r) > 0$, $h_1(r)h_2(r) = 1$. We fix some $r \in (b_0, b_1)$ or, if b_0 is not a singular point of H, $r \in [b_0, b_1)$ and introduce the increasing continuous functions

$$x(r) := \int_{r_0}^{r} h_1(s)\,\mathrm{d}s, \quad \mathscr{M}(r) := \int_{r_0}^{r} h_2(s)\,\mathrm{d}s \quad (b_0 < r < b_1)$$

and M: $M(x(r)) := \mathscr{M}(r)$ $(x(b_0) < x < x(b_1))$. Then we have

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{\mathrm{d}\mathscr{M}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}x} = h_2(r)h_1(r)^{-1} = h_2(r)^{-2},$$

hence M has a continuous derivative. The canonical system (11.12) takes the form

$$\begin{aligned} w'_{11}(r;z) &= -zh_2(r)w_{12}(r;z), & w'_{21}(r;z) &= -zh_2(r)w_{22}(r;z), \\ w'_{12}(r;z) &= zh_1(r)w_{11}(r;z), & w'_{22}(r;z) &= zh_1(r)w_{21}(r;z). \end{aligned} \tag{12.8}$$

If we introduce the functions $\hat{w}_{jk}(\cdot\,; z)$ on the interval $(x(b_0), x(b_1))$:

$$\hat{w}_{jk}(x(r);z) := w_{jk}(r;z) \quad (b_0 < r < b_1),$$

they satisfy the following equations:

$$\begin{aligned} D_M D_x \hat{w}_{12} &= -z^2 \hat{w}_{12}, & \hat{w}_{11} &= z^{-1}\hat{w}'_{12}, \\ D_M D_x \hat{w}_{22} &= -z^2 \hat{w}_{22}, & \hat{w}_{21} &= z^{-1}\hat{w}'_{22}. \end{aligned} \tag{12.9}$$

That is, the entries of the matrix function $\hat{W}$ are connected with a string on $(x(b_0), x(b_1))$ with mass distribution M (see, e.g., [46]). The point b_1 is singular for the canonical system (12.8), that is $\int^{b_1} \operatorname{tr} \mathscr{H}(r)\,\mathrm{d}r = \infty$, if and only if the string is singular at $x_1 = x(b_1)$, that is

$$x(b_1) + \mathscr{M}(b_1) = \infty. \tag{12.10}$$

Thus the condition (11.15) can be replaced by (12.10).

Now suppose additionally that the interval $[0, a]$ does not contain any singular point of H, that is $g \in \mathfrak{G}_{0;a}$ for the function g in (12.3). We choose $r_0 := 0$, $l := x(a)$ and introduce the solutions φ, ψ of the equation $D_M D_x u + zu = 0$ on $[0, l)$ with initial conditions

$$\varphi(0;z) = 1, \quad \varphi'(0;z) = 0, \quad \psi(0;z) = 0, \quad \psi'(0;z) = 1.$$

Then the matrix function $W(r, z)$ in Theorem 11.2 becomes (with $x = x(r)$).

$$W(r, z) = \hat{W}(x, z) = \begin{pmatrix} \psi'(x; z^2) & z\psi(x; z^2) \\ z^{-1}\varphi'(x; z^2) & \varphi(x; z^2) \end{pmatrix}.$$

This follows immediately from (12.9), if we only observe the initial condition $\hat{W}(0, z) = I_2$.

Now from Remark 2 after Theorem 9.1, similar considerations as in the proof of Theorem 12.1 and [47] we find the following result.

THEOREM 12.2. *Let $H \in L^1(-2a, 2a)$ be a real Hermitian function, $0 < a < \infty$, $I + \mathbf{H} > 0$, and let $g \in \mathfrak{G}_{0;a}$ be given by* (12.3). *Then the relation*

$$\int_0^\infty \frac{d\tau(\lambda)}{\lambda - z} = \frac{\psi'(l;z)S(z) + \psi(l;z)}{\varphi'(l;z)S(z) + \varphi(l;z)} \qquad (z \notin [0, \infty))$$

establishes a bijective correspondence between all reduced spectral measures τ of g and all functions $S \in \tilde{S}$. In other words, the reduced spectral measures of the real function $g \in \mathfrak{G}_{0;a}$ coincide with the spectral measures with support in $[0, \infty)$ of the string on $[0, l]$ with mass distribution M and initial condition $y'(0) = 0$.

REMARK 1. The length $l_a = x(a)$ and total mass $M_a = M(x(a)) = \mathcal{M}(a)$ of the string in Theorem 12.2 are given by

$$l_a = \frac{1}{2}\int_0^a E(r, 0)^2 \, dr, \quad M_a = 2\int_0^a D(r, 0)^2 \, dr.$$

REMARK 2. If a real function H is defined on the real axis, $H(-t) = H(t)$ $(t \in \mathbf{R})$, $H_a := H|_{[-2a,2a]}$ belongs to $L^1(-2a, 2a)$ and $I + \mathbf{H}_a > 0$ for each $0 < a < \infty$, then with H can be associated a singular string with mass distribution M on the interval $0 \leqslant x < x(\infty) = \lim_{a \uparrow \infty} x(a)$, such that the reduced spectral measure τ_g of g in (2.2) coincides with the spectral measure with support in $[0, \infty)$ of this string (under the initial condition $y'(0) = 0$). The singularity of this string follows from the relation

$$x(\infty) + \mathcal{M}(\infty) = \int_0^\infty (h_1(r) + h_1(r)^{-1}) \, dr \geqslant 2\int_0^\infty dr = \infty.$$

The fact that with the function H in Remark 2 there can be associated such a string is a particular case of a result of M. G. Kreĭn about the inverse spectral problem for strings. According to these results, also the relation $\mathcal{M}(\infty)^{-1} = \tau_g(\{0\})$ holds. On the other hand, the representation (12.7) for g (with $a = \infty$, $\tau = \tau_g$) implies

$$\tau_g(\{0\}) = \lim_{t \uparrow \infty} (-g(t)/t^2) = \lim_{t \uparrow \infty} \frac{1}{t^2}\int_0^t (t - s)H(s) \, ds.$$

Thus the total mass $\mathcal{M}(\infty)$ ($\leqslant \infty$) can be expressed by the accelerant as follows:

$$\mathcal{M}(\infty)^{-1} = \lim_{t\uparrow\infty} \frac{1}{t^2} \int_0^t (t-s)H(s)\,\mathrm{d}s.$$

In the same way, the total length $l(\infty)$ can be expressed by the dual accelerant.

12.3. Now we suppose additionally to the conditions at the beginning of n° 2 that H is continuous on $[-2a, 2a]$. Then we can introduce the function A of (10.1): $A(r) = 2\Gamma_r(2r, 0)$ $(b_0 < r < b_1)$ and the relations (10.2) and (10.3) imply

$$\frac{\mathrm{d}D(r;0)}{\mathrm{d}r} = -\,A(r)D(r;0), \quad \frac{\mathrm{d}E(r;0)}{\mathrm{d}r} = A(r)E(r;0).$$

Thus we get with $r_0 \in (b_0, b_1)$ or, if b_0 is not a singular point of H, $r_0 \in [b_0, b_1)$:

$$D(r;0) = D(r_0;0)\,\mathrm{e}^{-\int_{r_0}^{r} A(s)\,\mathrm{d}s}, \quad E(r;0) = E(r_0;0)\,\mathrm{e}^{\int_{r_0}^{r} A(s)\,\mathrm{d}s},$$

and it follows that

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = 4D(r_0;0)^4\,\mathrm{e}^{-4\int_{r_0}^{r} A(s)\,\mathrm{d}s}.$$

Therefore M has a continuous second derivative. In the same way, stronger smoothness properties of H imply stronger smoothness properties of M.

If H is continuous we can also consider the canonical system (11.8). As H is real we find $\beta(r) = 0$, $A(r) = \alpha(r)$ and (11.8) reduces to

(12.11) $$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \alpha\right) v_{j2} = z v_{j1}, \quad \left(\frac{\mathrm{d}}{\mathrm{d}r} - \alpha\right) v_{j1} = -z v_{j2}, \quad j = 1, 2$$

or

(12.12) $$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \alpha\right)\left(-\frac{\mathrm{d}}{\mathrm{d}r} + \alpha\right) v_{j1} = z^2 v_{j1}, \quad \left(-\frac{\mathrm{d}}{\mathrm{d}r} + \alpha\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} + \alpha\right) v_{j2} = z^2 v_{j2}, \quad j = 1, 2.$$

Suppose now additionally that H is absolutely continuous or has a continuous derivative. Then the function α has the same properties, and we can write the equations (12.12) as

(12.13) $$-v_{j1}'' + (\alpha^2 + \alpha')\,v_{j1} = z^2 v_{j1}, \quad -v_{j2}'' + (\alpha^2 - \alpha')v_{j2} = z^2\,v_{j2}, \quad j = 1, 2.$$

Recall that these relations hold on each interval $(b_0, b_1) \subset [0, a]$ which contains no singular point of H. At the singular points the function α and hence also the "potentials"

(12.14) $$q(r) := \alpha(r)^2 - \alpha'(r) \quad \text{or} \quad q_1(r) := \alpha(r)^2 + \alpha'(r)$$

are in general not defined. In the next section we shall show that, nevertheless, under an additional assumption on H the potential q makes sense on the whole interval $[0, a]$.

12.4. Let H be a real Hermitian function on $[-2a, 2a]$ and suppose that
(i) H is absolutely continuous,

or

(ii) H has a continuous derivative.

With the real function g in (12.3) we introduce the symmetric kernel

$$R_g(t, s) := g(t-s) - g(t+s) \quad (-a \leqslant s,\ t \leqslant a).$$

Recall that for an arbitrary real Hermitian and continuous function g on $[-2a, 2a]$, $g(0) = 0$, this kernel was introduced in [48], and it was shown there that the kernel R_g is nonnegative definite if and only if g admits a representation

$$g(t) = \int_{-\infty}^{\infty} \frac{\cos\sqrt{\lambda}t - 1}{\lambda}\, \mathrm{d}\tau(\lambda) \quad (-2a \leqslant t \leqslant 2a)$$

with some nonnegative measure τ on the real axis such that the integral exists.

LEMMA 12.1. *Suppose that the real Hermitian function $H \in L^1(-2a, 2a)$ satisfies the condition* (i) *(*(ii) *resp.), $-1 \notin \sigma(\mathbf{H})$ and that the kernel R_g is nonnegative definite. Then the potential $q = \alpha^2 - \alpha'$ has a summable (continuous, respectively) continuation to the whole interval* $[0, a]$.

Proof. If r is not a singular point of H we have from (11.6), (11.1) and (1.22)

$$q(r) = \alpha(r)^2 - \alpha'(r) = 4\Gamma_r(2r, 0)^2 - 2\frac{\mathrm{d}}{\mathrm{d}r}\Gamma_r(2r, 0) =$$

(12.15)

$$= -2\frac{\mathrm{d}}{\mathrm{d}r}(\Gamma_r(0, 0) + \Gamma_r(2r, 0)).$$

The operator $\dot{\mathbf{H}}$ in $L^2(-a, a)$ maps the set of all even functions $\varphi \in L^2(-a, a)$ into itself. Indeed, we have for such φ:

$$\int_{-a}^{a} H(t-s)\varphi(s)\,\mathrm{d}s = \int_{-a}^{a} H(t-s)\varphi(-s)\,\mathrm{d}s =$$

$$= \int_{-a}^{a} H(t+s)\varphi(s)\,\mathrm{d}s = \int_{-a}^{a} H(-t-s)\varphi(s)\,\mathrm{d}s \quad (-a \leqslant t \leqslant a).$$

With R_g also the kernel

$$\frac{\partial^2 R_g(t, s)}{\partial t \partial s} = -g''(t-s) - g''(t+s) = \delta_0(t-s) + \delta_0(t+s) + \hat{H}(t, s),$$

$\hat{H}(t, s) := H(t-s) + H(t+s)$ $(-a \leqslant s, t \leqslant a)$, is nonnegative definite. Then we have for even functions $\varphi \in L^2(-a, a)$:

$$\frac{1}{2}\int_{-a}^{a} |\varphi(s)|^2 \mathrm{d}s + \frac{1}{2}\int_{-a}^{a}\int_{-a}^{a} H(t-s)\varphi(s)\overline{\varphi(t)}\,\mathrm{d}s\,\mathrm{d}t =$$

$$= \int_{0}^{a} |\varphi(s)|^2 \mathrm{d}s + \int_{0}^{a}\int_{0}^{a} \hat{H}(t, s)\varphi(s)\overline{\varphi(t)}\,\mathrm{d}s\,\mathrm{d}t \geqslant 0.$$

As $(I + \dot{\mathbf{H}})^{-1}$ exists, the sign $=$ is excluded for nonzero functions φ. Thus for the integral operator $\hat{\mathbf{H}}$ with kernel $\hat{H}$ in $L^2(0, a)$ the inverse $(I + \hat{\mathbf{H}})^{-1}$ exists. It is not hard to check that the resolvent kernel $\hat{\Gamma}_a(t, s)$ of $\hat{\mathbf{H}}$ is given by

$$\hat{\Gamma}_a(t, s) := \dot{\Gamma}_a(t, s) + \dot{\Gamma}_a(t, -s) \quad (0 \leqslant s, t \leqslant a),$$

that is we have

$$\int_0^a \hat{H}(t, s)\hat{\Gamma}_a(s, u)\,\mathrm{d}s + \hat{\Gamma}_a(t, u) = \hat{H}(t, u) \quad (0 \leqslant t, u \leqslant a).$$

If, e.g., H has a continuous derivative, the same holds for the kernel $\hat{H}$, and hence also for the function $r \to \hat{\Gamma}_r(r, r)$ $(0 \leqslant r \leqslant a)$. On the other hand

$$\hat{\Gamma}_r(r, r) = \dot{\Gamma}_r(r, r) + \dot{\Gamma}_r(r, -r) = \dot{\Gamma}_r(2r, 2r) + \Gamma_r(2r, 0) = \Gamma_r(0, 0) + \Gamma_r(2r, 0)$$

$(0 \leqslant r \leqslant a)$. That is, by (12.15), the function $-2\frac{\mathrm{d}}{\mathrm{d}r}\hat{\Gamma}_r(r, r)$ is a continuous extension of q to the whole interval $[0, a]$. If H satisfies (i), also $\hat{\Gamma}_r(r, r)$ is absolutely continuous and hence $-2\frac{\mathrm{d}}{\mathrm{d}r}\hat{\Gamma}_r(r, r)$ is the summable extension of q to the interval $[0, a]$. The lemma is proved.

REMARK. The condition $-1 \notin \sigma(\mathbf{H})$ can be dropped if we replace the condition about the kernel R_g by the following: The kernel

$$\hat{R}_g(t, s) := \delta_0(t-s) + H(t-s) + H(t+s) \quad (0 \leqslant s, t \leqslant a) \tag{12.16}$$

is positive definite, that is, it is nonnegative definite and the integral equation

$$\varphi(t) + \int_0^a \hat{H}(t, s)\varphi(s)\,\mathrm{d}s = 0$$

has only the obvious solution $\varphi = 0$ (in $C(0, a)$ or in $L^2(0, a)$).

Let H be as in Lemma 12.1 or as in the above remark. Then the function q in (12.14) can be extended to the interval $[0, a]$. We consider the Sturm-Liouville equation *)

$$-y'' + qy = zy \quad \text{on } [0, a], \tag{12.17}$$

and its solutions $u(r, z)$, $v(r, z)$, satisfying the initial conditions

$$u(0) = 0, \quad u'(0) = 1, \quad v(0) = 1, \quad v'(0) = -2H(0). \tag{12.18}$$

The relations (12.11) and (12.13) imply that

$$v_{11}(r; z) = u'(r; z^2) + \alpha(r)u(r; z^2), \quad v_{12}(r; z) = zu(r; z^2),$$

$$v_{21}(r; z) = -\frac{1}{z}(v'(r, z^2) + \alpha(r)v(r; z^2)), \quad v_{22}(r; z) = v(r; z^2).$$

Thus (9.1) becomes

$$-\mathrm{i}z^2\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = \frac{1}{2}\,\frac{(u'(a; z^2) + \alpha(a)u(a; z^2))T(z) + zu(a; z^2)}{z^{-1}(v'(a; z^2) + \alpha(a)v(a; z^2))T(z) + v(a; z^2)} \quad (\operatorname{Im} z < -\gamma)$$

which gives a description of all the continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of $g \in \mathfrak{G}_{\varkappa;0}$ if T runs through the class $\tilde{N}_0$. For the real extensions $\tilde{g}$ the relation (12.2) must hold which implies $-T(-\bar{z}) = \overline{T(z)}$, hence $T(z) = zS(z^2)$ with some function $S \in \tilde{S}$. Thus we have proved the first part of the following theorem.

THEOREM 12.3. *Suppose that the real function $g \in \mathfrak{G}_{\varkappa;a}$ admits a representation* (12.3) *with an accelerant H satisfying* (i) *or* (ii).

(1) *If $-1 \notin \sigma(\mathrm{H})$ and the kernel R_g is nonnegative definite, the relation*

$$-\mathrm{i}z\int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = \frac{1}{2}\,\frac{(u'(a; z^2) + \alpha(a)u(a; z^2))S(z^2) + u(a; z^2)}{(v'(a; z^2) + \alpha(a)v(a; z^2))S(z^2) + v(a; z^2)} \quad (\operatorname{Im} z < -\gamma) \tag{12.19}$$

establishes a bijective correspondence between all real continuations $\tilde{g} \in \mathfrak{G}_\varkappa$ of g and all $S \in \tilde{S}$.

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*) This is the second equation in (12.13). Similar results hold for the first equation.

(2) *If* $-1 \in \sigma(\mathbf{H})$ *and the kernel* $\hat{R}_g$ *in* (12.16) *is positive definite the unique continuation* $\tilde{g} \in \mathfrak{G}_\varkappa$ *of* g *is given by the relation*

$$-\mathrm{i}z \int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = \frac{1}{2} u(a; z^2)(v(a; z^2))^{-1} \quad (\operatorname{Im} z < -\gamma).$$

Here u *and* v *are the solutions of the initial problem* (12.17), (12.18).

By $\Sigma_\varkappa$ we denote the set of all spectral measures τ (see [46]) of the Sturm--Liouville problem

$$(12.20) \qquad -y'' + qy = zy \quad \text{on } [0, a], \quad y'(0) + 2H(0)y(0) = 0,$$

for which $(-\infty, 0) \cap \operatorname{supp} \tau$ consists of exactly $\varkappa$ points. Then the Theorem 12.3 can be reformulated as follows.

THEOREM 12.4. *Under the conditions of Theorem* 12.3, (1), *the relation*

$$(12.21) \quad \int_0^\infty \frac{\mathrm{d}\tau(t)}{t - z} = \frac{(u'(a; z) + \alpha(a)u(a; z))S(z) + u(a; z)}{(v'(a; z) + \alpha(a)v(a; z))S(z) + v(a; z)} \quad (z \notin [0, \infty))$$

establishes a bijection between all $\tau \in \Sigma_\varkappa$ *and all* $S \in \tilde{S}$; *under the conditions of Theorem* 12.3, (2), $\Sigma_\varkappa$ *consists of exactly one element* τ *given by*

$$\int_{-\infty}^\infty \frac{\mathrm{d}\tau(t)}{t - z} = \frac{u(a; z)}{v(a; z)} \quad (z \notin [0, \infty)).$$

Proof. Let $S \in \tilde{S}$ be given and define τ by (12.21). The right hand side of (12.21) is of the form

$$\left(u'(a; z) \frac{S(z)}{\alpha(a)S(z) + 1} + u(a; z)\right)\left(v'(a; z) \frac{S(z)}{\alpha(a)S(z) + 1} + v(a; z)\right)^{-1}.$$

As $S \in \tilde{S}$ implies $S(\alpha(a)S + 1)^{-1} \in \tilde{N}_0$, according to [46] τ is a spectral measure of (12.20). Moreover, comparing (12.21) and (12.19) it follows that

$$\tilde{Q}(z) := -2\mathrm{i}z^2 \int_0^\infty \mathrm{e}^{-\mathrm{i}zt}\tilde{g}(t)\,\mathrm{d}t = z \int_{-\infty}^\infty \frac{\mathrm{d}\tau(t)}{t - z^2} =$$

$$= \frac{1}{2}\left(\int_0^\infty \mathrm{d}\hat{\tau}(t)\left(\frac{1}{t - z} + \frac{1}{-t - z}\right) + \int_{0+}^\infty \mathrm{d}\hat{\tau}(t)\left(\frac{1}{it - z} + \frac{1}{-it - z}\right)\right),$$

where $d\tau(t) = d\hat{\tau}(\sqrt{t})\ (t \geqslant 0)$, $d\tau(t) = -d\hat{\tau}(\sqrt{|t|})\ (t < 0)$. As $\tilde{Q} \in N_\varkappa$ (see Proposition 5.1) and the first integral on the right hand side belongs to N_0, the function $\hat{\hat{\tau}}$ has exactly $\varkappa$ points of increase, hence $\tau \in \Sigma_\varkappa$.

Conversely, suppose that we are given a spectral measure $\tau \in \Sigma_\varkappa$ of (12.20). With τ we define a real function $\tilde{g}_\tau$ on $\mathbf{R}$ as follows:

$$\tilde{g}_\tau(t) := \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos\sqrt{\lambda}t - 1}{\lambda}\, d\tau(\lambda) \quad (t \in \mathbf{R}). \tag{12.22}$$

This integral exists as $\int_1^\infty t^{-1} d\tau(t) < \infty$ and τ has only $\varkappa$ points of increase on $(-\infty, 0)$. It follows that

$$\tilde{g}_\tau(t - s) - \tilde{g}_\tau(t) - \tilde{g}_\tau(s) =$$

$$= \frac{1}{2}\int_0^\infty [(\cos\sqrt{\lambda}t - 1)(\cos\sqrt{\lambda}s - 1) + \sin\sqrt{\lambda}t \sin\sqrt{\lambda}s]\lambda^{-1} d\tau(\lambda) +$$

$$+ \frac{1}{2}\int_{-\infty}^{0-} [(\cosh\sqrt{\lambda}t - 1)(\cosh\sqrt{\lambda}s - 1) - \sinh\sqrt{\lambda}t \sinh\sqrt{\lambda}s]\lambda^{-1} d\tau(\lambda),$$

hence $\tilde{g}_\tau \in \mathfrak{G}_\varkappa$. On the other hand, according to [46] there exists a $T \in \tilde{N}_0$ such that

$$\int_{-\infty}^{\infty} \frac{d\tau(t)}{t - z} = \frac{u'(a; z)T(z) + u(a; z)}{v'(a; z)T(z) + v(a; z)}.$$

Observing the relation

$$\int_{-\infty}^{\infty} \frac{d\tau(t)}{t - z^2} = -2iz\int_0^\infty e^{-izt}\tilde{g}_\tau(t)\, dt \tag{12.23}$$

this implies

$$-2iz\int_0^\infty e^{-izt}\tilde{g}_\tau(t)\, dt = \frac{u'(a; z^2)T(z^2) + u(a; z^2)}{v'(a; z^2)T(z^2) + v(a; z^2)}.$$

Recall that $u(a; z^2)$, $v(a; z^2)$, $u'(a; z^2)$, $v'(a; z^2)$ are entire functions of exponential type $\leqslant a$. Moreover, we have

$$-2iz\int_0^\infty e^{-izt}\tilde{g}(t)\, dt = u(a, z^2)(v(a, z^2))^{-1}$$

for some continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ of $g \in \mathfrak{G}_{\varkappa;a}$. Repeating the arguments of Part (2) of the proof of Theorem 9.1 it follows that $\tilde{g}_\tau(t) = g(t)$ $(-2a \leqslant t \leqslant 2a)$, hence $\tilde{g}_\tau$ is a real continuation of g. According to Theorem 12.3, (1), there exists a function $S \in \tilde{S}$ such that

$$-2iz \int_0^\infty e^{-izt}\tilde{g}_\tau(t)\,dt = \frac{(u'(a;z^2) + \alpha(a)u(a;z^2))S(z^2) + u(a;z^2)}{(v'(a;z^2) + \alpha(a)v(a,z^2))S(z^2) + v(a,z^2)} \qquad (\operatorname{Im} z < -\gamma)$$

holds, and, using (12.23) the relation (12.21) follows. The proof of the second statement of the theorem is similar.

REMARK. It follows from this proof that an arbitrary real continuation $\tilde{g} \in \mathfrak{G}_\varkappa$ of g is a $\tilde{g}_\tau$ for some $\tau \in \Sigma_\varkappa$, see (12.22). Hence for an arbitrary such $\tilde{g}$ the kernel $R_{\tilde{g}}$ is nonnegative definite.

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CONCERNING THE PAPER

"ON POINT INTERACTIONS IN ONE DIMENSION"

by S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN, W. KIRSCH

(*J. Operator Theory*, **12**(1984), 101—126)

P. 109 and 110. Replace Theorem 3.1 by

THEOREM 3.1. *Suppose that V_j, $1 \leqslant j \leqslant N$ have compact support.*

a) *If* $\operatorname{n-lim}_{\varepsilon\to 0}(H_{\varepsilon,N} - k^2)^{-1} = (H_{\{\alpha_j\},N} - k^2)^{-1}$ *has eigenvalues* $E_m = k_m^2 < 0$, $1 \leqslant m \leqslant M$, $M \leqslant N$ *then, for* $\varepsilon > 0$ *small enough,* $H_{\varepsilon,N}$ *has* M *negative and simple eigenvalues* $E_{\varepsilon,m}$ *which are analytic in* ε *near* $\varepsilon = 0$

$$k_{\varepsilon,m} = \sqrt{-E_{\varepsilon,m}} = k_m + \mathrm{O}(\varepsilon), \quad 1 \leqslant m \leqslant M. \tag{3.11}$$

Moreover $E_{\varepsilon,m}$ *are the only eigenvalues of* $H_{\varepsilon,N}$ *near* E_m.

b) *If* $\operatorname{n-lim}_{\varepsilon\to 0}(H_{\varepsilon,N} - k^2)^{-1} = (H_{\{\alpha_j\},N} - k^2)^{-1}$ *and* $H_{\{\alpha_j\},N}$ *has no eigenvalues, then all eigenvalues of* $H_{\varepsilon,N}$ *tend to zero* i.e. *are absorbed into the essential spectrum as* $\varepsilon \to 0$.

P. 110. 9th and 14th line from below: Delete "$\operatorname{Im} k > -a$".

P. 110. Replace 6th line from below by: "Moreover, since E_m are simple [10]".

P. 111. Delete the rest of the proof of Theorem 3.1 (i.e. from line 1 until Remark 3.1).

P. 111. Delete "part a)" of Remark 3.1.