J. OPERATOR THEORY
14(1985), 87–111

SPECTRAL PROPERTIES OF L^p TRANSLATIONS

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0. INTRODUCTION

In [6], U. Fixman showed that the bilateral unit shift in $l^p(\mathbf{Z})$, where $1 < p < \infty$, $p \neq 2$, and $\mathbf{Z}$ is the additive group of integers, is not a spectral operator in the sense of N. Dunford, [3]. This example is in fact typical of translation operators in general locally compact abelian groups. For, if G is such a group, then except in trivial cases, translations in $L^p(G)$, $1 < p < \infty$, $p \neq 2$, are not spectral; see [9] and [2; Chapter 20], for example. However, it is natural to expect translations to be spectral in some sense, because they are isometries in $L^p(G)$ and, hence, analogues of unitary operators. This point was taken up in [9] and also in the recent article [1; §4], where it is shown that translations can indeed be expressed in the form

$$\int_{\mathbf{R}} e^{i\lambda} \, dQ(\lambda), \tag{1}$$

where $\{Q(\lambda) \,;\, \lambda \in \mathbf{R}\}$ is an associated spectral family of commuting projection operators satisfying certain properties [1; §4], and the integral (1), which can be interpreted as being over the unit circle $\mathbf{T}$ of the complex plane $\mathbf{C}$, exists in a certain well defined sense. But, it should be stressed that in general the spectral family does not generate a σ-additive, projection-valued spectral measure.

However, as suggested in the note [18], an alternative interpretation of (1) is possible. Namely, an operator may fail to be spectral solely because its domain space is "too small" to accommodate the projections needed to form its resolution of the identity. Accordingly, if interpreted as acting in a suitable space containing the domain space, it happens often that such an operator is spectral in the sense of Dunford. This has the advantage that the operator then has associated with it a rich functional calculus.

It is shown in [18; Example 2.8] that the bilateral unit shift in $l^p(\mathbf{Z})$, $1 < p < 2$, which has spectrum equal to $\mathbf{T}$, is a scalar-type spectral operator in this wider sense with resolution of the identity supported in $\mathbf{T}$. Earl Berkson posed the natural question of whether every translation in $L^p(G)$, where G is an arbitrary locally compact abelian group (all such operators have spectrum in $\mathbf{T}$), is a scalar-type spectral operator in this sense.

Our aim is to show that this is indeed the case if $1 < p < 2$; see Theorem 2.1 below. Hence, the failure of spectrality of translations in $L^p(G)$ for the case $1 < p < 2$ is only apparent; it is due solely to the fact that $L^p(G)$ is "too small" to accommodate the resolution of the identity of the translation operator (this is made precise in §1).

In contrast, the situation is fundamentally different when $2 < p < \infty$. For, although there are certain types of locally compact abelian groups for which non-trivial translation operators are scalar-type spectral operators in this wider sense (see §4), there are, nevertheless, a large class of groups for which this is not the case. It is shown for example (cf. Theorem 3.3 below), that already for the case of the line group $\mathbf{R}$, a non-trivial translation in $L^p(\mathbf{R})$, $2 < p < \infty$, can never be interpreted as a scalar type spectral operator with resolution of the identity supported in $\mathbf{T}$, in *any* space containing $L^p(\mathbf{R})$. In turns out that the line group is in a certain sense a paradigm for this phenomenon. For, if G is a locally compact abelian group and $g \in G$ is any element which generates a subgroup of G isomorphic to $\mathbf{Z}$, then the operator of translation by g in the space $L^p(G)$ can never be treated as a scalar-type operator with resolution of the identity supported in $\mathbf{T}$, in any space containing $L^p(G)$; see Theorem 4.2. Accordingly, the failure of spectrality for the case $2 < p < \infty$ is often genuine; no change of domain space will save the situation.

The authors wish to thank Professor I. Kluvánek for valuable discussions.

1. PRELIMINARIES AND NOTATION

Let X be a locally convex Hausdorff space, always assumed to be quasi-complete, X' its continuous dual and $L(X)$ the space of all continuous linear operators on X equipped with the topology of pointwise convergence on X. The identity operator is denoted by I. The adjoint of an operator T in X is denoted by T'.

A *spectral measure in* X is an $L(X)$-valued, σ-additive and multiplicative map $P : \mathscr{M} \to L(X)$, whose domain $\mathscr{M}$ is a σ-algebra of subsets of a set Ω, such that $P(\Omega) = I$. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)P(F)$, for every $E \in \mathscr{M}$ and $F \in \mathscr{M}$. It follows from the Orlicz-Pettis lemma that an $L(X)$-valued function P on a σ-algebra $\mathscr{M}$ is σ-additive if and only if the complex-valued set function

$$\langle Px, x' \rangle : E \mapsto \langle P(E)x, x' \rangle, \quad E \in \mathscr{M},$$

is σ-additive for each $x \in X$ and $x' \in X'$.

Let $P : \mathscr{M} \to L(X)$ be a spectral measure. An $\mathscr{M}$-measurable function f on Ω is said to be *P-integrable* if it is $\langle Px, x'\rangle$-integrable for every $x \in X$ and $x' \in X'$, and for each $E \in \mathscr{M}$ there is an operator $\int_E f\,\mathrm{d}P$ in $L(X)$ such that

$$\left\langle \left(\int_E f\,\mathrm{d}P\right) x, x' \right\rangle = \int_E f\,\mathrm{d}\langle Px, x'\rangle,$$

for every $x \in X$ and $x' \in X'$. This definition of integrability agrees with that for more general vector measures, [14].

An operator $T \in L(X)$ is called a *scalar-type spectral operator* if there exists a spectral measure P in X with range an equicontinuous part of $L(X)$ and a P-integrable function f such that $T = \int_\Omega f\,\mathrm{d}P$. This is the classical definition introduced by N. Dunford, [3]. The equicontinuity of the range of P ensures that all bounded measurable functions are P-integrable, [18; Proposition 1.1].

The class of operators relevant to this note are those whose spectrum, in the sense of [19; §3], is a part of the unit circle $\mathbf{T}$. The spectral properties of such operators have been well studied; see for example [11], [12], 16], [17] and the references therein. Of particular relevance are the *pseudo-unitary operators* [12], [15], [19], that is, those scalar-type spectral operators $T \in L(X)$ for which there exists a spectral measure P in X, defined on the σ-algebra $\mathscr{B}$ of Borel subsets of $\mathbf{T}$, such that

$$T = \int_{\mathbf{T}} z\,\mathrm{d}P(z).$$

The spectrum of such operators is a part of $\mathbf{T}$, [19; Theorem 4]. They are a natural generalization of unitary operators in a Hilbert space.

Let $T \in L(X)$. A locally convex Hausdorff space Y is said to be *admissible* for T, [18; p. 275], if there exist a continuous linear injection $\iota : X \to Y$ such that Y is the completion or quasi-completion of $\iota(X)$, and an operator T_Y in $L(Y)$, necessarily unique, such that

$$T_Y(\iota x) = \iota T x, \quad x \in X. \tag{2}$$

In this case the dual space Y' can be identified with the subspace $\{y' \circ \iota : y' \in Y'\}$ of X'. Therefore, we write $Y' \subseteq X'$. The subspace Y' of X' separates the points of X. Furthermore, Y' is an invariant subspace for T' that is, $T'(Y') \subseteq Y'$. Sets bounded in X remain bounded in Y but, more importantly, sets which are unbounded in X may be bounded in Y.

LEMMA 1.1. *Let T be an invertible operator in $L(X)$. Let Y be an admissible space for T such that the operator T_Y is invertible in $L(Y)$. Then for each trigonometric polynomial* $\Phi(z) = \sum_{n=-N}^{N} \alpha_n z^n$, *the space Y is admissible for the operator* $\Phi(T) = \sum_{n=-N}^{N} \alpha_n T^n$ *and*

$$(\Phi(T))_Y = \Phi(T_Y).$$

Proof. If $\iota : X \to Y$ is the continuous injection such that (2) is satisfied, then for each $x \in X$ it is clear that

$$T_Y^n \iota x = \iota T^n x, \quad n = 0, 1, 2, \ldots,$$

and, hence, that $(T^n)_Y = (T_Y)^n$ for each $n = 0, 1, 2, \ldots$. Furthermore, for each $x \in X$,

$$\iota x = \iota T T^{-1} x = T_Y \iota T^{-1} x,$$

and, hence,

$$(T_Y)^{-1} \iota x = \iota T^{-1} x.$$

Accordingly, Y is an admissible space for T^{-1}. It follows that

$$(T^{-n})_Y = (T_Y)^{-n}, \quad n = 1, 2, \ldots.$$

The result is now obvious.

REMARK. It is worth noting that if Y is an admissible space for an invertible operator $T \in L(X)$, then it need not follow that T_Y has an inverse in $L(Y)$ and, hence, Y may not be an admissible space for T^{-1}. For example, if X is the (strict) inductive limit of the spaces $l^1(\{1, 2, \ldots, n\})$, $n = 1, 2, \ldots$, then the operator T given by $Tx = y$, $x \in X$, where $y_n = n^{-1} x_n$ for each $n = 1, 2, \ldots$, is continuous and has a continuous inverse; the space $Y = l^1(\mathbf{N})$ is admissible for T but not admissible for T^{-1}.

An operator $T \in L(X)$ is said to be an *extended pseudo-unitary operator* if there exists an admissible space Y for T such that the operator T_Y is pseudo-unitary. The phenomenon of a non scalar-type spectral operator (with spectrum in $\mathbf{T}$) being an extended pseudo-unitary operator occurs often; see Examples 2.3 and 2.8 of [18] and Theorem 2.1 below.

In the remainder of this section we state some results from harmonic analysis which are needed in the sequel.

Let G be a locally compact abelian group and Γ its dual group. The value of $\gamma \in \Gamma$ at $g \in G$ is written as (g, γ). The Haar measures on G and Γ are assumed to be normalized so that the Fourier transform $f \mapsto \hat{f}$ is an isometry of $L^2(G)$ onto $L^2(\Gamma)$.

If $\varphi : \Gamma \to \mathbf{C}$ is a bounded measurable function, we define $S[\varphi]$ to be the continuous linear mapping of $L^2(G)$ into itself for which

$$(S[\varphi]f)^\wedge = \varphi\hat{f}, \quad f \in L^2(G).$$

Given $p \in [1, \infty]$, φ is said to be a *p-multiplier* if there is a number B such that

$$\|S[\varphi]f\|_p \leqslant B\|f\|_p, \quad f \in L^2(G) \cap L^p(G),$$

where $\|\cdot\|_p$ denotes the standard norm in $L^p(G)$. The smallest such B is denoted by $|||S[\varphi]|||_p$ or $|||\varphi|||_p$. Denote by $M_p(\Gamma)$ the space of all p-multipliers on Γ. If $p \in$ $\in [1, \infty]$, then q denotes the conjugate index of p.

Let $1 < p < 2$. Then the Fourier transform $f \mapsto \hat{f}$ is a norm-decreasing mapping of $L^p(G)$ into $L^q(\Gamma)$ which agrees with the ordinary Fourier transform

$$\hat{f}(\gamma) = \int_G (-g, \gamma) f(g)\, \mathrm{d}g, \quad \gamma \in \Gamma,$$

on $L^1(G) \cap L^p(G)$. In this case, the condition that φ belongs to $M_p(\Gamma)$ can be expressed by the condition that $\varphi\hat{f}$ be the Fourier transform of a function in $L^p(G)$ for every $f \in L^p(G)$. The operator $S[\varphi]$ then has a natural extension to all of $L^p(G)$.

The following result, permitting the transfer of Fourier multipliers from one group to another, is known as the extended deLeeuw theorem, [4; Appendix B].

LEMMA 1.2. *Let G and H be locally compact abelian groups with duals Γ and Λ respectively, and π a continuous homomorphism from Λ to Γ. Then if $p \in [1, \infty]$ and φ is a p-multiplier on Γ which is a continuous function, it follows that $\varphi \circ \pi \in M_p(\Lambda)$ and*

$$|||\varphi \circ \pi|||_p \leqslant |||\varphi|||_p.$$

LEMMA 1.3. *Let $1 < p < \infty$. Let $\{\varphi_r\}_{r=1}^\infty$ be a sequence of p-multiplier functions on Γ and $S_r = S[\varphi_r]$, $r = 1, 2, \ldots$, the corresponding sequence of operators on $L^p(G)$. Assume that*

(i) $\sup\{|||S_r|||_p : r = 1, 2, \ldots\} = M < \infty$, *and*

(ii) $\varphi_r \to \varphi$ a.e. *on* Γ.

Then φ is a p-multiplier and the sequence $\{S_r\}_{r=1}^\infty$ converges to $S[\varphi]$ in the weak operator topology.

Proof. Since $|||S_r|||_p \geqslant |||S_r|||_2 = \|\varphi_r\|_\infty$ for each $r = 1, 2, \ldots$, it follows from (i) that

$$\sup\{\|\varphi_r\|_\infty \,;\, r = 1, 2, \ldots\} < \infty.$$

It is now evident that if $f, h \in L^1(G) \cap (L^1(\Gamma))^\wedge$, then

$$\langle S_r f, h\rangle = \int_\Gamma (S_r f)^\wedge(t)\hat{h}(-t)\,\mathrm{d}t = \int_\Gamma \varphi_r(t)\hat{f}(t)\hat{h}(-t)\,\mathrm{d}t$$

and, hence, by the Dominated Convergence Theorem

$$\lim_{r\to\infty} \langle S_r f, h\rangle = \int_\Gamma \varphi(t)\hat{f}(t)\hat{h}(-t)\,\mathrm{d}t. \tag{3}$$

Furthermore,

$$\left|\int_\Gamma \varphi(t)\hat{f}(t)\hat{h}(-t)\,\mathrm{d}t\right| \leqslant \varlimsup_{r\to\infty}\left|\int_\Gamma \varphi_r(t)\hat{f}(t)\hat{h}(-t)\,\mathrm{d}t\right| = \tag{4}$$

$$= \varlimsup_{r\to\infty} |\langle S_r f, h\rangle| \leqslant M\|f\|_p\|h\|_q,$$

by (i). It follows from (4) and the density of the space $L^1(G) \cap (L^1(\Gamma))^\wedge$ in $L^1(G) \cap \cap L^p(G)$ that φ is a p-multiplier, [4; p. 7], and $|||S[\varphi]|||_p \leqslant M$.

Now if $F \in L^p(G)$, $H \in L^q(G)$ and $\varepsilon > 0$, we can choose $f, h \in L^1(G) \cap (L^1(\Gamma))^\wedge$ so that $\|F - f\|_p < \varepsilon$ and $\|H - h\|_q < \varepsilon$. So

$$|\langle (S[\varphi] - S_r)F, H\rangle - \langle (S[\varphi] - S_r)f, h\rangle| \leqslant |\langle (S[\varphi] - S_r)(F-f), H\rangle| +$$

$$+ |\langle (S[\varphi] - S_r)f, H - h\rangle| \leqslant$$

$$\leqslant |||S[\varphi] - S_r|||_p \varepsilon\|H\|_q + |||S[\varphi] - S_r|||_p(\|F\|_p + \varepsilon)\varepsilon \leqslant 2M\varepsilon\|H\|_q + 2M\varepsilon(\|F\|_p + \varepsilon).$$

Letting $r \to \infty$ it follows (using (3)) that

$$\varlimsup_{r\to\infty} |\langle (S[\varphi] - S_r)F, H\rangle| \leqslant 2M\varepsilon(\|H\|_q + \|F\|_p + \varepsilon),$$

for every $\varepsilon > 0$. It follows that $S_r \to S[\varphi]$ in the weak operator topology. This completes the proof.

An *arc* in $\mathbf{T}$ is any subset of the form $\{e^{it}; t \in J\}$ where J is an interval in $\mathbf{R}$. The collection of all arcs in $\mathbf{T}$ will be denoted by $\mathscr{A}$.

The proof of the next lemma is included for the sake of completeness. The result is not new (cf. [9; Lemma 6] and [2; Lemma 20.15]).

LEMMA 1.4. *Suppose* $1 < p < \infty$, G *is a locally compact abelian group, and* $g \in G$. *Let* χ_E *denote the characteristic function of an arc* E *in* $\mathbf{T}$, *and define*

$$\varphi_E : \gamma \to \chi_E((g, \gamma)), \quad \gamma \in \Gamma. \tag{5}$$

Then $\varphi_E \in M_p(\Gamma)$ *and*

$$\sup\{|||\varphi_E|||_p\,;\ E \in \mathscr{A}\} < \infty.$$

Proof. Let e^{ia} and e^{ib} be the end points of the arc E. Denote by $\{F_r\}_{r=1}^{\infty}$ the sequence of Fejér kernels on $\mathbf{T}$. Then the sequence of functions $\{\chi_E * F_r\}_{r=1}^{\infty}$ has the following properties:

(i) $(\chi_E * F_r)(e^{it}) \to \chi_E(e^{it})$ if $e^{it} \neq e^{ia}, e^{ib}$,

(ii) $\chi_E * F_r \to 1/2$ at the points e^{ia} and e^{ib}, and

(iii) $\sup\{|||\chi_E * F_r|||_p\,;\ E \in \mathscr{A},\ r = 1, 2, \ldots\} < \infty$.

Statements (i) and (ii) are standard. Property (iii) can be proved as follows.

Let h and k be finitely supported in $\mathbf{Z}$. Then it follows by Fubini's Theorem that for each $r = 1, 2, \ldots,$

$$\left|\int_{\mathbf{T}} \chi_E * F_r(z)\hat{h}(z)\hat{k}(-z)\,dz\right| = \left|\int_{\mathbf{T}} F_r(\xi)\int_{\mathbf{T}} \chi_E(z-\xi)\hat{h}(z)\hat{k}(-z)\,dz\,d\xi\right| \leqslant$$

$$\leqslant \|F_r\|_1 \sup_{\xi\in\mathbf{T}}\left|\int_{\mathbf{T}} \chi_E(z-\xi)\hat{h}(z)\hat{k}(-z)\,dz\right| \leqslant |||\chi_E|||_p\|h\|_p\|k\|_q,$$

and, hence, (iii) follows by the M. Riesz theorem [4; Chapter 6].

To complete the proof of the lemma, let

$$\varphi_r(\gamma) = (\chi_E * F_r)((g, \gamma)), \quad \gamma \in \Gamma,$$

for each $r = 1, 2, \ldots$. Then, by Lemma 1.2 and (iii)

$$\sup\{|||\varphi_r|||_p\,;\ E \in \mathscr{A},\ r = 1,2, \ldots\} < \infty$$

Also, (i) and (ii) imply that

$$\varphi_r(\gamma) \to \varphi'_E(\gamma), \quad r \to \infty,$$

for each $\gamma \in \Gamma$, where

$$\varphi'_E(\gamma) = \begin{cases} \varphi_E(\gamma) & \text{if } (g, \gamma) \neq e^{ia}, e^{ib}. \\ 1/2 & \text{otherwise.} \end{cases}$$

By applying Lemma 1.3, we see that φ'_E has the properties asserted for φ_E. Yet φ'_E differs from φ_E only on the sets

$$\Gamma_a = \{\gamma \in \Gamma\,;\ (g, \gamma) = e^{ia}\}$$

and

$$\Gamma_b = \{\gamma \in \Gamma \; ; (g, \gamma) = e^{ib}\},$$

on which each function is constant, equal to 0, 1/2 or 1. The sets Γ_a and Γ_b are cosets modulo the closed subgroup

$$\Gamma_0 = \{\gamma \in \Gamma \; ; (g, \gamma) = 1\}$$

of Γ. If Γ_0 is open, each of the characteristic functions of Γ_a and Γ_b is the Fourier-Stieljes transform of a measure of total mass 1; each therefore belongs to $M_p(\Gamma)$ and is of norm at most 1. If Γ_0 us not open (hence null) then φ_E and φ'_E agree a.e.. The properties claimed for φ_E therefore follow in both cases from the ones already established for φ'_E.

2. TRANSLATION IN L^p FOR $1 < p < 2$

We assume throughout this section that $p \in (1, 2)$. Let G be a locally compact abelian group and $g \in G$. Then ${}_gT$ denotes the translation operator in $L^p(G)$ defined by

$$({}_gTf)(s) = f(s + g), \quad s \in G,$$

where $f \in L^p(G)$. Since the operator ${}_gT$ is an isometry of $L^p(G)$ onto $L^p(G)$, its spectrum, which is completely determined in [8], is a part of $\mathbf{T}$. If g has finite order, then ${}_gT$ is a pseudo-unitary operator in the space $L^p(G)$, [9; Theorem 2(i)]. Otherwise however, ${}_gT$ is not a pseudo-unitary operator in $L^p(G)$, [9; Theorem 2(ii)]. The aim of this section is to show, however, that every translation operator corresponding to an element of infinite order is an extended pseudo-unitary operator.

Fix an element $g \in G$ of infinite order. For each arc $E \subseteq \mathbf{T}$, the function φ_E defined by (5) belongs to $M_p(\Gamma)$; see Lemma 1.4. Since $\varphi_E^2 = \varphi_E$ for each $E \in \mathscr{A}$, the corresponding operators $P(E) = S[\varphi_E]$ are projections, which clearly commute with ${}_gT$. Furthermore, P is easily seen to be multiplicative on $\mathscr{A}$. Since the element $g \in G$ is assumed fixed, the notation $P(E)$, $E \in \mathscr{A}$, which suppresses the dependence of these operators on g, should not lead to any confusion.

Let $\mathscr{S}$ denote the semiring of all arcs of the form $\{e^{it}; a \leqslant t < b\}$, where $-\infty < a < b < \infty$, and $\mathscr{R}$ denote the ring of sets generated by $\mathscr{S}$. It is easily verified that P is finitely additive on $\mathscr{S}$ and hence has a finitely additive (multiplicative) extension to $\mathscr{R}$, again denoted by P. Furthermore,

$$P(E)_gT = {}_gTP(E), \quad E \in \mathscr{R}.$$

Hence, P is a prospective resolution of the identity for ${}_gT$. As noted above, the family of operators $\{P(E) \; ; E \in \mathscr{A}\}$ is uniformly bounded. However, it is not true in general that $\{P(E)\ ;\ E \in \mathscr{R}\}$ is uniformly bounded; see [18; Example 2.8] and also the construction of §3. This is enough to suggest that P is not extendable to an $L(L^p(G))$-valued spectral measure on $\mathscr{B}$ and, hence, that ${}_gT$ is not pseudo-unitary [9]. Nevertheless we do have:

THEOREM 2.1. *Let $1 < p \leqslant 2$. Let G be a locally compact abelian group and $g \in G$ have infinite order. Then the translation operator ${}_gT$ is an extended pseudo--unitary operator, in the admissible space $L^q(\Gamma)$.*

Proof. Let $Y = L^q(\Gamma)$ and $\iota : L^p(G) \to Y$ denote the Fourier transform map. Then ι is continuous, injective and $\iota(L^p(G))$ is dense in Y, [10; Chapter 8].

Let ${}_gT_Y$ denote the operator given by

$$({}_gT_Y h)(\gamma) = (g, \gamma)h(\gamma), \quad \gamma \in \Gamma,$$

where $h \in Y$. A simple calculation shows that

$${}_gT_Y(\iota f) = \iota({}_gTf), \quad f \in L^p(G).$$

Accordingly, Y is an admissible space for ${}_gT$.

It is clear from the calculation

$$\iota(P(E)f) = (P(E)f)^\wedge = \chi_E((g, \cdot))\hat{f} = \chi_E((g, \cdot))\iota f,$$

valid for each $E \in \mathscr{R}$ and $f \in L^p(G)$, that Y is also an admissible space for each of the operators $P(E)$, $E \in \mathscr{R}$, and that $P_Y(E) \in L(Y)$ is the operator

$$P_Y(E) : h \mapsto \chi_E((g, \cdot))h, \quad h \in Y, \tag{6}$$

for every $E \in \mathscr{R}$. Accordingly, define a set function $P_Y : \mathscr{B} \to L(Y)$ by (6), but now for every Borel set $E \in \mathscr{B}$. Clearly P_Y is multiplicative and $P_Y(\mathbf{T}) = I$. If $h \in Y$ and $\xi \in Y'$, then it is easily verified that

$$\langle P_Y(E)h, \xi \rangle = \int_{(g, \cdot)^{-1}(E)} h(\gamma)\xi(\gamma)\,\mathrm{d}\gamma, \quad E \in \mathscr{B}. \tag{7}$$

Since $h\xi \in L^1(\Gamma)$ it follows that $\langle P_Y h, \xi \rangle$ is σ-additive. Hence P_Y is a spectral measure.

The identity function on $\mathbf{T}$, being bounded and measurable, is certainly P_Y--integrable. It follows from (7) that for each $h \in Y$ and $\xi \in Y'$, the identity

$$\int_{\mathbf{T}} s(t)\mathrm{d}\langle P_Y(t)h, \xi \rangle = \int_{\Gamma} s((g, \gamma))h(\gamma)\xi(\gamma)\,\mathrm{d}\gamma$$

holds for all simple functions s on $\mathbf{T}$ and, hence, for all continuous functions. In particular,

$$\int_{\mathbf{T}} z\,\mathrm{d}\langle P_Y(z)h, \xi\rangle = \int_{\Gamma} (g, \gamma)h(\gamma)\xi(\gamma)\,\mathrm{d}\gamma = \langle {}_gT_Y h, \xi\rangle,$$

for each $h \in Y$ and $\xi \in Y'$. This shows that ${}_gT_Y = \int_{\mathbf{T}} z\,\mathrm{d}P_Y(z)$ is a pseudo-unitary operator with P_Y as its resolution of the identity.

REMARKS. (i) Once it is established that Y is an admissible space for ${}_gT$, then the pseudo-unitary property of ${}_gT_Y$, clearly a scalar-type spectral operator, follows from [3, XVII, Corollary 2.11(ii)]. In this particular case we have chosen to check the relevant properties of P_Y directly.

(ii) We note that if the group G is discrete, then it is possible to treat ${}_gT$ as an extended pseudo-unitary operator in an admissible Hilbert space. This is seen from the fact that the natural inclusion of $L^q(\Gamma)$ into $L^2(\Gamma)$ is continuous, injective and its range is dense in $L^2(\Gamma)$. Furthermore, in the notation of Theorem 2.1, it is clear that the space $L^2(\Gamma)$ is admissible for each of the operators ${}_gT_Y$ and $P_Y(E)$, $E \in \mathscr{B}$, and hence also for the operators ${}_gT$ and $P(E)$, $E \in \mathscr{R}$. It is easily verified that the extension of ${}_gT$ to $L^2(\Gamma)$ so determined is pseudo-unitary.

3. TRANSLATION IN $L^p(\mathbf{R})$ FOR $2 < p < \infty$

Throughout this section we assume that $2 < p < \infty$. Consider the general setting of a locally compact abelian group G and an element $g \in G$, which we may as well suppose to be of infinite order. Otherwise the translation operator ${}_gT$ is pseudo-unitary in $L^p(G)$ anyway, [9; Theorem 2(i)].

For each arc $E \subseteq \mathbf{T}$, let $P(E)$ be the extension of the operator $S[\varphi_E]$ to $L^p(G)$ associated with the p-multiplier φ_E (Lemma 1.4). Then the family of projections $\{P(E)\,;\, E \in \mathscr{A}\}$ so determined is uniformly bounded, multiplicative and commutes with ${}_gT$. Furthermore, P has a finitely additive extension, again denoted by P, from $\mathscr{S}$ to $\mathscr{R}$. Hence, P is a prospective resolution of the identity for ${}_gT$. However, as noted previously, P is not in general extendable to a spectral measure on $\mathscr{B}$ with values in $L(L^p(G))$.

It is natural to expect, in view of the proof of Theorem 2.1, that ${}_gT$ might be an extended pseudo-unitary operator in the space of *quasi-measures* $D'(\Gamma)$, equipped with its weak-star topology (see [7] for the definition and notation), since the Fourier transform map can be defined as a continuous, injective map of $L^p(G)$ into $D'(\Gamma)$. However, since the operation of multiplication of quasi-measures by characteristic functions of Borel subsets of Γ is not defined in general, it seems likely that $D'(\Gamma)$

is not an admissible space for the operators $P(E)$, $E \in \mathscr{R}$, and accordingly, that T is not pseudo-unitary in $D'(\Gamma)$. In fact, we are going to show that if $G = \mathbf{R}$, then the translation operator ${}_gT$ is not an extended pseudo-unitary operator in *any* admissible space; see Theorem 3.3.

The proof of Theorem 2.1 demonstrates that, in the case $1 < p \leqslant 2$, the spectral measure associated with the extended pseudo-unitary operator ${}_gT_Y$ is obtained by extension of the set function $E \mapsto P(E)$ from the ring $\mathscr{R}$ to the σ-algebra $\mathscr{B}$. We wish to show now for $2 < p < \infty$ that if ${}_gT$ is an extended pseudo-unitary operator in any admissible space for ${}_gT$, then the associated spectral measure must arise by extension, from $\mathscr{R}$, of the set function P.

Let g be a fixed element of a locally compact abelian group G. If $F = \{z_1, \ldots, z_n\}$ is a finite set of distinct points in $\mathbf{T}$, then φ_F denotes the characteristic function of the disjoint union of cosets

$$\bigcup_{j=1}^{n} \Gamma_j,$$

where $\Gamma_j = \{\gamma \in \Gamma\ ; (g, \gamma) = z_j\}$, $1 \leqslant j \leqslant n$. We remark that φ_F is a p-multiplier; see the proof of Lemma 1.4.

Let E belong to the ring of sets $\mathscr{R}$. Then there exists a (unique) finite family of pairwise disjoint intervals $I_j = [a_j, b_j)$, $1 \leqslant j \leqslant n$, with $I_j \subseteq [0, 2\pi)$ for each $j = 1, 2, \ldots, n$ and $a_j \neq b_k$ if $j \neq k$, such that

$$E = \bigcup_{j=1}^{n} \{e^{it}\ ; t \in I_j\}. \tag{8}$$

The associated set of left-hand endpoints, $\{e^{ia_j}; 1 \leqslant j \leqslant n\}$, is denoted by $E(l)$ and the right-hand endpoints, $\{e^{ib_j}; 1 \leqslant j \leqslant n\}$, by $E(r)$.

LEMMA 3.1. *Let $2 < p < \infty$, G be a locally compact abelian group and $g \in G$ be an element of infinite order. Suppose that ${}_gT$ is an extended pseudo-unitary operator in an admissible space Y, with associated spectral measure $Q\colon \mathscr{B} \to L(Y)$. Let E be an element of the ring of sets $\mathscr{R}$. If $\iota\colon L^p(G) \to Y$ is the associated imbedding satisfying* (2) *for the operator ${}_gT$, then*

$$\begin{aligned}\langle Q(E)\iota f, \xi\rangle + \frac{1}{2}\langle Q(E(r))\iota f, \xi\rangle - \frac{1}{2}\langle Q(E(l))\iota f, \xi\rangle = \\ = \langle P(E)f, \xi\circ\iota\rangle + \frac{1}{2}\langle S[\varphi_{E(r)}]f, \xi\circ\iota\rangle - \frac{1}{2}\langle S[\varphi_{E(l)}]f, \xi\circ\iota\rangle,\end{aligned} \tag{9}$$

for each $f \in L^p(G)$ *and* $\xi \in Y'$. *Moreover, for each* $\lambda \in \mathbf{T}$,

$$Q(\{\lambda\})\iota f = \iota S[\varphi_{\{\lambda\}}]f, \quad f \in L^p(G),$$

and hence,

$$Q(E)\iota f = \iota P(E)f, \quad f \in L^p(G). \tag{10}$$

Proof. Since both P and Q are finitely additive on $\mathscr{R}$, it suffices to verify (9) for the case of a single arc $E = \{e^{it}; t \in [a, b)\}$.

Since ${}_gT_Y$ is pseudo-unitary with Q as resolution of the identity, we have

$$ {}_gT_Y = \int_{\mathbf{T}} z \, \mathrm{d}Q(z).$$

Then for any integer k, positive or negative, the bounded measurable function $z \mapsto z^k$, $z \in \mathbf{T}$, is Q-integrable and

$$\int_{\mathbf{T}} z^k \, \mathrm{d}Q(z) = {}_gT_Y^k ;$$

see [18; Proposition 1.1] for example. So if $\Phi(z) = \sum_{n=-N}^{N} \alpha_n z^n$ is a trigonometric polynomial on $\mathbf{T}$, then

$$\Phi({}_gT_Y) = \int_{\mathbf{T}} \Phi(z) \, \mathrm{d}Q(z). \tag{11}$$

An application of Lemma 1.1 shows that

$$\Phi({}_gT_Y) = \Phi({}_gT)_Y. \tag{12}$$

Suppose now that $f \in L^p(G)$ and $\xi \in Y'$. Then

$$\langle Q(E)\iota f, \xi \rangle = \int_{\mathbf{T}} \chi_E \mathrm{d}\langle Q\iota f, \xi \rangle. \tag{13}$$

Let $\{F_r\}_{r=1}^{\infty}$ be the sequence of Fejér kernels on $\mathbf{T}$. Then the uniformly bounded sequence of trigonometric polynomials

$$\Phi_r = \chi_E * F_r, \quad r = 1, 2, \ldots,$$

converges pointwise on $\mathbf{T}$ to the function

$$\psi = \chi_E + \frac{1}{2}\chi_{\{e^{ib}\}} - \frac{1}{2}\chi_{\{e^{ia}\}}.$$

Hence, it follows from (11), (12), (13) and the Dominated Convergence Theorem, that

$$\langle Q(E)\imath f, \xi\rangle + \frac{1}{2}\langle Q(\{e^{ib}\})\imath f, \xi\rangle - \frac{1}{2}\langle Q(\{e^{ia}\})\imath f, \xi\rangle =$$

(14) $$= \int_{\mathbf{T}} \psi(z)\, d\langle Q(z)\imath f, \xi\rangle = \lim_{r\to\infty} \int_{\mathbf{T}} \Phi_r(z)\, d\langle Q(z)\imath f, \xi\rangle =$$

$$= \lim_{r\to\infty} \langle \Phi_r({}_gT_\gamma)\imath f, \xi\rangle = \lim_{r\to\infty} \langle \Phi_r({}_gT) f, \xi \circ \imath\rangle.$$

A simple calculation shows that $\Phi_r({}_gT)$ is the operator corresponding to the p-multiplier

$$\varphi_r : \gamma \mapsto \Phi_r((g, \gamma)), \quad \gamma \in \Gamma,$$

for each $r = 1, 2, \ldots$. Since $\sup\{|||\varphi_r|||_p \,;\, r = 1, 2, \ldots\} < \infty$ and $\{\varphi_r\}_{r=1}^{\infty}$ converges pointwise on Γ to the function

$$\varphi_E' = \varphi_E + \frac{1}{2}\chi_{\{e^{ib}\}} - \frac{1}{2}\chi_{\{e^{ia}\}},$$

(see Lemma 1.4 and its proof), it follows from Lemma 1.3 that the last limit in (14) is equal to

$$\langle S[\varphi_E'] f, \xi \circ \imath\rangle = \langle P(E) f, \xi \circ \imath\rangle + \frac{1}{2}\langle S[\varphi_{\{e^{ib}\}}] f, \xi \circ \imath\rangle - \frac{1}{2}\langle S[\varphi_{\{e^{ia}\}}] f, \xi \circ \imath\rangle.$$

Therefore,

$$\langle Q(E)\imath f, \xi\rangle + \frac{1}{2}\langle Q(\{e^{ib}\})\imath f, \xi\rangle - \frac{1}{2}\langle Q(\{e^{ia}\})\imath f, \xi\rangle =$$

$$= \langle P(E) f, \xi \circ \imath\rangle + \frac{1}{2}\langle S[\varphi_{\{e^{ib}\}}] f, \xi \circ \imath\rangle - \frac{1}{2}\langle S[\varphi_{\{e^{ia}\}}] f, \xi \circ \imath\rangle.$$

This proves (9).

It remains to show that

(15) $$Q(\{\lambda\})\imath f = \imath S[\varphi_{\{\lambda\}}] f, \quad f \in L^p(G),$$

for each point $\lambda \in \mathbf{T}$. We shall, for simplicity, treat the case $\lambda = 1$.

Let $f \in C_c * C_c$, where C_c denotes the space of continuous functions on G with compact support. It follows from the Dominated Convergence Theorem for vector measures [14; Theorem II 4.2], that

$$Q(\{1\})\iota f = \lim_{N\to\infty} \int_{\mathbf{T}} (N+1)^{-1} F_N(z)\, \mathrm{d}Q(z)\iota f.$$

It follows from Lemma 1.1 that

$$(N+1)^{-1} \int_{\mathbf{T}} F_N(z)\, \mathrm{d}Q(z)\iota f = (N+1)^{-1} F_N({}_gT_\gamma)\iota f = \iota((N+1)^{-1} F_N({}_gT)f),$$

for each $N = 1, 2, \ldots$. Since ι is continuous, it suffices to show that

$$\lim_{N\to\infty} (N+1)^{-1} F_N({}_gT)f = S[\varphi_{\{1\}}]f, \tag{16}$$

in $L^p(G)$. Now the Fourier transform of

$$(N+1)^{-1} F_N({}_gT)f = (N+1)^{-1} \sum_{n=-N}^{N} (1 - |n|(N+1)^{-1})_g T^n f$$

is the function

$$\begin{aligned} \gamma \mapsto (N+1)^{-1} \sum_{n=-N}^{N} (1 - |n|(N+1)^{-1})(g,\gamma)^n \hat{f}(\gamma) &= \\ = (N+1)^{-1} F_N((g,\gamma))\hat{f}(\gamma), \quad \gamma \in \Gamma, \end{aligned} \tag{17}$$

for each $N = 1, 2, \ldots$. Notice that if $(g,\gamma) = 1$, then each function (17) has the value $\hat{f}(\gamma)$ at γ. For all other values of γ,

$$\lim_{N\to\infty} (N+1)^{-1} F_N((g,\gamma))\hat{f}(\gamma) = 0.$$

Observe that $\hat{f} \in L^q(\Gamma)$ since $f \in C_c * C_c$ and so

$$((N+1)^{-1} F_N({}_gT)f)^\wedge \to \chi_{\{\gamma;\,(g,\gamma)=1\}}\hat{f}$$

in $L^q(\Gamma)$ by the Dominated Convergence Theorem (cf. (17)). It follows from the Hausdorff-Young inequality that (16) is satisfied and hence, that (15) is valid (for $\lambda = 1$) whenever $f \in C_c * C_c$. A routine approximation argument shows that (15) is valid for any $f \in L^p(G)$. This completes the proof of the lemma.

REMARK. Although it is not needed, we remark that Lemma 3.1 is actually valid for all $1 \leq p < \infty$. Of course, a different argument for the validity of (10) is needed when $1 < p < 2$. Furthermore, for any $1 < p < \infty$, if the closed subgroup of G generated by g is isomorphic to $\mathbf{Z}$, then it can be shown that

$$Q(\{\lambda\})\iota f = 0 = \iota S[\varphi_{\{\lambda\}}]f, \quad f \in L^p(G),$$

for each $\lambda \in \mathbf{T}$. Hence, in this case, Q is necessarily a continuous measure on $\mathbf{T}$.

The next lemma is needed in §4. We place it here since it is closely related to Lemma 3.1.

LEMMA 3.2. *Let ${}_gT$, Y and Q be as in Lemma* 3.1. *Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of sets in $\mathscr{R}$, and $\{f_j\}_{j=1}^{\infty} \subset L^p(G)$ be any bounded sequence. Then*

$$\sup_{j \in \mathbf{N}} \left| \langle Q(E_j)\iota f_j, \xi\rangle + \frac{1}{2}\langle Q(E_j(r))\iota f_j, \xi\rangle - \frac{1}{2}\langle Q(E_j(l))\iota f_j, \xi\rangle \right| < \infty,$$

for each $\xi \in Y'$.

Proof. By the triangle inequality, it is clear that the given supremum is not greater than

$$2\sup\{|\langle \iota f_j, Q(E)'\xi\rangle| \; ; E \in \mathscr{B}, j = 1, 2, \ldots\}. \tag{18}$$

But, (18) is finite as $\{\iota f_j \; ; j = 1, 2, \ldots\}$ is a bounded set in Y and $\{Q(E)'\xi \; ; E \in \mathscr{B}\}$ is an equicontinuous part of Y' (since Q has equicontinuous range; see §1).

For the remainder of this section let $G = \mathbf{R}$ and, without loss of generality, take $g = -1$. Then the corresponding translation operator ${}_{-1}T$, which we write simply as T, is given by

$$(Tf)(s) = f(s-1), \quad s \in \mathbf{R},$$

for each $f \in L^p(\mathbf{R})$.

THEOREM 3.3. *Suppose that $2 < p < \infty$. The operator T is not an extended pseudo-unitary operator.*

Proof. If T were an extended pseudo-unitary operator in an admissible space Y, with associated spectral measure $Q : \mathscr{B} \to L(Y)$, then it follows from (10) that

$$\langle Q(E)\iota f, \xi\rangle = \langle P(E)f, \xi \circ \iota\rangle, \quad E \in \mathscr{R}, \tag{19}$$

for each $f \in L^p(\mathbf{R})$ and $\xi \in Y'$, where $\iota : L^p(\mathbf{R}) \to Y$ is the associated imbedding satisfying (2). Since $\{\xi \circ \iota \; ; \xi \in Y'\}$ is a total subspace of $L^q(\mathbf{R})$ and Q is equicontinuous, it is clear from (19) that Theorem 3.3 is proved if we verify the following construction.

LEMMA 3.4. *If h is a non-zero element of $L^q(\mathbf{R})$, there exists a sequence of sets $E_j \in \mathscr{R}$, and a sequence of elements $f_j \in L^p(\mathbf{R})$, $j = 1, 2, \ldots$, satisfying $\|f_j\|_p \leqslant 1$ for each j, such that*

$$\lim_{j \to \infty} |\langle P(E_j)f_j, h\rangle| = \infty.$$

Construction. Let ψ denote the kernel on $\mathbf{Z}$ given by $\psi(0) = 1/2$ and

$$\psi(n) = \begin{cases} 0 & \text{if } n \neq 0 \text{ is even} \\ \mathrm{i}/\pi n & \text{if } n \text{ is odd.} \end{cases}$$

Then $\hat{\psi}$, which is the characteristic function of the arc $\{e^{it} ; 0 \leqslant t \leqslant \pi\}$, is an l-multiplier on $\mathbf{T}$ for all $l \in (1, \infty)$.

Let N be a positive integer. Define a function $K_0 : \mathbf{Z} \to \mathbf{C}$ by

$$K_0(j) = (e^{\mathrm{i}2\pi j 2^{-N}} - 1)\psi(j), \quad j \in \mathbf{Z}.$$

Then,

$$|K_0(j)| = 2|\sin \pi j 2^{-N}| \cdot |\psi(j)|, \quad j \in \mathbf{Z}.$$

Furthermore, K_0 is the function given by

$$\hat{K}_0(e^{it}) = \begin{cases} 1 & \text{if } \pi \leqslant t \leqslant \pi + 2\pi 2^{-N} \\ -1 & \text{if } 0 \leqslant t \leqslant 2\pi 2^{-N} \\ 0 & \text{elsewhere.} \end{cases}$$

Beginning with the pair of functions K_0 and $L_0 = K_0$, we carry out a *Rudin-Shapiro construction*. If γ denotes the character of $\mathbf{Z}$ given by

$$\gamma(j) = e^{\mathrm{i}2\pi j 2^{-N}}, \quad j \in \mathbf{Z},$$

then γ_n denotes the character γ^{2^n} for each $n = 0, 1, 2, \ldots$. Define functions K_n and L_n on $\mathbf{Z}$ inductively by

$$K_{n+1} = K_n + \gamma_n L_n$$

and

$$L_{n+1} = K_n - \gamma_n L_n,$$

for each $n = 0, 1, 2, \ldots$. It follows (from the parallelogram law) that

$$|K_{n+1}(j)|^2 + |L_{n+1}(j)|^2 = 2(|K_n(j)|^2 + |L_n(j)|^2), \quad j \in \mathbf{Z},$$

for each $n = 0, 1, 2, \ldots$, and hence, that

$$|K_{n+1}(j)|^2 + |L_{n+1}(j)|^2 = 2^{n+1}|K_0(j)|^2, \quad j \in \mathbf{Z},$$

for each $n = 0, 1, 2, \ldots$. In particular,

$$(20) \qquad |K_N(j)|^2 + |L_N(j)|^2 = 2^{N+3}|\psi(j)|^2 \sin^2(\pi j 2^{-N}), \quad j \in \mathbf{Z}.$$

LEMMA 3.5. *There exists a positive constant β (independent of N) such that*

$$\sum_{j \in \mathbf{Z}} (|K_N(j)|^q + |L_N(j)|^q) \geqslant \beta 2^{N(1-(1/2)q)}.$$

Proof. It follows from (20) and the fact that $q < 2$, that

$$(|K_N(j)|^q + |L_N(j)|^q)^{1/q} \geqslant 2^{(N+3)/2}|\psi(j)|\,|\sin \pi j 2^{-N}|, \quad j \in \mathbf{Z},$$

and, hence, that

$$|K_N(j)|^q + |L_N(j)|^q \geqslant 2^{Nq/2}|\psi(j)|^q\,|\sin \pi j 2^{-N}|^q, \quad j \in \mathbf{Z}.$$

It follows from the inequality $t^{-1}\sin t \geqslant 2/\pi$, $t \in [0, \pi/2]$, that

$$\sum_{j \in \mathbf{Z}} (|K_N(j)|^q + |L_N(j)|^q) \geqslant \sum_{j=1}^{\infty} 2^{Nq/2}|\psi(j)|^q\,|\sin \pi j 2^{-N}|^q \geqslant$$

$$\geqslant \pi^{-q} 2^{Nq/2} \sum_{j=1}^{2^{N-2}} (2j-1)^{-q}|\sin(2j-1)\pi 2^{-N}|^q \geqslant$$

$$\geqslant \pi^{-q} 2^{Nq/2} \sum_{j=1}^{2^{N-2}} (2j-1)^{-q}(2\pi^{-1}\cdot(2j-1)\pi 2^{-N})^q =$$

$$= \beta 2^{N(1-(1/2)q)},$$

where $\beta = \pi^{-q}2^{q-2}$. This completes the proof of Lemma 3.5.

It follows from Lemma 3.5 that we may construct a sequence $\{U_N\}_{N=1}^{\infty}$ of kernels in such a way that each U_N is either K_N or L_N and

$$\Big(\sum_{j \in \mathbf{Z}} |U_N(j)|^q\Big)^{1/q} \geqslant \left(\frac{1}{2}\beta\right)^{1/q} 2^{N(q^{-1}-1/2)}.$$

Since $q^{-1} > 1/2$, it follows that the kernels $W_N = (1/2)(U_N + \delta_0)$, $N \geqslant 1$, satisfy

$$\Big(\sum_{j \in \mathbf{Z}} |W_N(j)|^q\Big)^{1/q} \to \infty, \quad N \to \infty.$$

Furthermore, the function $\hat{W}_N$ takes only the values 0, 1 and is constant on arcs corresponding to intervals of length $(2\pi)2^{-N}$, for each $N \geqslant 1$. This is because the translates of the function $\hat{K}_0$ by the amounts $e^{i2\pi k/2^N}$, $k = 0, 1, 2, \ldots, 2^{N-1} - 1$, have disjoint supports.

LEMMA 3.6. *Let N be a positive integer and* $2 < p < \infty$. *For each* $s = 1, 2, \ldots$ *denote by* $W_{N,s}$ *the kernel on* $\mathbf{R}$

$$W_{N,s} = \sum_{n \in \mathbf{Z}} W_N(n)\delta_{ns}.$$

Denote by $\Phi_{N,s}$ *the function defined pointwise on* $\mathbf{R}$ *by the formula*

$$\Phi_{N,s}(y) = \lim_{r \to \infty} (\hat{W}_N * F_r)(e^{isy}), \tag{21}$$

where $\{F_r\}_{r=1}^{\infty}$ *is the sequence of Fejér kernels on* $\mathbf{T}$. *Then*

(i) $\Phi_{N,s} \in M_q(\mathbf{R})$ *and there exits a constant* K_N *such that*

$$|||\Phi_{N,s}|||_q \leqslant K_N,$$

for all s.

(ii)
$$S[\Phi_{N,s}](f) = W_{N,s} * f,$$

for all continuous functions f with compact support.

(iii) *If h is an arbitrary element of* $L^q(\mathbf{R})$, *then*

$$\lim_{s \to \infty} \| S[\Phi_{N,s}](h) \|_q = \|h\|_q \left(\sum_{n \in \mathbf{Z}} |W_N(n)|^q \right)^{1/q}.$$

Proof. That the pointwise limit in (21) exists follows from classical properties of Cesàro summability. Statement (i) follows from Lemmas 1.2—1.4. Indeed the statement (i) is valid not just for the index q but for all indices in the range $(1, \infty)$, since $\hat{W}_N$ is a Fourier multiplier on the circle group for all such indicies.

It is now a simple calculation using Parseval's formula and the Dominated Convergence Theorem to verify (ii).

To prove (iii), suppose that h_0 is a continuous function of compact support. Then by (ii),

$$\| S[\Phi_{N,s}](h_0) \|_q = \| W_{N,s} * h_0 \|_q = \|h_0\|_q \left(\sum_{n \in \mathbf{Z}} |W_N(n)|^q \right)^{1/q}$$

for all sufficiently large s, so (iii) is immediate when $h = h_0$. In the general case, let $\varepsilon > 0$ be given. Then we can choose a function h_0 as above satisfying $\|h - h_0\|_q < \varepsilon$. By (i),

$$\| S[\Phi_{N,s}](h - h_0) \|_q \leqslant \varepsilon |||\Phi_{N,s}|||_q \leqslant K_N \varepsilon.$$

Then (iii) follows by a routine approximation argument.

Conclusion of proof of Lemma 3.4. It follows from Lemma 3.6 that, given a nonzero element $h \in L^q(\mathbf{R})$, we can choose a sequence of kernels $\{V_j\}_{j=1}^{\infty}$ on $\mathbf{R}$, each of the form $W_{N,s}$, s being appropriately large for the chosen N, such that:

(a) for each j, there is a set E_j which is a finite disjoint union of arcs in $\mathbf{T}$ such that

$$\varphi'_{E_j}(y) \equiv \hat{V}_j(y) = \lim_{r\to\infty} (\chi_{E_j} * F_r)(\mathrm{e}^{iy})$$

pointwise a.e. (cf. proof of Lemma 1.4 for the notation); and

$$\text{(b)} \qquad \|V_j * \check{h}\|_q \to \infty \quad \text{as } j \to \infty,$$

where $\check{h}(x) = h(-x)$. Hence, there exists a sequence $\{f_j\}_{j=1}^{\infty}$ in the unit ball of $L^p(\mathbf{R})$ such that

$$|\langle S[\varphi'_{E_j}]f_j, h\rangle| = \left|\int_{\mathbf{R}} f_j(-x) V_j * \check{h}(x)\,\mathrm{d}x\right| \to \infty$$

as $j \to \infty$. But

$$S[\varphi'_{E_j}]f_j = P(E_j)f_j$$

since, if λ is an end-point of an arc of E_j (i.e. $\lambda \in E_j(l) \cup E_j(r)$) the set $\{y\,;\, \mathrm{e}^{iy} = \lambda\}$ is of measure zero. Therefore, $|\langle P(E_j)f_j, h\rangle| \to \infty$. Lemma 3.4 is thus established, and the proof of Theorem 3.3 is complete.

4. TRANSLATIONS IN ARBITRARY GROUPS FOR $2 < p < \infty$

The proof that the translation operator $T = {}_{-1}T$ of §3 is not an extended pseudo-unitary operator clearly exploits the noncompactness of the group $\mathbf{R}$. We assemble in this section a number of observations which show that what really counts is the compactness or otherwise of the closed group $\langle \bar{g} \rangle$ generated by the element g of G: the operator ${}_gT$ is extended pseudo-unitary precisely when $\langle \bar{g} \rangle$ is compact. Our treatment deals with the case where $\langle \bar{g} \rangle$ is metrizable, but that restriction can probably be removed. In any event, our intention is not to strive for the utmost generality but to illustrate the phenomena of significance: the distinction between the cases $2 < p < \infty$ and $1 < p \leqslant 2$, and the effect of compactness of the group $\langle \bar{g} \rangle$ on the extended pseudo-unitariness of the operator ${}_gT$.

LEMMA 4.1. *Let G be a locally compact abelian group and $g \in G$. Then the group generated (algebraically) by g is either closed and topologically isomorphic to $\mathbf{Z}$, or has compact closure.*

This lemma is a particular case of Theorem (9.1) of [10].

THEOREM 4.2. *Let G be a locally compact abelian group, and suppose that $g \in G$ generates a subgroup isomorphic to* $\mathbf{Z}$. *If $2 < p < \infty$, then the translation operator ${}_gT$ is not extended pseudo-unitary.*

Proof. This follows the same lines as that given in §3 for the line group $\mathbf{R}$, but with some new features. We remark that the construction in §3 is in the first instance a construction for the integer group $\mathbf{Z}$, and it is only in the final stage that the kernels carried by $\mathbf{Z}$ are transferred to the group $\mathbf{R}$ (cf. Lemma 3.6). In the more general case, the transfer Lemma 3.6 has to be modified in a natural way. The kernels $W_{N,s}$ are defined as

$$W_{N,s} = \sum_{n \in \mathbf{Z}} W_N(n)\delta_{nsg}.$$

The formula (21) then has the form

$$\Phi_{N,s}(\gamma) = \lim_{r \to \infty} (\hat{W}_N * F_r)((sg, \gamma)), \quad \gamma \in \Gamma.$$

The statements (i) – (iii) of Lemma 3.6 then hold in the general setting.

The remainder of the construction following Lemma 3.6 goes as before. Notice however that, with the same notation as used there, we may *not* have that $S[\varphi'_{E_j}]f_j$ is equal to $P(E_j)f_j$ since end points of arcs in $\mathbf{T}$ may have pre-images in Γ, under the mapping $\gamma \mapsto (g, \gamma)$, $\gamma \in \Gamma$, that are of infinite measure. However, this possibility is taken care of by reference to the formula (9) and Lemma 3.2. Notice that for a given $h = \xi \circ \imath \in L^q(G)$, the sequence of sets $\{E_j\}_{j=1}^{\infty}$ and the functions f_j, $j = 1, 2, \ldots$, were constructed so that (cf. (9))

$$|\langle S[\varphi'_{E_j}]f_j, h\rangle| = |\langle Q(E_j)\imath f_j, \xi\rangle + \frac{1}{2}\langle Q(E_j(r))\imath f_j, \xi\rangle - \frac{1}{2}\langle Q(E_j(l))\imath f_j, \xi\rangle|$$

tends to ∞ as $j \to \infty$. This contradicts Lemma 3.2.

REMARK. As noted earlier, the bilateral unit shift in $l^p(\mathbf{Z})$, $1 < p \leqslant 2$, is an extended pseudo-unitary operator [18; Example 2.8]. Of course, this follows also from Theorem 2.1. The situation when $2 < p < \infty$ is a simple consequence of Theorem 4.2; the bilateral unit shift is not an extended pseudo-unitary operator in this case.

A partial converse to Theorem 4.2 is the following:

THEOREM 4.3. *Let $2 < p < \infty$. Let G be a compact abelian group and g be an element of G. Then the Hilbert space $Y = L^2(G)$ is admissible for each of the operators ${}_gT$ and $P(E)$, $E \in \mathscr{R}$, and ${}_gT$ is an extended pseudo-unitary operator in Y; its resolution of the identity in Y is the extension to $\mathscr{B}$ of the set function*

$$P_Y : E \mapsto P(E)_Y, \quad E \in \mathscr{R}.$$

Proof. If $\iota : L^p(G) \to L^2(G)$ denotes the natural inclusion map, then clearly $Y = L^2(G)$ is an admissible space for ${}_gT$; the operator ${}_gT_Y$ given by

$${}_gT_Y f = f(\cdot + g), \quad f \in Y,$$

certainly satisfies (2) for ${}_gT$. Since each translation operator ${}_gT_Y$ is unitary in the Hilbert space $L^2(G)$, the proof is complete.

The remainder of our treatment deals with the case where the group $\langle \overline{g} \rangle$ is metrizable.

Let G be a locally compact abelian group and G_0 be a closed subgroup of G. A subset B of G is said to be a *Borel section* for the quotient group G/G_0 if B is a Borel measurable subset of G and each coset of G_0 in G contains precisely one point of B. We call the *associated transversal mapping* τ the 1-1 mapping of G/G_0 onto B such that

$$\tau(b + G_0) = b, \quad b \in B.$$

The following result is a corollary of [5; Theorem 1].

LEMMA 4.4. *Let G be a locally compact abelian group and G_0 a compact metrizable subgroup of G. Then there is a Borel section B for G/G_0 whose associated transversal mapping is Borel measurable from G/G_0 onto B.*

Suppose now that G and G_0 are as in Lemma 4.4. The associated transversal mapping τ induces an identification of the Haar measure on G/G_0 with a measure on B which we denote by db. Consider then the mapping ρ of $B \times G_0$ onto G defined by the formula

$$\rho(b, h) = b + h, \quad (b, h) \in B \times G_0.$$

The mapping ρ is a Borel isomorphism of $B \times G_0$ onto G since the transversal mapping τ is Borel measurable.

Let $2 < p < \infty$. It follows from [10; Theorem (28.54)] that if f is Borel measurable on G and belongs to $L^p(G)$, then the composition $f \circ \rho \in L^p(B \times G_0)$, and

$$\int_G |f|^p \, dg = \int_B \int_{G_0} |f(b + h)|^p \, dh \, db. \tag{22}$$

Since G_0 is compact, and hence of finite measure,

$$\int_B \left(\int_{G_0} |f(b + h)|^2 \, dh \right)^{p/2} db < \infty.$$

Let Y denote the space of (equivalence classes of) Borel measurable functions F on $B \times G_0$ such that

(23) $$\|F\| = \left(\int_B \left(\int_{G_0} |F(b, h)|^2 \, dh\right)^{p/2} db\right)^{1/p} < \infty.$$

Let $\imath$ be the mapping of $L^p(G)$ into Y given by the formula

$$(\imath f)((b, h)) = f(b + h), \quad (b, h) \in B \times G_0,$$

for each $f \in L^p(G)$.

Finally, if $g \in G$ and $G_0 = \langle \bar{g} \rangle$ is a compact metrizable subgroup of G, let ${}_g t$ be the translation operator by amount g acting in $L^2(G_0)$, and write ${}_g T_Y$ for the operator on Y given by

(24) $$[{}_g T_Y(F)](b, h) = F(b, h + g) = [{}_g t F(b, \cdot)](h), \quad (b, h) \in B \times G_0,$$

for each $F \in Y$.

LEMMA 4.5. *Let $2 < p < \infty$, G be a locally compact abelian group and $g \in G$ such that the subgroup $G_0 = \langle \bar{g} \rangle$ is compact and metrizable. Then the space Y with norm defined by* (23) *is an admissible Banach space for the operator ${}_g T$, and the operator ${}_g T_Y$ defined in* (24) *satisfies*

$$({}_g T_Y)\imath f = \imath\, {}_g T f, \quad f \in L^p(G).$$

Proof. The formula (23) defines a norm on Y and Y can be shown to be a Banach space by using standard measure theory arguments. It remains only to show that $\imath(L^p(G))$ is dense in Y.

If $F \in Y$ and $\varepsilon > 0$, there exist a set B_0 of finite measure in B and a bounded measurable function F_0 on $B \times G_0$ such that F_0 vanishes off $B_0 \times G_0$ and $\|F - F_0\| < \varepsilon$. Let $f_0 = F_0 \circ \rho^{-1}$. This is a Borel measurable function on G such that $\imath(f_0) = F_0$. It is clear from the formula (22) that $f_0 \in L^p(G)$. This completes the proof.

THEOREM 4.6. *Let G be a locally compact abelian group and suppose $2 < p < \infty$. Let $g \in G$ be an element of infinite order such that $G_0 = \langle \bar{g} \rangle$ is compact and metrizable. Then the operator ${}_g T$ is extended pseudo-unitary.*

Proof. Consider first of all the unitary operator ${}_g t$ on $L^2(G_0)$. So there exists a spectral measure Q in $L^2(G_0)$ such that

(25) $${}_g t = \int_{\mathbf{T}} z \, dQ(z).$$

The set function Q has an explicit description: if E is a Borel subset of $\mathbf{T}$, then $Q(E)$ is the operator corresponding to the 2-multiplier $\chi_E((g, \cdot))$.

Let B be a Borel section for G/G_0 (cf. Lemma 4.4) and Y the Banach space of Lemma 4.5. Now define the set function Q_Y, with values in $L(Y)$, by stipulating that

$$[Q_Y(E)F](b, h) = [Q(E)F(b, \cdot)](h), \quad (b, h) \in B \times G_0,$$

for each $F \in Y$ and $E \in B$. In other words, the action of $Q_Y(E)$ on $F \in Y$ corresponds to holding fixed each $b \in B$ for which $F(b, \cdot) \in L^2(G_0)$ and acting with $Q(E)$ on the resulting function on G_0. It is necessary to show that this component-wise definition of $Q_Y(E)$ has sense. This involves checking a number of measure theoretic statements. The kind of arguments involved are written out in detail for a similar procedure in Lemma 1.3.2 of [4], and will not be repeated here.

Observe that each operator $Q_Y(E)$, $E \in \mathscr{B}$, is of norm at most 1; that if $H \in Y'$, then H can be identified with a measurable function on $B \times G_0$ for which

$$\int_B \left(\int_{G_0} |H(b, h)|^2 \, dh \right)^{q/2} db < \infty;$$

and finally that

$$\langle Q_Y(E)F, H \rangle = \int_B \langle Q(E)F(b, \cdot), H(b, \cdot) \rangle \, db, \quad F \in Y. \tag{26}$$

It follows from (26), the uniform boundedness of the family of operators $\{Q(E) ; E \in \mathscr{B}\}$ and the Orlicz-Pettis lemma, that Q_Y is a spectral measure, necessarily equicontinuous as Y is a Banach space.

Finally we must show that if ${}_gT_Y$ is defined by (24), then

$${}_gT_Y = \int_{\mathbf{T}} z \, dQ_Y(z).$$

It follows from (24) that this involves proving

$$\int_B \langle {}_gtF(b, \cdot), H(b, \cdot) \rangle \, db = \int_{\mathbf{T}} z \, d\langle Q_Y(z)F, H \rangle,$$

for each $F \in Y$ and $H \in Y'$. For the operator ${}_gt$ we have the spectral representation (25). So we have to prove that

$$\int_B \left(\int_{\mathbf{T}} z \, d\langle Q(z)F(b, \cdot), H(b, \cdot) \rangle \right) db = \int_{\mathbf{T}} z \, d\langle Q_Y(z)F, H \rangle. \tag{27}$$

Now if E is an arc in $\mathbf{T}$, then it follows from (26) that

$$\int_B \left(\int_{\mathbf{T}} \chi_E \mathrm{d}\langle QF(b, \cdot), H(b, \cdot)\rangle \right) \mathrm{d}b = \int_B \langle Q(E)F(b, \cdot), H(b, \cdot)\rangle \, \mathrm{d}b = \int_{\mathbf{T}} \chi_E \mathrm{d}\langle Q_Y F, H\rangle.$$

Hence,

$$\int_B \left(\int_{\mathbf{T}} s \, \mathrm{d}\langle QF(b, \cdot), H(b, \cdot)\rangle \right) \mathrm{d}b = \int_{\mathbf{T}} s \, \mathrm{d}\langle Q_Y F, H\rangle, \tag{28}$$

for all functions s that are simple relative to the ring of sets $\mathscr{R}$. If $s = \sum \alpha_j \chi_{E_j}$, then the left-hand side of (28) equals

$$\int_B \sum \alpha_j \langle Q(E_j)F(b, \cdot), H(b, \cdot)\rangle \, \mathrm{d}b.$$

Assuming that $\|s\|_\infty \leqslant 1$, we have

$$\begin{aligned} |\textstyle\sum \alpha_j \langle Q(E_j)F(b, \cdot), H(b, \cdot)\rangle| &= |\langle \textstyle\sum \alpha_j Q(E_j)F(b, \cdot), H(b, \cdot)\rangle| \leqslant \\ &\leqslant \|F(b, \cdot)\|_2 \|H(b, \cdot)\|_2 \end{aligned} \tag{29}$$

since $\sum \alpha_j Q(E_j)$ is an L^2-multiplier operator of norm at most 1. Choose a sequence $\{s_n\}_{n=1}^\infty$ of simple functions based on $\mathscr{R}$ that converges uniformly to the identity function on $\mathbf{T}$ and satisfies $\|s_n\|_\infty \leqslant 1$, $n = 1, 2, \ldots$. Since

$$\int_{\mathbf{T}} s_n \mathrm{d}\langle QF(b, \cdot), H(b, \cdot)\rangle \to \int_{\mathbf{T}} z \, \mathrm{d}\langle Q(z)F(b, \cdot), H(b, \cdot)\rangle, \quad n \to \infty,$$

for almost all b, we conclude from (29) and the Dominated Convergence Theorem that the left-hand side of (28), with s replaced by s_n, converges to the left-hand side of (27). At the same time the right-hand side of (28), with s replaced by s_n, converges to the right-hand side of (27). This completes the proof.

REMARK. It is clear from the results presented that in many cases certain non-trivial translations in $L^p(G)$, although not pseudo-unitary in the space $L^p(G)$ itself, can nevertheless be treated as pseudo-unitary operators in some natural admissible space for the operator, containing $L^p(G)$. As such, they still fall into the class of operators which can be treated by the classical methods of spectral theory introduced by N. Dunford. However, it was also established that when $2 < p < \infty$ there is a large class of translation operators which is not covered by the classical theory of spectral operators. Any integral representation of the form (1) for such operators can never be interpreted as being with respect to some σ-additive measure on a σ-algebra. A more extensive theory of integration is needed to give a satisfactory treatment of the spectral properties for operators of this type, such as that based on the notion of spectral family rather than spectral measure, [1; §4], [9]. The theory of integration with respect to measures of infinite variation, recently developed in [13], appears also to provide a suitable framework for such problems.

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Received December 15, 1983; revised August 8, 1984.