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WIENER-HOPF OPERATORS WITH PIECEWISE CONTINUOUS ALMOST PERIODIC SYMBOL

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INTRODUCTION

Several authors have studied the index theories of different types of singular integral operators with piecewise continuous symbol, see [11], [12], [14], [15]. Basically, their results are generalizations of corresponding theory for the same type of operators with continuous symbol. What makes significant difference is that in the noncontinuous case the discontinuities of the symbol represent a part of the essential spectrum for the corresponding operator and become a part of the curve in the index counting. In the context of von Neumann algebra, these results may be categorized as type I_∞ theory. In the present paper, we shall develop a type II_∞ analogue.

It is well known that there is an index theory for Wiener-Hopf operators with continuous almost periodic symbol where the analytical index is obtained via a faithful representation in a type II_∞ factor $\mathcal{N}$ (see [6], [7], [4]). We are interested in adding discontinuities to these almost periodic symbols. We know that every continuous almost periodic function is identified with its Gelfand transform on $\mathbf{R}^{\mathrm{B}}$, the Bohr compactification of $\mathbf{R}$. Also, there is an obvious way to identify each piecewise continuous periodic function with an element in $L^\infty(\mathbf{R}^{\mathrm{B}})$. We simply mix these two types of functions to produce a C^*-algebra $\mathcal{S}$, which is also considered as the uniform closure of all the piecewise continuous almost periodic functions. The main object of the study is the C^*-algebra $\mathcal{A}(\mathcal{S})$ generated by Wiener-Hopf operators on $H^2(\mathbf{R})$ with symbols in $\mathcal{S}$. For each $W_\varphi \in \mathcal{A}(\mathcal{S})$, the symbol φ is regarded as a function on $\mathbf{R}^{\mathrm{B}}$. Intuitively, since the von Neumann algebra $\mathcal{N}$ comes from the group measure space construction on $\mathbf{R}\times\mathbf{R}_{\mathrm{d}}$, it should be large enough to accommodate a faithful representation of $\mathcal{A}(\mathcal{S})$. Indeed it does. What we shall do is to extend the representation ρ used in [7] to $\mathcal{A}(\mathcal{S})$. Then the Breuer index for $\mathcal{N}$ serves as an analytical index for $\mathcal{A}(\mathcal{S})$.

In order to characterize the Fredholmness of $A = \sum_j \prod_k W_{\varphi_{jk}} \in \mathscr{A}(\mathscr{S})$, we shall introduce the symbol $\mathfrak{s}(A) = \sum_j \prod_k \varphi^{\#}_{jk}$, defined on $\mathbf{R} \times [0, 1]$ for A. Thus the discontinuities of the individual symbols φ_{jk} become visible as a part of the range of $\mathfrak{s}(A)$. The calculation of the essential spectrum is carried out through the localization of C^*-algebra introduced in [9]. We shall show in the text that if φ, ψ have discontinuities of different periods then W_φ and W_ψ represent elements of completely different types in the local algebra. So where the discontinuities are mixed, the spectrum of local operator is extremely complicated. But our interest at this stage is not to tackle the kind of complication but just to explore possible generalizations of existing index theory. Therefore we shall restrict the attention to certain subalgebras of $\mathscr{A}(\mathscr{S})$ in the present paper and defer the difficult cases for further investigation. In Section 7, we fully explain the remaining problems.

The rest of the paper is arranged as follows. In Section 1, we explain in detail the function algebra $\mathscr{S}$ whose elements we shall regard as piecewise continuous almost periodic function. We construct the C^*-algebra $\mathscr{A}(\mathscr{S})$ and extend the representation ρ in Sections 2 and 3. Then we prove that the commutator ideal of $\mathscr{A}(\mathscr{S})$ is mapped by ρ into the ideal in $\mathscr{N}$ generated by the trace class elements in Section 4. Section 5 is devoted to the calculation of the local spectra and essential spectra for certain operators. In Section 6, we prove the existence of mean motion for certain symbols and identify minus the mean motion with analytical index.

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1. ALMOST PERIODIC FUNCTIONS

We shall denote by $\mathrm{CAP}(\mathbf{R})$ the continuous almost periodic functions. The maximal ideal space $\mathbf{R}^{\mathrm{B}}$ of $\mathrm{CAP}(\mathbf{R})$ is called the Bohr compactification of $\mathbf{R}$. $\mathbf{R}^{\mathrm{B}}$ has a natural group structure induced by that of $\mathbf{R}$ and $\mathbf{R}$ with its original topology and additions is a dense subgroup of $\mathbf{R}^{\mathrm{B}}$ (see [16], §1). For each $f \in \mathrm{CAP}(\mathbf{R})$, its Gelfand transform $\hat{f} \in C(\mathbf{R}^{\mathrm{B}})$ is considered as the extension of f to the whole $\mathbf{R}^{\mathrm{B}}$ and, for simplicity, we shall write $f(x)$ $(=\hat{f}(x))$ for $x \in \mathbf{R}^{\mathrm{B}}$. Also, for each $f \in \mathrm{CAP}(\mathbf{R})$, the limit

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, \mathrm{d}t = M(f)$$

exists and is called the mean of f. $M(\cdot)$ is a positive functional on $\mathrm{CAP}(\mathbf{R}) = \mathrm{C}(\mathbf{R}^{\mathrm{B}})$. As a matter of fact, the measure associated with $M(\cdot)$ is the normalized Haar measure on $\mathbf{R}$, which we denote by m. Thus the Gelfand transform of $\{f_n\} \subset \mathrm{CAP}(\mathbf{R})$

is a Cauchy sequence in $L^2(\mathbf{R}^B, m)$ if and only if

$$\lim_{L\to\infty} \sup\{M(|f_k - f_n|^2) : n, k \geqslant L\} = 0$$

(see [1]). The dual group of $\mathbf{R}^B$ is identified with $\mathbf{R}_d$, the real line with discrete topology. In other words, each $\lambda \in \mathbf{R}_d$ is identified with $\lambda(t)$ ($= e^{i\lambda t}$ for $t \in \mathbf{R}$).

A function η defined on $\mathbf{R}$ is said to be a piecewise continuous periodic function if

(i) there exists $a \in \mathbf{R}$, $a \neq 0$, such that $\eta(t + a) = \eta(t)$ for all $t \in \mathbf{R}$;

(ii) η is continuous on $[0, a + 1]$ except possibly a finite number of points $\tau_1, \ldots, \tau_k$;

(iii) at these exceptional points, $\lim_{t\to\tau_j^{\pm}} \eta(t)$ exist, $j = 1, \ldots, k$, and, as a convention, $\lim_{t\to\tau_j^{+}} \eta(t) = \eta(\tau_j)$.

The algebra of symbols, $\mathscr{S}$, is the uniform closure of the algebra generated by CAP($\mathbf{R}$) and all the piecewise continuous periodic functions described above. We shall describe a dense subalgebra of $\mathscr{S}$ whose elements are relatively simple.

It is easy to see that a piecewise continuous periodic function η is the sum $\eta_1 + \ldots + \eta_k$ where each η_j, $j = 1, \ldots, k$, is a piecewise continuous periodic function such that each period contains only one jump discontinuity. Let η be a piecewise continuous periodic function with period $2a$ and on $(-a-\varepsilon, a+\varepsilon)$ η has only one discontinuity at 0. Straightforward calculation shows that η has a Fourier series expansion

$$\eta(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt\pi/a} + c \sum_{n\neq 0} \frac{1}{n} e^{int\pi/a}$$

where $\sum a_k e^{ikt\pi/a}$ is the Fourier series of a continuous periodic function. For $\omega, b \in \mathbf{R}$, let

$$\eta_{\omega,b}(t) = \sum_{n\neq 0} \frac{1}{n} e^{i2n\pi(t-b)/\omega}$$

for $t \in (b - \omega,\ b + \omega)\setminus\{b\}$ and, as definition,

$$\eta_{\omega,b}(b) = \lim_{t\to b^+} \eta_{\omega,b}(t).$$

By the Fourier series expansion argument used above, we can also conclude that there exists a constant c such that

$$\eta^2_{\omega,b}(t) - c\eta_{\omega,b}(t)$$

is continuous. For $\omega, b \in \mathbf{R}$, $\omega/2 > \delta > 0$, let $\xi_{\omega,b,\delta}$ be a continuous periodic function such that

$$\xi_{\omega,b,\delta} \Big| \bigcup_{n\in\mathbf{Z}} [n\omega + b - \delta/2, n\omega + b + \delta/2] = 1,$$

$$\xi_{\omega,b,\delta} \Big| \{\mathbf{R} \setminus \bigcup_{n\in\mathbf{Z}} [2n\omega + b - \delta,\ 2n\omega + b + \delta]\} = 0$$

and

$$0 \leqslant \xi_{\omega,b,\delta} \leqslant 1.$$

Thus for any $p \in \mathbf{Z}$, it is easy to see that $\eta_{p\omega,b} - \eta_{\omega,b}\xi_{p\omega,b,\omega/4}$ is a continuous function. Therefore $\eta_{p\omega,b} \times \eta_{q\omega,c} = \eta_{\omega,b} \times \eta_{q\omega,c} \times f_1 + \eta_{q\omega,c} f_2 = \eta_{\omega,b}\eta_{\omega,c} g_1 + \eta_{\omega,b} g_2 + \eta_{q\omega,c} f_2$ where f_1, f_2, g_1 and g_2 are continuous periodic functions. If $n\omega + b = m\omega + c$ for some $n, m \in \mathbf{Z}$, then $\eta_{\omega,b}(t) = \eta_{\omega,0}(t-b) = \eta_{\omega,0}(t - m\omega + n\omega - c) = \eta_{\omega,0}(t-c) = \eta_{\omega,c}(t)$. In this case $\eta_{\omega,b}\eta_{\omega,c} = \eta^2_{\omega,b} = h_0 + h_1\eta_{\omega,b}$ where h_1 and h_2 are continuous periodic functions. If $(\mathbf{Z}\omega + b) \cap (\mathbf{Z}\omega + c) = \emptyset$, then the distance between these two sets is positive. Let δ be one half of that distance. Then $\eta_{\omega,b}\eta_{\omega,c} = \eta_{\omega,b}\eta_{\omega,c}\xi_{\omega,c,\delta} + \eta_{\omega,b}\eta_{\omega,c}(1 - \xi_{\omega,c,\delta})$. By the definition of $\xi_{\omega,c,\delta}$, $\eta_{\omega,b}\xi_{\omega,c,\delta}$ and $\eta_{\omega,c}(1 - \xi_{\omega,c,\delta})$ are continuous. Hence in any case, $\eta_{\omega,b}\eta_{\omega,c} = h_0 + h_1\eta_{\omega,b} + h_2\eta_{\omega,c}$ where h_0, h_1 and h_2 are continuous periodic functions. Using induction, we have derived:

LEMMA 1.1. *Let $p_1, \ldots, p_k \in \mathbf{Z}$ and $b_1, \ldots, b_k \in \mathbf{R}$, then*

$$\prod_{j=1}^{k} \eta_{p_j\omega,b_j} = \sum_{j=1}^{k} \eta_{\omega,b_j} f_j + f_0$$

where $f_0, f_1, \ldots, f_k$ are continuous periodic functions.

By the definition, functions of the form $\sum_j f_j \prod_k \eta_{\omega_{jk},b_{jk}}$ are dense in $\mathcal{S}$. By Lemma 1.1, we can easily show that

$$\sum_j f_j \prod_k \eta_{\omega_{jk},b_{jk}} = \sum_{p=1}^{N} g_p \prod_{q=1}^{L_p} \eta_{\lambda_{pq},b_{pq}},$$

where $g_1, \ldots, g_N \in \mathrm{CAP}(\mathbf{R})$, and for each p, $\lambda_{p1}, \ldots, \lambda_{pL_p}$ are pairwise linearly independent over $\mathbf{Q}$, the rational numbers.

Let $\mathcal{S}_0 = \{\sum_j f_j \prod_k \eta_{\lambda_{j,k},b_{j,k}} : f_j \in \mathrm{CAP}(\mathbf{R})$ for each j, $\lambda_{j1}, \ldots, \lambda_{jk}, \ldots$ are pairwise linearly independent over $\mathbf{Q}\}$. It is clear that we have:

PROPOSITION 1.2. *$\mathcal{S}_0$ is a dense subalgebra of $\mathcal{S}$.*

Since $\eta_{\omega,b}$ is a periodic function, it has the usual Fejér polynomials approximating sequence $\{\sigma_n\}$. Recall that σ_n has the following properties: $\|\sigma_n\|_\infty \leqslant \|\eta_{\omega,b}\|_\infty$,

$$\sigma_n(t) = \sum_{k\neq 0} d_k^n \frac{1}{k} e^{ik\pi(t-b)/\omega} \text{ where } 0 \leqslant d_k^n \leqslant 1,$$

$$\lim_{n\to\infty} \int_a^{a+\omega} |\sigma_n(t) - \eta_{\omega,b}(t)|^2 \, dt = 0$$

for any $a \in \mathbf{R}$, and $\lim_{n\to\infty} d_k^n = 1$ for each k. Since

$$\frac{1}{2n} \int_{-n\omega}^{n\omega} |\sigma_n(t) - \eta_{\omega,b}(t)|^2 \, dt = \int_0^{\omega} |\sigma_n(t) - \eta_{\omega,b}(t)|^2 \, dt,$$

we also have

$$\lim_{n\to\infty} \lim_{T\to\infty} \left[\frac{1}{2T} \int_{-T}^{T} |\sigma_n(t) - \eta_{\omega,b}(t)|^2 \, dt \right] = 0.$$

Another important property of the Fejér approximating polynomials is that for any $\varepsilon > 0$, $\{\sigma_n\}$ converges to $\eta_{\omega,b}$ uniformly on $\mathbf{R} \setminus \bigcup_{k\in\mathbf{Z}} (k\omega + b - \varepsilon, k\omega + b + \varepsilon)$.

For each $\eta_{\omega,b}$, it is not hard to construct a sequence of periodic functions $\{v_n\}$ of period ω so that $v_n(k\omega + b) = 0, k \in \mathbf{Z}, 0 \leqslant v_n \leqslant 1, v_n \leqslant v_{n+1}$ and $\lim_{n\to\infty} v_n(t) = 1$ pointwise. Since v_n is periodic, the pointwise convergence also means that $\lim_{n\to\infty} \| v_n - 1 \|_{L^2(\mathbf{R}^\mathbf{B})} = 0$. This construction can be generalized. Let $\{\eta_{\omega_1,b_1}, \ldots, \eta_{\omega_k,b_k}\}$ be a finite family. Then we can produce a sequence $\{v_n\} \subset \mathrm{CAP}(\mathbf{R})$ such that $v_n(p\omega_j + b_j) = 0$, $p \in \mathbf{Z}$, $1 \leqslant j \leqslant k$, $0 \leqslant v_n \leqslant 1$, $v_n \leqslant v_{n+1}$, $\lim_{n\to\infty} v_n(t) = 1$ and $\lim_{n\to\infty} \| v_n - 1 \|_{L^2(\mathbf{R}^\mathbf{B})} = 0$. Note that if $f_0, \ldots, f_m \in \mathrm{CAP}(\mathbf{R})$, η_j is the product of some elements in $\{\eta_{\omega_1,b_1}, \ldots, \eta_{\omega_k,b_k}\}$, $j = 1, \ldots, m$, then $v_n(f_0 + f_1\eta_1 + \ldots + f_m\eta_m) \in \mathrm{CAP}(\mathbf{R})$.

Functions $\eta_{\omega,b}$, $\omega, b \in \mathbf{R}$, serve as the building blocks of discontinuities for functions on $\mathscr{S}$. But in some sense, it seems that a function as simple as $\eta_{\omega,b}\eta_{\lambda,c}$ should not be viewed as being "piecewise continuous". The following is the reason. Suppose $(\mathbf{Z}\omega + b) \cap (\mathbf{Z}\lambda + c) = \emptyset$ and λ and ω are linearly independent over $\mathbf{Q}$. Then $(\mathbf{Z}\omega + b) \cup (\mathbf{Z}\lambda + c)$, the discontinuities of $\eta_{\omega,b}\eta_{\lambda,c}$, has accumulation points in $\mathbf{R}^\mathbf{B}$. In fact, let δ and σ be continuous periodic functions such that $0 \leqslant \delta \leqslant 1$, $0 \leqslant \sigma \leqslant 1$ and the zeros of δ and σ are, respectively, exactly $\mathbf{Z}\omega + b$ and $\mathbf{Z}\lambda + c$. Since the distance between $\mathbf{Z}\omega + b$ and $\mathbf{Z}\lambda + c$ is 0, $\inf\{\sigma(t) + \delta(t) : t \in \mathbf{R}\} = 0$. On the other hand, $\sigma(t) + \delta(t)$ does not have zeros on $\mathbf{R}$. Thus the compactness of $\mathbf{R}^\mathbf{B}$ forces $\sigma + \delta$ to have zeros on $\mathbf{R}^\mathbf{B} \setminus \mathbf{R}$, which are, obviously, accumulation points of $(\mathbf{Z}\omega + b) \cup (\mathbf{Z}\lambda + c)$. And indeed, $\eta_{\omega,b}\eta_{\lambda,c}$ should be regarded as a function on $\mathbf{R}^{\mathbf{B}}$. Therefore it is conceivable that the investigation of the C^*-algebra generated by Wiener-Hopf operators with symbol in $\mathcal{S}$ will face certain technical complexities beyond those caused only by "piecewise" discontinuity. As we explained in the introduction, our interest at this preliminary stage of the investigation is not to tackle the technical details, but rather to explore the new phenomena that do not exist if the symbols are continuous. Hence in this paper we shall develop the full index theory only for the C^*-algebras generated by Wiener-Hopf operators whose symbol is in certain subalgebras of $\mathcal{S}$ in which the algebraic operations do not generate new accumulation points of discontinuity.

The following are the subalgebras that we are particularly interested in. Let $\Lambda = \{\eta_{\omega_i,b_i}\delta_i : i \in I\}$, where $\delta_i \in \mathrm{CAP}(\mathbf{R})$ and $d(\operatorname{supp}\delta_i \cap (\mathbf{Z}\omega_i + b_i), \operatorname{supp}\delta_j \cap (\mathbf{Z}\omega_j + b_j)) > 0$ for $i \neq j$. Let S_Λ be the algebra generated by $\mathrm{CAP}(\mathbf{R})$ and Λ. We shall develop the index theory for the C^*-algebra generated by Wiener-Hopf operators with symbols in S_Λ. As a convention, when we use notation S_Λ, we shall mean an algebra of functions described as above.

LEMMA 1.3. *Let $\eta_1, \ldots, \eta_k \in \Lambda$, then there exist $f_1, \ldots, f_k \in \mathrm{CAP}(\mathbf{R})$ such that*

$$\eta_1 \ldots \eta_k = \sum_{j=1}^{k} \eta_j f_j .$$

Proof. Let $\eta_i = \eta_{\omega_i,b_i}\delta_i$, $i = 1, 2$, and let $c = d(\operatorname{supp}\delta_1 \cap (\mathbf{Z}\omega_1 + b_1), \operatorname{supp}\delta_2 \cap (\mathbf{Z}\omega_2 + b_2)) > 0$. Then it is easy to see that both $\xi_{\omega_1,b_1,c}\eta_2$ and $(1 - \xi_{\omega_1,b_1,c})\eta_1$ are in $\mathrm{CAP}(\mathbf{R})$ (see the paragraph preceding Lemma 1.1 for the definition of $\xi_{\omega_1,b_1,c}$). Hence $\eta_1\eta_2 = \eta_1[\eta_2\xi_{\omega_1,b_1,c}] + \eta_2[\eta_1(1 - \xi_{\omega_1,b_1,c})]$. The rest of the proof is the routine induction procedure, which we omit.

The following are examples of S_Λ.

i) Fix $\omega > 0$ and let $\Lambda = \{\eta_{\omega,b} : b \in \mathbf{R}\}$. Then S_Λ is the algebra generated by $\mathrm{CAP}(\mathbf{R})$ and piecewise continuous functions with the fixed period ω. But note that the discontinuities of functions in S_Λ may not be of period ω. For example, it is easy to see that if m is a positive integer, then $\eta_{m\omega,b} = \eta_{\omega,b}f + g$, where f and g are continuous periodic functions.

Let ω and λ be linearly independent over $\mathbf{Q}$. For $\varepsilon > 0$, let $\eta_2 = (1 - \xi_{\omega,b,\varepsilon}\xi_{\lambda,c,\varepsilon})\eta_{\lambda,c}$. If $k \in \mathbf{Z}$ is such that $d(k\lambda + c, \mathbf{Z}\omega + b) < \varepsilon/2$, then η_2 vanishes on a neighborhood of $k\lambda + c$. If $t \in \mathbf{R}$ is such that $d(t, \mathbf{Z}\omega + b) > \varepsilon$, then $\eta_2(t) = \eta_{\lambda,c}(t)$.

ii) Let $\{\omega_j : j \in \mathbf{Z}\}$ be pairwise linearly independent over $\mathbf{Q}$. By the discussion above, there are $\{\delta_j : j \in \mathbf{Z}\} \subset \mathrm{CAP}(\mathbf{R})$ such that $0 \leqslant \delta_j \leqslant 1$, δ_j is not identically zero on $\mathbf{Z}\omega_j + b_j$ and

$$d(\operatorname{supp}\delta_j \cap \mathbf{Z}\omega_j + b_j, \operatorname{supp}\delta_i \cap \mathbf{Z}\omega_i + b_i) > 0$$

if $i \neq j$. Let $\Lambda = \{\eta_{\omega_j, b_j}\delta_j : j \in \mathbf{Z}\}$, then S_Λ contains functions whose discontinuities do not have any periodicity.

iii) Obviously we can construct various mixtures of the two kinds of algebras described above. We shall omit the details.

2. WIENER-HOPF OPERATORS WITH PIECEWISE CONTINUOUS SYMBOLS

Now we introduce the C^*-algebras of Wiener-Hopf operators with piecewise continuous almost periodic symbols.

Let $H^2(\mathbf{R})$ be the Hardy space of analytic functions on the upper half plane, which is considered as a subspace of $L^2(\mathbf{R})$. For each $\varphi \in L^\infty(\mathbf{R})$, let $W_\varphi = PM_\varphi | H^2(\mathbf{R})$ where M_φ is the multiplication by φ and P is the orthogonal projection from $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$. Let S be any subalgebra of $\mathscr{S}$, we denote by $\mathscr{A}(S)$ the C^*-algebra generated by all W_φ with $\varphi \in S$. The algebra $\mathscr{A}(\mathrm{CAP}(\mathbf{R}))$ is simply denoted by $\mathscr{A}$. The commutator ideal of $\mathscr{A}(S)$ is denoted by $\mathscr{C}(S)$ and that of $\mathscr{A}$ by $\mathscr{C}$. This setting $L^2(\mathbf{R})$, $H^2(\mathbf{R})$ and $\mathscr{A}(\mathscr{S})$ gives a clear picture of the operators we study, but it is not convenient to work with. Let $\mathscr{F}$ be the Fourier transform on $L^2(\mathbf{R})$, then $\mathscr{F}H^2(\mathbf{R}) = L^2(\mathbf{R}_+)$, $\mathscr{F}P\mathscr{F}^{-1} = \chi_{\mathbf{R}_+}$ and $\mathscr{F}M_{e^{i\lambda t}}\mathscr{F}^{-1} = \hat{T}_\lambda$ where $(\hat{T}_\lambda f)(x) = f(x-\lambda)$. The analysis will be carried out mostly in the setting $L^2(\mathbf{R})$, $L^2(\mathbf{R}_+)$ and $\mathscr{F}\mathscr{A}(S)\mathscr{F}^{-1}$.

3. TYPE II_∞ FACTOR AND REPRESENTATION

We need a commonly used type II_∞ von Neumann algebra which comes from the group-measure space construction due to Murray and von Neumann (see [8], [7], [4]). Let $\mathscr{L}$ be the Hilbert space $L^2(\mathbf{R}) \otimes L^2(\mathbf{R}_d) = L^2(\mathbf{R} \times \mathbf{R}_d)$, where $\mathbf{R}_d$ is the discrete reals and the measure on it is the counting measure. On $\mathscr{L}$, we define

$$(\tilde{M}_\varphi f)(x, t) = \varphi(x) f(x, t)$$

and

$$(\tilde{T}_\lambda f)(x, t) = f(x - \lambda, t - \lambda)$$

for $\varphi \in L^\infty(\mathbf{R})$ and $\lambda \in \mathbf{R}$. $\{\tilde{M}_\varphi, \tilde{T}_\lambda : \varphi \in L^\infty(\mathbf{R}), \lambda \in \mathbf{R}\}$ generates a type II_∞ factor $\tilde{\mathscr{N}}$ (see [8], p. 136) into which the algebra $\mathscr{A}(\mathscr{S})$ will be represented. As usual, the dimension function is normalized so that for $\sum \tilde{M}_{\varphi_j}\tilde{T}_{\lambda_j} \in \tilde{\mathscr{N}}$ and $\varphi_j \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, the normal faithful trace τ is given by $\tau(\sum \tilde{M}_{\varphi_j}\tilde{T}_{\lambda_j}) = \int_{\mathbf{R}} \varphi_0(t)\,\mathrm{d}t$ where $\lambda_0 = 0$.

Let $\mathscr{B}$ be the C^*-algebra of operators on $L^2(\mathbf{R})$ generated by finite sums $\sum M_{\varphi_j}\hat{T}_{\lambda_j}$, $\varphi_j \in L^\infty(\mathbf{R})$, $\lambda_j \in \mathbf{R}$. According to [7], there is a faithful representation $\tilde{\rho}$

of $\mathscr{B}$ into $\tilde{\mathscr{N}}$. Clearly, $\mathscr{F}\mathscr{A}\mathscr{F}^{-1}$ is contained in $\mathscr{B}$. We shall extend $\tilde{\rho}$ to a faithful representation of $\mathscr{F}\mathscr{A}(\mathscr{S})\mathscr{F}^{-1}$ into $\tilde{\mathscr{N}}$. Let $\mathscr{B}_0$ be the closure of finite sums $\{\sum \chi_{(a_j,b_j)}\hat{T}_{\lambda_j} : \lambda_j \in \mathbf{R},\ -\infty < a_j < b_j < \infty\}$, where $\chi_{(a_j,b_j)}$ is identified with the multiplication operator. $\mathscr{B}_0$ is a subalgebra and contains $\mathscr{F}\mathscr{C}\mathscr{F}^{-1}$. Indeed, it is easy to see that $\mathscr{F}\mathscr{C}\mathscr{F}^{-1}$ is the norm closure of finite sums $\{\sum \chi_{(a_j,b_j)}\hat{T}_{\lambda_j}\chi_{\mathbf{R}_+} : \lambda_j \in \mathbf{R}, b_j > a_j > 0\}$. Let $Q : L^2(\mathbf{R}) \to H^2(\mathbf{R})^\perp$ be the orthogonal projection.

LEMMA 3.1. *For* $f \in \mathrm{CAP}(\mathbf{R})$, $\mathscr{F}QM_fP\mathscr{F}^{-1} \in \mathscr{B}_0$.

Proof. If $f(t) = \sum a_j e^{i\lambda_j t}$ is a trigonometric polynomial, then it is easy to see that

$$\mathscr{F}QM_fP\mathscr{F}^{-1} = \chi_{\mathbf{R}_-}\sum_{\lambda_j<0} a_jT_{\lambda_j}\chi_{\mathbf{R}_+} = \chi_{\mathbf{R}_-}\sum_{\lambda_j<0} a_j\hat{T}_{\lambda_j}\chi_{(0,-\lambda_j)} = \sum_{\lambda_j<0} a_j\chi_{(\lambda_j,0)}\hat{T}_{\lambda_j}.$$

The assertion then follows from the usual limiting argument.

LEMMA 3.2. *For* $\omega, b \in \mathbf{R}$, $\mathscr{F}M_{\eta_{\omega,b}}\mathscr{F}^{-1}\mathscr{B}_0 \subset \mathscr{B}_0$.

Proof. Indeed it suffices to show that for any $a < c$, $\mathscr{F}M_{\eta_{\omega,b}}\mathscr{F}^{-1}\chi_{(a,c)} \in \mathscr{B}_0$. Let $\{\sigma_n\}$ be the Fejér approximating polynomials of $\eta_{\omega,b}$ (see §1), then obviously $\operatorname{s-lim}_{n\to\infty}\mathscr{F}M_{\sigma_n}\mathscr{F}^{-1}\chi_{(a,c)} = \mathscr{F}M_{\eta_{\omega,b}}\mathscr{F}^{-1}\chi_{(a,c)}$. It is equally obvious that $\mathscr{F}M_{\sigma_n}\mathscr{F}^{-1}\chi_{(a,c)} \in \mathscr{B}_0$. Therefore we only need to show that $\{\mathscr{F}M_{\sigma_n}\mathscr{F}^{-1}\chi_{(a,c)}\}$ forms a Cauchy sequence in the operator norm topology. Since $\chi_{(a,c)} = \chi_{(a,c_1)} + \chi_{(c_1,c_2)} + \ldots + \chi_{(c_p,c)}$ for any partition $a < c_1 < c_2 < \ldots < c_p < c$, we may assume that $c - a < \pi/\omega$. Let $\{a_k : -\infty < k < \infty\}$ be a sequence which has only a finite number of nonzero terms. Then for $f \in L^2(\mathbf{R})$,

$$\int_{\mathbf{R}} |\sum a_k(\hat{T}_{2k\pi/\omega}\chi_{(a,c)}f)(t)|^2\,dt = \int_{\mathbf{R}} |\sum a_k\chi_{(a+k\pi/\omega,c+k\pi/\omega)}f(t-2k\pi/\omega)|^2\,dt =$$

$$= \sum_k \int_{a+k\pi/\omega}^{c+k\pi/\omega} |a_k|^2 f(t-2k\pi/\omega)|^2\,dt = \sum_k |a_k|^2 \int_a^c |f(t)|^2\,dt \leqslant \sum_k |a_k|^2\|f\|^2_{L^2(\mathbf{R})}.$$

Hence $\|\sum a_k\hat{T}_{2k\pi/\omega}\chi_{(a,c)}\|^2 \leqslant \sum_k |a_k|^2$. Since $\mathscr{F}M_{\sigma_n}\mathscr{F}^{-1} = \sum_{k\neq 0} d_k^n \frac{1}{k} e^{-ik\pi/\omega}\hat{T}_{2k\pi/\omega}$, we have

$$\|\mathscr{F}M_{\sigma_n}\mathscr{F}^{-1}\chi_{(a,c)} - \mathscr{F}M_{\sigma_m}\mathscr{F}^{-1}\chi_{(a,c)}\| \leqslant \left[\sum_{k\neq 0} |d_k^n - d_k^m|\frac{1}{k^2}\right]^{1/2}.$$

This completes the proof.

Actually what we have shown is that if $\{\sigma_n\}$ is the Fejér approximating polynomials for $\eta_{\omega,b}$, then for $B \in \mathscr{B}_0$,

$$\lim_{n\to\infty} \|[\mathscr{F} M_{\sigma_n} \mathscr{F}^{-1} - \mathscr{F} M_{\eta_{\omega,b}} \mathscr{F}^{-1}]B\| = 0.$$

This can obviously be generalized to:

Corollary 3.3. *If $\{\sigma_n^j\}$ is the Fejér approximating polynomials for η_{ω_j,b_j}, $j = 1, \ldots, k$, then for each $B \in \mathscr{B}_0$,*

$$\lim_{n\to\infty} \left\| \left(\mathscr{F} \prod_{j=1}^{k} M_{\sigma_n^j} \mathscr{F}^{-1} - \mathscr{F} \prod_{j=1}^{k} M_{\eta_{\omega_j,b_j}} \mathscr{F}^{-1} \right) B \right\| = 0.$$

Corollary 3.4. *For $\varphi \in \mathscr{S}$, $W_\varphi \mathscr{C} \subset \mathscr{C}$.*

Proof. It suffices to present the proof for $\varphi \in \mathscr{S}_0$. By Lemma 3.2, $\mathscr{F} W_\varphi \mathscr{C} \mathscr{F}^{-1} \subset \mathscr{B}_0$ since $\mathscr{F} \mathscr{C} \mathscr{F}^{-1} \subset \mathscr{B}_0$. Then we only need to note that $\chi_{\mathbf{R}_+} \mathscr{B}_0 \chi_{\mathbf{R}_+} = \mathscr{F} \mathscr{C} \mathscr{F}^{-1}$.

Corollary 3.5. *Let $\eta = \eta_{\omega_1,b_1} \cdots \eta_{\omega_k,b_k}$ then for $f \in$ CAP(**R**),*

$$W_{\eta f} - W_\eta W_f \in \mathscr{C}.$$

Proof. $\mathscr{F} W_{\eta f} \mathscr{F}^{-1} - \mathscr{F} W_\eta W_f \mathscr{F}^{-1} = \mathscr{F} P \mathscr{F}^{-1} (\mathscr{F} M_\eta \mathscr{F}^{-1})(\mathscr{F} Q M_f P \mathscr{F}^{-1}) \in \mathscr{B}_0$.

Now we start extending $\tilde{\rho}$. Let $F : L^2(\mathbf{R}_d) \to L^2(\mathbf{R}^B)$ be the Fourier transform, then the unitary operator $\mathscr{F}^{-1} \otimes F : L^2(\mathbf{R} \times \mathbf{R}_d) \to L^2(\mathbf{R}) \otimes L^2(\mathbf{R}^B)$ induces a faithful represnetation $\tilde{\rho}' = (\mathscr{F}^{-1} \otimes F) \tilde{\rho} (\mathscr{F}^{-1} \otimes F)^{-1}$ of $\mathscr{B}$ in $\tilde{\mathscr{N}}' = (\mathscr{F}^{-1} \otimes F) \tilde{\mathscr{N}} (\mathscr{F}^{-1} \otimes F)^{-1}$. It is easy to see that $(\mathscr{F}^{-1} \otimes F) \tilde{T}_\lambda (\mathscr{F}^{-1} \otimes F)^{-1} = M_{e^{i\lambda \cdot}} \otimes \overline{M}_{e^{i\lambda \cdot}}$ and that for $f \in L^2(\mathbf{R}) \otimes L^2(\mathbf{R}^B)$, $(M_{e^{i\lambda \cdot}} \otimes \overline{M}_{e^{i\lambda \cdot}} f)(x) = e^{i\lambda x} [\overline{M}_{e^{i\lambda \cdot}} f(x)] = \overline{M}_{e^{i\lambda(\cdot + x)}} f(x)$, where $f(x)$ is regarded as an element in $L^2(\mathbf{R}^B)$ and $\overline{M}_\varphi$ the multiplication by φ on $L^2(\mathbf{R}^B)$. Thus if $g \in$ CAP(**R**), then $[(\mathscr{F}^{-1} \otimes F) \tilde{\rho} (\mathscr{F} W_g \mathscr{F}^{-1})(\mathscr{F}^{-1} \otimes F)^{-1} \mathscr{U}](x) = (P \otimes 1) \overline{M}_{g(x+\cdot)} [(P \otimes 1) \mathscr{U}](x)$, where $P \otimes 1$ is the orthogonal projection from $L^2(\mathbf{R}) \otimes L^2(\mathbf{R}^B)$ onto $H^2(\mathbf{R}) \otimes L^2(\mathbf{R}^B)$, $\mathscr{U} \in L^2(\mathbf{R}) \otimes L^2(\mathbf{R}^B)$, and the exact meaning of the right hand side of the equality is the value of $(P \otimes 1) f$ at x where $f(y) = \overline{M}_{g(y+\cdot)} [P \otimes 1 \mathscr{U}](y)$, $y \in \mathbf{R}$.

Lemma 3.6. *Let $\{g_n\} \subset$ CAP(**R**) be such that $\|g_n\|_\infty \leqslant L$ and*

$$(*) \qquad \lim_{n\to\infty} \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g_n(t) - g(t)|^2 \, dt \right] = 0.$$

Then $\text{s-}\lim_{n\to\infty}\tilde{\rho}(\mathscr{F}W_{g_n}\mathscr{F}^{-1})$ *exists and is independent of the choice of* $\{g_n\}$. *If, in particular,* $g \in \mathrm{CAP}(\mathbf{R})$, *then*

$$\text{s-}\lim_{n\to\infty}\tilde{\rho}(\mathscr{F}W_{g_n}\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}W_g\mathscr{F}^{-1}).$$

Proof. Since $\|g_n\|_\infty \leqslant L$, g can be regarded as an element in $L^2(\mathbf{R^B}) \cap L^\infty(\mathbf{R^B})$. $(*)$ implies that on $L^2(\mathbf{R^B})$, $\text{s-}\lim_{n\to\infty}\overline{M}_{g_n(x+\cdot)} = \overline{M}_{g(x+\cdot)}$ for $x \in \mathbf{R}$. Thus the existence of $\text{s-}\lim_{n\to\infty}(\mathscr{F}^{-1}\otimes F)\tilde{\rho}(\mathscr{F}W_{g_n}\mathscr{F}^{-1})(\mathscr{F}^{-1}\otimes F)^{-1}$ follows immediately and therefore so does $\text{s-}\lim_{n\to\infty}\tilde{\rho}(\mathscr{F}W_{g_n}\mathscr{F}^{-1})$. It is clear that if $g = 0$ then the above strong limit is zero too. The second assertion is obvious.

Recall that an element in $\mathscr{S}_0$ has the form $\varphi = f_0 + f_1\eta_1 + \ldots + f_k\eta_k$ where $f_0, \ldots, f_k \in \mathrm{CAP}(\mathbf{R})$ and each η_j is the product of a finite number of elements in $\{\eta_{\omega,b} : \omega, b \in \mathbf{R}\}$. By the properties of Fejér approximating polynomials of $\eta_{\omega,b}$, for each η_j we have a sequence $\{p_n^j\} \subset \mathrm{CAP}(\mathbf{R})$ such that $\|p_n^j\| \leqslant L$ for certain $L > 0$ and $\lim_{n\to\infty}\|p_n^j - \eta_j\|_{L^2(\mathbf{R^B})} = 0$. We define

$$\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1}) = \lim_{n\to\infty}\tilde{\rho}(\mathscr{F}(W_{f_0+f_1p_n^1 \ \ldots+f_kp_n^k})\mathscr{F}^{-1}).$$

Lemma 3.6 guarantees that the definition of $\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1})$ does not depend on the particular choice of $\{p_n^j\}$. Elements $\sum_j\prod_k W_{\varphi_{jk}}$ with $\varphi_{jk} \in \mathscr{S}_0$ are dense in $\mathscr{A}(\mathscr{S})$. So if $\tilde{\rho}$ is to extend to a representation of $\mathscr{F}\mathscr{A}(\mathscr{S})\mathscr{F}^{-1}$, then naturally

$$\tilde{\rho}(\mathscr{F}\sum_j\prod_k W_{\varphi_{jk}}\mathscr{F}^{-1}) = \sum_j\prod_k\tilde{\rho}(\mathscr{F}W_{\varphi_{jk}}\mathscr{F}^{-1}).$$

At this stage, however, it is not at all obvious that this $\tilde{\rho}$ is even well defined. Nevertheless, we shall proceed to prove that $\tilde{\rho}$ really extends to a faithful representation of $\mathscr{F}\mathscr{A}(\mathscr{S})\mathscr{F}^{-1}$.

LEMMA 3.7. *If* $B \in \mathscr{C}$, *then* $BW_\varphi \in \mathscr{C}$ *and* $\tilde{\rho}(\mathscr{F}B\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}BW_\varphi\mathscr{F}^{-1})$.

Proof. We define function φ_n by replacing every $\eta_{\omega,b}$ in the definition of φ by its n^{th} Fejér approximating polynomial. Then by Corollary 3.3, $\lim_{n\to\infty}\|BW_{\varphi_n} - BW_\varphi\| = 0$. So $\tilde{\rho}(\mathscr{F}B\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1}) = \lim_{n\to\infty}\tilde{\rho}(\mathscr{F}B\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_{\varphi_n}\mathscr{F}^{-1}) = \lim_{n\to\infty}\tilde{\rho}(\mathscr{F}BW_{\varphi_n}\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}BW_\varphi\mathscr{F}^{-1})$.

LEMMA 3.8. *Let* $v \in \mathrm{CAP}(\mathbf{R})$ *vanish at the discontinuities of* $\varphi \in \mathscr{S}_0$. *Then* $W_vW_\varphi \in \mathscr{A}$ *and*

$$\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}W_vW_\varphi\mathscr{F}^{-1}).$$

Proof. $W_vW_\varphi = W_vW_\varphi - W_{v\varphi} + W_{v\varphi}$. By Corollary 3.5, $W_vW_\varphi - W_{v\varphi} \in \mathscr{C} \subset \mathscr{A}$. Note that $v\varphi \in \mathrm{CAP}(\mathbf{R})$, so $W_vW_\varphi \in \mathscr{A}$. Let φ_n be the same as in the proof of Lemma 3.7, then

$$\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1}) = \lim_{n\to\infty}\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_{\varphi_n}\mathscr{F}^{-1}) =$$

$$= \lim_{n\to\infty}[\tilde{\rho}(\mathscr{F}(W_vW_{\varphi_n} - W_{v\varphi_n})\mathscr{F}^{-1}) + \tilde{\rho}(\mathscr{F}W_{v\varphi_n}\mathscr{F}^{-1})].$$

Corollary 3.3 implies $\lim_{n\to\infty}\|[W_vW_{\varphi_n} - W_{v\varphi_n}] - [W_vW_\varphi - W_{v\varphi}]\| = 0$ (in this case, take $B = \mathscr{F}QM_vP\mathscr{F}^{-1}$). Since v vanishes at the discontinuities of φ, $\lim_{n\to\infty}\|v\varphi_n - v\varphi\|_\infty = 0$. Thus

$$\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_\varphi\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}W_vW_\varphi\mathscr{F}^{-1}).$$

LEMMA 3.9. *Let $\varphi_1, \ldots, \varphi_k \in \mathscr{S}_0$ and let $v \in \mathrm{CAP}(\mathbf{R})$ vanish at the discontinuities of $\varphi_1, \ldots, \varphi_k$. Then $W_vW_{\varphi_1} \ldots W_{\varphi_k} \in \mathscr{A}$ and*

$$\rho(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_{\varphi_1}\mathscr{F}^{-1}) \ldots \tilde{\rho}(\mathscr{F}W_{\varphi_k}\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}W_vW_{\varphi_1} \ldots W_{\varphi_1}\mathscr{F}^{-1}).$$

Proof. We use the induction. By Lemma 3.8, we have $\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_{\varphi_1}\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}W_{v\varphi_1}\mathscr{F}^{-1} + \mathscr{F}K\mathscr{F}^{-1})$ where $v\varphi_1$ vanishes at the discontinuities of $\varphi_2, \ldots, \varphi_k$ and $K \in \mathscr{C}$. Thus by the induction hypothesis and Lemma 3.7 we have

$$\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})\tilde{\rho}(\mathscr{F}W_{\varphi_1}\mathscr{F}^{-1}) \ldots \tilde{\rho}(\mathscr{F}W_{\varphi_k}\mathscr{F}^{-1}) =$$

$$= \tilde{\rho}(\mathscr{F}W_{v\varphi_1}W_{\varphi_2} \ldots W_{\varphi_k}\mathscr{F}^{-1}) + \tilde{\rho}(\mathscr{F}KW_{\varphi_2} \ldots W_{\varphi_k}\mathscr{F}^{-1}) =$$

$$= \tilde{\rho}(\mathscr{F}[W_{v\varphi_1} + K]W_{\varphi_2} \ldots W_{\varphi_k}\mathscr{F}^{-1}) = \tilde{\rho}(\mathscr{F}W_vW_{\varphi_1} \ldots W_{\varphi_k}\mathscr{F}^{-1}).$$

COROLLARY 3.10. *If $\varphi_{jk} \in \mathscr{S}_0$ and v vanishes at the discontinuities of φ_{jk}, then*

$$[\sum_j \prod_k \tilde{\rho}(\mathscr{F}W_{\varphi_{jk}}\mathscr{F}^{-1})]^*\tilde{\rho}(\mathscr{F}W_v\mathscr{F}^{-1})[\sum_j \prod_k \tilde{\rho}(\mathscr{F}W_{\varphi_{jk}}\mathscr{F}^{-1})] =$$

$$= \tilde{\rho}(\mathscr{F}[\sum_j \prod_k W_{\varphi_{jk}}]^*W_v[\sum_j \prod_k W_{\varphi_{jk}}]\mathscr{F}^{-1}).$$

Before finally presenting the proof that $\tilde{\rho}$ preserves the operator norm, we need one more lemma.

LEMMA 3.11. *Let $\{A_n\}$ be a sequence of bounded self-adjoint operators on Hilbert space H such that $0 \leqslant A_1 \leqslant A_2 \leqslant \ldots \leqslant A_n \leqslant \ldots$ and $\text{s-}\lim_{n\to\infty} A_n = A$. Then $\lim_{n\to\infty}\|A_n\| = \|A\|$.*

This fact is perhaps known to every operator theorist. But since we are unable to find a standard reference, we include a proof here for completeness.

Proof. Suppose the contrary. Then it would be true that $\lim_{n\to\infty}\|A_n\| > \|A\|$. We may assume that $\lim_{n\to\infty}\|A_n\| = 1$ and $\|A\| = 1 - \varepsilon$ for certain $\varepsilon > 0$. Pick n_0 such that $\|A_{n_0}\| > 1 - \varepsilon/4$. We can find an $x_0 \in H$ such that $\|x_0\| = 1$ and $(A_{n_0}x_0, x_0) \geqslant 1 - \varepsilon/2$. Since $A_n \geqslant A_{n_0}$ for $n \geqslant n_0$, we have $(Ax_0, x_0) = \lim_{n\to\infty}(A_n x_0, x_0) \geqslant (A_{n_0}x_0, x_0) > 1 - \varepsilon/2$. This is a contradiction.

Now we fix $\{\varphi_{jk}\} \subset \mathscr{S}_0$ and let $\{v_n\} \subset \mathrm{CAP}(\mathbf{R})$ be a sequence such that each v_n vanishes at the discontinuities of φ_{jk}'s, $0 \leqslant v_n \leqslant 1$, $v_n \leqslant v_{n+1}$, $\lim_{n\to\infty} v_n(t) = 1$ and $\lim_{n\to\infty}\|v_n - 1\|_{L^2(\mathbf{R}^{\mathbf{B}})} = 0$ (see §1). Then $0 \leqslant W_{v_n} \leqslant W_{v_{n+1}}$, $0 \leqslant \tilde{\rho}(\mathscr{F} W_{v_n}\mathscr{F}^{-1}) \leqslant \tilde{\rho}(\mathscr{F} W_{v_{n+1}}\mathscr{F}^{-1})$, $\text{s-}\lim_{n\to\infty} W_{v_n} = 1$ and, by Lemma 3.6, $\text{s-}\lim_{n\to\infty}\tilde{\rho}(\mathscr{F} W_{v_{n+1}}\mathscr{F}^{-1}) = \chi_{\mathbf{R}_+\times\mathbf{R}}$. Let $T = \sum_j \prod_k W_{\varphi_{jk}}$ and $S = \sum_j \prod_k \tilde{\rho}(\mathscr{F} W_{\varphi_{jk}}\mathscr{F}^{-1})$, then since $S^*\tilde{\rho}(\mathscr{F} W_{v_n}\mathscr{F}^{-1})S \leqslant S^*\tilde{\rho}(\mathscr{F} W_{v_{n+1}}\mathscr{F}^{-1})S$, $T^*W_{v_n}T \leqslant T^*W_{v_{n+1}}T \in \mathscr{A}$, $\chi_{\mathbf{R}_+\times\mathbf{R}}S = S$ and $\tilde{\rho}$ preserves operator norm on $\mathscr{F}\mathscr{A}\mathscr{F}^{-1}$, $\|S\|^2 = \|S^*S\| = \lim_{n\to\infty}\|S^*\tilde{\rho}(\mathscr{F} W_{v_n}\mathscr{F}^{-1})S\| = \lim_{n\to\infty}\|\tilde{\rho}(\mathscr{F}(T^*W_{v_n}T)\mathscr{F}^{-1})\| = \lim_{n\to\infty}\|T^*W_{v_n}T\| = \|T^*T\| = \|T\|^2$.

This shows that $\tilde{\rho}$ does preserve the operator norm of T. Thus $\tilde{\rho}$ extends in a natural way to a faithful representation of $\mathscr{F}\mathscr{A}(\mathscr{S})\mathscr{F}^{-1}$ into $\mathscr{N} = \chi_{\mathbf{R}_+\times\mathbf{R}_d}\tilde{\mathscr{N}}\chi_{\mathbf{R}_+\times\mathbf{R}_d}$. Let ρ be the pullback of $\tilde{\rho}$ to $\mathscr{A}(\mathscr{S})$, then this is the desired faithful representation.

But the discussion above may yield a general result which we shall need in Section 5. Let H be a Hilbert space and let $\alpha : \mathscr{A} \to \mathscr{L}(H)$ be a faithful representation. By the process of extending $\tilde{\rho}$, it is easy to see that to extend α to a faithful representation of $\mathscr{A}(\mathscr{S})$, we only need

(3.12) if $\{f_n\} \subset \mathrm{CAP}(\mathbf{R})$, $\|f_n\|_\infty \leqslant M$, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on $\mathbf{R}$ and $\lim_{n\to\infty}\|f_n - f\|_{L^2(\mathbf{R}^{\mathbf{B}})} = 0$, then $\text{s-}\lim_{n\to\infty}\alpha(W_{f_n})$ exists and, in the event $f \in \mathrm{CAP}(\mathbf{R})$, equals $\alpha(W_f)$;

(3.13) for any $A = \sum_j \prod_k W_{\varphi_{jk}} \in \mathscr{A}(\mathscr{S})$, there exists an increasing sequence $v_1 \leqslant v_2 \leqslant \ldots \leqslant v_n \leqslant \ldots \leqslant 1$ in $\mathrm{CAP}(\mathbf{R})$ such that $\|v_n - 1\|_{L^2(\mathbf{R}^{\mathbf{B}})} \to 0$, $W_{v_n}\sum_j \prod_k W_{\varphi_{jk}} \in \mathscr{A}$ and $\alpha(W_{v_n})\sum_j \prod_k \alpha(W_{\varphi_{jk}}) = \alpha(W_{v_n}\sum_j \prod_k W_{\varphi_{jk}})$ if we define $\alpha(W_{\varphi_{jk}})$ as the strong limit of $\{\alpha(W_{f_n})\}$ where $\{f_n\}$ is an approximating sequence of φ_{jk}.

What we have actually proved is:

LEMMA 3.14. *Let $\alpha : \mathscr{A} \to \mathscr{L}(H)$ be a faithful representation which satisfy* (3.12) *and* (3.13). *Then α can be extended to a faithful representation $\mathscr{A}(\mathscr{S}) \to \mathscr{L}(H)$.*

4. THE COMMUTATOR IDEALS

The representation ρ carries $\mathscr{C}$ into the ideal of compact operators in $\mathscr{N}$ (see [7]). We shall prove that $\rho(\mathscr{C}(\mathscr{S}))$ is contained in the compact operators in $\mathscr{N}$. Denote the ideal of compact operators in $\mathscr{N}$ by $\mathscr{K}$, which is the norm closure of the trace ideal.

LEMMA 4.1. *Let* $\eta_j = \eta_{\omega_j, b_j}$, $j = 1, 2$ *where* ω_1 *and* ω_2 *are linearly independent over* $\mathbf{Q}$. *Then*

$$\rho(W_{\eta_1} W_{\eta_2} - W_{\eta_1 \eta_2})$$

belongs to the Hilbert-Schmidt class of $\mathscr{N}$.

Proof. Without loss of generality, we may assume that $\omega_1 = 1, \omega_2 = \omega$ is irrational and $b_1 = b_2 = 0$. Let $\{\sigma_k^j\}$ be the Fejér approximating polynomials of η_j, $j = 1, 2$, respectively. Then $A_k = W_{\sigma_k^1} W_{\sigma_k^2} - W_{\sigma_k^1 \sigma_k^2} = -\mathscr{F}^{-1} \left[\sum_{n,m>0} \frac{1}{nm} d_n^{k,1} d_m^{k,2} T_{n-m\omega} \chi_{(0,m\omega)} \right] \mathscr{F}$ where $T_\lambda = \chi_{\mathbf{R}_+} \hat{T}_\lambda \chi_{\mathbf{R}_+}$, $0 \leqslant d_n^{k,j} \leqslant 1$, $j = 1, 2$, and $\lim_{k\to\infty} d_n^{k,j} = 1$ (see §1). Hence

$$\mathscr{F} A_k^* A_k \mathscr{F}^{-1} = \sum_{n-m\omega = p-q\omega} \frac{1}{mnpq} d_n^{k,1} d_p^{k,1} d_m^{k,2} d_q^{k,2} \chi_{(0,q\omega)} T_{q\omega - p} T_{n-m\omega} \chi_{(0,m\omega)} + \sum_{\lambda \neq 0} M_{\varphi_\lambda} T_\lambda =$$

$$= \sum_{n > m\omega} \frac{1}{m^2 n^2} (d_n^{k,1})^2 (d_m^{k,2})^2 \chi_{(0,m\omega)} + \sum_{m\omega > n} \frac{1}{m^2 n^2} (d_n^{k,1})^2 (d_m^{k,2}) \chi_{(m\omega - n, m\omega)} + \sum_{\lambda \neq 0} M_{\varphi_\lambda} T_\lambda .$$

Thus

$$\tau\rho(A_k^* A_k) \leqslant \sum_{n > m\omega} \frac{1}{mn^2} + \sum_{m\omega > n} \frac{1}{nm^2} < \infty.$$

By the definition of ρ (see §3),

$$\rho([W_{\eta_1} W_{\eta_2} - W_{\eta_1\eta_2}]^* [W_{\eta_1} W_{\eta_2} - W_{\eta_1\eta_2}]) = \operatorname{s-lim}_{k\to\infty} \rho(A_k^* A_k).$$

Therefore $\rho(W_{\eta_1} W_{\eta_2} - W_{\eta_1\eta_2})$ belongs to the Hilbert-Schmidt class of $\mathscr{N}$.

REMARK. Observe that $W_{\eta_1} W_{\eta_2} - W_{\eta_1\eta_2} = PM_{\eta_1} QM_{\eta_2} P$. If we exchange the roles of $L^2(\mathbf{R}_+)$ and $L^2(\mathbf{R}_-)$, and P and Q, it is clear that the above proof also shows that $\tau\tilde{\rho}(\tilde{A}_k^* \tilde{A}_k) \leqslant M < \infty$ where $\tilde{A}_k = QM_{\sigma_k^1} PM_{\sigma_k^2} Q$. Hence for any $\varphi, \psi \in \mathscr{S}$, $\rho(PM_\varphi QM_{\eta_1} PM_{\eta_2} QM_\psi P)$ also belongs to the Hilbert-Schmidt class of $\mathscr{N}$.

LEMMA 4.2. *Let* $\eta_j = \eta_{\omega_j, b_j}$, $j = 1, 2, \ldots, k$ *where* $\omega_1, \ldots, \omega_k$ *are pairwise linearly independent over* $\mathbf{Q}$. *Then* $\rho(W_{\eta_1} \ldots W_{\eta_k} - W_{\eta_1 \ldots \eta_k}) \in \mathscr{K}$.

Proof. We use the induction. Suppose that the lemma is proved for $k-1 \geqslant 2$. By the induction hypothesis, it suffices to show that

$$\rho(W_{\eta_1\ldots\eta_k} - W_{\eta_1\ldots\eta_{k-1}}W_{\eta_k}) \in \mathscr{K}.$$

Let $A = W_{\eta_1\ldots\eta_k} - W_{\eta_1\ldots\eta_{k-1}}W_{\eta_k} = PM_{\eta_1}\ldots M_{\eta_{k-1}}QM_{\eta_k}P$. We have

$$W_{\eta_1\ldots\eta_{k-1}}W_{\eta_k} - W_{\eta_1\ldots\eta_{k-2}\eta_k}W_{\eta_{k-1}} =$$

$$= (W_{\eta_1\ldots\eta_{k-1}} - W_{\eta_1}\ldots W_{\eta_{k-1}})W_{\eta_k} + W_{\eta_1}\ldots W_{\eta_{k-2}}[W_{\eta_k}, W_{\eta_{k-1}}] +$$

$$+ (W_{\eta_1}\ldots W_{\eta_{k-2}}W_{\eta_k} - W_{\eta_1\ldots\eta_{k-2}\eta_k})W_{\eta_{k-1}} = K,$$

and by the hypothesis $\rho(K) \in \mathscr{K}$. Therefore by the remark preceding the lemma,

$$\rho(A(A+K)^*) = \rho(PM_{\eta_1}\ldots M_{\eta_{k-1}}QM_{\eta_k}PM_{\eta_{k-1}}QM_{\eta_k}M_{\eta_{k-2}}\ldots M_{\eta_1}P) =$$

$$= \rho(\ldots[QM_{\eta_k}PM_{\eta_{k-1}}Q]\ldots)$$

is a compact operator in $\mathscr{N}$. Hence $\rho(A)^*\rho(A) = \rho(A^*A)$ is compact. Obviously, so is $\rho(A)$. This completes the proof.

Note that this proof is the only place in the whole paper where we need the assumption that $\omega_1, \ldots, \omega_k$ are pairwise linear independent over $\mathbf{Q}$.

LEMMA 4.3. *Let* $\tilde{\eta}_j = \eta_{\omega_j,b_j}f_j$, $j = 1, \ldots, k$, *be such that* $f_j \in \mathrm{CAP}(\mathbf{R})$ *and* $d(\operatorname{supp} f_i \cap (\mathbf{Z}\omega_i + b_i), \operatorname{supp} f_j \cap (\mathbf{Z}\omega_j + b_j)) = c > 0$. *Then*

$$W_{\tilde{\eta}_1\ldots\tilde{\eta}_k} - W_{\tilde{\eta}_1}\ldots W_{\tilde{\eta}_k} \in \mathscr{C}.$$

Proof. Denote $\xi = \xi_{\omega_1,b_1,c/2}$ and $\eta_j = \eta_{\omega_j,b_j}$. Observe that $(1-\xi)\eta_1 \in \mathrm{CAP}(\mathbf{R})$ and if $d(k\omega_2 + b_2, \mathbf{Z}\omega_1 + b_1) > c$, then $f_1(k\omega_2 + b_2)f_2(k\omega_2 + b_2) = 0$. Hence $\eta_2 f_1 f_2 \xi \in \mathrm{CAP}(\mathbf{R})$. Therefore by Corollary 3.5,

$$W_{\tilde{\eta}_1}W_{\tilde{\eta}_2} = W_{\eta_1}W_{\eta_2 f_1 f_2}(\mathrm{mod}\,\mathscr{C}) = (W_{\eta_1\xi}W_{\eta_2 f_1 f_2} + W_{(1-\xi)\eta_1\eta_2 f_1 f_2})(\mathrm{mod}\,\mathscr{C}) =$$

$$= (W_{\eta_1}W_{\xi\eta_2 f_1 f_2} + W_{(1-\xi)\eta_1\eta_2 f_1 f_2})(\mathrm{mod}\,\mathscr{C}) = W_{\tilde{\eta}_1\tilde{\eta}_2}(\mathrm{mod}\,\mathscr{C}).$$

The rest of the proof is essentially the same as the induction procedure in Lemma 4.3. However, since this time we are working in the C^*-algebra $\mathscr{A}(\mathscr{S})$ in which elements may not have polar decomposition, we must make certain in the induction that $A^*A \in \mathscr{C}$ yields $A \in \mathscr{C}$. But since $\mathscr{C}$ is an ideal in $\mathscr{A}(\mathscr{S})$ (see

Corollary 3.4); in the quotient algebra $\mathscr{A}(\mathscr{S})/\mathscr{C}$, $\|A+\mathscr{C}\|^2 = \|(A+\mathscr{C})^*(A+\mathscr{C})\|$. Hence $A \in \mathscr{C}$ if $A^*A \in \mathscr{C}$ and the proof of the lemma follows.

Now we consider $[W_{\eta_{\omega,b}}, W_{\eta_{\lambda,c}}]$ where ω and λ are linearly dependent over $\mathbf{Q}$. By §1, $\eta_{\omega,b} = \eta_{\mu,b}f_1 + f_0$ and $\eta_{\lambda,c} = \eta_{\mu,c}g_1 + g_0$ where $f_i, g_i \in \mathrm{CAP}(\mathbf{R})$, $i = 1, 2$. If $\mathbf{Z}\mu + c = \mathbf{Z}\mu + b$, then $\eta_{\mu,c} = \eta_{\mu,b}$. Otherwise, $d(\mathbf{Z}\mu + c, \mathbf{Z}\mu + b) > 0$, therefore $[W_{\eta_{\mu,c}}, W_{\eta_{\mu,b}}] \in \mathscr{C}$. In any event, we have $[W_{\eta_{\omega,b}}, W_{\eta_{\lambda,c}}] \in \mathscr{C}$ if ω and λ are linearly dependent over $\mathbf{Q}$.

Combining this with Proposition 1.2, Corollary 3.5 and Lemma 4.2, we thus have proved:

THEOREM 4.4. *$\rho(\mathscr{C}(\mathscr{S}))$ is contained in the ideal of compact operators in $\mathscr{N}$.*

Let S_A be the subalgebra of $\mathscr{S}$ generated by a family A of functions such that the distance between the discontinuities of any two is greater than zero, as described in Section 1. Then Lemma 4.3 yields immediately:

THEOREM 4.5. *The commutator ideal of $\mathscr{A}(S_A)$ is the same as $\mathscr{C}$.*

5. THE MAXIMAL IDEAL SPACES OF QUOTIENT ALGEBRAS

Denote $\mathscr{N}_1 = \rho(\mathscr{A}(\mathscr{S}))$ and $\tilde{\mathscr{K}} = \mathscr{N}_1 \cap \mathscr{K}$. In the quotient algebra $\mathscr{N}_1/\tilde{\mathscr{K}}$, the subalgebra $\rho\mathscr{A}/\tilde{\mathscr{K}}$ is identified with CAP(**R**), whose maximal ideal space is $\mathbf{R}^{\mathbf{B}}$, see [7, Lemma 2.3, Remark 3.7]. Let $\mathscr{K}_x(\mathscr{C}_x)$ be the ideal of $\mathscr{N}_1$ (resp. $\mathscr{A}(\mathscr{S})$) generated by $\tilde{\mathscr{K}}$ (resp. $\mathscr{C}$) and all $\rho(W_\varphi)$ (resp. W_φ) where $\varphi \in \mathrm{CAP}(\mathbf{R}) = C(\mathbf{R}^{\mathbf{B}})$ such that $\varphi(x) = 0$. Corollary 3.5 says that $\mathscr{A}/\mathscr{C}$ is contained in the center of $\mathscr{A}(\mathscr{S})/\mathscr{C}$, hence we have the following localizations of quotient algebras: the sequences

$$0 \to \mathscr{C} \to \mathscr{A}(\mathscr{S}) \to \bigoplus_{x \in \mathbf{R}^{\mathbf{B}}} \mathscr{A}(\mathscr{S})/\mathscr{C}_x$$

and

$$0 \to \mathscr{K} \to \mathscr{N}_1 \to \bigoplus_{x \in \mathbf{R}^{\mathbf{B}}} \mathscr{N}_1/\mathscr{K}_x$$

are exact, see [10] and [9, §4]. We denote by $[A]_x$ the equivalent class of A in the quotient algebra $\mathscr{A}(\mathscr{S})/\mathscr{C}_x$, or that of A in $\mathscr{N}_1/\mathscr{K}_x$ in the event $A \in \mathscr{N}_1$. The main task of this section is the computation of the spectrum $\sigma([A]_x)$.

Obviously, if $\varphi \in \mathrm{CAP}(\mathbf{R})$, then $\sigma([W_\varphi]_x) = \sigma([\rho(W_\varphi)]_x) = \{\varphi(x)\}$. Now let us consider the case $A = W_{\eta_{\omega,b}}$. Note that the range of $-i\eta_{\omega,b}$ is $[-\pi, \pi]$. Let F be the closure of $\mathbf{Z}\omega + b$ in $\mathbf{R}^{\mathbf{B}}$. If $x \notin F$, then there exists $\delta \in \mathrm{CAP}(\mathbf{R}) = C(\mathbf{R}^{\mathbf{B}})$ such that $\delta(x) = 1$ and $\delta|F = 0$. Thus $\delta\eta_{\omega,b}$ can be regarded as a function in $C(\mathbf{R}^{\mathbf{B}})$ and $(\delta\eta_{\omega,b})(x)$ does not depend on the choice of δ. Hence we may define

$\eta_{\omega,b}(x) = (\delta\eta_{\omega,b})(x)$. It is easy to show that $m(F) = 0$ where m is the normalized Haar measure on $\mathbf{R}^{\mathbf{B}}$. Hence $\eta_{\omega,b}$ may be identified as an element in $L^\infty(\mathbf{R}^{\mathbf{B}})$. For $x \in \mathbf{R}^{\mathbf{B}}$, let C_x be the ideal of $\mathscr{S}$ generated by all $\varphi \in \mathrm{CAP}(\mathbf{R})$ such that $\varphi(x) = 0$. By Corollary 3.5, it is easy to show that if $\psi \in C_x$, then $W_\psi \in \mathscr{C}_x$.

Let $\tilde{\eta} = \eta_{\omega_1,b_1} \ldots \eta_{\omega_k,b_k}$ and $F_j = \overline{\mathbf{Z}\omega_j + b_j}$ for $j = 1, \ldots, k$.

LEMMA 5.1. *Let* $f \in \mathrm{CAP}(\mathbf{R})$. *If* $x \notin \left(\bigcup_{j=1}^{k} F_j\right) \cap (\mathrm{supp} f)$, *then* $\sigma([W_{\tilde{\eta}f}]_x) = \{\tilde{\eta}(x)f(x)\}$. *If* $x \in F_j \setminus \bigcup_{i \neq j} F_i$ *then* $\sigma([W_{\tilde{\eta}f}]_x) \subset f(x) \prod_{i \neq j} \eta_{\omega_i,b_i}(x)[-\pi, \pi]\sqrt{-1}.$ [1]

Proof. If $x \notin \left(\bigcup_{j=1}^{k} F_j\right) \cap (\mathrm{supp} f)$, then $\tilde{\eta}f - \tilde{\eta}(x)f(x) \in C_x$ and therefore the first assertion follows. If $x \in F_j \setminus \bigcup_{i \neq j} F_i$, then $\tilde{\eta}f - f(x) \prod_{i \neq j} \eta_{\omega_i,b_i}(x)\eta_{\omega_j,b_j} \in C_x$. The second assertion follows from the fact that the spectrum of $W_{\eta_{\omega,b}}$ is the range of $\eta_{\omega,b}$, which is $[-\pi, \pi]\mathrm{i}$.

LEMMA 5.2. *If* $x \notin F = \overline{\mathbf{Z}\omega + b}$, *then* $\sigma([\rho(W_{\eta_{\omega,b}})]_x) = \{\eta_{\omega,b}(x)\}$; *if* $x \in F$, *then* $\sigma([\rho(W_{\eta_{\omega,b}})]_x) \supset [-\pi, \pi]\mathrm{i}$.

Obviously, only the second assertion needs a proof, which, unfortunately, is rather complicated.

Let $\eta : [\omega + b, 2\omega + b) \to \{\pi e^{it} : -\pi/2 \leqslant t < \pi/2\} = C$ be a homeomorphism such that $\eta(\omega + b) = -\pi\mathrm{i}\ (= \eta_{\omega,b}(\omega + b))$ and $\lim_{t \to 2\omega+b-0} \eta(t) = \pi\mathrm{i} = \lim_{t \to 2\omega+b-0} \eta_{\omega,b}(t)$. Then we extend η periodically to $\mathbf{R}$. It is easy to see that $\eta - \eta_{\omega,b} \in \mathrm{CAP}(\mathbf{R})$ and $(\eta - \eta_{\omega,b})|\mathbf{Z}\omega + b = 0$. Hence $[\rho(W_\eta)]_x = [\rho(W_{\eta_{\omega,b}})]_x$ if $x \in F$. Thus it suffices to prove that $\sigma([\rho(W_\eta)]_x) \supset [-\pi, \pi]\mathrm{i}$. Let E be the region enclosed by the semicircle C and $[-\pi, \pi]\mathrm{i}$. For each z in the interior of E, there is $\varepsilon > 0$ such that if we define $\eta_z(t) = \eta(t)$ for $t \in [k\omega + b, (k + 1)\omega + b - \varepsilon]$ and $\eta_z(t)$ a linear function linking $\eta((k + 1)\omega + b - \varepsilon)$ and $-\pi\mathrm{i}$ on $[(k + 1)\omega + b - \varepsilon, (k + 1)\omega + b]$, then $|\eta_z(t) - z| \geqslant \sigma > 0$ on $\mathbf{R}$ and the mean motion of $\eta_z - z$ is not zero. The function η_z so constructed has the property that the range of $(\eta - z)/(\eta_z - z)$ is contained in a sector whose vertex is the origin and whose central angle is strictly less than π. Hence there is $\alpha \in \mathbf{C}$ such that $\|\alpha(\eta - z)/(\eta_z - z) - 1\|_\infty < 1$. This implies that $\rho(W_{(\eta-z)/(\eta_z-z)})$ is invertible and therefore $\mathrm{index}\,\rho(W_{\eta-z}) = \mathrm{index}\,\rho(W_{\eta_z-z})$ minus the mean motion of $\eta_z - z \neq 0$. On the other hand, if ζ is in the exterior of E, it is easy to see that $\mathrm{index}\,\rho(W_{\eta-\zeta}) = 0$. Hence the boundary of E, $C \cup [-\pi, \pi]\mathrm{i}$, is the essential spectrum of $\rho(W_\eta)$. Therefore by the second exact sequence,

$$C \cup [-\pi, \pi]\mathrm{i} = \bigcup_{x \in \mathbf{R}^{\mathbf{B}}} \sigma([\rho(W_\eta)]_x).$$

[1] Here we use $\sqrt{-1}$ instead of i to avoid confusion.

Like $\eta_{\omega,b}$, η can also be extended to a function on $\mathbf{R}^{\mathrm{B}}$, and it is clear that if $x \notin F$, $\sigma([\rho(W_\eta)]_x) = \{\eta(x)\}$. Since $\eta(\omega + b, 2\omega + b) \cap [-\pi, \pi]\mathrm{i} = \emptyset$, it is also clear that $\eta(x) \notin [-\pi, \pi]\mathrm{i}$ if $x \notin F$. Hence

$$[-\pi, \pi]\mathrm{i} \subset \bigcup_{x \in F} \sigma([\rho(W_\eta)]_x).$$

Thus to complete the proof of the lemma, we only need to show that for $x, y \in F$, $\sigma([\rho(W_\eta)]_x) = \sigma([\rho(W_\eta)]_y)$. This, however, is another long story. What we shall do is to show that the translation of symbols by $t \in \mathbf{R}^{\mathrm{B}}$ induces an automorphism on $\mathcal{N}_1$, or equivalently, on $\mathscr{A}(\mathscr{S})$. This automorphism will induce $\mathcal{N}_1/\mathscr{K}_x \cong \mathcal{N}_1/\mathscr{K}_y$ which maps $[W_\eta]_x$ to $[W_\eta]_y$. But the difficulty is that t may not belong to $\mathbf{R}$, thus this automorphism cannot be constructed spatially. Therefore we are forced to consider the harmonic analysis on the Bohr group $\mathbf{R}^{\mathrm{B}}$.

Let $H^2(\mathbf{R}^{\mathrm{B}})$ be the subspace of $L^2(\mathbf{R}^{\mathrm{B}})$ such that the Fourier transforms of its elements are supported in $\mathbf{R}_{\mathrm{d}+}$, the discrete nonnegative numbers. For each $\varphi \in L^\infty(\mathbf{R}^{\mathrm{B}})$, we define $W'_\varphi = P'M_\varphi|H^2(\mathbf{R}^{\mathrm{B}})$ where P' is the orthogonal projection from $L^2(\mathbf{R}^{\mathrm{B}})$ onto $H^2(\mathbf{R}^{\mathrm{B}})$. Let $\mathscr{A}'$ be the C^*-algebra generated by all W'_φ, $\varphi \in C(\mathbf{R}^{\mathrm{B}})$. Proposition 3.3 of [7] asserts that $\alpha : W_\varphi \mapsto W'_\varphi$ extends to an isomorphism from $\mathscr{A}$ to $\mathscr{A}'$. It is easy to check that, like $\tilde{\rho}$, α also satisfies (3.12) and (3.13). Therefore by Lemma 3.14, α can be extended to an isomorphism from $\mathscr{A}(\mathscr{S})$ to $\mathscr{A}'(\mathscr{S})$, the C^*-algebra generated by all W'_φ, $\varphi \in \mathscr{S}$. Since this is essentially a repetition of what we did in Section 3, all the technical details will be omitted. Let $(U_t f)(s) = f(s - t)$ for $f \in L^2(\mathbf{R}^{\mathrm{B}})$ and $s \in \mathbf{R}^{\mathrm{B}}$. Each element in $\mathbf{R}_{\mathrm{d}}$ is identified with an eigenvalue of U_t. Therefore $P'U_t = U_t P'$ and $U_t^{-1} W'_\varphi U_t = W'_{\varphi_t}$ where $\varphi_t(s) = \varphi(s + t)$. Hence $U_t^{-1}\mathscr{A}'(\mathscr{S})U_t = \mathscr{A}'(\mathscr{S})$ if we can prove that $\mathscr{S}$ is closed under the translation. Indeed we only need to show that the translation of an $\eta_{\omega,b}$ is still in $\mathscr{S}$. But for any $\lambda \in \mathbf{R}_{\mathrm{d}} = \widehat{\mathbf{R}^{\mathrm{B}}}$,

$$\int_{\mathbf{R}^{\mathrm{B}}} \eta_{\omega,b}(s + t)\bar{\lambda}(s)\,\mathrm{d}m(s) = \lambda(t) \int_{\mathbf{R}^{\mathrm{B}}} \eta_{\omega,b}(s + t)\overline{\lambda(s + t)}\,\mathrm{d}m(s) = \lambda(t)\,(\eta_{\omega,b}, \lambda)$$

and $(\eta_{\omega,b}, \lambda) \neq 0$ only if $\lambda = \lambda_0^n$, where $\lambda_0(s) = \exp(2\pi\mathrm{i}s/\omega)$ for $s \in \mathbf{R}$. Hence if $\lambda_0(t) = \exp(2\pi\mathrm{i}\tilde{t}/\omega)$ for some $\tilde{t} \in \mathbf{R}$, then the translation of $\eta_{\omega,b}$ by t is $\eta_{\omega,b-\tilde{t}}$. If, in particular, $t = t_1 - t_0$ where $t_1 \in \overline{\mathbf{Z}\omega + b}$ and $t_0 = k\omega + b$, then $\lambda_0(t) = \lim_{n\to\infty} \lambda_0(k_n\omega + b - k\omega - b) = 1$ and therefore the translation of $\eta_{\omega,b}$ by t is $\eta_{\omega,b}$ itself. Hence $\alpha'_t : W'_\varphi \to W'_{\varphi_t}$ extends to an automorphism on $\mathscr{A}'(\mathscr{S})$.

Let $\alpha_t = \rho^{-1} \circ \alpha^{-1} \circ \alpha'_t \circ \alpha \circ \rho$, then α_t is an automorphism on $\mathcal{N}_1$ and $\alpha_t(\rho(W_\varphi)) = \rho(W_{\varphi_t})$ for $\varphi \in \mathscr{S}$. Thus we have proved:

LEMMA 5.3. *For each $t \in \mathbf{R}^{\mathrm{B}}$, the translation $\varphi \mapsto \varphi_t = \varphi(\cdot + t)$ induces an automorphism α_t on $\mathcal{N}_1$. If, in particular, $t = t_0 - t_1$ and $t_0, t_1 \in \overline{\mathbf{Z}\omega + b}$, then $\alpha_t(\rho(W_{\eta_{\omega,b}})) = \rho(W_{\eta_{\omega,b}})$.*

But what we need is the isomorphism induced by α_t. Lemma 5.2 follows from the following lemma immediately.

LEMMA 5.4. *α_t induces an isomorphism*

$$\hat{\alpha}_t : \mathcal{N}_1/\mathcal{K}_x \to \mathcal{N}_1/\mathcal{K}_{x-t}$$

and, in particular, $\hat{\alpha}_{x-y}[\rho(W_\eta)]_x = [\rho(W_\eta)]_y$ *if* $x, y \in \mathbf{Z}\omega + \bar{b}$.

Proof. The second assertion follows from the second assertion of Lemma 5.3. Since α_{-t} obviously induces the inverse of $\hat{\alpha}_t$, we only need to show that α_t induces a homomorphism $\hat{\alpha}_t : \mathcal{N}_1/\mathcal{K}_x \to \mathcal{N}_1/\mathcal{K}_{x-t}$. Clearly, it suffices to show that $\alpha_t(\mathcal{K}_x) \subset \subset \mathcal{K}_{x-t}$, and this becomes obvious if we can show that $\alpha_t\tilde{\mathcal{K}} \subset \tilde{\mathcal{K}}$. Since α_t is continuous in the norm topology and trace class operators are dense in $\tilde{\mathcal{K}}$ (see [7, Lemma 2.3]), we only need to show that if $A \in \tilde{\mathcal{K}}$ is of trace class, then so is $\alpha_t(A)$. Since, for trace class operator A, $\|A\|_1 = \||A^*A|^{1/2}\|_1 = \||A^*A|^{1/4}\|_2^2$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ indicate the trace norm and the Hilbert-Schmidt norm respectively, the problem may be reduced to showing that if B belongs to the Hilbert-Schmidt class, then so does $\alpha_t(B)$ and $\|\alpha_t(B)\|_2 \leqslant \|B\|_2$. Let $B \in \tilde{\mathcal{K}}$ be of Hilbert-Schmidt class. Then by the construction of ρ, there exists a sequence $\{B_n\} \subset \mathcal{N}_1$ such that $\operatorname{s-lim}_{n\to\infty} B_n = B$, $\operatorname{s-lim}_{n\to\infty} \alpha_t(B_n) = \alpha_t(B)$, and each B_n is of the form $\rho(\sum_j \prod_k W_{p_{jk}})$ where p_{jk} are trigonometric polynomials. On $L^2(\mathbf{R}_+ \times \mathbf{R}_d)$, let $P_m = \chi_{(0,m)\times \mathbf{R}_d}$. For each fixed m, $\|B_nP_m\|_2 \leqslant \|B_n\|m \leqslant Mm$ and $\operatorname{s-lim}_{n\to\infty} B_nP_m = BP_m$. The Hilbert-Schmidt class of $\mathcal{N}$ can be regarded as a pre-Hilbert space with the inner product $\langle T, S\rangle = \tau(TS^*)$ which induces the norm $\|\cdot\|_2$. Hence $\{B_nP_m : n \geqslant 0\}$ is a bounded set in that pre-Hilbert space and converges weakly to BP_m. Thus by a convergence theorem due to Mazur, see [13], we can choose $B_{n_1^k}, \ldots, B_{n_k^k}$, $k = 0, 1, \ldots$, such that $\left\{\frac{1}{k}\sum_{j=1}^{k} B_{n_j^k}P_m : k \geqslant 0\right\}$ converges to BP_m in the norm $\|\cdot\|_2$. Hence we can choose a sequence $\{B^m\} \subset \mathcal{N}_1$ such that $\|BP_m - B^mP_m\|_2 \leqslant 1/m$, $\operatorname{s-lim}_{m\to\infty} B^m = B$, $\operatorname{s-lim}_{m\to\infty} \alpha_t(B^m) = \alpha_t(B)$ and each B^m is of the form $\rho(\sum_j \prod_k W_{p_{jk}})$ where p_{jk} are polynomials.

On the space $L^2(\mathbf{R}_+)$, each $\mathcal{F}\rho^{-1}(B^mP_m)\mathcal{F}^{-1}$ has the form of a finite sum $\sum T_{\lambda_j}M_{\varphi_j}$, where $\lambda_i \neq \lambda_j$ if $i \neq j$ and each φ_j is a finite linear combination of $\chi_{(a,b)}$, $0 \leqslant a < b < \infty$. Let β_t be the automorphism on $\mathcal{F}\mathcal{A}(\mathcal{S})\mathcal{F}^{-1}$ induced by α_t. It is easy to see that $\beta_t(T_\lambda) = \lambda(t)T_\lambda$, where $\lambda \in \mathbf{R}_d$ is identified with the corresponding character on $\mathbf{R}^\mathbf{B}$. Thus $\beta_t(\chi_{(0,a)}) = \beta_t(1 - T_aT_{-a}) = 1 - a(t)(-a)(t)T_aT_{-a} =$

$= 1 - T_a T_{-a} = \chi_{(0,a)}$, therefore we also have $\beta_t(M_{\varphi_j}) = M_{\varphi_j}$. Hence

$$\beta_t(\mathscr{F}\rho^{-1}(B^m P_m)^* \rho^{-1}(B^m P_m)\mathscr{F}^{-1}) =$$

$$= \sum_j M_{\overline{\varphi}_j}(-\lambda_j)(t) T_{-\lambda_j} T_{\lambda_j}(\lambda_j)(t) M_{\varphi_j} + \sum_{\lambda \neq 0} M_{\varphi_\lambda} T_\lambda \lambda(t),$$

and $\|\alpha_t(B^m P_m)\|_2 = \|B^m P_m\|_2 \leqslant \|B\|_2 + 1/m$. Since $\operatorname{s-lim}_{m\to\infty} \alpha_t(B^m P_m) = \alpha_t(B)$, we may conclude that $\alpha_t(B)$ is of Hilbert-Schmidt class and $\|\alpha_t(B)\|_2 \leqslant \|B\|_2$. This completes the proof.

To describe the spectrum of $[A]_x$ and $[\rho(A)]_x$ for $A \in \mathscr{A}(\mathscr{S})$, we need to introduce a symbol for A. For each $\varphi \in \mathscr{S}_0$, we define

$$\varphi^\#(t, s) = s\varphi(t-0) + (1-s)\varphi(t)$$

on $\mathbf{R}\times[0,1]$. We equip $\mathbf{R}\times[0,1]$ with the topology generated by open sets $\{t_1\} \times \times (a, b)$, $(t, t')\times[0,1] \cup \{t\}\times(a', 1]$, $(\tilde{t}, \hat{t})\times[0,1] \cup \{\hat{t}\}\times[0, b')$. $\mathbf{R}\times[0,1]$ with this topology will be denoted by X. If $\varphi \in \mathscr{S}_0$, then $\varphi^\#$ is continuous on X. In fact, this is the weakest topology on $\mathbf{R}\times[0,1]$ that makes $\varphi^\#$ continuous. For $A = \sum_j \prod_k W_{\varphi_{jk}}$ where $\varphi_{jk} \in \mathscr{S}_0$, we define $\mathfrak{s}(A) = \sum_j \prod_k \varphi^\#_{jk}$ to be the symbol of A. Here we would like to bring to the reader's attention the fact that in general $(\varphi\psi)^\# \neq \varphi^\#\psi^\#$.

We shall say that $\{F_1, \ldots, F_p\}$ is a collection of discontinuities associated with $A = \sum_j \prod_k W_{\varphi_{jk}} \in \mathscr{A}(\mathscr{S})$ if all φ_{jk} belong to the algebra generated by CAP($\mathbf{R}$) and $\{\eta_{\omega_1,b_1} f_1, \ldots, \eta_{\omega_p,b_p} f_p\}$, and $\overline{\mathbf{Z}\omega_j + b_j} \cap \operatorname{supp} f_j = F_j$. If it happens that $\varphi_{jk} \in S_A$, where S_A is a subalgebra of $\mathscr{S}$ described at the end of Section 1, then we can choose $\{F_1, \ldots, F_p\}$ for A such that $F_l \cap F_n = \emptyset$ if $l \neq n$. The following theorem holds for A with $\varphi_{jk} \in \mathscr{S}$.

THEOREM 5.5. *Let $x \in \mathbf{R}$. If x belongs to at most one F_l, then* $\sigma([A]_x) = \sigma([\rho(A)]_x) = \mathfrak{s}(A)(\{x\}\times[0,1])$.

Proof. The theorem is obviously true if $x \notin \bigcup_{j=1}^{p} F_j$. Suppose $x \in F_l \setminus \bigcup_{j\neq l} F_j$. Then by Lemma 5.1 and Lemma 5.2, $[A]_x = p([W_{\eta_l}]_x)$ and $[\rho(A)]_x = p([\rho(W_{\eta_l})]_x)$ where p is a polynomial such that $\mathfrak{s}(A)(x, s) = p(s)$ and $\eta_l = \eta_{\omega_l, b_l}$. The spectral mapping theorem and Lemma 5.2 say that

$$\sigma([A]_x) \subset p([-\pi, \pi]\mathrm{i}) \subset \sigma([\rho(A)]_x).$$

Because $\rho\mathscr{C} \subset \tilde{\mathscr{K}}$, it is obvious that $\rho\mathscr{C}_x \subset \mathscr{K}_x$. Therefore ρ induces a homomorphism from $\mathscr{A}(\mathscr{S})/\mathscr{C}_x$ onto $\mathscr{N}_1/\tilde{\mathscr{K}}$. Hence $\sigma([\rho(A)]_x) \subset \sigma([A]_x)$. Taking into account the other inclusion, we have $\sigma([\rho(A)]_x) = \sigma([A]_x) = p([-\pi, \pi]i)$. But $\mathfrak{s}(A)(x, s) =$ $= p(s\eta_l(x-0) + (1-s)\eta_l(x)) = p(s\pi i + (1-s)(-\pi i))$. This gives the proof.

Let $[A]$ ($[\rho(A)]$) denote the equivalent class of A in $\mathscr{A}(\mathscr{S})/\mathscr{C}$ (resp. $\mathscr{N}_1/\tilde{\mathscr{K}}$).

COROLLARY 5.6. *If $F_l \cap F_n = \emptyset$ for $l \neq n$, then $\sigma([A]) = \sigma([\rho(A)]) = \overline{\mathfrak{s}(A)(X)}$ and $\|[A]\| = \|[\rho(A)]\| = \|\mathfrak{s}(A)\|_\infty$.*

Proof. Let $\omega = \omega_l$ and $b = b_l$. Then we can write

$$\mathfrak{s}(A) = u_0 + u_1\eta^{\#}_{\omega,b} + \ldots + u_m(\eta^{\#}_{\omega,b})^m$$

where each u_i has the form $\sum_\alpha \prod_\beta v^{\#}_{\alpha\beta}$ for some $v_{\alpha\beta} \in \mathscr{S}$ and the discontinuities of $v_{\alpha\beta}$ are contained in $\bigcup_{n\neq l} F_n$. If $x \in F = F_l$, then there exist $\{k_n\} \subset \mathbb{Z}$ such that $k_n\omega +$ $+ tb \to x$ in $\mathbf{R}^{\mathrm{B}}$. $u_i(k_n\omega + b, \cdot)$, $i = 1, \ldots, m$, are constants on $[0, 1]$ and limits $u_i(x) = \lim_{m\to\infty} u_i(k_n\omega + b, s)$ exist. Let $p_n(z) = \sum_{i=0}^m u_i(k_n\omega + b, s)z^i$ and $p(z) = \sum_{i=0}^m u_i(x)z^i$. Then $[A]_{k_n\omega+b} = p_n([W_{\eta_{\omega,b}}]_{k_n\omega+b})$ and $[A]_x = p([W_{\eta_{\omega,b}}]_x)$. Therefore by Lemma 5.2 and Theorem 5.5, $\sigma([A]_x) = p([-\pi, \pi]i) \subset \overline{\bigcup_n p_n([-\pi, \pi]i)} = \overline{\bigcup_n \mathfrak{s}(A)(\{k_n\omega + b\} \times \times [0, 1])} \subset \overline{\mathfrak{s}(A)(X)}$. Since $\{F_1, \ldots, F_p\}$ are pairwise disjoint, we can apply this argument to other $\omega = \omega_n$ and $b = b_n$. Thus $\sigma([A]_x) \subset \overline{\mathfrak{s}(A)(X)}$ for $x \in F_1 \cup \ldots$ $\ldots \cup F_p$. If $x \notin \bigcup_{l=1}^p F_l$, then certainly $\sigma([A]_x) \subset \overline{\mathfrak{s}(A)(X)}$. Hence $\overline{\bigcup_{x\in\mathbf{R}^{\mathrm{B}}} \sigma([A]_x)} \subset$ $\subset \overline{\mathfrak{s}(A)(X)}$. But $[A]$ is invertible if and only if each $[A]_x$ is, see [9], Proposition 4.5, so $\sigma([\rho(A)]) \subset \sigma([A]) \subset \overline{\mathfrak{s}(A)(X)}$. But Theorem 5.5 implies that $\sigma([\rho(A)]) \supset \overline{\mathfrak{s}(A)(X)}$. This proves the spectral equality. Note that $\{F_1, \ldots, F_p\}$ is also a collection of discontinuities associated with A^*A. The norm equality follows if we apply the same argument to A^*A.

It was illustrated at the end of Section 1 that we can construct various subalgebras S_A of $\mathscr{S}$ with CAP(R) and $\Lambda = \{\eta_{\omega_i,b_i}\delta_i : i \in I\}$ such that if $A = \sum_j \prod_k W_{\varphi_{jk}}$ with $\varphi_{jk} \in S_A$, then the collection of discontinuities $\{F_1, \ldots, F_p\}$ associated with A has the property that $F_l \cap F_n = \emptyset$ if $l \neq n$. Let $C(S_A)$ be the C^*-subalgebra of $C(X)$, the continuous functions on X, generated by all $\varphi^{\#}$, $\varphi \in S_A$. Originally, $\mathfrak{s}(A)$ is defined only for $A = \sum_j \prod_k W_{\varphi_{jk}}$. But the norm equality of Corollary 5.6 says that $\mathfrak{s}(\cdot)$ can be naturally and uniquely extended to the whole $\mathscr{A}(S_A)$.

Combining Theorem 4.5 and Corollary 5.6, we have:

THEOREM 5.7. *$\mathscr{A}(S_A)/\mathscr{C}$ is isomorphic to $C(S_A)$ via the symbol map $A \mapsto \mathfrak{s}(A)$. Furthermore, $\rho(A)$ is Fredholm in $\mathscr{N}$ if and only if $0 \notin \overline{\mathfrak{s}(A)(X)}$.*

Proof. Only the last sentence needs proof. Obviously, the essential spectrum of $\rho(A)$ in $\mathscr{N}$ is contained in $\overline{\mathfrak{s}(A)(X)} = \sigma([\rho(A)])$. But on the other hand, if $\rho(A) - \lambda$ is Fredholm in $\mathscr{N}$, since $\mathscr{N}_1/\mathscr{K}$ is a C^*-subalgebra of $\mathscr{N}/\mathscr{K}$, there exist $T \in \mathscr{N}_1$ and $K_1, K_2 \in \mathscr{K}$ such that $T(\rho(A) - \lambda) = 1 + K_1$ and $(\rho(A) - \lambda)T = 1 + K_2$. This means that $K_1, K_2 \in \mathscr{K} \cap \mathscr{N}_1 = \tilde{\mathscr{K}}$, therefore $\lambda \notin \sigma([\rho(A)])$. This proves the theorem.

COROLLARY 5.8. *The sequence*

$$0 \to \mathscr{C} \to \mathscr{A}(S_A) \to \bigoplus_{x \in \mathbf{R}} \mathscr{A}(\mathscr{S})/\mathscr{C}_x$$

is exact.

REMARK 5.9. We could have defined symbol $\mathfrak{s}(A)$ as a function on $\mathbf{R}^{\mathbf{B}} \times [0, 1]$. In fact, if we let $\eta^{\#}_{\omega,b}(x, s) = \eta_{\omega,b}(x)$ for $x \notin F = \overline{\mathbf{Z}\omega + b}$ and $\eta^{\#}_{\omega,b}(x, s) = s(\pi_i) + (1 - s)(-\pi_i)$ for $x \in F$, then for each $\varphi \in \mathscr{S}_0$, $\varphi^{\#}$ is a well-defined function on $\mathbf{R}^{\mathbf{B}} \times [0, 1]$. For $A = \sum_j \prod_k W_{\varphi_{jk}}$, we define $\tilde{\mathfrak{s}}(A) = \sum_j \prod_k \varphi^{\#}_{jk}$ as a function on $\mathbf{R}^{\mathbf{B}} \times [0, 1]$. Then Theorem 5.5 can be restated as "*if* $x \in \mathbf{R}^{\mathbf{B}}$ *belongs to at most one* F_l, *then* $\sigma([A]_x) = \sigma([\rho(A)]_x) = \tilde{\mathfrak{s}}(A)(\{x\} \times [0, 1])$" and the proof is exactly the same. With this version of symbol the statement of Corollary 5.6 can be made stronger: $\sigma([A]) = \sigma([\rho(A)]) = \tilde{\mathfrak{s}}(A)(\mathbf{R}^{\mathbf{B}} \times [0,1])$ (see [9], Proposition 4.5). Also, $\rho(A)$ is Fredholm in $\mathscr{N}$ if and only if $0 \notin \tilde{\mathfrak{s}}(A)(\mathbf{R}^{\mathbf{B}} \times [0, 1])$. The reason we did not do so is that it is difficult to choose an appropriate topology on $\mathbf{R}^{\mathbf{B}} \times [0, 1]$ that makes $\eta^{\#}_{\omega,b}$ a continuous function. What causes the difficulty is the accumulation of $\mathbf{Z}\omega + b$ in $\mathbf{R}^{\mathbf{B}}$. Of course one can always choose the weakest topology on $\mathbf{R}^{\mathbf{B}} \times [0, 1]$ that makes all $\eta^{\#}_{\omega,b}$ continuous. But since one still does not know what the open sets are, this version of symbols hardly provides any information about the maximal ideal space of $\mathscr{A}(S_A)/\mathscr{C}$. This is why we defined the symbols as functions on X.

6. MEAN MOTION AND INDEX

For a piecewise continuous function φ_{jk} on $\mathbf{R}$, there are continuous determinations of $\arg \psi(x, s)$ on X if $\psi(x, s) = \sum_j \prod_k \varphi^{\#}_{jk}(x, s)$ does not vanish. We define mean motion of ψ to be the limit

$$\lim_{T \to \infty} \frac{1}{2T} (\arg \psi(T, s) - \arg \psi(-T, s))$$

if it exists. Obviously, if φ_{jk} are continuous almost periodic functions, then the mean motion of $\psi(x, s)$ exists and coincides with the mean motion of $\psi(x) = \sum_j \prod_k \varphi_{jk}(x)$. Let S_A be a subalgebra as described at the end of Section 5 and let $\operatorname{inv} C(S_A)$ denote the invertible elements in $C(S_A)$. Let $\psi \in \operatorname{inv} C(S_A)$, then there exist $\{\psi_n\} \subset \operatorname{inv} C(S_A)$ such that $\lim_{n\to\infty} \|\psi - \psi_n\|_\infty = 0$ where each ψ_n has the form $\sum_j \prod_k \varphi_{jk}^\#$, $\varphi_{jk} \in S_A$. We can choose $\arg\psi_n$ for each n such that at a fixed point x_0, $|\arg\psi_n(x_0, s) - \arg\psi_m(x_0, s)| < \pi$. It is then easy to see that $\{\arg\psi_n\}$ converges to a continuous function $\varphi(x, s)$ on X and, obviously, this function can be regarded as an $\arg\psi(x, s)$. Therefore we can extend the definition of mean motion to functions in $\operatorname{inv} C(S_A)$, subject to the existence of the limit.

THEOREM 6.1. *For each $\psi \in \operatorname{inv} C(S_A)$, the mean motion exists. Moreover, if $f \in \mathrm{CAP}(\mathbf{R})$ is invertible and has the same mean motion, then there exists a path in $\operatorname{inv} C(S_A)$ joining ψ and $f^\#(x, s) = f(x)$.*

Proof. Suppose that $\psi = \sum_j \prod_k \varphi_{jk}$, where $\varphi_{jk} \in S_A$. As before, we write

$$\psi = u_0 + u_1\eta^\#_{\omega,b} + \ldots + u_m(\eta^\#_{\omega,b})^m,$$

where $u_1, \ldots, u_m$ have the property that there exists an open set U containing $F = \overline{\mathbf{Z}\omega + b}$, such that for each $y \in U$, $u_i(y, \cdot)$ is constant on $[0, 1]$ and $u_i(y, s)$ is continuous on U, $i = 1, \ldots, m$. Let $\eta_n(x)$ be $\eta_{\omega,b}(x)$ if $x \notin \bigcup_{k\in\mathbf{Z}} [k\omega + b - 1/n, k\omega+b]$ and the linear function joining $\eta_{\omega,b}(k\omega + b - 1/n)$ and $-\pi\mathrm{i}$ $(=\eta_{\omega,b}(k\omega + b))$ if $x \in [k\omega + b - 1/n, k\omega + b]$. If $\psi \in \operatorname{inv} C(S_A)$, then there exists $\delta > 0$ such that $d(0, p_k([-\pi, \pi]\mathrm{i})) \geqslant \delta$, where $p_k(z) = u_0(k\omega + b, s) + u_1(k\omega + b, s)z + \ldots + u_m(k\omega + b, s)z^m$. By the continuity of u_i near F, there exists $\varepsilon > 0$ such that if $|y - (k\omega + b)| < \varepsilon$ for some k, then $|u_0(y, s) + \ldots + u_m(y, s)z^m| \geqslant (1/2)\delta$ for all $z \in [-\pi, \pi]\mathrm{i}$. Let $1/n < \varepsilon$. Since $\eta_n(\mathbf{R}) = [-\pi, \pi]\mathrm{i}$, it is obvious that for $t \in [0, 1]$,

$$|u_0(y, s) + u_1(y, s)(t\eta_{\omega,b} + (1 - t)\eta_n)^\#(y, s) + \ldots$$

$$\ldots + u_m(y, s)(t\eta_{\omega,b} + (1 - t)\eta_n)^{\# m}(y, s)| \geqslant \frac{1}{2}\delta$$

for all $y \in \mathbf{R}$. Hence we have actually presented a path in $\operatorname{inv} C(S_A)$ joining ψ and ψ_0 which does not have $\mathbf{Z}\omega + b$ in its discontinuities. Also note that $\arg\psi(x, s) = \arg(u_0(x, s) + \ldots + u_m(x, s)(\eta(x))^m) + 2h\pi$ for $x \notin [k\omega + b - 1/n, k\omega + b]$ and fixed $h \in \mathbf{Z}$. Therefore the theorem follows from a routine application of the induction.

By this theorem, for each element in $\operatorname{inv} C(S_A)$, the mean motion is a complete set of homotopy invariant. Thus, naturally for each $A \in \mathrm{Fred}(\mathscr{A}(S_A), \mathscr{C})$, we define the topological index t-ind(A) to be the minus mean motion of $\mathfrak{s}(A)$. As usual, we define the analytic index a-ind(A) = ind$(\rho(A))$, the index of $\rho(A)$ in $\mathcal{N}$. Since t-ind(A) = = a-ind(A) if $A \in$ Fred$(\mathcal{A}, \mathcal{C})$ (see [7], Theorem 2.2), we have:

THEOREM 6.2. *$A \in$ Fred$(\mathcal{A}(S_\Lambda), \mathcal{C})$ if and only if $\mathfrak{s}(A)$ is invertible in $C(S_\Lambda)$ and if this is the case*

$$\text{a-ind}(A) = \text{t-ind}(A).$$

We can derive the similar results for Wiener-Hopf operators with matrix symbol. For example, for each positive integer n we have a diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{C}\otimes M_n & \to & \mathcal{A}(S_\Lambda)\otimes M_n & \to & C(S_\Lambda)\otimes M_n & \to 0 \\ & \downarrow & & \downarrow \rho & & \downarrow & \\ 0 \to & \mathcal{K}\otimes M_n & \longrightarrow & \mathcal{N}\otimes M_n & \to & \mathcal{N}/\mathcal{K}\otimes M_n & \to 0 \end{array}$$

where M_n is the collection of all $n \times n$ matrices, the horizontal sequences are exact and the vertical arrows are injective. In particular, if $\varphi \in S_\Lambda \otimes M_n$ and $|\det \varphi^\#| \geqslant \geqslant \delta > 0$, then W_φ belongs to Fred$(\mathcal{A}(S_\Lambda)\otimes M_n, \mathcal{C}\otimes M_n)$ and the analytical index of $\rho(W_\varphi)$ is minus the mean motion of $\det \varphi^\#$. Since these are consequences of standard abstract nonsense plus Theorem 6.1, we omit the details.

7. REMARKS AND PROBLEMS

We have developed the index theory for algebras $\mathcal{A}(S_\Lambda)$. One would presume that it should be possible to develop the index theory for $\mathcal{A}(\mathcal{S})$ in a similar pattern. However, this one step further encounters a number of substantial difficulties. The following are the main problems.

First, we should decide which ideal $\mathcal{I}$ in $\mathcal{A}(\mathcal{S})$ gives the natural Fredholm structure. By Theorem 4.4, there are, at least apparently, three ideals in $\mathcal{A}(\mathcal{S}) : \mathcal{C} \subset \subset \mathcal{C}(\mathcal{S}) \subset \rho^{-1}\mathcal{K}$. The quotient algebra is expected to provide topological invariants. Intuitively, $\mathcal{I} = \mathcal{C}(\mathcal{S})$ is a natural choice. It is also reasonable to expect, no matter what $\mathcal{I}$ is, that if $\rho(W_\varphi W_\psi - W_{\varphi\psi}) \in \mathcal{K}$ then $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{I}$. On the other hand, we know that $\rho(W_{\eta_{\omega,b}} W_{\eta_{\lambda,c}} - W_{\eta_{\omega,b}\eta_{\lambda,c}}) \in \mathcal{K}$ if ω and λ are linearly independent over $\mathbf{Q}$; but we have not yet been able to prove that $W_{\eta_{\omega,b}} W_{\eta_{\lambda,c}} - - W_{\eta_{\omega,b}\eta_{\lambda,c}} \in \mathcal{C}(\mathcal{S})$ not its contrary. This seems to bring up the doubt that $\mathcal{C}(\mathcal{S})$ is the most suitable choice. However we have reason to believe that actually $\mathcal{C} = = \rho^{-1}\mathcal{K}$. But until this (or the contrary) is proved, which ideal to choose remains a question.

Second, how do we define symbols for elements in $\mathcal{A}(\mathcal{S})$. For $A = \sum_j \prod_k W_{\varphi_{jk}} \in \in \mathcal{A}(\mathcal{S})$, we cannot simply define $\mathfrak{s}(A) = \sum_j \prod_k \varphi_{jk}^\#$ with $\varphi_{jk}^\#(x, s)$ being $s\varphi_{jk}(x-0) +$

$+(1-s)\varphi_{jk}(x)$ any more. For example, let $A := W_{\eta_{\omega,0}} W_{\eta_{\lambda,0}} - W_{\eta_{\omega,0}\eta_{\lambda,0}}$ where ω and λ are linearly independent over $\mathbf{Q}$. Then simple calculation shows that

$$[\eta^{\#}_{\omega,0}\eta^{\#}_{\lambda,0} - (\eta_{\omega,0}\eta_{\lambda,0})^{\#}] = \begin{cases} 0 & \text{if } x \neq 0 \\ -\pi(2s-1)^2 + \pi i(2s-1) & \text{if } x = 0. \end{cases}$$

But $\rho(A) \in \mathscr{K}$, so the correct symbol of A should be identically zero however it is defined. What causes the difficulty here is that the collection of discontinuities $\{F_1, \ldots, F_p\}$ associated with A does not have pairwise empty intersections. But we believe that off the intersections $F_l \cap F_n$, the symbol of A is given by our formula in Section 5 and that on the intersections, we can "reduce" the symbol defined in Section 5 in certain appropriate ways to obtain the correct symbol for $A \in \mathscr{A}(\mathscr{S})$.

Third, the essential spectrum calculation for $A \in \mathscr{A}(\mathscr{S})$ is extremely complicated. Again, the difficulty comes from the intersections $F_l \cap F_n$. If, say, $x \in F_l \cap \cap F_n \setminus \bigcup_{j \neq n,l} F_j$ then simple calculation shows that $[\rho(A)]_x = p([W_\eta]_x, [W_\zeta]_x)$ where p is a polynomial in two variables and η and ζ have the discontinuities F_l and F_n respectively. So the only spectral information which we can obtain is that $\sigma([\rho(A)]_x) \supset \supset \{p(a(s), b(s)) : s \in \mathscr{M}_x\}$ where $\mathscr{M}_x$ is the maximal ideal space of $\mathscr{N}_1/\mathscr{K}_x$ and $a(\mathscr{M}_x) = b(\mathscr{M}_x) = [-\pi, \pi]i$. Unless $[W_\eta]_x$ commutes with $[W_\zeta]_x$, one would expect that $\sigma([A]_x)$ is even more complicated.

Finally, what is the appropriate topological index for $\mathscr{A}(\mathscr{S})$. We can prove that if $\varphi_{jk} \in \mathscr{S}_0$ is not in the closure of the range of $\psi = \sum_j \prod_k \varphi^{\#}_{jk}$, then the mean motion

$$\lim_{T\to\infty} \frac{1}{2T} (\arg\psi(T, s) - \arg\psi(-T, s))$$

still exists. Since we have not yet been able to connect this mean motion with any topological invariant when $\varphi_{jk} \in \mathscr{S}_0$, the proof of the existence of the above limit will be presented elsewhere. But the question here is that does this mean motion give any kind of index. It is unlikely that for $\varphi_{jk} \in \mathscr{S}_0$, the mean motion constitutes a complete set of homotopy invariant. Nevertheless, we still expect that minus the mean motion coincides with the analytical index of $A = \sum_j \prod_k W_{\varphi_{jk}}$ if it is Fredholm.

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