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TRANSITIVE ALGEBRA CONTAINING TRIANGULAR OPERATOR MATRICES

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1. INTRODUCTION

Arveson [1] introduced the notion of transitive operator algebra, that is an algebra with no nontrivial invariant subspaces. He proved that a transitive operator algebra containing either the unilateral shift or a maximal abelian von Neumann algebra is strongly dense in the algebra of all operators. A number of other authors using Arveson's Lemma obtained similar results for a transitive operator algebra satisfying some additional hypothesis [2], [4], [5], [6], [7], [8], [9], [10], and [12, Theorem 6].

By presenting a new technique in this paper, we generalize the results of [1] and [10] to prove that if a transitive operator algebra $\mathcal{U}$ either contains a perturbation of the unilateral shift by certain rank one operators, or contains a certain triangular operator matrix, then $\mathcal{U}$ is strongly dense in the algebra of all operators. We will conclude by listing a number of open question which arise from our work.

2. MAIN RESULTS

Let $\mathcal{H}$ denote an infinite dimensional, separable, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all operators on $\mathcal{H}$.

DEFINITION. A set $\mathcal{A}$ of operators on $\mathcal{H}$ is said to have the *transitive algebra property* (TAP) if whenever $\mathcal{U}$ is a transitive operator algebra such that $\mathcal{A} \subset \mathcal{U}$, $\mathcal{U}$ is strongly dense in $\mathcal{L}(\mathcal{H})$. The operator $A \in \mathcal{L}(\mathcal{H})$ is said to have the TAP if the set $\{A\}$ has the TAP. It is easy to prove that if $\mathcal{A} \cup \{1\}$ has the TAP, then $\mathcal{A}$ has the TAP.

If $\mathcal{U}$ is an operator algebra on $\mathcal{H}$, we will write $\mathcal{U}_s$ for the closure of $\mathcal{U}$ in the strong operator topology of $\mathcal{L}(\mathcal{H})$.

THEOREM 2.1. *Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces, and let $\mathcal{A}$ be a subset of $\mathcal{L}(\mathcal{H}\oplus\mathcal{K})$ consisting of operators of the form*

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

with the following condition

$$(\dagger) \qquad \bigvee \left\{ A\mathcal{H} + B\mathcal{K} : \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \in \mathcal{A} \right\} = \mathcal{H}.$$

If T is an operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{K})$, if $\mathcal{H} \in \operatorname{Lat} T$, and if $T|\mathcal{H}$ has the TAP, *then the set $\mathcal{A} \cup \{T\}$ has the* TAP.

Proof. Let $\mathcal{U}$ be a transitive operator algebra such that $\mathcal{A} \cup \{T\} \subset \mathcal{U}$. Define the set $\mathcal{B} = \left\{X \in \mathcal{L}(\mathcal{H}) : \text{there are operators } Y \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \text{ and } Z \in \mathcal{L}(\mathcal{K})\right.$ such that the operators $\left.\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{U}\right\}$. It is clear from the definition of $\mathcal{B}$ that $\mathcal{B}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ which contains the operator $T|\mathcal{H}$.

We will now prove that $\mathcal{B}$ is transitive. To this end, let $f, g \in \mathcal{H}$ with $f \neq 0$, and let ε be positive. It follows from condition ($\dagger$) that there exists a positive integer n, operators $C_i = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$, and vectors $h_i \in \mathcal{H}$, $k_i \in \mathcal{K}$ $(i = 1, \ldots, n)$ such that

$$(1) \qquad \left\| \sum_{i=1}^{n} (A_i h_i + B_i k_i) - g \right\| < \varepsilon/2.$$

Since $\mathcal{U}$ is transitive, there exist operators $D_i = (D_{jk}^{(i)}) \in \mathcal{U}$ such that

$$(2) \qquad \| D_i(f, 0) - (h_i, k_i) \| < (\varepsilon/2n)\|C_i\|$$

for $i = 1, \ldots, n$. It is easy to see that

$$C_i D_i = \begin{bmatrix} A_i D_{11}^{(i)} + B_i D_{21}^{(i)} & A_i D_{12}^{(i)} + B_i D_{22}^{(i)} \\ 0 & 0 \end{bmatrix};$$

therefore $E_i = (A_i D_{11}^{(i)} + B_i D_{21}^{(i)}) \in \mathcal{B}$ for $i = 1, \ldots, n$. From (2) we conclude that

$$(3) \qquad \| E_i f - (A_i h_i + B_i k_i) \| < \varepsilon/2n$$

for $i = 1, \ldots, n$. Now (1) and (3) together imply that

$$(4) \qquad \left\| \left(\sum_{i=1}^{n} E_i \right)(f) - g \right\| < \varepsilon;$$

therefore $\mathcal{B}$ is transitive.

Now $\mathscr{B}$ is a transitive operator algebra and contains $T|\mathscr{H}$, which has the TAP; hence

$$\mathscr{B}_s = \mathscr{L}(\mathscr{H}). \tag{5}$$

Nonzero compact operators have the TAP [11]; therefore to complete the proof it suffices to show that $\mathscr{U}_s$ contains a nonzero compact operator. To this end, let $C = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ be a fixed nonzero operator in $\mathscr{A}$, and let $K_0 \in \mathscr{L}(\mathscr{H})$ be a compact operator such that either

$$K_0 A \neq 0 \quad \text{or} \quad K_0 B \neq 0. \tag{6}$$

If we define operators $K_1 = K_0 \oplus 0$ and $K = K_1 C$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{K})$, then it follows from (6) that K is a nonzero compact operator. We will now show that $K \in \mathscr{U}_s$. To this end, let $(f, g) \in \mathscr{H} \oplus \mathscr{K}$ and let ε be positive. Since $\mathscr{B}_s = \mathscr{L}(\mathscr{H})$, there exists $X \in \mathscr{B}$ such that

$$\|((X - K_0)(Af + Bg), 0)\| < \varepsilon. \tag{7}$$

We now choose Y and Z such that $R = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathscr{U}$; hence $RC = \begin{bmatrix} XA & XB \\ 0 & 0 \end{bmatrix} \in \mathscr{U}$ and

$$(RC - K)(f, g) = (R - K_1)(Af + Bg, 0) = ((X - K_0)(Af + Bg), 0). \tag{8}$$

It now follows from (7) and (8) that

$$\|(RC - K)(f, g)\| < \varepsilon;$$

hence $K \in \mathscr{U}_s$. Therefore the set $\mathscr{A} \cup \{T\}$ has the TAP, as was to be shown. ▨

REMARK. It follows from the above theorem that $\mathscr{A}$ has the TAP if (†) is satisfied and if $\mathscr{A}$ contains an operator $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ in which A has the TAP.

An operator A is said to have the property (P) if A satisfies the conditions:

(i) A has the TAP.

(ii) A does not have eigenvalues.

We note that every injective weighted shift satisfies condition (ii) [4, Problem 78]. Therefore, unilateral shift of finite multiplicity and strictly cyclic injective weighted shift have the property (P). In particular, a weighted shift with the weight sequence $\{\exp(\sqrt{n+1} - \sqrt{n})\}_{n=0}^{\infty}$ has the property (P) [12, page 103].

THEOREM 2.2. *If $A \in \mathscr{L}(\mathscr{H})$ has the property* (P) *and if $B \in \mathscr{L}(\mathscr{K})$ is algebraic, then*

$$T = \begin{bmatrix} A & 0 \\ X & B \end{bmatrix}$$

has the TAP *for every $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$.*

Proof. It is easy to see that an operator has the TAP if and only if its adjoint does. Therefore, it suffices to show that T^* has the TAP. Since B is algebraic there exists a polynomial $P(\lambda) = c\prod_{i=1}^{n}(\lambda - \lambda_i)$ such that $P(B) = 0$; thus $P(T)$ has the matrix form $P(T) = \begin{bmatrix} P(A) & 0 \\ Y & 0 \end{bmatrix}$ for some $Y \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. Since A does not have eigenvalues, $P(A) = c\prod_{i=1}^{n}(A - \lambda_i)$ is injective; therefore $P(A)^*$ has dense range. Hence we have $P(A)^*\mathscr{H} \vee Y^*\mathscr{K} = \mathscr{H}$. It now follows from Theorem 2.1 that the set $\{T^*, P(T)^*\}$ has the TAP; hence T^* has the TAP. ▨

COROLLARY 2.3. *If U denotes a unilateral shift of finite multiplicity and if F is a finite rank operator, then $U \oplus F$ has the* TAP.

A completely non-unitary contraction A is a C_0-operator if there exists an inner function $f \in H^\infty(\mathbf{D})$ such that $f(A) = 0$.

THEOREM 2.4. *If $U \in \mathscr{L}(\mathscr{H})$ is a unilateral shift of finite multiplicity and if A is a C_0-operator, then $U \oplus A$ has the* TAP.

Proof. It suffices to show that $(U \oplus A)^*$ has the TAP. To this end, let $\mathscr{U}$ be a transitive operator algebra which contains $U \oplus A$. There exists an inner function $f \in H^\infty(\mathbf{D})$ such that $f(A) = 0$. Since $U \oplus A$ is a completely non-unitary contraction $f(U \oplus A)$ is well defined. It is easy to show that

$$f(U \oplus A) = f(U) \oplus 0 \in \mathscr{U}_s .$$

Since f is an inner function, it follows that $f(U)$ is an isometry; in particular $f(U)$ is injective; therefore $f(U)^*$ has dense range. It now follows from Theorem 2.1 that the set $\{(U \oplus A)^*, f(U \oplus A)^*\}$ has the TAP: hence $(U \oplus A)^*$ has the TAP. ▨

We will now prove that perturbations of the simple shift by certain rank one operators have the TAP. To this end, let $S \in \mathscr{L}(H^2)$ be the simple shift and let $\{e_n\}_{n=0}^{\infty}$ be the usual orthonormal basis for H^2. For $f, g \in H^2$, define the rank one operator $f \otimes g$ by

$$(f \otimes g)(h) = (h, g)f, \quad h \in H^2.$$

THEOREM 2.5. (i) *The operator $S + e_0 \otimes e_n$ has the* TAP *for every $n \geq 0$.* (ii) *The operator $S + f \otimes e_0$ has the* TAP *for every $f = \sum_{n=1}^{\infty} d_n e_n$ with $d_1 + 1 \neq 0$.*

Proof. (i) If m is a positive integer, then we have

$$(S + e_0 \otimes e_n)(e_m) = Se_m + (e_0 \otimes e_n)e_m =$$

$$= e_{m+1} + (e_m, e_n)e_0 = e_{m+1} + \delta_{mn}e_0 ,$$

where δ_{mn} is the Kronecker δ. Thus the subspace

$$H_n = \bigvee \{e_k : k \geqslant n+1\}$$

is invariant for $S + e_0 \otimes e_n$ and $(S + e_0 \otimes e_n)|H_n = S|H_n$. If $S_n = S|H_n$, then S_n is a simple shift. Now the operator $S + e_0 \otimes e_n$ has the matrix form

$$S + e_0 \otimes e_n = \begin{bmatrix} S_n & A \\ 0 & B \end{bmatrix}$$

with repect to the decomposition $H^2 = H_n \oplus H_n^\perp$, where

$$Ae_k = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant n-1 \\ e_{k+1} & \text{if } k = n \end{cases}$$

and

$$Be_k = \begin{cases} e_{k+1} & \text{if } 0 \leqslant k \leqslant n-1 \\ e_0 & \text{if } k = n \end{cases} .$$

It easily follows from the definition of B that $B^{n+1} = 1$; thus

$$(S + e_0 \otimes e_n)^{n+1} - 1 = \begin{bmatrix} S_n^{n+1} - 1 & c_n \\ 0 & 0 \end{bmatrix},$$

where $C_n = \sum_{i=0}^{n} S_n^{n-i} A B^i$. We will show that

$$(\dagger) \qquad (S_n^{n+1} - 1)(H_n) \vee C_n(H_n^\perp) = H_n .$$

We observe that if $0 \leqslant k \leqslant n$, then

$$B^i e_k = \begin{cases} e_{k+1} & \text{if } 0 \leqslant i \leqslant n-k \\ e_{k+i-n-1} & \text{if } n-k+1 \leqslant i \leqslant n. \end{cases}$$

Thus for $0 \leqslant i \leqslant n$ and $i \neq n-k$, we have

$$AB^i e_k = 0;$$

therefore $C_n e_k = e_{n+k+1}$ for $0 \leqslant k \leqslant n$. This fact, together with

$$(S_n^{n+1} - 1)e_{n+1} = e_{2n+2} - e_{n+1} ,$$

proves (†). Thus by Theorem 2.1, the set $\{S + e_0 \otimes e_n,\ (S + e_0 \otimes e_n)^{n+1} = 1\}$ has the TAP; hence T has the TAP.

(ii) If $H_1 = \bigvee \{e_k : k \geqslant 1\}$ denotes the range of S, then H_1 is invariant for $S + f \otimes e_0$, $(S + f \otimes e_0)|H_1 = S|H_1$, and $f \in H_1$. If $S_1 = S|H_1$, then S_1 is a simple shift. The operator $S + f \otimes e_0$ has the matrix form

$$S + f \otimes e_0 = \begin{bmatrix} S_1 & D \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $H^2 = H_1 \oplus H_1^{\perp}$, where $De_0 = e_1 + f$. Since $d_1 + 1 \neq 0$ and $S_1 e_k = e_{k+1}$ for $k \geqslant 1$, it follows that

$$S_1 H_1 \vee D H_1^{\perp} = H_1 .$$

Thus by Theorem 2.1, the operator $S + f \otimes e_0$ has the TAP, as was to be shown. ▨

Let $T \in \mathscr{L}(\mathscr{H})$ be an injective weighted shift, where $Te_n = w_n e_{n+1}$, $n \geqslant 0$, and $\mathscr{H}_n = \bigvee \{e_k : k \geqslant n\}$. The subspace $\mathscr{H}_n$ is invariant for T and $T_n = T|\mathscr{H}_n$ is an injective weighted shift with weight sequence $\{w_k\}_{k=n}^{\infty}$. It was proved by Allen Shields [12, Proposition 33] that if T is strictly cyclic and if T is bounded below, then T_n is strictly cyclic. We will now establish the following result.

THEOREM 2.6. *If the operator T_n has the* TAP *for some n, then T has the* TAP.

Proof. The operator T has the matrix form

$$T = \begin{bmatrix} T_n & A \\ 0 & B \end{bmatrix}$$

with respect to the decomposition $\mathscr{H} = \mathscr{H}_n \oplus \mathscr{H}_n^{\perp}$ where

$$Ae_k = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant n-2 \\ w_k e_{k+1} & \text{if } k = n-1 \end{cases}$$

and

$$Be_k = \begin{cases} w_k e_{k+1} & \text{if } 0 \leqslant k \leqslant n-2 \\ 0 & \text{if } k = n-1 \end{cases} .$$

It is easy to see that $B^n = 0$ and

$$T^n = \begin{bmatrix} T_n^n & C_n \\ 0 & 0 \end{bmatrix},$$

where $C_n = \sum_{i=0}^{n-1} T_n^{n-i-1}AB^i$. If $0 \leqslant k \leqslant n-1$, then an easy calculation shows

$$B^i e_k = \left(\prod_{j=k}^{k+i-1} w_j\right) e_{k+i} \quad \text{for } 0 \leqslant i \leqslant n-k-1,$$

$$AB^i e_k = \left(\prod_{j=k}^{k+i-1} w_j\right) Ae_{k+i} = 0 \quad \text{for } 0 \leqslant i \leqslant n-k-2,$$

and

$$B^i e_k = 0 \quad \text{for } n-k \leqslant i \leqslant n.$$

Thus

$$C_n e_k = T_n^k AB^{n-k-1} e_k = \left(\prod_{j=k}^{n+k-1} w_j\right) e_{n+k} \quad \text{for } 0 \leqslant k \leqslant n-1.$$

This fact, together with

$$T_n^n e_k = \left(\prod_{j=k}^{n+k-1} w_j\right) e_{n+k} \quad \text{for } k \geqslant n,$$

imply the condition

$$T_n^n \mathscr{H}_n \vee C_n \mathscr{H}_n^{\perp} = \mathscr{H}_n .$$

Now by Theorem 2.1, the set $\{T, T^n\}$ has the TAP; hence T has the TAP. ▨

3. OPEN QUESTIONS

An affirmative answer to either of the following questions would extend our results (Corollary 2.3 and Theorem 2.5).

If $U \in \mathscr{L}(\mathscr{H})$ is a unilateral shift of finite multiplicity and if $K \in \mathscr{L}(\mathscr{K})$ is a nonzero compact operator, must $U \oplus K \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ have the TAP? We have an affirmative answer for the case in which K is an algebraic compact operator.

If S is the simple shift and if F is an arbitrary rank one operator, must $S + F$ have the TAP? Is the analogous result true for finite rank operators, or even for compact operators?

It would be desirable to see if the converse of Theorem 2.6 is true: if $Te_n = w_n e_{n+1}$, $n \geqslant 0$, is an injective weighted shift and if T has the TAP, must $T_n = T|H_n$ have the TAP for some $n \geqslant 1$, where $H_n = \vee\{e_k : k \geqslant n\}$?

Finally, if A is a strictly cyclic operator, then $A \oplus A$ has strict multiplicity 2. Therefore, $A \oplus A$ has the TAP. This leads to the following question: If A and B are strictly cyclic operators, must $A \oplus B$ have the TAP?

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