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COMMUTATOR METHODS AND BESOV SPACE ESTIMATES FOR SCHRÖDINGER OPERATORS

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1. INTRODUCTION

In this note we show how Mourre's commutator methods [7, 8, 9] can be used to prove resolvent estimates of the "optimal" type introduced by Agmon and Hörmander [3] for a large class of Schrödinger operators. These estimates are "sharp" in a sense made precise below: in the case of N-body Schrödinger operators (see below), they are new.

To state the class of operators we will study, let $\triangle$ be the Laplace operator on $\mathbf{R}^n$ and let Π_i, $1 \leqslant i \leqslant M$, be projections onto subspaces $\mathscr{X}_i$ of $\mathbf{R}^n$. If $\triangle_i$ is the Laplacian on $\mathscr{X}_i$ and $V_i : \mathscr{X}_i \to \mathbf{R}$ is a measurable function such that the operator $V_i(-\triangle_i + 1)^{-1}$ is compact on $L^2(\mathscr{X}_i)$, the differential operator ("generalized N-body Schrödinger operator")

$$P = -\triangle + \sum_{i=1}^{M} V_i(\Pi_i x)$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^n)$. Letting H denote its self-adjoint extension, we want to study the behaviour of $R(z) = (H - z)^{-1}$, $\operatorname{Im} z \neq 0$, as z approaches points λ in the continuous spectrum of H.

For the class of Schrödinger operators considered in Theorem 1.1 below, it is known (cf. [11] and references therein) that $R(z)$ is bounded as a map from $L_s^2(\mathbf{R}^n)$ to $L_{-s}^2(\mathbf{R}^n)$ for any $s > 1/2$, with bound uniform in z with $\operatorname{Im} z \neq 0$ and $\operatorname{Re} z \in \mathbf{R} \setminus \mathscr{E}(H)$. Here $\mathscr{E}(H)$ is a closed, countable set [7, 11] consisting of eigenvalues and thresholds of H (see e.g. [11] for a discussion of thresholds), and $L_s^2(\mathbf{R}^n) = := \left\{ u \in L_{\mathrm{loc}}^2(\mathbf{R}^n) : \int (1 + |x|^2)^s |u(x)|^2 \, \mathrm{d}x < \infty \right\}$ with the obvious norm. It is known moreover that the resolvent has Hölder continuous boundary values $R(\lambda + \mathrm{i}0)$ for $\lambda \notin \mathscr{E}(H)$ as maps from $L_s^2(\mathbf{R}^n)$ to $L_{-s}^2(\mathbf{R}^n)$, $s > 1/2$. These boundary values are basic objects in the stationary scattering theory and the theory of eigenfunction expansions for H.

For the case of two-body Schrödinger operators, i.e., $M = 1$ and $\Pi_1 =$ identity, Agmon and Hörmander [3] introduced an optimal framework in which to study boundary values of $R(z)$. They defined the space $B(\mathbf{R}^n)$ and its dual $B^*(\mathbf{R}^n)$ as follows. Let $R_j = 2^j$ for $j = 0, 1, \ldots$, and let $\Omega_j = \{x \in \mathbf{R}^n : 2^{j-1} \leqslant |x| \leqslant 2^j\}$, $j \geqslant 1$, and $\Omega_0 = \{x \in \mathbf{R}^n : |x| \leqslant 1\}$. Then

$$B(\mathbf{R}^n) = \left\{u \in L^2_{\text{loc}}(\mathbf{R}^n) : \sum_{j=0}^{\infty} R_j^{1/2} \|u\|_{\Omega_j} < \infty\right\}$$

with the obvious norm, where

$$\|u\|_{\Omega_j} = \left(\int_{\Omega_j} |u|^2 \, \mathrm{d}x\right)^{1/2}.$$

Its dual $B^*(\mathbf{R}^n)$ is given by

$$B^*(\mathbf{R}^n) = \{u \in L^2_{\text{loc}}(\mathbf{R}^n) : \sup_j R_j^{-1/2} \|u\|_{\Omega_j} < \infty\}$$

with the obvious norm. The Fourier transform of B is the Besov space $B_2^{1/2,1}$ cf. Peetre [12]) and that of B^* is the Besov space $B_2^{-1/2,\infty}$. They satisfy the inclusions

$$L^{2,s}(\mathbf{R}^n) \subset B(\mathbf{R}^n) \subset L^{2,1/2}(\mathbf{R}^n)$$

and

$$L^{2,-1/2}(\mathbf{R}^n) \subset B^*(\mathbf{R}^n) \subset L^{2,-s}(\mathbf{R}^n)$$

for any $s > 1/2$. The spaces B and B^* arise naturally in the study of Fourier restriction maps from $L^2(\mathbf{R}^n)$ to $L^2(M, \mathrm{d}\mu)$ where $M \subset \mathbf{R}^n$ is a compact, C^1 manifold of codimension 1 and $\mathrm{d}\mu$ is its natural surface measure induced from Euclidean measure on $\mathbf{R}^n$. In [3], Agmon and Hörmander study the resolvent of a constant coefficient, symmetric differential operator $P(D)$ where the symbol $P(\xi)$ is a polynomial having only simple zeros. The existence of boundary values of the resolvent of $P(D)$ is naturally connected with trace theorems for B. Agmon [1, 2] (see also Hörmander [6]) has proven the existence and uniqueness of boundary values for "two-body" perturbations of elliptic operators (including the Laplace operator) with short- and long-range coefficients in the $B - B^*$ framework. More precisely, he shows that for $f \in B$ and $\lambda = \operatorname{Re} z$ outside a closed countable set, $R(z)f$ has a unique limit in the weak-* topology on B^* as $\pm \operatorname{Im} z \downarrow 0$. Agmon's result relies in the short-range case on perturbation theory and the results of [3] (cf. also Hörmander [6]) and in the long-range case on a detailed microlocal analysis of the resolvent. This result is optimal in the sense that for each λ for which boundary values exist, there is a dense open subset of $B(\mathbf{R}^n)$ for which convergence to the weak limit cannot be improved. Murata [10] also studied differentiability of boundary values of the resolvent for certain elliptic operators in a Besov space setting, in order to study spectral properties and time-decay of solutions to the associated Schrödinger equation.

Finally, we note that Hörmander (independently of Mourre) has used commutator methods and a pseudodifferential operator analysis to study long-range, two-body scattering in the $B - B^*$ framework; details will appear in volume 3 of the series [6].

Here we would like to show how Mourre's commutator method [7, 8, 9] can be used to recover $B - B^*$ estimates for the class of generalized N-body Schrödinger operators considered in [11]. In what follows, let $x_i = \Pi_i x$ and let ∇_i be the gradient on $\mathscr{X}_i$, where $\mathscr{X}_i$ and Π_i are the subspaces and projections introduced above.

THEOREM 1.1. *Let H be a generalized N-body Schrödinger operator, let $W_i = x_i \cdot \nabla_i V_i$, and let $Q_i = x_i \cdot \nabla_i W_i$ (distributional gradient). Suppose that:*

(1) *$V_i(-\Delta_i + 1)^{-1}$ is a compact operator on $L^2(\mathscr{X}_i)$.*

(2) *$W_i(-\Delta_i + 1)^{-1}$ is a compact operator on $L^2(\mathscr{X}_i)$.*

(3) *$(-\Delta_i + 1)^{-1} Q_i (-\Delta_i + 1)^{-1}$ is a bounded operator on $L^2(\mathscr{X}_i)$.*

Let $R(z) = (H - z)^{-1}$ for $\operatorname{Im} z \neq 0$ and let $\mathscr{E}(H)$ be the set of eigenvalues and thresholds of H. Then for $\lambda \in \mathbf{R}\setminus\mathscr{E}(H)$, the estimate

$$\sup_{\eta \neq 0} \|R(\lambda + i\eta) f\|_{B^*(\mathbf{R}^n)} \leqslant c(\lambda) \|f\|_{B(\mathbf{R}^n)}$$

holds, where $c(\lambda)$ can be chosen uniform in λ running over a fixed compact subset of $\mathbf{R}\setminus\mathscr{E}(H)$.

REMARK 1.2. This theorem establishes the existence and the uniqueness of the weak-* limit in $B^*(\mathbf{R}^n)$ for $R(\lambda \pm i\eta)f$ as $\eta \downarrow 0$, when $f \in B(\mathbf{R}^n)$, and $\lambda \in \mathbf{R}\setminus\mathscr{E}(H)$. This result follows from the above $B - B^*$-estimate, the density of $L^2_s(\mathbf{R}^n)$ in $B(\mathbf{R}^n)$ for $s > 1/2$, and the existence of the boundary values $R(\lambda \pm i0)$ in the $L^2_s - L^2_{-s}$ -topology for $s > 1/2$ (see above). (The authors are indebted to Professor A. Devinatz for this remark.)

REMARK 1.3. By using the refined version of Mourre's abstract theory presented in [11], we can relax condition (2) to

(2)′ $(-\Delta_i + 1)^{-1/2} W_i (-\Delta_i + 1)^{-1}$ *is compact as an operator on* $L^2(\mathscr{X}_i)$.

Examples of potentials allowed by hypotheses (1) — (3) are:

(1) "short range" V_i with $V_i, W_i \in L^p(\mathscr{X}_i) + L^\infty_\varepsilon(\mathscr{X}_i)$, $Q_i \in L^p(\mathscr{X}_i) + L^\infty(\mathscr{X}_i)$ where $p > \sup(2, n/2)$. Here we say that a measurable function $f \in L^p + L^\infty$ if $= f_1 + f_2$ where $f_1 \in L^p$, $f_2 \in L^\infty$, and $f \in L^p + L^\infty_\varepsilon$ if for each $\varepsilon > 0$ there is a ecomposition $f = f_{1\varepsilon} + f_{2\varepsilon}$ with $f_{1\varepsilon} \in L^p$, $f_{2\varepsilon} \in L^\infty$, and $\|f_{2\varepsilon}\|_\infty < \varepsilon$.

(2) "Long range", $C^2(\mathscr{X}_i)$ functions V_i with $|D^\alpha V_i(X_i)| \leqslant C_\alpha(1 + |x_i|)^{-\epsilon-|\alpha|}$ for multi-indices α with $|\alpha| \leqslant 2$.

Under hypotheses (1), (2)′, and (3), we can relax the regularity conditions on V_i; cf. [11] for discussion.

The theorem follows from abstract results of Eric Mourre. Mourre develops an abstract theory for pairs of self-adjoint operators H, A which obey technical hypotheses together with the crucial "Mourre estimate" on $\mathrm{i}[H, A]$. In what follows, we denote by $\mathscr{D}(C)$ the domain of a densely defined operator C on a Hilbert space $\mathscr{H}$ with its graph norm $\|u\|_{\mathscr{D}(C)} = \|u\| + \|Cu\|$, where $\|\cdot\|$ is the norm in $\mathscr{H}$. $\mathscr{D}(C)^*$ is its dual under the $\mathscr{H}$ inner product. Given a pair of self-adjoint operators H and A on $\mathscr{H}$, we say that A is *conjugate to* H *in the interval* $I \subset \sigma(H)$ if:

(i) $\mathscr{D}(A) \cap \mathscr{D}(H)$ is dense in $\mathscr{D}(H)$ in graph norm,

(ii) the unitary group $\exp(\mathrm{i}\theta A)$ is a bounded map of $\mathscr{D}(H)$ into itself and

$$\sup_{|\theta|<1} \|\exp(\mathrm{i}\theta A)u\|_{\mathscr{D}(H)} < \infty$$

for each $u \in \mathscr{D}(H)$,

(iii) the quadratic form $\mathrm{i}[H, A]$ defined on $\mathscr{D}(H) \cap \mathscr{D}(A)$ is bounded from below and extends to a bounded operator, B, from $\mathscr{D}(H)$ to $\mathscr{H}$,

(iv) the form defined on $\mathscr{D}(H) \cap \mathscr{D}(A)$ by $[B, A]$ extends to a bounded operator from $\mathscr{D}(H)$ to $\mathscr{D}(H)^*$, and

(v) the estimate

$$E_I(H)\,\mathrm{i}[H, A]\,E_I(H) \geqslant C_0(I)E_I(H) + K \tag{1.1}$$

holds, where $E_I(H)$ is the spectral projection for H onto I, $C_0(I)$ is a strictly positive constant, and K is a compact operator.

This estimate (together with the technical hypotheses) implies that H has no singular continuous spectrum in I, and the set D of eigenvalues of H contained in I is finite counting multiplicity: this aspect of the theory is developed in [7] (some refinements may be found in [11], Theorem 1.2). It is applied to N-body Schrödinger operators including those satisfying the hypotheses of Theorem 1.1 in [7] ($N = 3$) and [11] (any N) (cf. also [4] for an elegant proof of (1.1) for generalized N-body Schrödinger operators). In the application, $A = -(1/2\mathrm{i})(x\cdot\nabla + \nabla\cdot x)$, the generator of dilations on $L^2(\mathbf{R}^n)$, and (1.1) holds for any sufficiently small interval away from thresholds of H. In the application, it is shown that the set of eigenvalues and thresholds of H, $\mathscr{E}(H)$, is closed and countable.

In [9], Mourre proves the following abstract results which we apply to prove Theorem 1.1. Let $f_1, f_2 : \mathbf{R} \to \mathbf{R}$ be measurable functions belonging to $\ell^2(L^\infty(\mathbf{R}))$, the space of real-valued measurable functions $g(t)$ with

$$\|g\|_{\ell^2(L^\infty)} = \left\{\sum_{n=0}^{\infty} s_n(g)^2\right\}^{1/2} < \infty$$

where

$$s_n(g) = \operatorname{ess\,sup}\{|g(x)| : n \leqslant |x| \leqslant n+1\}.$$

Suppose (i)—(v) hold in I and D is the set of eigenvalues of H in I. Then for $\lambda \in I \setminus D$,

$$\sup_{\eta \neq 0} \|f_1(A) R(\lambda + i\eta) f_2(A)\| \leqslant C_1(\lambda) \|f_1\|_{\ell^2(L^\infty)} \|f_2\|_{\ell^2(L^\infty)} \tag{1.2}$$

holds with $C_1(\lambda)$ uniform in compacts of $I \setminus D$ ([9], Theorem 1.2 (III)). Although a stronger norm on f_1, f_2 appears in [9], a close examination shows that the $\ell^2(L^\infty)$ norm is sufficient.

REMARK 1.4. By combining Mourre's analysis with that of Perry, Sigal and Simon [11], one can show that (1.2) in fact holds under the weaker hypotheses of Theorem 1.2 in [11]. This leads to the improvement in Theorem 1.1 discussed in Remark 1.3 above.

Below, we show how this estimate implies Theorem 1.1. We first define "abstract" spaces B_A and B_A^* using the spectral representation for the operator A. We then recover a $B_A - B_A^*$ estimate in the framework of Mourre's abstract theory. Next, we show that, "locally", the abstract spaces look like the concrete ones: for any $\varphi \in C_0^\infty(\mathbf{R})$ and H obeying the hypotheses of Theorem 1.1, the operator $\varphi(H)$ is a bounded mapping from $B(\mathbf{R}^n)$ to B_A and by duality from B_A^* to $B^*(\mathbf{R}^n)$. Combining this result with the abstract estimate, we obtain Theorem 1.1.

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2. PROOF OF THEOREM 1

First, we define suitable abstract analogues of the spaces $B(\mathbf{R}^n)$ and $B^*(\mathbf{R}^n)$. If A is a self-adjoint operator on a separable Hilbert space $\mathscr{H}$ with norm $\|\cdot\|$, we define the Banach space

$$B_A = \left\{ u \in \mathscr{H} : \sum_{j=0}^{\infty} R_j^{1/2} \|F(A \in \Omega_j) u\| < \infty \right\}$$

where $F(A \in \Omega_j)$ is the spectral projection for A onto the set $\Omega_j = \{t \in \mathbf{R} : 2^{j-1} \leqslant \leqslant |t| \leqslant 2^j\}$, $j \geqslant 1$, $\Omega_0 = \{t \in \mathbf{R} : |t| \leqslant 1\}$, and $R_j = 2^j$. We write $\|\cdot\|_{B_A}$ for the obvious norm on B_A. Its dual B_A^* with respect to the inner product on $\mathscr{H}$ is the Banach space obtained by completing $\mathscr{H}$ in the norm

$$\|u\|_{B_A^*} = \sup_j R_j^{-1/2} \|F(A \in \Omega_j) u\|.$$

The case $A = |x|, \mathscr{H} = L^2(\mathbf{R}^n)$ gives the usual spaces B and B^*.

The estimate (1.2) implies the key

PROPOSITION 2.1. *Let A be a conjugate operator for H at λ_0 and let I, D be defined as above. Then for $\lambda \in I \setminus D$*

$$\sup_{\eta \neq 0} \|R(\lambda + i\eta)f\|_{B_A^*} \leqslant C_1(\lambda)\|f\|_{B_A}$$

holds with $C_1(\lambda)$ uniform in λ in compacts of $I \setminus D$.

Proof. Using (1.2), we estimate, for $\lambda \in I \setminus D$, $\eta \neq 0$, $z = \lambda + i\eta$,

$$R_j^{-1/2}\|F(A \in \Omega_j)(H - z)^{-1}f\| \leqslant$$

$$\leqslant R_j^{-1/2} \sum_{k=0}^{\infty} \|F(A \in \Omega_j)(H - z)^{-1}F(A \in \Omega_k)\| \, \|F(A \in \Omega_k)f\| =$$

$$= R_j^{-1/2} \sum_{k=0}^{\infty} C_1(\lambda)R_j^{1/2}R_k^{1/2}\|F(A \in \Omega_k)f\| = C_1(\lambda)\|f\|_{B_A},$$

so that $\|(H - z)^{-1}f\|_{B_A^*} \leqslant C_1(\lambda)\|f\|_{B_A}$ as claimed. ▨

We now consider the "concrete" case where H is an N-body Schrödinger operator satisfying the hypotheses of Theorem 1.1 and $A = \frac{-i}{2}(x \cdot \nabla + \nabla \cdot x)$ is the generator of dilations on $L^2(\mathbf{R}^n)$. A is a conjugate operator for H for sufficiently small open intervals about every point $\lambda \in \mathbf{R} \setminus \mathscr{E}(H)$ where $\mathscr{E}(H)$ is a closed countable set consisting of eigenvalues and thresholds of H (cf. [4, 7, 11]). By Proposition 2.1 and an obvious covering argument, we immediately get:

PROPOSITION 2.2. *Let H obey the hypotheses of Theorem 1.1. Then for all $\lambda \in \mathbf{R} \setminus \mathscr{E}(H)$, the estimate*

$$\sup_{\eta \neq 0} \|R(\lambda + i\eta)f\|_{B_A^*} \leqslant C(\lambda)\|f\|_{B_A}$$

holds with $C(\lambda)$ uniform in compacts of $\mathbf{R} \setminus \mathscr{E}(H)$.

Next, we show that the abstract spaces B_A and B_A^* look "locally" like $B(\mathbf{R}^n)$ and $B^*(\mathbf{R}^n)$:

PROPOSITION 2.3. *Let H satisfy the hypothesis of Theorem 1.1. Then for any $\varphi \in C_0^\infty(\mathbf{R})$, the operator $\varphi(H)$ is a bounded mapping from $B(\mathbf{R}^n)$ to B_A and from B_A^* to $B^*(\mathbf{R}^n)$.*

Proof. We show $\varphi(H) : B(\mathbf{R}^n) \to B_A$ since the other assertion follows by duality. To do this, we use a minor variant of the interpolation Lemma 2.5 in [3]: let $T : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ be a linear operator with $T : L^{2,N}(\mathbf{R}^n) \to \mathscr{D}(|A|^N)$ for some $N > 1/2$. Then $T : B(\mathbf{R}^n) \to B_A$. A proof of this interpolation result is readily obtained by mimicking the proof of Lemma 2.5 in [3].

Since $\varphi(H) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, we need only show that $(|A|+1)^N\varphi(H)(1+x^2)^{-N/2}$ is a bounded operator, and hence $\varphi(H) : L^{2,N}(\mathbf{R}^n) \to \mathscr{D}(|A|^N)$, for some $N > 1/2$. But, by [11], Lemma 8.2, $(|A| + 1)(H + i)^{-1}(1 + x^2)^{-1/2}$ is bounded, while by a simple commutation argument (cf. [5], Appendix, Lemmas A.2 and A.3), $(1 + x^2)^{1/2}\psi(H)(1 + x^2)^{-1/2}$ is bounded for any $\psi \in C_0^\infty(\mathbf{R})$.

Hence

$$(|A| + 1)\varphi(H)(1 + x^2)^{-1/2} =$$

$$= [(|A| + 1)(H + i)^{-1}(x^2 + 1)^{-1/2}][(x^2 + 1)^{1/2}(H + i)\varphi(H)(1 + x^2)^{-1/2}]$$

is bounded. ▨

Combining Propositions 2.1—2.3, we can give the

Proof of Theorem 1.1. Pick $\eta \neq 0$, $\lambda \in \mathbf{R}\setminus\mathscr{E}(H)$, and pick $\varphi \in C_0^\infty(\mathbf{R})$ with $\varphi = 1$ near λ. Then for $f \in B(\mathbf{R}^n)$,

$$\|R(\lambda + i\eta)f\|_{B^*(\mathbf{R}^n)} \leqslant \|\varphi^2(H)R(\lambda + i\eta)f\|_{B^*(\mathbf{R}^n)} +$$

$$+ \|(1 - \varphi^2(H))R(\lambda + i\eta)f\|_{B^*(\mathbf{R}^n)}. \tag{2.1}$$

The second term in (2.1) satisfies

$$\|(1 - \varphi^2(H))R(\lambda + i\eta)f\|_{B^*(\mathbf{R}^n)} \leqslant \|(1 - \varphi^2(H))R(\lambda + i\eta)f\|_{L^2(\mathbf{R}^n)} \leqslant$$

$$\leqslant b_1\|f\|_{L^2(\mathbf{R}^n)} \leqslant b_1\|f\|_{B(\mathbf{R}^n)}, \tag{2.2}$$

where b_1 depends on λ and $\operatorname{supp}\varphi$. The first term in (2.1) is estimated as follows:

$$\|\varphi^2(H)R(\lambda + i\eta)f\|_{B^*(\mathbf{R}^n)} \leqslant b_2\|R(\lambda + i\eta)\varphi(H)f\|_{B_A^*} \leqslant \qquad \text{(by Proposition 2.4)}$$

$$\leqslant b_2C_1(\lambda)\|\varphi(H)f\|_{B_A} \leqslant \qquad \text{(by Proposition 2.3)} \tag{2.3}$$

$$\leqslant b_2^2C_1(\lambda)\|f\|_{B(\mathbf{R}^n)}, \qquad \text{(by Proposition 2.4)}$$

where b_2 is the norm of $\varphi(H)$ as a map from $B(\mathbf{R}^n)$ to B_A. (2.2) and (2.3) together give

$$\|R(\lambda + i\eta)f\|_{B^*(\mathbf{R}^n)} \leqslant b_3(1 + C_1(\lambda))\|f\|_{B(\mathbf{R}^n)},$$

where $b_3 := \sup(b_1, b_2^2)$. Since a fixed φ suffices for λ in a small interval, b_3 has the same uniformity in λ as $C_1(\lambda)$ by an obvious covering argument. This gives Theorem 1.1. ▨

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