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SAW^*-ALGEBRAS AND CORONA C^*-ALGEBRAS, CONTRIBUTIONS TO NON-COMMUTATIVE TOPOLOGY

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INTRODUCTION

One aspect of non-commutative topology is to view a general C^*-algebra as a "non-commutative $C_0(X)$". Each property concerning a locally compact Hausdorff space X can (in principle) be formulated in terms of the function algebra $C_0(X)$, and will then usually make sense (and hopefully be true) for any non-commutative C^*-algebra. Instead of this translation one may look directly for the objects, that in a non-commutative C^*-algebra A replaces the open and closed sets from the case $A = C_0(X)$. It is generally agreed (by Chuck Akemann and me) that these objects are the open and the closed projections in the enveloping von Neumann algebra A'' of A. The open projections are in a bijective correspondence with the hereditary C^*-subalgebras of A (of the form $L \cap L^*$ for some closed left ideal L of A), see [11, 1.5.2, 3.10.7, 3.11.9]. Unfortunately their complements, the closed projections, only correspond to quotients when they are central in A'' (so that the complementary open projection corresponds to a closed ideal of A). In this paper, the results are framed exclusively in terms of hereditary algebras and ideals.

We shall be much concerned with C^*-algebras or C^*-subalgebras that are *σ-unital*, which by definition means that they contain a strictly positive element. As shown by Aarnes and Kadison, A is σ-unital if and only if it contains a countable approximate unit (e_n). Indeed, any sequence $e_n = f_n(h)$ will do, provided that h is strictly positive and the functions f_n increase pointwise to 1 on $\mathrm{Sp}(h)$, [11, 3.10.5]. Furthermore, the approximate unit (e_n) may be chosen quasi-central with respect to any fixed countable subset of the multiplier algebra $M(A)$ of A, [11, 3.12.14]. Clearly a C^*-algebra $C_0(X)$ is σ-unital precisely when X is σ-compact. The σ-unital algebras occur quite frequently in C^*-algebra theory now, and a proper terminology is long overdue.

In this paper we introduce a new class of C^*-algebras, the SAW^*-algebras. In the commutative case they have the form $C_0(X)$ for some sub-Stonean space X. Such spaces were studied in [9], and this work is an attempt to generalize the properties of sub-Stonean spaces to SAW^*-algebras. We then consider corona C^*-algebras, of the form $M(A)/A$ for some σ-unital C^*-algebra A, and show that they are SAW^*-algebras. Finally we show that in a corona algebra one has a double annihilator theorem $B = (B^{\perp})^{\perp}$, for each hereditary, σ-unital C^*-subalgebra B. These last two results are closely related to work of G. G. Kasparov [10, §3] and D. Voiculescu [12, 1.9].

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SAW^*-ALGEBRAS

A C^*-algebra A is called an *SAW^*-algebra* if for any two orthogonal elements x and y in A_+ there is an element e in A_+ such that $ex = x$ and $ey = 0$. Applying the condition to the pair e, y we obtain an element d in A_+ such that $dy = y$ and $de = 0$. In this more symmetric setting we say that e, d is an orthogonal pair of *local units* for x and y. It follows from [9, 1.1] that if $A = C_0(X)$, then A is an SAW^*-algebra if and only if X is a sub-Stonean space, i.e. any two disjoint, open, σ-compact subsets of X have disjoint, compact closures.

To clarify the relations between SAW^*-algebras, Rickart algebras and AW^*-algebras we need the notion of (two-sided) annihilators. For every subset B of a C^*-algebra A, set

$$B^{\perp} = \{x \in A \mid xB = Bx = 0\}.$$

PROPOSITION 1. *Consider the following condition: Given two orthogonal, hereditary C^*-subalgebras B and C of A, there is an e in A_+ which is a unit for B and annihilates C. If this condition applies to all pairs B, C, then A is an AW^*-algebra. If it applies when B is σ-unital, then A is a Rickart C^*-algebra. If it applies when both B and C are σ-unital, then A is an SAW^*-algebra.*

Proof. Note first that an hereditary C^*-subalgebra B of A is σ-unital if and only if it has the form $B = (xAx)^-$ for some x in A_+, or, equivalently $B = (Ay)^- \cap (y^*A)^-$ for some y in A. The existence of a unit for B in $C^{\perp}$, when both B and A are σ-unital, is therefore equivalent with the SAW^*-condition.

Now let $B = (Ay)^- \cap (y^*A)^-$ and note that if R is the right annihilator of y then $R \cap R^* = B^{\perp}$. Consider the pair $B, B^{\perp}$. If there is an e in A_+, which is a unit for B and belongs to $(B^{\perp})^{\perp}$, then $e(1-e) = 0$, because $(1-e)A(1-e) \subset B^{\perp}$. Thus e is a projection in A such that $R = A(1-e)$; equivalently, e is the support projection of y. But the fact that right annihilators of single elements are principal right ideals is the defining property of Rickart C^*-algebras (alias B_p^*-algebras).

Finally, if the condition is valid for all pairs (so that A is unital, because of the pair A, $\{0\}$), take any subset E of A and consider the hereditary C^*-subalgebra $B = (AE)^- \cap (E^*A)^-$. Note that if R is the right ideal of right annihilators of E then $R \cap R^* = B^\perp$. For the pair $B, B^\perp$ we find an element e in $(B^\perp)^\perp$ which is a unit for B. As above we see that e is a projection and that $1-e$ is a unit for $B^\perp$. Consequently $B^\perp = (1-e)A(1-e)$, whence $R = A(1-e)$. We have shown that the right annihilator of any subset E is a principal right ideal, and this is Kaplansky's original definition of an AW^*-algebra.

Despite the formal similarity between AW^*-algebras and SAW^*-algebras described in Proposition 1, they are really quite different. One reason is that the SAW^*-condition, forcing the existence of a local unit e, does not require e to be unique. Applied in Rickart algebras or AW^*-algebras to pairs of the form $B, B^\perp$, the local unit e is unique, and is a projection. By contrast, there are commutative examples of SAW^*-algebras with no non-trivial projections (connected sub--Stonean spaces [9, 3.5]).

An SAW^*-algebra need not be unital. However, every σ-unital SAW^*--algebra is unital, because a local unit for a strictly positive element is in fact a unit. That SAW^*-algebras tend to be rather large is documented by our next result.

COROLLARY 2. *If A is a separable SAW^*-algebra then it is finite dimensional.*

Proof. In a separable C^*-algebra, every hereditary C^*-subalgebra is separable, hence σ-unital. A separable SAW^*-algebra is therefore an AW^*-algebra by Proposition 1, and hence finite dimensional.

PROPOSITION 3. (cf. [9, 1.4]). *Every quotient of an SAW^*-algebra is again an SAW^*-algebra.*

Proof. If $\pi : A \to B$ is a morphism (i.e. a $*$-homomorphism) from the SAW^*--algebra A onto the C^*-algebra B, and x, y are orthogonal elements in B_+, we can choose orthogonal counter-images x_1, y_1 in A_+ by [4, 2.4]. By assumption there are orthogonal local units d_1 and e_1 in A_+ for x_1 and y_1, and their images $d = \pi(d_1)$ and $e = \pi(e_1)$ will then be orthogonal local units for x and y in B.

In general, an hereditary C^*-subalgebra of an SAW^*-algebra need not itself be an SAW^*-algebra. In the commutative case the necessary and sufficient condition for this to happen is that the open set corresponding to the hereditary C^*-subalgebra has a basically isolated complement [9, p. 128]. This result translates into non-commutative topological language via the notion of annihilators.

PROPOSITION 4. *An hereditary C^*-subalgebra I of an SAW^*-algebra A is itself an SAW^*-algebra if and only if each element in I_+ has a local unit in I_+.*

In particular, if I is an ideal, it will be an SAW^*-ideal if and only if $B^\perp + I = A$ for every hereditary, σ-unital C^*-subalgebra $B \subset I$.

Proof. The first condition is clearly necessary for I to be an SAW^*-subalgebra. Now assume the condition and consider an orthogonal pair x, y in I_+. There is then an element d in I_+, such that $d(x+y) = x+y$, which implies that $dx = x$ and $dy = y$. Since A is an SAW^*-algebra, there is also an element e in A_+ such that $ex = x$ and $ey = 0$. Take $e_0 = ded$, so that $e_0 \in I_+$; and compute $e_0 x = x$, $e_0 y = 0$, as desired.

If I is an SAW^*-ideal in A, and B is an hereditary, σ-unital C^*-subalgebra of I, there is a unit e for B in I_+. Assuming, as we may, that $0 \leq e \leq 1$, it follows that $(1-e)A(1-e) \subset B^\perp$, whence

$$A = (1 - e + e)A(1 - e + e) \subset B^\perp + I.$$

Conversely, if we always have $B^\perp + I = A$, consider an element x in I_+ and its associated, hereditary C^*-subalgebra $B = (xAx)^-$. Let e be a unit for B in A_+ and, by assumption, write it in the form $e = y + z$ with y in I and z in $B^\perp$. Then both y and y^* are units for B, hence for x; so that y^*y is a unit for x in I_+. It follows from the first part of the proof that I is an SAW^*-ideal.

REMARK 5. The simplest way to obtain hereditary SAW^*-subalgebras of an SAW^*-algebra A is to take $I = B^\perp$, for some hereditary, σ-unital C^*-subalgebra B of A. Indeed, if $x \in I_+$, and h is a strictly positive element in B, there is an e in A_+ which is a unit for x and annihilates h. Thus $e \in B^\perp = I$, whence I is an SAW^*-subalgebra of A by Proposition 4. Other SAW^*-ideals arise from the corona construction, see Theorem 23.

We now prove the analogue of [9, 1.12].

THEOREM 6. *Let π be a morphism of an SAW^*-algebra A and put $I = \ker \pi$. Then I is an SAW^*-ideal if and only if*

$$\pi(B^\perp) = (\pi(B))^\perp$$

for every hereditary, σ-unital C^*-subalgebra B of A.

Proof. If the condition is satisfied, take $B \subset I$. Then $\pi(B) = 0$, so that $\pi(B^\perp) = \pi(A)$, i.e. $A = B^\perp + I$. By Proposition 4, I is an SAW^*-ideal.

To prove the converse, note that $\pi(B^\perp) \subset (\pi(B))^\perp$. Thus it suffices to consider an element x in A_+ such that $\pi(x) \in (\pi(B))^\perp$, i.e. $xB \subset I$, and then show that there is an element y in $B^\perp$ such that $x - y \in I$. Towards this end, let h be a strictly positive element in B. Since $xB \subset I$, the same is true for any continuous function of x, in particular $x^{1/4}B \subset I$. Let D be the hereditary (σ-unital) C^*-subalgebra of I generated by $x^{1/4}h^2x^{1/4}$. Then $D^\perp + I = A$ by Proposition 4, so $x^{1/4} = a + b$ for some a in $D^\perp$ and b in I. Put $y = x^{1/4}a^*ax^{1/4}$ and note that $x - y \in I$. Furthermore,

$$yh = x^{1/4}a^*(ax^{1/4}h) = 0,$$

since $a(x^{1/4}h) = 0$, so that $y \in B^\perp$.

If B is a subset of a C^*-algebra A, let her(B) denote the hereditary C^*-subalgebra of A generated by B. If B is a C^*-algebra, it contains an approximate unit for her(B), and thus by [3, 1.2], her$(B)_+$ can be characterized as the set of elements in A_+ dominated by elements in B_+.

We say that a morphism $\pi : A_1 \to A_2$ between C^*-algebras A_1 and A_2 is a *morphismo* if

$$(*) \qquad \pi(B_1)^\perp = \mathrm{her}(\pi(B_1^\perp))$$

for every hereditary C^*-subalgebra B_1 of A_1. We say that π is a *σ-morphismo* if the condition (*) holds only for σ-unital, hereditary subalgebras B_1 of A_1.

These concepts, as well as the ensuing discussion of the commutative case, are due to L. G. Brown. Note that in this language Theorem 6 says that I is an SAW^*-ideal if and only if the quotient map is a σ-morphismo.

Assume that A_1 and A_2 are commutative C^*-algebras, i.e. $A_i = C_0(X_i)$, X_i a locally compact Hausdorff space, $i = 1, 2$. In this case we have a morphism $\pi : A_1 \to A_2$ if and only if there is a proper continuous map $\rho : X_2 \to X_1$ such that $\pi(a_1)(t_2) = a_1(\rho(t_2))$ for all a_1 in A_1 and t_2 in X_2. A proper map between locally compact Hausdorff spaces is necessarily a closed map; in particular, $\rho(X_2)$ is closed in X_1. Most problems therefore quickly reduce to the case where ρ is surjective (so that π is injective).

PROPOSITION 7. *If $\rho : X_2 \to X_1$ is a proper, continuous map between locally compact Hausdorff spaces, and $\pi : C_0(X_1) \to C_0(X_2)$ denotes the transposed morphism, then the following conditions are equivalent:*

(i) *π is a morphismo (respectively a σ-morphismo);*

(ii) *For each open, σ-compact subset E_2 of X_2 and every open (respectively open, σ-compact) subset G_1 of X_1 disjoint from $\rho(E_2)$, there is an open σ-compact subset E_1 of X_1, such that $\rho(E_2) \subset E_1$ and $E_1 \cap G_1 = \emptyset$;*

(iii) *ρ is an open map.*

Proof. (i) $\Rightarrow$ (ii). There is a bijective correspondence between open, σ-compact subsets of X and co-zero sets for functions in $C_0(X)$. Given E_2 we can therefore find f_2 in $C_0(X_2)$ such that $E_2 = \{t \in X_2 \mid f_2(t) \neq 0\}$. We let B denote the closed ideal (= hereditary C^*-subalgebra) of $C_0(X_1)$ corresponding to the given open set G_1

(so that $B = C_0(G_1)$), and we note that B is σ-unital precisely when G_1 is σ-compact. The conditions $\rho(E_2) \cap G_1 = \emptyset$ and $f_2 \in \pi(B)^\perp$ are equivalent, and we conclude that $f_2 \in \mathrm{her}(\pi(B^\perp))$; so that $|f_2| \leqslant \pi(f_1)$ for some $f_1 \geqslant 0$ in $B^\perp$. Taking $E_1 = \{s \in X_1 \mid f_1(s) \neq 0\}$ we see that $E_2 \subset \rho^{-1}(E_1)$, which is equivalent to $\rho(E_2) \subset E_1$; and clearly $E_1 \cap G_1 = \emptyset$.

(ii) $\Rightarrow$ (i). The arguments in the proof above can all be reversed.

(ii) $\Rightarrow$ (iii). If G_2 is open in X_2, choose for each t in G_2 an open, σ-compact neighborhood E_2 of t such that $\bar{E}_2 \subset G_2$. Put $G_1 = X_1 \diagdown \rho(\bar{E}_2)$, and apply the assumption to obtain an open (σ-compact) subset E_1 with $\rho(E_2) \subset E_1 \subset \rho(\bar{E}_2)$. Thus $\rho(G_2)$ contains an open neighborhood around each of its points, and is consequently open.

(iii) $\Rightarrow$ (ii). Given E_2 and G_1 as in (ii), take $E_1 = \rho(E_2)$ which is open (and σ-compact) by assumption, and note that $\rho(E_2) \subset E_1$ and $E_1 \cap G_1 = \emptyset$.

There seems to be no natural σ-analogue of condition (iii) in Proposition 7. This is irrelevant as long as we are dealing with the separable case; but for the spaces and algebras under consideration it matters a great deal: Let X_1 be a compact sub-Stonean space, and $\rho : X_2 \to X_1$ the injection map of a closed subset X_2 of X_1. Then ρ is an open map if and only if X_2 is open in X_1. However, ρ will satisfy the weaker σ-version of condition (ii) in Proposition 7 (which states that X_2 is basically isolated in X_1, see [9, p. 128]) if and only if $X_1 \diagdown X_2$ is a sub-Stonean space.

The non-commutative analogue of gluing two spaces together along a continuous map between closed subsets is easy to formulate in C^*-algebraic terms. The next result shows when such a "gluing" between SAW^*-algebras again produces an SAW^*-algebra. It is the analogue of [9, 1.13], in which we unfortunately forgot to mention that the map should be proper. The present proof is due to L. G. Brown.

THEOREM 8. *Let π_1 and π_2 be morphisms of SAW^*-algebras A_1 and A_2, respectively, such that $\pi_1(A_1) \subset \pi_2(A_2)$. Consider the C^*-algebra*

$$A = \{(x_1, x_2) \in A_1 \oplus A_2 \mid \pi_1(x_1) = \pi_2(x_2)\}.$$

If both π_1 and π_2 are σ-morphismos (in particular their kernels are SAW^*-ideals) then A is an SAW^*-algebra.

Proof. Take orthogonal elements $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in A_+. Since both A_1 and A_2 are SAW^*-algebras, there are positive elements e_1 and e_2 in A_1 and A_2, respectively, such that $x_i e_i = x_i$ and $y_i e_i = 0$, $i = 1, 2$. Note that

$$\pi_1(x_1 + y_1)(\pi_1(e_1) - \pi_2(e_2)) = \pi_1((x_1 + y_1)e_1) - \pi_2((x_2 + y_2)e_2) = \pi_1(x_1) - \pi_2(x_2) = 0.$$

If therefore $h = (\pi_1(e_1) - \pi_2(e_2))^2$, then $\pi_1(x_1 + y_1)h = 0$. Since π_1 is a σ-morphismo, there is an element $k \geqslant 0$ in A_1, such that $(x_1 + y_1)k = 0$ and $h \leqslant \pi_1(k)$. By the SAW^*-condition we find $d_1 \geqslant 0$ in A_1 with $(x_1 + y_1)d_1 = x_1 + y_1$ and $kd_1 = 0$. This means that $\pi_1(d_1)\pi_2(x_2 + y_2) = 0$ and that $\pi_1(d_1)h = 0$.

Since A_2 is an SAW^*-algebra and π_2 is surjective there is an element e in A_2 with $0 \leqslant e \leqslant 1$, such that e is a unit for $x_2 + y_2$ and $\pi_2(e)$ is a unit for $\pi_1(d_1)$. Assuming, as we may, that $d_1 \leqslant 1$ we see that

$$\pi_2(e) - \pi_1(d_1) \in \{\pi_2(x_2 + y_2)\}^\perp.$$

Since π_2 is a surjective σ-morphismo, this implies that $\pi_2(e) - \pi_1(d_1) = \pi_2(z)$ for some z in A_2 with $0 \leqslant z \leqslant 1$, such that $z(x_2 + y_2) = 0$. Set $d_2 = |e - z|$ and note that $d_2(x_2 + y_2) = x_2 + y_2$, since $(e - z)^n$ is a unit for $x_2 + y_2$ for every n. Moreover,

$$\pi_2(d_2) = |\pi_2(e) - (\pi_2(e) - \pi_1(d_1))| = \pi_1(d_1).$$

Now define $c_i = e_i d_i e_i$ for $i = 1, 2$, and compute

$$\pi_1(c_1) = (\pi_1(e_1) - \pi_2(e_2) + \pi_2(e_2))\pi_1(d_1)(\pi_1(e_1) - \pi_2(e_2) + \pi_2(e_2)) =$$

$$= \pi_2(e_2)\pi_1(d_1)\pi_2(e_2) = \pi_2(c_2),$$

since $h\pi_1(d_1) = 0$. Thus $c = (c_1, c_2) \in A$. Moreover, for $i = 1, 2$ we have

$$x_i c_i = x_i e_i d_i e_i = x_i, \quad y_i c_i = y_i e_i d_i e_i = 0,$$

so that $xe = x$ and $ye = 0$ in A. Hence A is an SAW^*-algebra.

REMARK 9. As a special case of Theorem 8 one may take SAW^*-algebras A_1 and A_2 with SAW^*-ideals I_1 and I_2 respectively, such that $A_1/I_1 = A_2/I_2$. The resulting SAW^*-algebra A corresponds to an honest gluing with a local homeomorphism. At the other extreme one may take $A_1 = \mathbf{C}$ and A_2 a unital SAW^*--algebra with an SAW^*-ideal I_2. The morphism π_1 is the embedding of $\mathbf{C}$ into A_2/I_2, and the resulting SAW^*-algebra is

$$A = \{(\lambda, x) \in \mathbf{C} \oplus A_2 \mid \lambda - x \in I_2\} = \mathbf{C} + I_2 .$$

This is the non-commutative analogue of pinching the space corresponding to A_2 (collapsing A_2/I_2 to $\mathbf{C}$). Note that A has an SAW^*-ideal (viz. I_2) of co-dimension one, corresponding to the commutative construction in [9, 1.14] of sub-Stonean spaces with a basically isolated point.

Recall that for a C^*-algebra A, the multiplier algebra $M(A)$ is defined as the idealizer of A in its enveloping von Neumann algebra A''. Identifying $M(A)$ with the set of double centralizers of A, it is easy to show that $M(A)$ is (isomorphic to) the idealizer of A in any von Neumann algebra containing A as a strongly dense sub--algebra [11, 3.12]. If $A = C_0(X)$, then $M(A) = C(\beta(X))$, where $\beta(X)$ denotes the Stone-Čech compactification of the locally compact Hausdorff space X.

The next result does not really belong to the theory of SAW^*-algebras, but certainly to that of non-commutative topology. Its proof is an easy modification of the argument for the separable case, [5, 4.2] or [11, 3.12.10], and, as pointed out by J. Cuntz, there is another short argument which reduces the theorem to the separable case. Nevertheless we present here the proof in full detail. After all, it constitutes in the commutative case a proof of Tietze's extension theorem for locally compact, σ-compact Hausdorff spaces, in particular a proof that such spaces are normal.

THEOREM 10. *Let $\pi : A \to B$ be a surjective morphism between σ-unital C^*-algebras A and B. Then π extends to a surjective morphism between $M(A)$ and $M(B)$.*

Proof. By [11, 3.7.7] there is a unique normal extension $\pi'' : A'' \to B''$, and clearly $\pi''(M(A)) \subset M(B)$. Take now an element z in $M(B)_{sa}$. By [11, 3.12.9] there are nets (x_i) and (y_j) in B_{sa} such that $x_i \nearrow z$ and $y_j \searrow z$ strongly in B''. If h is a strictly positive element in A, then $k = \pi(h)$ is strictly positive in B; and we see that the net with elements $k(y_j - x_i)k$ decreases strongly to zero in B. Regarding B_{sa} as continuous (affine) functions on the quasi-state space of B (see [11, 3.10.3]), it follows from Dini's lemma that the net converges uniformly to zero. We can therefore find sequences $(x_n) \subset (x_i)$ and $(y_n) \subset (y_j)$ such that (x_n) is increasing, (y_n) is decreasing, $x_n \leqslant z \leqslant y_n$ and $\|k(y_n - x_n)k\| < n^{-1}$ for every n.

We claim that there are sequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$ in A_{sa} such that $(\tilde{x}_n)$ is increasing, $(\tilde{y}_n)$ is decreasing, $\tilde{x}_n \leqslant \tilde{y}_m$ for all n, m, $\pi(\tilde{x}_n) = x_n$, $\pi(\tilde{y}_n) = y_n$ and

$$\|h(\tilde{y}_n - \tilde{x}_n)h\| < n^{-1},$$

for every n. To prove this, let (u_λ) be an approximate unit for $\ker\pi$, and assume that we have found elements $\tilde{x}_k$ and $\tilde{y}_k$ satisfying the conditions for all $k < n$. Thus

$$\pi(\tilde{x}_{n-1}) = x_{n-1} \leqslant x_n \leqslant y_n \leqslant y_{n-1} = \pi(\tilde{y}_{n-1}),$$

and by [11, 1.5.10] there are elements $\tilde{x}_n$ and $\tilde{y}_n$ in A_{sa} such that $\pi(\tilde{x}_n) = x_n$, $\pi(\tilde{y}_n) = y_n$ and

$$\tilde{x}_{n-1} \leqslant \tilde{x}_n \leqslant \tilde{y}_n \leqslant \tilde{y}_{n-1}.$$

Applying [11, 1.5.4] we see that

$$\|h(\tilde{y}_n - \tilde{x}_n)^{1/2}(1 - u_\lambda)(\tilde{y}_n - \tilde{x}_n)^{1/2}h\| \searrow \|k(y_n - x_n)k\|.$$

Replacing if necessary $\tilde{x}_n$ by

$$\tilde{x}_n + (\tilde{y}_n - \tilde{x}_n)^{1/2}u_\lambda(\tilde{y}_n - \tilde{x}_n)$$

for a suitably large λ, we may therefore assume that $\|h(\tilde{y}_n - \tilde{x}_n)h\| < n^{-1}$, and the claim is established by induction.

The sequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$ converge strongly in A''_{sa} to elements x and y with $x \leqslant y$. However, the norm is strongly lower semi-continuous, so $h(y - x)h = 0$, whence $(y - x)h = 0$. Since h is strictly positive in A, its support projection in A'' is 1, and it follows that $x = y$. Consequently,

$$x \in (A_{sa})^m \cap (A_{sa})_m = M(A)_{sa}$$

by [11, 3.12.9], and since π'' is normal we see that $\pi''(x) = z$, so that $\pi''(M(A)) = M(B)$.

THECREM 11. (cf. [9, 1.10]). *Let I be a closed, σ-unital ideal in an SAW^*-algebra A. Then*

$$M(I) = A/I^{\perp}.$$

Proof. Since I is isomorphically embedded in $A/I^{\perp}$, we may assume that $I^{\perp} = 0$, so that I is essential in A. There is then a natural embedding $I \subset A \subset M(I)$ by 11, 3.12.8]. If $A \neq M(I)$ there is a self-adjoint functional in $M(I)^*$, of norm one, that annihilates A; so its Jordan decomposition produces two orthogonal states φ and ψ of $M(I)$, such that $\varphi - \psi$ annihilates A, [11, 3.2.5]. By [11, 3.2.3] we may choose, for any $\varepsilon > 0$, a z in $M(I)$ with $0 \leqslant z \leqslant 1$, such that $\varphi(1 - z) < \varepsilon^2$ and $\psi(z) < \varepsilon^2$. Let f and g be continuous piecewise linear functions on $\mathrm{Sp}(z)$ with $0 \leqslant f \leqslant 1$ and $0 \leqslant g \leqslant 1$, such that $f(1) = 1$ but $f(t) = 0$ for $t < 1 - \varepsilon$, and $g(0) = 1$ but $g(t) = 0$ for $t > \varepsilon$. Put $x = f(z)$ and $y = g(z)$ and note that $xy = 0$. Moreover, $1 - x \leqslant \varepsilon^{-1}(1 - z)$ and $1 - y \leqslant \varepsilon^{-1}z$. Consequently $\varphi(1 - x) < \varepsilon$ and $\psi(1 - y) < \varepsilon$, so that $\varphi(x) > 1 - \varepsilon$ and $\psi(y) > 1 - \varepsilon$.

Let h be a strictly positive element in I. Since xh^2x and yh^2y are orthogonal elements in the SAW^*-algebra A, there are orthogonal local units d and e in A_+ for xh^2x and yh^2y. Thus $(1 - d)xh^2x(1 - d) = 0$, whence $(1 - d)xh = 0$. Since h is strictly positive in I, its support projection in I'' is 1; and we conclude that $(1 - d)x = 0$. In particular, $d \geqslant x$. Similarly we show that $e \geqslant y$, and it follows from the above that

$$\varphi(d + e) = \varphi(d) + \psi(e) \geqslant \varphi(x) + \psi(y) > 2(1 - \varepsilon).$$

Having chosen $\varepsilon < 1/2$ we reach a contradiction, since $\|d + e\| = \max\{\|d\|, \|e\|\} = 1$.

COROLLARY 12. *If I is a closed, σ-unital ideal in an SAW^*-algebra, then M(I) is an SAW^*-algebra.*

CORONA C^*-ALGEBRAS

Let A be a non-unital, σ-unital C^*-algebra. The *corona* of A is defined as the quotient $\mathrm{C}(A) = M(A)/A$ (where the C conveniently enough also may indicate Calkin). The commutative origin is the corona sets $\chi(X) = \beta(X)\setminus X$, where $\beta(X)$ is the Stone-Čech compactification of the non-compact, σ-compact space X, see [9].

THEOREM 13. (cf. [9, 3.2]). *For each σ-unital C^*-algebra A, its corona $C(A)$ is an SAW^*-algebra.*

Proof. Let $\pi : M(A) \to C(A)$ denote the quotient map and consider a pair x, y of orthogonal elements in $C(A)_+$. By [4, 2.4] there are orthogonal elements a, b in $M(A)_+$ with $\pi(a) = x$ and $\pi(b) = y$. Let D be the C^*-algebra generated by a, b and A, so that $A \subset D \subset M(A)$. If now $z \in M(D)$, $c \in A$ and (u_λ) is an approximate unit for A, then $zc \in D$ (since $A \subset D$), whence $czu_\lambda \in A$ (since $D \subset M(A)$); moreover $cu_\lambda \to c$, so that $zc \in A$. Similarly $cz \in A$, and we conclude that there is a natural embedding $D \subset M(D) \subset M(A)$.

If $\rho : D \to D/A$ is the quotient map, and ρ'' denotes its natural extension $\rho'' : M(D) \to M(D/A)$, then it is easily verified that

$$\ker \rho'' = \{z \in M(D) \mid zD \subset A\}.$$

(We return to this set-up in Theorem 23.) In our case, D/A is a commutative C^*--algebra of the form $C_0(X) \oplus C_0(Y)$, where $X \subset \mathrm{Sp}(a)\setminus\{0\}$ and $Y \subset \mathrm{Sp}(b)\setminus\{0\}$, so that $M(D/A) = C_b(X) \oplus C_b(Y)$. Since ρ'' is surjective by Theorem 9, there are by [4, 2.4] orthogonal elements d_0 and e_0 in $M(D)_+$ with $\rho''(d_0) = 1_X$ and $\rho''(e_0) = 1_Y$. Thus $d_0a - a \in \ker \rho''$. On the other hand $d_0a - a \in D$, and since $\ker \rho'' \cap D = A$ we conclude that $d_0a - a \in A$. Similarly $e_0b - b \in A$. Consequently, with $e = \pi(e_0)$ and $d = \pi(d_0)$ we have found a pair of orthogonal local units for x and y in $C(A)$.

REMARK 14. The proof given above is due to J. Cuntz, and replaces a much more complicated argument.

As pointed out by G. Skandalis the result is similar to the two "technical lemmas", Theorems 3 and 4 in [10, § 3]. Indeed, one may derive Theorem 12 from Kasparov's Theorem 3 assuming (in his terminology) that $B = D$, and then construct orthogonal local units for B_1 and B_2 (modulo D) as $f(1 - M)$ and $f(1 - N)$, where $f(1) = 1$ and $f(t) = 0$ for $t \leqslant 1/2$. Conversely, one may obtain a weak form of Kasparov's Theorem (the case $B = D$ and ignoring the maps F_x) by defining $M = b(a + b)^{-1}$ and $N = a(a + b)^{-1}$, where $a = 1 - d$, $b = 1 - e$ and d, e are orthogonal local units for B_1 and B_2 modulo D. In fact, the proof of Theorem 12, that we have borrowed from Cuntz, originated in an attempt to simplify Kasparov's work.

This raises the question whether the ultra-technical part of KK-theory could use the terminology of SAW^*-algebras and corona algebras as structural signposts. Certainly the educated public (including the present author) would welcome any attempt to map out this rather terrifying territory.

THEOREM 15. *Let A be a σ-unital C^*-algebra with corona algebra $C(A)$. Then $B = (B^\perp)^\perp$ for each hereditary, σ-unital C^*-subalgebra B of $C(A)$.*

Proof. Since obviously $B \subset (B^{\perp})^{\perp}$, it suffices to show that for each x_0 not in B, with $0 \leqslant x_0 \leqslant 1$, there is a y_0 in $B^{\perp}$ such that $x_0 y_0 \neq 0$. Towards this end let $\pi : M(A) \to C(A)$ denote the quotient map, and choose a positive element k in $M(A)$ such that $\pi(k)$ is strictly positive for B. Now let h be a strictly positive element for A, and let D be the hereditary C^*-subalgebra of $M(A)$ generated by $h + k$. Thus $D \supset A$ (as $h \leqslant h + k$), and $\pi(D) = B$ by [11, 1.5.11]. Choose x in $M(A)$ with $0 \leqslant x \leqslant 1$ such that $\pi(x) = x_0$, and let L denote the closed left ideal of $M(A)$ such that $D = L \cap L^*$, [11, 1.5.2]. Since $x \notin D$ it follows that $x \notin L$, so that we can find $\alpha > 0$ with $\alpha < \operatorname{dist}(x, L)$.

Let (e_n) and (d_n) be countable approximate units for A and D, respectively, such that $e_{n+1}e_n = e_n$ and $d_{n+1}d_n = d_n$ for every n. If for some n and m we had

$$\|x(e_i - e_n)(d_j - d_m)\| \leqslant \alpha$$

for infinitely many i and j, then, since $e_i \nearrow 1$ and $d_j \nearrow 1$ strongly in A'', it would follow that

$$\|x(1 - e_n)(1 - d_m)\| = \|x - x(e_n + d_m - e_n d_m)\| \leqslant \alpha.$$

However, $x(e_n + d_m - e_n d_m) \in L$ in contradiction with $\operatorname{dist}(x, L) > \alpha$. Consequently we always have

$$\|x(e_i - e_n)(e_j - e_m)\| > \alpha$$

for i and j sufficiently large. Passing if necessary to a sequence in $\operatorname{conv}(e_n)$ we may assume from the outset that (e_n) is quasi-central with respect to the set $\{d_j\}$, [11, 3.12.14]. Thus for a given m and $\varepsilon > 0$ we may choose n such that $\|e_n d_m - d_m e_n\| < \varepsilon$. Since (d_j) is an approximate unit for $D \supset A$, we know that $\|d_j e_n - e_n d_j\| < \varepsilon$ for j sufficiently large. Finally we may choose i so large that $\|e_i(d_j - d_m) - (d_j - d_m)e_i\| < \varepsilon$. Working by induction and passing if necessary to subsequences of (e_n) and (d_n) we may therefore assume the conditions

(i) $e_{n+1}e_n = e_n, \quad d_{n+1}d_n = d_n$;

(ii) $\|xa_n\| \geqslant \alpha, \quad$ where $a_n = (e_{n+1} - e_n)(d_{n+1} - d_n)$;

(iii) $\|a_n - a_n^*\| \leqslant 2^{-n}$;

for all n.

Define $y = \sum a_{2n}^* a_{2n}$ and $z = \sum a_{2n} a_{2n}^*$, and note that the sums converge strongly in A'', since the summands are bounded and pairwise orthogonal by (i). Since $(a_n) \subset A$ it follows from (iii) that $y - z \in A$. We claim that $z \in M(A)$, whence also $y \in M(A)$. To see this, note that for each j

$$e_j z = \sum_{2n \leqslant j} e_j(e_{2n+1} - e_{2n})(d_{2n+1} - d_{2n})^2(e_{2n+1} - e_{2n}) \in A,$$

since the sum is finite by (i). As (e_j) is an approximate unit for A it follows that $Az \subset A$, whence $z \in M(A)$. We further claim that $Dy \subset A$. To prove this, note that for each j,

$$d_j y = \sum_{2n \leqslant j} d_j(d_{2n+1} - d_{2n})(e_{2n+1} - e_{2n})^2(d_{2n+1} - d_{2n}) \in A$$

since the sum is finite by (i) and A is an ideal in $M(A)$. As (d_j) is an approximate unit for D it follows that $Dy \subset A$, as desired. Finally we claim that $xz \notin A$, whence also $xy \notin A$. Indeed, if $xz \in A$ we would have $\|xz(1 - e_j)\| < \alpha^4$ for some large j. But for $2n > j$ we have

$$\|xz(1 - e_j)\| \geqslant \|xz(1 - e_j)a_{2n}a_{2n}^*\| = \|xza_{2n}a_{2n}^*\| =$$
$$= \|x(a_{2n}a_{2n}^*)^2\| \geqslant \|x(a_{2n}a_{2n}^*)^2x\| = \|xa_{2n}a_{2n}^*\|^2 \geqslant$$
$$\geqslant \|xa_{2n}a_{2n}^*x\|^2 = \|xa_{2n}\|^4 \geqslant \alpha^4$$

by (ii), a contradiction.

Define $y_0 = \pi(y) = \pi(z)$ in $C(A)$. Since $Dy \subset A$ and $\pi(D) = B$ we see that $y_0 \in B^\perp$. On the other hand $x_0 y_0 \neq 0$ since $xz \notin A$, and the proof is complete.

REMARK 16. The preceding result is the non-commutative generalization of [9, 3.3], namely that in a corona set $\chi(X) = \beta(X)\setminus X$, every open, σ-compact subset is the interior of its closure. The theorem has an uncanny resemblance to the bi-commutant theorem of Voiculescu, [12, 1.9], that says that $B = B''$ for every separable C^*-subalgebra B of the Calkin algebra. However, the two theorems seem to be independent, although their proofs (especially in the version given in [6]) borrow from the same techniques. Both results testify to the fact that the Calkin algebra and other corona algebras belong to a radically different species than the C^*-algebras one encounters in daily life.

Instead of working in the rather intractable realms of corona algebras, one may of course reformulate the results in terms of elements in $M(A)$ and perturbations by elements in A. Let us agree on the terminology that y is *contained in* x, where x and y are positive elements in a C^*-algebra M, if y belongs to the hereditary C^*-subalgebra of M generated by x. This condition may be reformulated in the following different ways:

(i) For each $\varepsilon > 0$ there is an $\alpha > 0$ such that $y \leqslant \alpha x + \varepsilon 1$.

(ii) $\operatorname{Lim\,Sup}\|y(1 + nx)^{-1}\| = 0$.

(iii) There is a positive function f in $C_0(\operatorname{Sp}(x)\setminus\{0\})$ such that $y \leqslant f(x)$.

To obtain condition (iii) from the others, apply [3, 1.2] to the C^*-algebra A_0 generated by x, and the hereditary C^*-subalgebra A of M generated by x. The other implications are straightforward. Now Theorem 15 applies to give a result in single operator theory which appears to be new.

COROLLARY 17. *Let H be a separable Hilbert space and denote by $\mathbf{K}$ the algebra of compact operators in $\mathbf{B}(H)$. If x, y are positive operators and x is non-singular, then there is an element z in $\mathbf{B}(H)_+$ such that $xz \in \mathbf{K}$ but $yz \notin \mathbf{K}$, unless y is contained in x.*

Proof. Let B denote the hereditary C^*-subalgebra of $\mathbf{B}(H)$ generated by x. Since x is non-singular, $\mathbf{K} \subset B$. If y is not contained in x, then $y \notin B$; since $M(\mathbf{K}) = \mathbf{B}(H)$ the result is now immediate from Theorem 15.

PROPOSITION 18. (cf. [9, 3.4]). *No corona C^*-algebra is monotone sequentially complete.*

Proof. If $\mathrm{C}(A)$ was monotone sequentially complete for some σ-unital C^*-algebra A, then for a given positive element h in $\mathrm{C}(A)$ with $h \leqslant 1$ there would be a smallest projection p in $\mathrm{C}(A)$ majorizing h, viz. the least upper bound of the sequence $(h^{1/n})$. Let B denote the hereditary, σ-unital C^*-subalgebra of $\mathrm{C}(A)$ generated by h. We see that $B^{\perp} = (1-p)\mathrm{C}(A)(1-p)$ and it follows from Theorem 14 that $B = p\mathrm{C}(A)p$. In particular, $p \in B$, so that $\|p - h^{1/n}\| \to 0$. Since $\mathrm{C}(A)$ is infinite dimensional (in fact non-separable), it contains self-adjoint elements with infinite spectrum, so we may choose h such that 0 is an accumulation point in $\mathrm{Sp}(h)$. Consequently $(h^{1/n})$ is not a Cauchy sequence and we have reached a contradiction.

Recall from [8, § 6] that a projection p in A'' is *regular* in Tomita's sense if

$$\|xp\| = \mathrm{Inf}\{\|x-y\| \mid y \in A,\ yp = 0\}$$

for every x in A. As shown by Akemann [2, II.12], this condition is equivalent with having $\|xp\| = \|x\overline{p}\|$ for every x in A, where $\overline{p}$ denotes the smallest closed projection in A'' majorizing p. We offer the following order-theoretic characterization of regularity.

PROPOSITION 19. *If A is unital C^*-algebra and p is projection in A'', the following conditions are equivalent:*

(i) *p is regular;*

(ii) *If $x \in A_+$ and $p \leqslant x$, then $\overline{p} \leqslant x$;*

(iii) *If $x \in A_+$ and $p \leqslant x$, there is for each $\varepsilon > 0$ a y in A with $0 \leqslant y \leqslant 1$, such that $yp = 0$ and $x + y \geqslant 1 - \varepsilon$.*

Proof. (i) $\Rightarrow$ (ii). If $p \leqslant x$, set $a = (x+\delta)^{-1/2}$ for $\delta > 0$. Then $p \leqslant a^{-2}$, whence $apa \leqslant 1$, cf. [11, 1.3.5], so $\|ap\|^2 = \|apa\| \leqslant 1$. By assumption this implies that $\|a\overline{p}a\| = \|a\overline{p}\|^2 \leqslant 1$, so that $a\overline{p}a \leqslant 1$. Again by [11, 1.3.5] this means that $\overline{p} \leqslant a^{-2} = x + \delta$. Since δ is arbitrary, $\overline{p} \leqslant x$, as desired.

(ii) $\Rightarrow$ (iii). If $p \leqslant x$ then $\overline{p} \leqslant x$, or $1 \leqslant 1 - \overline{p} + x$ by assumption. Since $1 - \overline{p}$ is an open projection, cf. [11, 3.11.9], there is an increasing net (y_λ) in A_+ converging strongly to $1 - \overline{p}$ in A''. This means that the net $(y_\lambda + x)$, regarded as a net of continuous affine functions on the state space S of A, increases pointwise to a function dominating 1. Since S is compact, it follows that for each $\varepsilon > 0$ we have $y_\lambda + x \geqslant 1 - \varepsilon$, eventually. Take $y = y_\lambda$ and check that $0 \leqslant y \leqslant 1 - \overline{p} \leqslant 1 - p$, as we wished.

(iii) $\Rightarrow$ (i). Since $\|ap\| = \|\,|a|p\,\|$, it suffices to check the regularity condition for each element a in A_+. Set $\alpha = \|ap\|$ and note that $\|(a+\varepsilon)p\| \leqslant \alpha + \varepsilon$, so that

$(a+\varepsilon)p(a+\varepsilon) \leqslant (\alpha+\varepsilon)^2$. With $x = (a+\varepsilon)^{-2}(\alpha+\varepsilon)^2$, this implies that $p \leqslant x$. By assumption there is an element y in A with $0 \leqslant y \leqslant 1-p$ such that $x+y \geqslant \geqslant 1-\varepsilon$, i.e. $1-y \leqslant x+\varepsilon$. Since $\bar{p} \leqslant 1-y$ this implies that $\bar{p} \leqslant x+\varepsilon$. Consequently

$$(a+\varepsilon)\bar{p}(a+\varepsilon) \leqslant (a+\varepsilon)(x+\varepsilon)(a+\varepsilon) = (\alpha+\varepsilon)^2 + \varepsilon(a+\varepsilon)^2,$$

so that $\|(a+\varepsilon)\bar{p}\|^2 \leqslant (\alpha+\varepsilon)^2 + \varepsilon(\|a\|+\varepsilon)^2$. Since ε is arbitrary, $\|a\bar{p}\| \leqslant \alpha$, as desired.

REMARK 20. The closure $\bar{p}$ of a projection p in A'' is the largest projection in A'', dominated by every element x in A_+ such $p \leqslant x \leqslant 1$. We see from Proposition 19, that regularity of p means that $\bar{p}$ is actually the largest projection in A'', dominated by every element in A_+ such that $p \leqslant x$. In the absense of regularity, the majorants for p in A_+ will not have a minimum in A''.

In a von Neumann algebra every open projection in its double dual is regular [2, II. 14]. Presumably the same is true for open, σ-unital projections in the second dual of an SAW^*-algebra, but for the moment we can only establish this result for corona algebras.

THEOREM 21. *Let A be a σ-unital C^*-algebra with corona algebra $C(A)$. Then every open, σ-unital projection in $C(A)''$ is regular.*

Proof. If p is an open, σ-unital projection in $C(A)''$, there is by definition an increasing sequence (x_n) in $C(A)_+$, such that $x_n \nearrow p$. Multiplying the sequence with suitable scalars we may assume that $\|x_n\| \leqslant 1-n^{-1}$ for every n. Suppose that $x \in C(A)_+$ with $p \leqslant x$, and choose y in $M(A)_+$ with $\pi(y) = x$, where $\pi : M(A) \to C(A)$ as usual denotes the quotient map. Let h be a strictly positive element in A. We claim that there is an increasing sequence (y_n) in $M(A)_+$, such that for every $n \geqslant 1$,

(i) $\pi(y_n) = x_n$;
(ii) $y_n \leqslant y$;
(iii) $\|(y_n - y_{n-1})h\| \leqslant 2^{-n}$ (with $y_0 = 0$);
(iv) $\|y_n\| \leqslant 1-(2n)^{-1}$.

To prove this by induction, assume that we have found $y_1, \ldots, y_{n-1}$ satisfying (i) — (iv). By [11, 1.5.10] there is an element a in $M(A)_+$, with $y_{n-1} \leqslant a \leqslant y$, such that $\pi(a) = x_n$. Let (u_λ) be an approximate unit for A and put

$$a_\lambda = y_{n-1} + (a-y_{n-1})^{1/2}(1-u_\lambda)(a-y_{n-1})^{1/2}.$$

Then $y_{n-1} \leqslant a_\lambda \leqslant a$ and $\pi(a_\lambda) = x_n$, so that each a_λ satisfies (i) and (ii). Now

$$\|(a_\lambda - y_{n-1})h\| = \|(a-y_{n-1})^{1/2}(1-u_\lambda)(a-y_{n-1})^{1/2}h\|,$$

which tends to zero because $(a - y_{n-1})^{1/2}h \in A$ and (u_λ) is an approximate unit for A. Thus $\|(a_\lambda - y_{n-1})h\| \leqslant 2^{-n}$ for λ sufficiently large. To prove that also condition (iv) is satisfied for λ large, note that the family of sets F_λ consisting of states φ of $M(A)$ such that $\varphi(a_\lambda) \geqslant 1 - (2n)^{-1}$, form a decreasing net of closed compact sets. If $\varphi \in \cap F_\lambda$, let q denote the open, central projection in $M(A)''$, corresponding to the ideal A, and note that (a_λ) converges strongly to $y_{n-1}q + a(1 - q)$ in $M(A)''$. Since φ is a normal state of $M(A)''$, this means that

$$\varphi(y_{n-1}q + a(1 - q)) \geqslant 1 - (2n)^{-1}.$$

However, $\|y_{n-1}q\| \leqslant 1 - (2(n - 1))^{-1}$ by hypothesis, and $\|a(1 - q)\| = \|\pi(a)\| = \|x_n\| \leqslant 1 - n^{-1}$. Since the operators $y_{n-1}q$ and $a(1 - q)$ are orthogonal, the norm of their sum cannot exceed the maximum of $1 - (2(n - 1))^{-1}$ and $1 - n^{-1}$, which is strictly smaller than $1 - (2n)^{-1}$; and we have reached a contradiction. Consequently, $\cap F_\lambda = \emptyset$, which means that $F_\lambda = \emptyset$ for λ sufficiently large; and thus $\varphi(a_\lambda) < 1 - (2n)^{-1}$ for all states φ of $M(A)$, i.e. $\|a_\lambda\| < 1 - (2n)^{-1}$. We can therefore take $y_n = a_\lambda$ for λ large, and obtain a sequence (y_n) in $M(A)_+$ satisfying (i) – (iv).

Since (y_n) is increasing and bounded there is a b in A'' such that $y_n \nearrow b$, and we see from (ii) and (iv) that $b \leqslant y$ and $b \leqslant 1$. It follows from (iii) that $bh \in A$, being the limit of the norm converging sequence $(y_n h)$. But this means that $b \in M(A)$, since hA is dense in A. Take $z = 1 - \pi(b)$ in $C(A)_+$ and note that $x + z \geqslant 1$, since $\pi(b) \leqslant x$. Furthermore $zp = 0$, because (i) implies that $\pi(b) \geqslant \pi(y_n) = x_n$ for all n, so that $1 - z \geqslant p$. Since x was an arbitrary majorant for p in $C(A)_+$ it follows from condition (iii) in Proposition 19 that p is regular.

LEMMA 22. *Let I be a closed ideal in a σ-unital C^*-algebra A and define in $M(A)$ the closed ideal*

$$M(A, I) = \{x \in M(A) \mid xA \subset I\}.$$

Furthermore, for each hereditary, σ-unital C^*-subalgebra B of $M(A, I)$ define the hereditary C^*-subalgebra

$$(B^\perp, I) = \{x \in M(A) \mid xB + Bx \subset I\}.$$

Then $M(A) = (B^\perp, I) + M(A, I)$.

Proof. Let h be a strictly positive element in A and k a strictly positive element in B, and denote by D the separable C^*-subalgebra of $M(A)$ generated by h and k. Put $J_1 = I \cap D$ and $J_2 = J_1^\perp \cap D$, so that the ideal $J = J_1 + J_2$ is essential in D. We then have a natural embedding $J \subset D \subset M(J)$, [11, 3.12.8].

Since $k \in M(A, I)$ we see that $hk \in J_1 \subset J$. By Theorem 13 (applied to the separable algebra J), there are orthogonal elements d, e in $M(J)$, such that $(1 - d)h \in J$ and $(1 - e)k \in J$. Note now that for each x in $(J_2)_+$ we have $hkx = 0$, since

$J_1J_2 = \{0\}$. Since h is strictly positive in A, its annihilator in A'' is zero, whence $kx = 0$. Thus $J_2 \subset B^{\perp}$. The direct sum $J = J_1 + J_2$ means that we have a direct sum $M(J) = M(J_1) + M(J_2)$, so we may write $d = d_1 + d_2$ and $e = e_1 + e_2$. Now considering $M(J)$ as a subset of A'' we have

$$e_1h \in e_1(dh + J_1 + J_2) = e_1J_1 \subset J_1 \subset I.$$

It follows that $e_1hA \subset I$, and since h is strictly positive in A this implies that $e_1A \subset I$, i.e. $e_1 \in M(A, I)$. On the other hand $J_2k = 0$ so

$$(1 - e_1)k = (1 - (e_1 + e_2))k \in J,$$

whence $(1 - e_1)k^2 \in J_1 \subset I$. Since k^2 is strictly positive in B this implies that $1 - e_1 \in (B^{\perp}, I)$, and the proof is complete.

THEOREM 23. (cf. [9, 3.1]). *Each morphism $\rho : A \to B$ between σ-unital C^*-algebras A and B, such that $\rho(A)$ contains an approximate unit for B, induces via its canonical extension $\rho'' : M(A) \to M(B)$ a morphism $\tilde{\rho} : \mathrm{C}(A) \to \mathrm{C}(B)$; and* $\ker\tilde{\rho}$ *is an SAW^*-ideal in* $\mathrm{C}(A)$ *isomorphic with $M(A, \ker\rho)/\ker\rho$, where*

$$M(A, \ker\rho) = \{x \in M(A) \mid xA \subset \ker\rho\}.$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
\ker\rho & \xrightarrow{\iota} & A & \xrightarrow{\rho} & B \\
\downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\
\ker\rho'' & \xrightarrow{\iota} & M(A) & \xrightarrow{\rho''} & M(B) \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\ker\tilde{\rho} & \xrightarrow{\iota} & \mathrm{C}(A) & \xrightarrow{\tilde{\rho}} & \mathrm{C}(B)
\end{array} .$$

Here the maps denoted by ι are the obvious embeddings, and the maps denoted by π are the canonical quotient maps. The map ρ'' is the restriction to $M(A)$ of the unique normal morphism of A'' into B'' that extends ρ, see [11, 3.7.7]. Since $\rho(A)$ contains an approximate unit for B it follows from [11, 3.12.12] that $\rho''(M(A)) \subset M(B)$. Clearly

$$\ker\rho'' = \{x \in M(A) \mid xA \subset \ker\rho\} = M(A, \ker\rho);$$

in particular, the upper half of the diagram is commutative. The map $\tilde{\rho}$ is defined, by setting $\tilde{\rho}(\pi(x)) = \pi(\rho''(x))$ for each x in $M(A)$. Since $\pi(x) = 0$ implies that $x \in A$, so that $0 = \pi(\rho(x)) = \pi(\rho''(x))$, this is an admissible definition. Now

$$\pi^{-1}(\ker\tilde{\rho}) = \{x \in M(A) \mid \rho''(x) \in B\}.$$

But if $\rho''(x) \in B$, let (e_n) be an approximate unit for A. Then $(\rho(e_n))$ is an approximate unit for B so that $\rho''((1 - e_n)x) \to 0$. Thus $x \in A + \ker\rho''$, whence

$$\pi^{-1}(\ker\tilde{\rho}) = A + M(A, \ker\rho).$$

We can therefore define the map σ to be $\iota^{-1} \circ \pi \circ \iota$, so that the whole diagram is commutative. Furthermore,

$$\ker \tilde{\rho} = \pi(A + M(A, \ker \rho)) =$$
$$= M(A, \ker \rho)/M(A, \ker \rho) \cap A = M(A, \ker \rho)/\ker \rho.$$

To show that $\ker \tilde{\rho}$ is an SAW^*-ideal in $C(A)$ we consider an hereditary, σ-unital C^*-subalgebra D of $\ker \tilde{\rho}$. Choose an hereditary, σ-unital C^*-subalgebra E of $\ker \rho''$ such that $\sigma(E) = D$, and apply Lemma 22 (with $\ker \rho$ and E in place of I and B) to show that $M(A) = (E^\perp, \ker \rho) + \ker \rho''$. Consequently $C(A) = D^\perp + \ker \tilde{\rho}$, whence $\ker \tilde{\rho}$ is an SAW^*-ideal of $C(A)$ by Proposition 4.

REMARK 24. Every closed ideal I of a corona algebra $C(A)$ inherits the double annihilator property. Indeed, if B is an hereditary, σ-unital C^*-subalgebra of I and $x \in I \setminus B$, then $xy \neq 0$ for some $y \geq 0$ in $B^\perp$ (computed in $C(A)$) by Theorem 15. If (u_λ) is an approximate unit for I, then $yu_\lambda y \in B^\perp \cap I$ for all λ, and since $xyu_\lambda yx \to xy^2x$ we see that $xyu_\lambda y \neq 0$ for some λ.

This observation applies in particular to the SAW^*-ideal $\ker \rho$ in Theorem 23.

EXAMPLE 25. Let H be a separable, infinite-dimensional Hilbert space, and denote by $\mathbf{K}$ the algebra of compact operators in $\mathbf{B}(H)$. Let A be a non-unital, σ-unital C^*-subalgebra of $\mathbf{B}(H)$ containing $\mathbf{K}$ and denote by B its image in the Calkin algebra $\mathbf{C}(H)$. Take

$$M(A, \mathbf{K}) = \{x \in \mathbf{B}(H) \mid xA + Ax \subset \mathbf{K}\}.$$

By Theorem 23 and Remark 24 the image of $M(A, \mathbf{K})$ in $\mathbf{C}(H)$ is an SAW^*-algebra with the double annihilator property, and we have a short exact sequence

$$M(A, \mathbf{K})/\mathbf{K} \to C(A) \to C(B).$$

To exemplify the above construction, let B be the C^*-algebra generated by a sequence (p_n) of pairwise orthogonal, infinite-dimensional projections with sum 1, and set $A = B + \mathbf{K}$. Thus $x \in M(A, \mathbf{K})$ precisely when xp_n and $p_n x$ are compact for every n. Writing the Hilbert space H in the form $\ell^2 \otimes \ell^2$, we have $B = c_0 \otimes 1$, so that $A = c_0 \otimes 1 + \mathbf{K} \otimes \mathbf{K}$ (C^*-tensor products). Even so, the algebras $M(A)$ and $M(A, \mathbf{K})$ are not easy to describe. The latter will obviously contain the C^*-tensor product $\mathbf{B}(\ell^2) \otimes \mathbf{K}$, but the inclusion is strict. Note the commutative diagram:

$$\begin{array}{ccccc}
\mathbf{K} & \longrightarrow & A & \longrightarrow & c_0 \\
\downarrow & & \downarrow & & \downarrow \\
M(A, \mathbf{K}) & \longrightarrow & M(A) & \longrightarrow & \ell^\infty \\
\downarrow & & \downarrow & & \downarrow \\
M(A, \mathbf{K})/\mathbf{K} & \longrightarrow & C(A) & \longrightarrow & C(\beta\mathbf{N} \setminus \mathbf{N})
\end{array}.$$

An even more interesting example arises by taking $H = L^2(\mathbf{R})$, with $B = C_0(\mathbf{R})$ regarded as multiplication operators, and again $A = B + \mathbf{K}$. Here the bottom line of the diagram shows that $C(A)$ is an extension of the form

$$M(A, \mathbf{K})/\mathbf{K} \to C(A) \to C(\beta\mathbf{R}/\mathbf{R}).$$

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