J. OPERATOR THEORY
15(1986), 53–81

ON FACTORIAL STATES OF OPERATOR ALGEBRAS. III

R. J. ARCHBOLD and C. J. K. BATTY

1. INTRODUCTION

In this paper we extend and clarify results from [6, 8, 19] concerning the pure state space $\overline{\mathrm{P}(A)}$ and the factorial state space $\overline{\mathrm{F}(A)}$ of a C^*-algebra A. Crucial to this programme is the notion of a primal ideal of a C^*-algebra, defined in Section 3. Subsequently, we investigate the associated notion of a (weakly) primal face of a compact convex set.

To describe our results, a convenient starting point is the following result (A) of Glimm, Tomiyama and Takesaki [30; Theorem 2], in which $\mathrm{P}(A)$ denotes the set of pure states in the state space $\mathrm{S}(A)$ and the bar denotes w*-closure:

(A) $$\overline{\mathrm{P}(A)} \supseteq \mathrm{S}(A) \Leftrightarrow \begin{cases} \text{either } A \text{ is one-dimensional} \\ \text{or } A \text{ is prime and antiliminal.} \end{cases}$$

This theorem has been split by the discovery in [6 ,8] of the following results (B) and (C), in which $\mathrm{F}(A)$ denotes the set of factorial states of A (recall that $\varphi \in \mathrm{S}(A)$ is *factorial* if the GNS representation π_φ gives rise to a commutant $\pi_\varphi(A)'$ which is a factor):

(B) $$\overline{\mathrm{F}(A)} \supseteq \mathrm{S}(A) \Leftrightarrow A \text{ is a prime } C^*\text{-algebra;}$$

(C) $$\overline{\mathrm{P}(A)} \supseteq \mathrm{F}(A) \Leftrightarrow \begin{cases} \text{either } A \text{ is abelian} \\ \text{or } A \text{ contains an abelian ideal } I \text{ such that } A/I \text{ is antiliminal.} \end{cases}$$

We shall denote by $\mathrm{F}_{\mathrm{I}}(A)$ (respectively $\mathrm{F}_{\mathrm{f}}(A)$, $\mathrm{F}_{\mathrm{II}}(A)$, $\mathrm{F}_{\mathrm{III}}(A)$) the set of factorial states φ such that $\pi_\varphi(A)'$ is type I (respectively finite type I, type II, type III). Result (D) below is closely related to (B) and it was used in the proof of (C) (see [8]):

(D) $$\text{For any } C^*\text{-algebra } A,\quad \overline{F_{\mathrm{f}}(A)} \supseteq \mathrm{F}(A).$$

Finally, result (E) was obtained in [6] as a corollary of (B):

(E) If $\ker \pi_\varphi$ contains a prime ideal of A then $\varphi \in \overline{\mathrm{F}(A)}$. The converse is false.

Since it follows from (D) that $\overline{F_I(A)} \supseteq F(A)$, the possibility that $\overline{F_{III}(A)} \supseteq F(A)$ was touched upon in [6]. Our first main result (Theorem 2.1) uses (C) to show that

$$\overline{F_{III}(A)} \supseteq F(A) \Leftrightarrow A \text{ is antiliminal.}$$

Indeed, this is analogous to (C), with $F_{III}(A)$ taking the role of P(A).

At the start of Section 3 we define the term *primal ideal*, and then in Theorem 3.3 we obtain the following generalization of (B), in which I is an ideal of the C^*-algebra A:

$$I \text{ is primal} \Leftrightarrow S(A/I) \subseteq \overline{F_f(A)} \Leftrightarrow S(A/I) \subseteq \overline{F(A)}.$$

Here $S(A/I)$ is identified with a subset of S(A) (see below). Result (E) is then improved and clarified by Theorem 3.5:

$$\varphi \in \overline{F(A)} \Leftrightarrow \ker \pi_\varphi \text{ is a primal ideal.}$$

It follows (Corollary 3.6) that if A is unital:

$$\overline{F(A)} \text{ is a union of (w*-) closed split faces of } S(A).$$

In Theorem 3.8 a simple argument shows that various subsets of S(A), including $\overline{F(A)}$ and $\overline{P(A)}$, are unions of (w*-) closed faces of S(A). The result for $\overline{P(A)}$ has been previously obtained by Shultz [27]. The faces for $\overline{F(A)}$ may be taken to be the annihilators of the primal ideals; only in special cases have we been able to characterize the faces for $\overline{P(A)}$. In Theorems 3.10 and 3.11 we offer alternative proofs of Glimm's Stone-Weierstrass theorem and Glimm's characterization of the vector state space of a C^*-algebra of operators [19].

Section 4 is concerned with $\overline{P(A)}$ and $\overline{F(A)}$ for a von Neumann algebra A with centre Z. Theorem 4.2 extends [6; Theorem 4.3(1)] and shows (in particular) that

$$\varphi \in \overline{F(A)} \Leftrightarrow \varphi \mid Z \in P(Z).$$

This was implicit in [6] but not actually stated there. This result and (C) immediately lead to Theorem 4.4, a result of Glimm [19; Theorem 3]:

If A has no central summand of type I then $\overline{P(A)} = \{\varphi \in S(A) : \varphi \mid Z \in P(Z)\}$.

Suppose that B is an ultraweakly dense C^*-subalgebra of a von Neumann algebra A. Glimm [19; Theorem 5] proved that

(F) $$\overline{P(B)} = \overline{P(A)} \mid B.$$

Using this result and tensor products, it was shown in [6, 8] that

(G) $$\overline{F(B)} = \overline{F(A)} \mid B.$$

In Theorem 4.5 we prove (G), without recourse to (F), by using the theory of primal ideals. We then use (G) to simplify, in part, the proof of (F) (see 4.6—4.9), thus fulfilling a programme suggested by Professor R. V. Kadison during the 1983 OATE Conference in Romania.

In Section 5, the duality between ideals of a C^*-algebra A and faces of $S(A)$ (see below) leads to the definition of a *(weakly) primal face* of an arbitrary compact convex subset K of a locally convex space. This is related to the existing notion of (weak) primeness of K introduced by Chu [12] and Ellis [18] in much the same way as primality of ideals is related to primeness. A face F of $S(A)$ is (weakly) primal if and only if F is contained in the annihilator of a primal ideal. In Theorem 5.4, we show that the weakly primal faces F determine the space $A(K)$ of continuous affine functions on K in the sense that

(H) $A(K)\,|\,X = \{f \in C_{\mathbf{R}}(X) : f \,|\, F\cap X \in A(F)\,|\, F \cap X$ for each weakly primal face $F\}$

where X is the closure of the extreme boundary of K. This improves results obtained in [11, 17, 18] where either larger faces F are considered or restrictions are imposed on K. On the other hand, if K is the complex state space of a function algebra, it was shown in [17] that the weakly prime faces (which are smaller than the weakly primal faces) determine $A(K)$. In Theorem 5.8, it is shown that this is also true for a unital C^*-algebra A. Indeed the prime ideals determine A in a sense stronger than that of (H), namely:

(I) $A = \{f \in C_{\mathbf{C}}(\overline{P(A)}) : f \,|\, P(A/I) \in A \,|\, P(A/I)$ for each prime ideal I of $A\}$

(where A acts on $\overline{P(A)}$ in the obvious way). As a consequence of (H) or (I),

$$A = \{f \in C_{\mathbf{C}}(\overline{F(A)}) : f \,|\, C \text{ is affine for each convex subset } C \text{ of } \overline{F(A)}\}.$$

It follows that $\overline{F(A)}$, with its topological and convex structure inherited from $S(A)$, determines A up to Jordan isomorphism (Theorem 5.10).

Finally, in Section 6 we suggest a possible characterizaton of $\overline{P(A)}$ in terms of the geometry of $S(A)$.

We conclude this secton with some more notation and preliminaries. For a Hilbert space H, we shall denote by $L(H)$ (respectively $LC(H)$) the C^*-algebra of all bounded linear (respectively compact linear) operators on H. If S is a subset of $L(H)$, we shall denote by $\bar{S}$ the closure of S with respect to the weak operator topology. If ξ is a unit vector in H, we shall denote by ω_ξ the state of $L(H)$ given by $\omega_\xi(a) = \langle a\xi, \xi\rangle$ $(a \in L(H))$. Suppose that a C^*-algebra A acts non-degenerately on H. Then $\omega_\xi \,|\, A$ is called a *vector state* of A. The w*-closure of the set of all such states is called the *vector state space* of A.

Suppose that A is a C^*-algebra (with or without identity). For $\varphi \in S(A)$, $(H_\varphi, \pi_\varphi, \xi_\varphi)$ will denote the Hilbert space, the representation, and the cyclic vector, associated with φ by the GNS construction. For a non-degenerate representation π

of A and a unit vector $\eta \in H_\pi$, let ω_η^π be the state defined by $\omega_\eta(a) = \langle \pi(a)\eta, \eta \rangle$. If $\pi = \pi_\varphi$ for some $\varphi \in S(A)$, we shall write ω_η^φ instead of ω_η^π. If $\eta = \|\pi_\varphi(b)\xi_\varphi\|^{-1}\pi_\varphi(b)\xi_\varphi$ for some $b \in A$ with $\varphi(b^*b) > 0$, we shall write φ_b instead of ω_η^φ, so that $\varphi_b(a) = \varphi(b^*ab)/\varphi(b^*b)$.

By an *ideal* of A we shall mean a closed two-sided ideal, unless stated otherwise. For an ideal I of A, we shall identify $S(A/I)$ with $\{\varphi \in S(A) : \varphi(I) = \{0\}\}$, and $S(I)$ with $\{\varphi \in S(A) : \|\varphi \mid I\| = 1\}$ (see [24; 3.1.6]). Then $P(A) = P(A/I) \cup P(I)$ [15; 2.11.8] and $F(A) = F(A/I) \cup F(I)$ (see [6; § 2]). Similarly, for the spectrum and primitive ideal space of A we have $\hat{A} = (A/I)^\wedge \cup \hat{I}$ and $\mathrm{Prim}(A) = \mathrm{Prim}(A/I) \cup \mathrm{Prim}(I)$ [15; 3.2.1].

For a compact convex subset K of a locally convex space, the set of all extreme (respectively primary [31]) points will be denoted by $\partial_e K$ (respectively $\partial_{pr} K$). For a subset E of K, the convex hull (respectively closed convex hull) of E will be denoted by $\mathrm{co}(E)$ (respectively $\overline{\mathrm{co}}(E)$).

The dual of a C^*-algebra A will be considered in the w*-topology, except where stated otherwise. If A is unital, $S(A)$ is a compact convex set, $P(A) = \partial_e S(A)$, $F(A) = \partial_{pr} S(A)$, and there is a bijective correspondence between closed faces F of $S(A)$ and closed left ideals L of A given by

$$F \to F_\perp = \{a \in A : \varphi(a^*a) = 0 \text{ for all } \varphi \in F\}$$

$$L \to L^\perp = \{\varphi \in S(A) : \varphi \mid L = 0\}.$$

In particular, all closed faces are semi-exposed. Furthermore, F is split if and only if L is a (closed two-sided) ideal I, so that $F = S(A/I)$. There is also an isometric isomorphism between the self-adjoint part of A and $A(S(A))$ given by $a \to \hat{a}$, where $\hat{a}(\varphi) = \varphi(a)$. See [7; 5.2], [24; 3.10], [28; III.6], [31; 4.5] for further details of these correspondences.

2. FACTORIAL STATES OF TYPE III

In [8; Theorem 3.4] it was shown that if A is a C^*-algebra then $\overline{P(A)} \supseteq F(A)$ if and only if either A is abelian or A contains an abelian ideal I such that A/I is antiliminal. In the following result $F_{III}(A)$ replaces $P(A)$ at the expense of the abelian ideal.

THEOREM 2.1. *Let A be a C^*-algebra. The following conditions are equivalent.*
(1) $\overline{F_{III}(A)} \supseteq F(A)$;
(2) *A is antiliminal.*

Proof. (1) $\Rightarrow$ (2). Suppose that J is a nonzero liminal ideal of A. Let $\varphi \in P(J)$ have extension $\psi \in P(A)$. By (1) there is a net (ψ_α) in $F_{III}(A)$ such that $\psi_\alpha \to \psi$.

Hence $\psi_\alpha \mid J \to \varphi$. So eventually $\psi_\alpha \mid J$ is nonzero and then $\psi_\alpha \mid J \in \mathrm{F}_{\mathrm{III}}(J)$, a contradiction since $\mathrm{F}_{\mathrm{III}}(J)$ is empty. It follows that A is antiliminal.

(2) $\Rightarrow$ (1). Suppose that A is unital. Let I be the intersection of the kernels of the type III factor representations of A. Suppose that I has a type III factor representation π. By [15; 2.10.4] π extends to a type III factor representation σ of A. This gives a contradiction, since $I \nsubseteq \ker \sigma$. Thus I has no type III factor representation and so I is postliminal by the Glimm-Sakai theorem [26; Theorem 2]. By (2), $I = \{0\}$. Thus if $a \in A^+ \setminus \{0\}$ there exists a type III factor representation π_a of A on a Hilbert space H_a such that $\pi_a(a) \neq 0$. Let

$$S = \{\omega_\xi \circ \pi_a : a \in A^+ \setminus \{0\},\ \xi \in H_a,\ \|\xi\| = 1\}.$$

If $\varphi \in S$ then π_φ is a subrepresentation of some π_a and hence $\varphi \in \mathrm{F}_{\mathrm{III}}(A)$ (see [16; I.2.1, Proposition 2]). Since $\oplus \{\pi_a : a \in A^+ \setminus \{0\}\}$ is faithful, $\bar{S} \supseteq \mathrm{P}(A)$ by [15; 3.4.1]. Thus $\overline{\mathrm{F}_{\mathrm{III}}(A)} \supseteq \bar{S} \supseteq \overline{\mathrm{P}(A)} \supseteq \mathrm{F}(A)$, the last inclusion holding since A is antiliminal [8; Theorem 3.4].

Now suppose that A is not unital and let $\tilde{A}$ be obtained by adjoining an identity. By (2), $\tilde{A}$ is antiliminal and so $\overline{\mathrm{F}_{\mathrm{III}}(\tilde{A})} \supseteq \mathrm{F}(\tilde{A})$ by the previous paragraph. Let $\varphi \in \mathrm{F}(A)$ have extension $\psi \in \mathrm{F}(\tilde{A})$. There is a net (ψ_α) in $\mathrm{F}_{\mathrm{III}}(\tilde{A})$ such that $\psi_\alpha \to \psi$ and hence $\psi_\alpha \mid A \to \varphi$. Eventually $\psi_\alpha \mid A$ is nonzero and then $\psi_\alpha \mid A \in \mathrm{F}_{\mathrm{III}}(A)$. Thus $\varphi \in \overline{\mathrm{F}_{\mathrm{III}}(A)}$. This completes the proof.

Since $\overline{\mathrm{F}(A)} \supseteq \mathrm{S}(A)$ if and only if A is prime [6; Theorem 3.3], we obtain the following result which is analogous to the theorem of Glimm, Tomiyama and Takesaki (see [30; Theorem 2]).

COROLLARY 2.2. *Let A be C^*-algebra. The following conditions are equivalent.*

(1) $\overline{\mathrm{F}_{\mathrm{III}}(A)} \supseteq \mathrm{S}(A)$;

(2) *A is prime and antiliminal.*

Glimm proved that if A is a separable C^*-algebra which is not postliminal then A has a type II factor representation [24; 6.8.7]. It follows by the methods of this section that if A is separable then

(i) $\overline{\mathrm{F}_{\mathrm{II}}(A)} \supseteq \mathrm{F}(A)$ if and only if A is antiliminal;

(ii) $\overline{\mathrm{F}_{\mathrm{II}}(A)} \supseteq \mathrm{S}(A)$ if and only if A is prime and antiliminal.

It is also easy to see that the general conjectures (a) and (b) below are either both true or both false:

(a) Every non-postliminal C^*-algebra has a type II factor representation;

(b) For every antiliminal C^*-algebra A, $\overline{\mathrm{F}_{\mathrm{II}}(A)} \supseteq \mathrm{F}(A)$.

COROLLARY 2.3. *Let A be a C^*-algebra.*

(1) *Every type* II *factor state is a* w*-*limit of type* III *factor states.*

(2) *If A is separable, every type* III *factor state is a* w*-*limit of type* II *factor states.*

Proof. Let $\varphi \in \mathrm{F}_{\mathrm{II}}(A)$ and let J be the largest postliminal ideal of A. Then, by Theorem 2.1,

$$\varphi \in \mathrm{F}(A/J) \subseteq \overline{\mathrm{F}_{\mathrm{III}}(A/J)} = \overline{\mathrm{F}_{\mathrm{III}}(A)}.$$

The proof of (2) is similar, using (i) above.

We complete this section with the following strengthening of [6; Corollary 3.4].

THEOREM 2.4. *Let A be a C^*-algebra. Then*

$$\mathrm{F}_{\mathrm{II}}(A) \cup \mathrm{F}_{\mathrm{III}}(A) \subseteq \overline{\mathrm{P}(A)} \cap \mathrm{S}(A).$$

Proof. Suppose that $\varphi \in \mathrm{F}_{\mathrm{II}}(A) \cup \mathrm{F}_{\mathrm{III}}(A)$, and let J be the largest postliminal ideal of A. Then $\varphi(J) = \{0\}$ (for, otherwise, $\varphi \in \mathrm{F}(J) \subseteq \mathrm{F}_{\mathrm{I}}(A)$), and so

$$\varphi \in \mathrm{F}(A/J) \subseteq \overline{\mathrm{P}(A/J)} \subseteq \overline{\mathrm{P}(A)}.$$

Alternatively, one may observe that $\pi_\varphi(A)$ is a prime, antiliminal C^*-algebra, so that

$$\varphi \in \mathrm{S}(A/\ker \pi_\varphi) \subseteq \overline{\mathrm{P}(A/\ker \pi_\varphi)} \subseteq \overline{\mathrm{P}(A)}.$$

3. THE FACTORIAL STATE SPACE

In this section we require a weakening of the notion of primeness, as defined below.

DEFINITION 3.1. An ideal I of a C^*-algebra A is said to be *primal* if whenever $n \geqslant 1$ and $J_1, J_2, \ldots, J_n$ are ideals of A such that $J_1 J_2 \ldots J_n = \{0\}$ then $J_i \subseteq I$ for at least one value of i.

Such ideals in Gelfand rings have been recently described as "weakly prime" [9], but we prefer the term "primal" since we wish to avoid confusion in Section five with the notion of weak primeness in convexity theory (see [18]). It is immediate from Definition 3.1 that the zero ideal is primal if and only if it is prime, and that any ideal containing a primal ideal is itself primal. Clearly, every prime ideal is primal; an elementary example of an ideal which is primal but not prime may be obtained as follows. Let $A = LC(H) + \mathbf{C}E_1 + \mathbf{C}E_2$ where E_1 and E_2 are infinite dimensional projections with sum 1 on the Hilbert space H and $LC(H)$ is the algebra of compact operators. Since $\{0\}$ is a primitive ideal, $LC(H)$ is primal. However, $LC(H)$ is not prime since it is the intersection of the two maximal ideals which properly contain it.

It is remarkable that, whereas only prime C^*-algebras can occur as quotients by prime ideals, any C^*-algebra can occur as a quotient by a primal ideal. To see this, let A be a C^*-algebra acting on a Hilbert space H and let B be the C^*-algebra of all sequences $x = (x_n)_{n \geqslant 1}$ of elements in $L(H)$ which are norm-convergent to an element $a(x)$ in A. Let I be the ideal of null sequences, and for each n let q_n be the $*$-homomorphism of B onto $L(H)$ defined by $q_n(x) = x_n$. If J is an ideal of B such that $J \nsubseteq I$ then eventually $q_n(J)$ is a non-zero ideal of $L(H)$. Since $\{0\}$ is a prime ideal of $L(H)$ it follows easily that I is a primal ideal of B. Clearly $A \cong B/I$.

In view of both the definition of primeness and the second part of [19; Lemma 11], it is natural to ask whether the variable integer n in Definition 3.1 may be replaced by a fixed integer, perhaps $n = 2$. The following example, which adapts an idea from [22, § 6.2], shows that the answer is negative.

Example. Let $n \geqslant 2$ and let $H = H_1 \oplus H_2 \oplus \dots \oplus H_{n+1}$ where each H_i is an infinite dimensional Hilbert space. Let $K_i = LC(H_i)$ and let $e^{(i)}$ be the projection from H onto H_i. For each i let $\{e_j^{(i)} : 1 \leqslant j \leqslant n+1, j \neq i\}$ be a set of infinite dimensional projections with sum $e^{(i)}$. Let A_0 be the abelian C^*-algebra consisting of those bounded linear operators a_0 on H for which there exist (necessarily unique) scalars $\alpha_1(a_0)$, $\alpha_2(a_0)$, $\dots, \alpha_{n+1}(a_0)$ such that

$$a_0 = \bigoplus_{i=1}^{n+1} \bigoplus_{j \neq i} \alpha_j(a_0) e_j^{(i)} .$$

Let $I = \bigoplus_{i=1}^{n+1} K_i$ and let $A = I + A_0$. Then I is an ideal of the C^*-algebra A (see [15; 1.8.4]).

Let J be an ideal of A such that $J \nsubseteq I$. Then there exists $a = (a_1 \oplus a_2 \oplus \dots \oplus a_{n+1}) + a_0 \in J$ with $a_i \in K_i$ $(1 \leqslant i \leqslant n+1)$, $a_0 \in A_0$ and $\alpha_j(a_0) \neq 0$ for some j. Let $i \neq j$ $(1 \leqslant i \leqslant n+1)$. Since $a_i \in K_i$ there exists a unit vector $\eta_{ij} \in e_j^{(i)}(H)$ such that $\|a_i \eta_{ij}\| < (1/2)|\alpha_j(a_0)|$. Let p_{ij} be the projection from H onto the linear span of η_{ij}. Then $p_{ij} \in A$ and so

$$0 \neq a_i p_{ij} + \alpha_j(a_0) p_{ij} = a p_{ij} \in J \cap K_i .$$

Since K_i is a minimal ideal of A, $J \supseteq K_i$ and so $J \supseteq \bigoplus_{i \neq j} K_i$. Thus if $J_1, J_2, \dots, J_n$ are ideals of A with $J_r \nsubseteq I$ $(1 \leqslant r \leqslant n)$ then at least one of the ideals K_i is contained in every J_r and so $J_1 J_2 \dots J_n \neq \{0\}$.

On the other hand, for $1 \leqslant i \leqslant n+1$ we may define

$$J_i = (\bigoplus_{j \neq i} K_j) + \{a_0 \in A_0 : \alpha_j(a_0) = 0 \text{ for all } j \neq i\}.$$

Then $J_i \nsubseteq I$ $(1 \leqslant i \leqslant n+1)$ but $J_1 J_2 \dots J_{n+1} = \{0\}$.

If Prim(A) is Hausdorff then every proper primal ideal of A is maximal. This is an immediate consequence of the following proposition which describes primal ideals in terms of the hull-kernel topology on Prim(A).

PROPOSITION 3.2. *Let I be an ideal of a C^*-algebra A. The following conditions are equivalent.*

(1) *I is a primal ideal.*

(2) *Whenever $n \geqslant 1$ and $U_1, U_2, \ldots, U_n$ are open subsets of* Prim(A) *with $U_i \cap$ Prim(A/I) non-empty $(1 \leqslant i \leqslant n)$ then $\bigcap_{i=1}^{n} U_i$ is non-empty.*

(3) *There is a net (P_α) in* Prim(A) *which converges to every point of* Prim(A/I).

Proof. (1) $\Rightarrow$ (2). Let U_i be an open subset of Prim(A) intersecting Prim(A/I), and let J_i be the ideal such that $U_i =$ Prim(J_i) $(1 \leqslant i \leqslant n)$. Then $J_i \nsubseteq I$ $(1 \leqslant i \leqslant n)$ and so $J_1 J_2 \ldots J_n \neq \{0\}$ by (1). Hence $\bigcap_{i=1}^{n} U_i$ is non-empty.

(2) $\Rightarrow$ (3). Let $\alpha = (U_Q)_{Q \in \mathrm{Prim}(A/I)}$ where each U_Q is an open neighbourhood of Q and $U_Q =$ Prim(A) for all except finitely many Q. By (2), there exists $P_\alpha \in \bigcap \{U_Q : Q \in \mathrm{Prim}(A/I)\}$. The set of all such α is directed by the definition that $(U_Q) \geqslant (V_Q)$ if and only if $U_Q \subseteq V_Q$ for all $Q \in$ Prim(A/I). Then the net (P_α) is convergent to each Q in Prim(A/I).

(3) $\Rightarrow$ (1). Let $J_1, \ldots, J_n$ be ideals of A such that $J_i \nsubseteq I$ $(1 \leqslant i \leqslant n)$. Then there exists $Q_i \in$ Prim(A/I) such that $J_i \nsubseteq Q_i$. By (3), there exists α_i such that $J_i \nsubseteq P_\alpha$ for all $\alpha \geqslant \alpha_i$. Fix $\alpha \geqslant \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $J_i \nsubseteq P_\alpha$ $(1 \leqslant i \leqslant n)$. Since P_α is prime, $J_1 J_2 \ldots J_n \nsubseteq P_\alpha$. In particular, $J_1 J_2 \ldots J_n \neq \{0\}$.

The equivalence of (2) and (3) above is simply a matter of general topology. It is also the case that Prim(A) and Prim(A/I) may be replaced in Proposition 3.2 by $\hat{A}$ and $(A/I)^\wedge$.

Note that if (P_α) is a net in Prim(A) and S is a non-empty subset of Prim(A) such that $P_\alpha \to Q$ for each $Q \in S$ (and hence for each Q in the closure of S) then it follows from Proposition 3.2 ((3) $\Rightarrow$ (1)) that $\bigcap \{Q \mid Q \in S\}$ is a prima lideal of A. If Prim(A) is not Hausdorff then there exists a net (P_α) with distinct limits Q_1 and Q_2 in Prim(A). Then $Q_1 \cap Q_2$ is primal but not maximal. Thus, for any C^*-algebra A, every proper primal ideal is maximal if and only if Prim(A) is Hausdorff. In particular, these equivalent conditions are satisfied if A is abelian.

We turn now to the connection between primal ideals and the factorial state space $\overline{\mathrm{F}(A)}$.

THEOREM 3.3. *Let I be an ideal of a C^*-algebra A. The following conditions are equivalent.*

(1) *I is a primal ideal.*

(2) $\mathrm{S}(A/I) \subseteq \overline{\mathrm{F}_{\mathrm{f}}(A)}$.

(3) $\mathrm{S}(A/I) \subseteq \overline{\mathrm{F}(A)}$.

Proof. (1) $\Rightarrow$ (2). We adapt the proof of [8; Proposition 2.2]. Let $\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i$ be a convex combination of pure states of A/I. Since the set of such combinations is w*-dense in $S(A/I)$ (even if A/I is non-unital), it suffices to show that $\varphi \in \overline{F_f(A)}$. Let U be an open convex w*-neighbourhood of 0 in A^* and let

$$V_i = \{[\pi_\psi] : \psi \in P(A),\ \psi - \varphi_i \in U\} \quad (1 \leqslant i \leqslant n).$$

Since $\psi \to [\pi_\psi]$ is an open map of $P(A)$ into $\hat{A}$, $V_i = \hat{J}_i$ for some ideal J_i. Since $\varphi_i(I) = \{0\}$ but $\varphi_i(J_i) \neq \{0\}$, J_i is not contained in I. By (1), $J_1 J_2 \ldots J_n \neq \{0\}$ and so there exists an irreducible representation π of A such that $[\pi] \in \hat{J}_i$ $(1 \leqslant j \leqslant n)$. For each i there exists $\psi_i \in (\varphi_i + U) \cap P(A)$ such that $[\pi_{\psi_i}] = [\pi]$. Then $\psi_1, \psi_2, \ldots, \ldots, \psi_n$ are equivalent, so $\psi = \sum_{i=1}^{n} \lambda_i \psi_i \in F_f(A)$ (see [8; Section 2]). Since

$$\psi - \varphi = \sum_{i=1}^{n} \lambda_i(\psi_i - \varphi_i) \in U$$

and U was arbitrary, $\varphi \in \overline{F_f(A)}$ as required.

(2) $\Rightarrow$ (3). This follows from the fact that $F_f(A) \subseteq F(A)$.

(3) $\Rightarrow$ (1). Suppose that $J_1, J_2, \ldots, J_n$ are ideals of A such that $J_i \nsubseteq I$ $(1 \leqslant i \leqslant n)$. For each i there exists $a_i \in J_i^+$ and $\varphi_i \in S(A/I)$ such that $\varphi_i(a_i) > 0$. Let $\varphi = n^{-1} \sum_{i=1}^{n} \varphi_i$. Then $\varphi(a_i) > 0$ for each i, so by (3) there exists $\psi \in F(A)$ such that $\psi(a_i) > 0$ for each i. It follows that $\pi_\psi(J_i) \neq \{0\}$ and so $\overline{\pi_\psi(J_i)} = \overline{\pi_\psi(A)}$ for each i. Hence

$$\overline{\pi_\psi(J_i)}\, \overline{\pi_\psi(J_2)} \ldots \overline{\pi_\psi(J_n)} = \overline{\pi_\psi(A)}$$

and so $J_1 J_2 \ldots J_n \neq \{0\}$. Thus I is primal.

COROLLARY 3.4. (see [6; Theorem 3.3] and [8; Proposition 2.2]). *Let A be a C^*-algebra. The following conditions are equivalent.*

(1) *A is a prime C^*-algebra.*

(2) $S(A) \subseteq \overline{F_f(A)}$.

(3) $S(A) \subseteq \overline{F(A)}$.

Proof. Take $I = \{0\}$ in Theorem 3.3.

THEOREM 3.5. *Let A be a C^*-algebra and let $\varphi \in S(A)$. The following conditions are equivalent.*

(1) $\varphi \in \overline{F(A)}$.

(2) $\ker \pi_\varphi$ *is a primal ideal of A.*

Proof. (1) $\Rightarrow$ (2). Let $J_1, J_2, \ldots, J_n$ be ideals of A such that $J_i \nsubseteq \ker \pi_\varphi$ for each i. Then there exists $a_i \in J_i$ such that $\varphi(a_i) \neq 0$ $(1 \leqslant i \leqslant n)$. By (1), there exists

$\psi \in F(A)$ such that $\psi(a_i) \neq 0$ and hence $J_i \nsubseteq \ker \pi_\psi$ $(1 \leqslant i \leqslant n)$. Since $\ker \pi_\psi$ is prime, $J_1 J_2 \ldots J_n \nsubseteq \ker \pi_\psi$. In particular, $J_1 J_2 \ldots J_n \neq \{0\}$.

(2) $\Rightarrow$ (1). Write $I = \ker \pi_\varphi$. By (2) and Theorem 3.3, we have $\varphi \in S(A/I) \subseteq \subseteq \overline{F(A)}$.

Theorem 3.5 clarifies the results of [6; 4.1] where, in a particular example, a state $\varphi \in \overline{F(A)}$ is exhibited with $\ker \pi_\varphi$ containing no prime ideal. In this case, it is easy to verify that $\ker \pi_\varphi$ satisfies condition (3) of Proposition 3.2.

In view of Theorem 3.5, it is natural to ask whether, given a proper primal ideal I of A, there is a state $\varphi \in \overline{F(A)}$ such that $\ker \pi_\varphi = I$. By Theorem 3.3, this amounts to asking whether A/I has a state φ with $\ker \pi_\varphi = \{0\}$. The answer is affirmative if Prim(A/I) has a countable dense subset (this holds if A/I is separable, for example). On the other hand, since we have seen that any C^*-algebra can occur as a quotient by a primal ideal, it is possible to construct A and I so that A/I is an inseparable C^*-algebra with no state φ satisfying $\ker \pi_\varphi = \{0\}$.

COROLLARY 3.6. *Let A be a unital C^*-algebra. Then*

$$\overline{F(A)} = \bigcup \{S(A/I) : I \text{ a primal ideal}\},$$

a union of closed split faces of $S(A)$.

Proof. If I is primal then $S(A/I) \subseteq \overline{F(A)}$ by Theorem 3.3. If $\varphi \in \overline{F(A)}$ then $\varphi \in S(A/\ker \pi_\varphi)$ and $\ker \pi_\varphi$ is primal by Theorem 3.5.

Unlike the case of $\overline{F(A)}$ in Theorem 3.5, it is not possible in general to characterize states φ in $\overline{P(A)}$ purely in terms of $\ker \pi_\varphi$. For, if A is the C^*-algebra of $n \times n$ complex matrices then $S(A) \neq \overline{P(A)}$ $(= P(A))$ but $\ker \pi_\varphi = \{0\}$ for all $\varphi \in S(A)$. However, Shultz [27] has proved that $\overline{P(A)}$ is a union of closed faces, and we enlarge on this in Theorem 3.8 below.

Recall that a subset E of a compact convex set K of a locally convex space is said to be *extremal* (in K) if, whenever $x, y \in K$, $0 < \lambda < 1$ and $\lambda x + (1 - \lambda) y \in E$, then $x, y \in E$ (equivalently, E is a union of faces of K). The following lemma is fairly standard (see, for example, [27; proof of Lemma 11] and [29]).

LEMMA 3.7. *Let E be a compact extremal subset of K. Then E is a union of compact faces of K.*

THEOREM 3.8. *Let A be a unital C^*-algebra and let Π be a collection of non-degenerate representations of A. Let*

$$V_\Pi(A) = \{\omega_\eta^\pi : \pi \in \Pi,\ \eta \in H_\pi,\ \|\eta\| = 1\}.$$

Then $\overline{V_{II}(A)}$ is a compact extremal subset of S(A), *and there is a family $\mathscr{L}$ of closed left ideals L of A such that*

$$\overline{V_{II}(A)} = \bigcup\{L^{\perp} : L \in \mathscr{L}\} =$$
$$= \{\varphi \in S(A) : \varphi(L) = \{0\} \text{ for some } L \in \mathscr{L}\}.$$

Proof. Let (φ_α) be a net in $V_{II}(A)$ which is w*-convergent to $\varphi \in \overline{V_{II}(A)}$. If $\varphi_\alpha = \omega_{\eta_\alpha}^{\pi_\alpha}$ and $\varphi_\alpha(b^*b) > 0$, then $(\varphi_\alpha)_b = \omega_{\eta'_\alpha}^{\pi_\alpha}$ where $\eta'_\alpha = \|\pi_\alpha(b)\eta_\alpha\|^{-1}\pi_\alpha(b)\eta_\alpha$, and $((\varphi_\alpha)_b)$ is w*-convergent to φ_b (assuming $\varphi(b^*b) > 0$). Hence $\varphi_b \in \overline{V_{II}(A)}$. For any unit vector $\eta \in H_\varphi$, let (b_n) be a sequence in A such that $\|\pi_\varphi(b_n)\xi_\varphi - \eta\| \to 0$. Then $\|\varphi_{b_n} - \omega_\eta^\varphi\| \to 0$, so $\omega_\eta^\varphi \in \overline{V_{II}(A)}$.

Now suppose that $\varphi = \lambda\psi + (1 - \lambda)\rho$ where $\psi, \rho \in S(A)$ and $0 < \lambda < 1$. There exists $x' \in \pi_\varphi(A)'$ such that $\|x'\xi_\varphi\| = 1$ and $\psi = \omega_{x'\xi_\varphi}^\varphi \in \overline{V_{II}(A)}$. Similarly, $\rho \in \overline{V_{II}(A)}$. Thus $\overline{V_{II}(A)}$ is extremal.

The final statement now follows immediately from Lemma 3.7 and the correspondence between faces and left ideals described in Section 1.

In the case where Π is a collection of irreducible representations, Theorem 3.8 was proved in [27; Lemma 11]. In particular, if Π consists of all irreducible representations, it follows that $\overline{P(A)}$ is a union of closed faces of S(A). Similarly, if Π consists of all factorial representations, Theorem 3.8 shows that $\overline{F(A)}$ is a union of closed faces. However, Theorem 3.5 gives extra information about $\overline{F(A)}$ in that it shows that the faces may be taken to be the annihilators of the primal (two-sided) ideals.

Unfortunately, in general it does not seem easy to describe the family $\mathscr{L}$ of left ideals L such that $L^{\perp} \subseteq \overline{P(A)}$. If A is a von Neumann algebra, it follows from [19; Theorem 4′] that one may take $\mathscr{L}$ to be the set of left ideals of the form

$$\text{norm-closure}(x[A] + L'_p)$$

where $x[A]$ is a Glimm ideal of A (see Section 4), p is a non-zero abelian projection in A, and L'_p is the left ideal of A generated algebraically by $\{q \in A : q \text{ is an abelian projection and } pq = 0\}$. The structure of $\overline{P(A)}$ can also be described if A is antiliminal or if A acts on H with $A \supseteq LC(H)$. In the first case $\overline{P(A)} = \overline{F(A)}$, and in the second case $\overline{P(A)} = \bigcup\{F_\xi : \xi \in H, \|\xi\| = 1\}$ where

$$F_\xi = \{\lambda\omega_\xi \mid A + (1 - \lambda)g : 0 \leqslant \lambda \leqslant 1,\ g \in S(A/LC(H))\},$$

a closed face of S(A) (see [19; Theorem 2]). The fact that $\overline{P(A)}$ is a union of split faces if A is antiliminal has already been observed by Shultz in the proof of [27; Theorem 17].

We conclude this section by offering alternative proofs of two theorems from [19]. Our proof of Glimm's Stone-Weierstrass Theorem is essentially a reorganization of existing methods.

LEMMA 3.9. *Let B be a C^*-subalgebra of a unital C^*-algebra A. Suppose that B contains the identity element of A and that B separates $\overline{\mathrm{F}(A)}$. Then $B = A$.*

Proof. Suppose that $B \neq A$. By the Hahn-Banach and Kreĭn-Milman theorems there is a non-zero extreme point m of the compact convex set

$$\{h \in A^* : \|h\| \leqslant 1,\ h(B) = \{0\},\ h \text{ self-adjoint}\}.$$

Let I be the largest ideal of A that is contained in $\ker m$. Then I is prime [1; Lemma III.5].

Let $\pi : A \to A/I$ be the quotient map. Since $\mathrm{S}(A/I) \subseteq \overline{\mathrm{F}(A)}$ (by Theorem 3.3), $\pi(B)$ separates $\mathrm{S}(A/I)$ and so $\pi(B) = A/I$ [15; 11.3.2]. This contradicts the fact that $m(B) = \{0\}$ and so $B = A$.

Recent work of Anderson, Bunce, Longo and Popa [4, 23, 25] shows that if A is separable then one may replace $\overline{\mathrm{F}(A)}$ by $\mathrm{F}(A)$ in the above lemma (and hence $\overline{\mathrm{P}(A)}$ by $\mathrm{F}(A) \cap \overline{\mathrm{P}(A)}$ in the following).

THEOREM 3.10. [19; Theorem 1]. *Let B be a C^*-subalgebra of a unital C^*-algebra A. Suppose that B contains the identity element of A and that B separates $\overline{\mathrm{P}(A)}$. Then $B = A$.*

Proof. Let J be the largest postliminal ideal of A and let $\pi : A \to A/J$ be the quotient map. By [8, Theorem 3.4], $\overline{\mathrm{F}(A/J)} = \overline{\mathrm{P}(A/J)} \subseteq \overline{\mathrm{P}(A)}$. So $\pi(B)$ separates $\overline{\mathrm{F}(A/J)}$. By Lemma 3.9, $\pi(B) = A/J$. Since $B \supseteq J$ [15; 11.1.5, 11.1.7], we conclude that $B = A$.

The next theorem is Glimm's characterization of the vector state space of a C^*-algebra of operators [19]. We simplify one part of the proof by using a technique which Glimm himself developed later in [19].

THEOREM 3.11. [19; Theorem 2]. *Let A be a C^*-algebra of operators on a Hilbert space H and suppose that $1 \in A$. Let $\varphi \in \mathrm{S}(A)$. The following conditions are equivalent.*

(1) *φ is a w^*-limit of vectors states of A.*

(2) *$\varphi = \lambda(\omega_\xi \mid A) + (1-\lambda)\psi$ where $0 \leqslant \lambda \leqslant 1$, ξ is a unit vector in H and ψ is a state of A which annihilates $A \cap LC(H)$.*

Proof. (1) $\Rightarrow$ (2). Let $\varphi = \lim(\omega_{\xi_\alpha} \mid A)$ where each ξ_α is a unit vector in H. Suppose first of all that $A \supseteq LC(H)$. By [15; 2.11.7], $\varphi = \lambda\varphi_1 + (1-\lambda)\psi$ where $0 \leqslant \lambda \leqslant 1$, $\varphi_1 \in \mathrm{S}(LC(H))$ and $\psi \in \mathrm{S}(A/LC(H))$. If $\lambda = 0$ there is nothing to prove, so let us assume $\lambda \neq 0$.

By [15; 4.1.3], $\varphi_1 \mid LC(H) = \sum_{i=1}^{\infty} \mu_i(\omega_{\eta_i} \mid LC(H))$ where $\{\eta_1, \eta_2, \ldots\}$ is an orthonormal set of vectors in H, $\mu_i \geqslant 0$ $(1 \leqslant i < \infty)$ and $\sum_{i=1}^{\infty} \mu_i = 1$. Hence $\varphi_1 = \sum_{i=1}^{\infty} \mu_i(\omega_{\eta_i} \mid A)$ [24; 3.1.6]. We now adapt an argument from the proof of [19; Theorem 4]. Let p and q be distinct positive integers. Let E_p (respectively E_q) be the projection from H onto the linear span of η_p (respectively η_q). Let W be the unique partial isometry in $L(H)$ such that $W(\eta_q) = \eta_p$ and $W(1 - E_q) = 0$. Note that $E_p, E_q, W \in LC(H) \subseteq A$. For each α, $\xi_\alpha = s_\alpha \eta_p + t_\alpha \eta_q + (1 - E_p - E_q)\xi_\alpha$ for some scalars s_α and t_α. Hence

$$\begin{bmatrix} \langle E_p \xi_\alpha, \xi_\alpha \rangle & \langle W \xi_\alpha, \xi_\alpha \rangle \\ \langle W^* \xi_\alpha, \xi_\alpha \rangle & \langle E_q \xi_\alpha, \xi_\alpha \rangle \end{bmatrix} = \begin{bmatrix} |s_\alpha|^2 & t_\alpha \bar{s}_\alpha \\ s_\alpha \bar{t}_\alpha & |t_\alpha|^2 \end{bmatrix},$$

which has determinant zero. Thus

$$0 = \det \begin{bmatrix} \varphi(E_p) & \varphi(W) \\ \varphi(W^*) & \varphi(E_q) \end{bmatrix} = \det \begin{bmatrix} \lambda \mu_p & 0 \\ 0 & \lambda \mu_q \end{bmatrix}.$$

Since $\lambda \neq 0$, at least one of μ_p and μ_q is zero. It follows that φ has the required form.

Now suppose that $A \nsupseteq LC(H)$. Let ρ be a w*-limit point of the ne $(\omega_{\xi_\alpha} \mid (A + LC(H)))$. By the first part of the proof,

$$\rho = \lambda(\omega_\xi \mid (A + LC(H))) + (1 - \lambda)\psi$$

where $0 \leqslant \lambda \leqslant 1$, ξ is a unit vector in H and ψ is a state of $A + LC(H)$ which annihilates $LC(H)$. But $\rho \mid A = \varphi$, so

$$\varphi = \lambda(\omega_\xi \mid A) + (1 - \lambda)(\psi \mid A)$$

and $\psi \mid A$ is a state of A which annihilates $A \cap LC(H)$.

(2) $\Rightarrow$ (1). See the proof of [19; Theorem 2].

4. VON NEUMANN ALGEBRAS

In this section we study $\overline{\mathrm{F}(A)}$ and $\overline{\mathrm{P}(A)}$ for a von Neumann algebra A, paying particular attention to the relationships with $\overline{\mathrm{F}(B)}$ and $\overline{\mathrm{P}(B)}$ when B is a unital ultraweakly dense C^*-subalgebra of A. We begin by recalling some definitions and notation from [19; Section 4].

Let A be a von Neumann algebra with centre Z and let X be the maximal ideal space of Z. For $x \in X$, let $x[A]$ denote the smallest ideal of A which contains x. We shall refer to $x[A]$ as the *Glimm ideal* of A corresponding to x. Let $A(x) = A/x[A]$ and let $\psi_x : A \to A(x)$ be the quotient map.

LEMMA 4.1. *Let A be a von Neumann algebra and let B be an ultraweakly dense C^*-subalgebra of A. Let $x \in X$ (as above). Then*

(1) *$B \cap x[A]$ is a primal ideal of B.*

(2) *$x[A]$ is a primal ideal of A.*

Proof. (1) Suppose that $J_1, J_2, \ldots, J_n$ are ideals of B such that $J_1 J_2 \ldots J_n = \{0\}$. Then $\bar{J}_1 \bar{J}_2 \ldots \bar{J}_n = \{0\}$. For each i, there is a central projection q_i in A such that $\bar{J}_i = Aq_i$, and so $q_1 q_2 \ldots q_n = 0$. Since $\psi_x(Z) \cong \mathbf{C}$, there exists i such that $\psi_x(q_i) = 0$ and hence $J_i \subseteq B \cap x[A]$.

(2) This follows from (1) if we take $B = A$.

Of course, $x[A]$ is actually a prime ideal of A but this requires a slightly deeper argument [19; Lemma 11]. The next result extends [6; Theorem 4.3.(1)].

THEOREM 4.2. *Let A be a von Neumann algebra and let $\varphi \in \mathrm{S}(A)$. The following conditions are equivalent.*

(1) $\varphi \in \overline{\mathrm{F}(A)}$.

(2) $\ker \pi_\varphi$ *contains a primitive ideal of A.*

(3) $\ker \pi_\varphi$ *contains a prime ideal of A.*

(4) $\ker \pi_\varphi$ *is primal.*

(5) $\ker \pi_\varphi$ *contains a Glimm ideal of A.*

(6) $\varphi \mid Z \in \mathrm{P}(Z)$.

Proof. It is immediate that (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1). This follows from Theorem 3.5.

(1) $\Rightarrow$ (6). If $\varphi \in \mathrm{F}(A)$ then $\pi_\varphi(Z)$ consists of scalar operators and so $\varphi \mid Z$ is multiplicative. Thus, by continuity, if $\varphi \in \overline{\mathrm{F}(A)}$ then $\varphi \mid Z$ is multiplicative.

(6) $\Rightarrow$ (5). Suppose $\varphi \mid Z \in \mathrm{P}(Z)$. Let $x = \ker(\varphi \mid Z) \in X$. It follows from the Cauchy-Schwarz inequality that $\varphi(x[A]) = \{0\}$ (see [19; p. 232, Remarks]), so $\ker \pi_\varphi$ contains the Glimm ideal $x[A]$.

(5) $\Rightarrow$ (2). This follows from the fact that any Glimm ideal is a primitive ideal of A [20; 4.7].

The equivalence of (4) and (5) in Theorem 4.2 is a special case of the following result.

PROPOSITION 4.3. *Let I be an ideal of a von Neumann algebra A. Then I is primal if and only if I contains a Glimm ideal of A.*

Proof. If I contains a Glimm ideal of A then, by Lemma 4.1, I contains a primal ideal of A and hence is itself primal. Conversely, suppose that I is a primal ideal of A. Then $I \cap Z$ is a primal ideal of the centre Z of A. Since $\mathrm{Prim}(Z)$ $(= X)$ is Hausdorff, $I \cap Z$ is maximal (see the remark preceding Proposition 3.2). Thus $I \cap Z = x$ for some $x \in X$ and hence $I \supseteq x[A]$.

We now give a new proof of a result due to Glimm.

THEOREM 4.4. [19; Theorem 3]. *Let A be a von Neumann algebra with no central summand of type* I. *Then*

$$\overline{\mathrm{P}(A)} = \{\varphi \in \mathrm{S}(A) : \varphi \mid Z \in \mathrm{P}(Z)\}.$$

Proof. Since A is antiliminal, $\overline{\mathrm{P}(A)} = \overline{\mathrm{F}(A)}$ [8; Theorem 3.4]. The result now follows from Theorem 4.2 ((1) $\Leftrightarrow$ (6)).

Suppose that B is a C^*-algebra of operators acting on a Hilbert space H and that $1 \in B$. Let A be the von Neumann algebra generated by B. Glimm proved that $\overline{\mathrm{P}(B)} = \overline{\mathrm{P}(A)} \mid B$ [19; Theorem 5]. Using this result, it has been shown that $\overline{\mathrm{F}(B)} = \overline{\mathrm{F}(A)} \mid B$ (see [6; Theorem 4.6] and [8; Theorem 5.3]). Motivated by a question posed by Professor R. V. Kadison during the 1983 OATE Conference in Romania, we now prove the factorial result directly and then use it to simplify the proof of Glimm's theorem.

THEOREM 4.5. *With the above notation,* $\overline{\mathrm{F}(B)} = \overline{\mathrm{F}(A)} \mid B$.

Proof. It follows from [6; Proposition 4.4] that $\overline{\mathrm{F}(B)} \subseteq \overline{\mathrm{F}(A)} \mid B$.

Conversely, suppose that $\psi \in \overline{\mathrm{F}(A)}$. By Theorem 4.2, $\ker \pi_\psi \supseteq x[A]$ for some $x \in X$ (the maximal ideal space of the centre of A). Let $\varphi = \psi \mid B \in \mathrm{S}(B)$ and let $I = B \cap x[A]$, a primal ideal of B by Lemma 4.1. Then $\varphi(I) = \{0\}$ and so $\varphi \in \mathrm{S}(B/I) \subseteq \overline{\mathrm{F}(B)}$ by Theorem 3.3.

COROLLARY 4.6. *Suppose that B is an antiliminal C^*-algebra of operators acting on a Hilbert space H and that $1 \in B$. Let A be the von Neumann algebra generated by B. Then* $\overline{\mathrm{P}(A)} \mid B \subseteq \overline{\mathrm{P}(B)}$.

Proof. By Theorem 4.5 and [8; Theorem 3.4] we have

$$\overline{\mathrm{P}(A)} \mid B \subseteq \overline{\mathrm{F}(A)} \mid B = \overline{\mathrm{F}(B)} = \overline{\mathrm{P}(B)}.$$

In proving that $\overline{\mathrm{P}(A)} \mid B \subseteq \overline{\mathrm{P}(B)}$ in general, Glimm made a reduction to two special cases. In the proof of Theorem 4.9 below we shall make a similar reduction using a different technique. Then the first case will be dealt with by applying Corollary 4.6 (in place of [19; Lemma 13]). To handle the second case, in Lemma 4.8, we modify Glimm's approach. For the sake of completeness we shall give the details. The following lemma will be used in place of [19; Lemma 12].

LEMMA 4.7. *Let B a postliminal C^*-algebra acting non-degenerately on a Hilbert space H and let A be the von Neumann algebra generated by B. Let q be a non-zero central projection in A. Then there is a non-zero central projection z in A and $b \in B$ such that $z \leqslant q$ and bz is a non-zero abelian projection in A.*

Proof. The algebra Bq is isomorphic to a quotient of B and is therefore postliminal. Thus, since Bq generates the von Neumann algebra Aq, we may assume that $q = 1$.

Let b_1 be any abelian element of B^+ with $\|b_1\| = 1$ (see [24; 6.1]). Define $f, g : \mathbf{R} \to \mathbf{R}$ by

$$f(t) = \begin{cases} 1 & \left(t \geqslant \frac{1}{2}\right) \\ 0 & \left(t < \frac{1}{2}\right) \end{cases}, \quad g(t) = \begin{cases} 1 & \left(t \geqslant \frac{1}{2}\right) \\ 2t & \left(t < \frac{1}{2}\right) \end{cases}.$$

Let $p = f(b_1)$, a non-zero abelian projection in A, and let $b = g(b_1)$, an abelian element of B. Then $p = bp = pb$. Let $c \in B$. Since b is abelian

$$pcb^2 = p(bcb)b = pb(bcb) = pcb.$$

Hence $I(b^2 - b) = \{0\}$, where I is the ultraweakly closed two-sided ideal of A generated by p. There exists a non-zero central projection z in A such that $I = Az$. Hence $z(b^2 - b) = 0$ and so zb is a projection which is abelian in A since b is abelian in B. Since $zbp = zp = p \neq 0$, zb is non-zero.

LEMMA 4.8. *Let B be a C^*-algebra, acting non-degenerately on a Hilbert space H, containing a postliminal ideal J which generates the same von Neumann algebra A as is generated by B (that is, J is ultraweakly dense in B). Then*

$$\overline{\mathrm{P}(A)} \mid B \subseteq \overline{\mathrm{P}(B)}.$$

Proof. Let $x \in X$ (the maximal ideal space of the centre Z of A). Since A is a type I von Neumann algebra it follows from [19; Theorem 4] that there is an irreducible representation π_x of A on a Hilbert space H_x such that $\ker \pi_x = x[A]$ and $\pi_x(A) \supseteq LC(H_x)$. Let ξ be a unit vector in H_x and let $\varphi = \omega_\xi \circ \pi_x \in \mathrm{P}(A)$. We shall show that $\varphi \mid B \in \overline{\mathrm{P}(B)}$ (examples show that $\varphi \mid B$ need not lie in $\mathrm{P}(B)$).

Since A is type I, there is an abelian projection e in A such that $\pi_x(e)$ is the rank one projection onto $\mathbf{C}\xi$ (see the proof of [19; Theorem 4]). Let q be any projection in Z with $\pi_x(q) = 1$ and $q \leqslant c(e)$ (the central carrier of e). By Lemma 4.7 there exists a non-zero projection z_q in Z and $b_q \in J$ such that $z_q \leqslant q$ and $b_q z_q$ is a non-zero abelian projection in A. Let w_q be a non-zero partial isometry in A such that $w_q^* w_q \leqslant b_q z_q$ and $w_q w_q^* \leqslant e$ [28; V.1.7]. Note that $ew_q = w_q = w_q b_q z_q$. By Kaplansky's density theorem there exists $v_{q,n} \in B$ such that $\| v_{q,n} \| = 1$ and $\|ev_{q,n} b_q z_q\| > 1 - n^{-1}$ $(n = 1, 2, \ldots)$. Let $\pi_{q,n}$ be an irreducible representation of A such that $\| \pi_{q,n}(ev_{q,n} b_q z_q) \| >$ $> 1 - n^{-1}$. Then $\pi_{q,n}(z_q) = 1$ (since it is not zero) and so $\pi_{q,n}(b_q) = \pi_{q,n}(b_q z_q)$ which is an abelian, hence rank one, projection in $\pi_{q,n}(B)$. Let $\xi_{q,n}$ be a unit vector in the range of this projection. Then $1 \geqslant \|\pi_{q,n}(v_{q,n})\xi_{q,n}\| > 1 - n^{-1}$. Let

$$\eta_{q,n} = \|\pi_{q,n}(v_{q,n})\xi_{q,n}\|^{-1}\pi_{q,n}(v_{q,n})\xi_{q,n}$$

and let $\varphi_{q,n}$ (respectively $\psi_{q,n}$) be the pure state of A defined by $\pi_{q,n}$ and $\xi_{q,n}$ (respectively $\eta_{q,n}$). Since the projection onto $\mathbf{C}\xi_{q,n}$ belongs to $\pi_{q,n}(B)$, $\varphi_{q,n} \mid B \in \mathrm{P}(B)$. Since $\eta_{q,n} \in \pi_{q,n}(B)\xi_{q,n}$, $\psi_{q,n} \mid B \in \mathrm{P}(B)$.

Let $\psi \in \overline{\mathrm{P}(A)}$ be a limit point of the net $(\psi_{q,n})$ (where $(q_1, n_1) \geqslant (q, n)$ if and only if $q_1 \leqslant q$ and $n_1 \geqslant n$). Let r be a projection in Z with $\varphi(r) = 1$. Then $rc(e) \neq 0$ and if q is a projection in Z with $\pi_x(q) = 1$ and $q \leqslant rc(e)$ we have

$$1 = \psi_{q,n}(z_q) \leqslant \psi_{q,n}(q) \leqslant \psi_{q,n}(r) \leqslant 1 \qquad (n = 1, 2, \ldots)$$

and so $\psi(r) = 1$. Since $\psi \mid Z$, $\varphi \mid Z \in \mathrm{P}(Z)$, it follows that $\psi \mid Z = \varphi \mid Z$. Thus $\psi(x) = \{0\}$ and hence $\psi(x[A]) = 0$ (see [19; p. 232, Remarks]). So there is a state $\tilde{\psi}$ of $\pi_x(A)$ such that $\psi = \tilde{\psi} \circ \pi_x$. But

$$\psi_{q,n}(e) \geqslant \|\pi_{q,n}(ev_{q,n})\xi_{q,n}\|^2 =$$

$$= \|\pi_{q,n}(ev_{q,n}b_q z_q)\|^2 > (1 - n^{-1})^2$$

and so $1 = \psi(e) = \tilde{\psi}(\pi_x(e))$. Since $\pi_x(e)$ is the rank one projection onto $\mathbf{C}\xi$, we have $\tilde{\psi} = \omega_\xi \mid \pi_x(A)$ and hence $\psi = \varphi$. Thus

$$\varphi \mid B = \psi \mid B = \lim \psi_{q,n} \mid B \in \overline{\mathrm{P}(B)}.$$

Finally, let $S = \{\rho \in \mathrm{S}(A) : \rho \mid B \in \overline{\mathrm{P}(B)}\}$, a w*-closed subset of $\mathrm{S}(A)$. Let $a = a^* \in A$ and suppose $\rho(a) \geqslant 0$ for all $\rho \in S$. Since $\varphi \in S$, $\langle \pi_x(a)\xi, \xi\rangle \geqslant 0$. But x and ξ were arbitrary and so $(\bigoplus_{x \in X} \pi_x)(a) \geqslant 0$. Since $\bigcap_{x \in X} x[A] = \{0\}$ (see [19; p. 232, Remarks]), $\bigoplus_{x \in X} \pi_x$ is a $*$-isomorphism and so $a \geqslant 0$. It follows from [15; 3.4.1] that $S \supseteq \overline{\mathrm{P}(A)}$.

The above proof is based on Glimm's argument in the proof of [19; Theorem 5], but we have avoided his use of the penultimate line of the statement of [19; Theorem 4], and our representations $\pi_{q,n}$ need not be of his special kind.

THEOREM 4.9. [19; Theorem 5]. *Let B be a C^*-algebra of operators acting on a Hilbert space H and suppose that $1 \in B$. Let A be the von Neumann algebra generated by B. Then*

$$\overline{\mathrm{P}(A)} \mid B = \overline{\mathrm{P}(B)}.$$

Proof. Since $P(A) \mid B \supseteq P(B)$, a simple compactness argument shows that $\overline{P(A)} \mid B \supseteq \overline{P(B)}$ (see the proof of [19; Theorem 5]).

Let $\varphi \in \overline{P(A)}$. In order to prove that $\varphi \mid B \in \overline{P(B)}$ we may assume, by the continuity of restriction, that $\varphi \in P(A)$, and we may also assume, by a routine argument (see the proof of [6; Theorem 4.6], that B is acting in its universal representation. Let J be the largest postliminal ideal of B. Then $\bar{J} = Ap$ for some central projection p in A. Since $\varphi \in P(A)$, either $\varphi(p) = 0$ or $\varphi(p) = 1$.

Suppose that $\varphi(p) = 0$. Since B is acting in its universal representation, $\bar{J} \cap B = J$, and so there is a well-defined $*$-isomorphism of B/J onto $B(1 - p)$ given by $b + J \to b(1 - p)$ $(b \in B)$. Applying Corollary 4.6 to the antiliminal algebra $B(1 - p)$ and the pure state $\varphi \mid A(1 - p)$ of $A(1 - p)$, we obtain a net (φ_α) of pure states of $B(1 - p)$ such that

$$\varphi(b) = \varphi(b(1 - p)) = \lim \varphi_\alpha(b(1 - p)) \quad (b \in B).$$

Define $\psi_\alpha \in P(B)$ by $\psi_\alpha(b) = \varphi_\alpha(b(1 - p))$ $(b \in B)$. Then $\varphi \mid B = \lim \psi_\alpha \in \overline{P(B)}$.

Suppose that $\varphi(p) = 1$. Applying Lemma 4.8 to the C^*-algebra Bp (with postliminal ideal J satisfying $\bar{J} = Ap = \overline{Bp}$) and the pure state $\varphi \mid Ap$ of Ap, we obtain a net (φ'_α) of pure states of Ap such that

$$\varphi(b) = \varphi(bp) = \lim \varphi'_\alpha(bp) \quad (b \in B).$$

Define $\psi'_\alpha \in P(B)$ by $\psi'_\alpha(b) = \varphi'_\alpha(bp)$ $(b \in B)$. Then $\varphi \mid B = \lim \psi'_\alpha \in \overline{P(B)}$.

If B is a non-unital C^*-algebra then $0 \in \overline{P(B)}$ [15; 2.12.13]. It follows that the conclusions of Theorems 4.5 and 4.9 are still true if B is a non-unital C^*-algebra of operators acting non-degenerately on H (see the proof of [6; Theorem 4.6.(2)]).

5. PRIMAL FACES

Let K be a compact convex subset of a locally convex space. Recall that K is said to be (*weakly*) *prime* [12, 18] if, whenever F_1 and F_2 are closed (split) semi--exposed faces of K with $K = \mathrm{co}(F_1 \cup F_2)$, then either $K = F_1$ or $K = F_2$. If A is a unital C^*-algebra, then $S(A)$ is (weakly) prime if and only if A is a prime C^*-algebra — equivalently, $\{0\}$ is a prime ideal in A [12, 13]. When these properties are compared with Definition 3.1, a notion of (weak) primality of faces emerges.

DEFINITION 5.1. A closed face F of a compact convex set K is said to be (*weakly*) *primal* in K if, whenever $n \geqslant 1$ and $F_1, F_2, \ldots, F_n$ are closed (split) semi-exposed faces of K such that $K = \mathrm{co}(F_1 \cup F_2 \cup \ldots \cup F_n)$, then $F \subseteq F_i$ for at least one value of i.

Note that any closed split face is semi-exposed, any semi-exposed face is closed, and the convex hull of a finite union of closed split faces is a closed split face (see

[7; Chapter 2] and [2; II.5, II.6] where the term "relatively exposed" is used in place of "semi-exposed"). The following observations are more or less immediate.

(1) Any primal face is weakly primal.

(2) K is weakly primal in itself if and only if K is weakly prime.

(3) If F_2 is a closed face of F_1 and F_1 is (weakly) primal in K, then F_2 is (weakly) primal in K.

(4) If F is (weakly) primal in K, then the smallest closed (split) semi-exposed face of K containing F is (weakly) primal in K.

A planar k-gon K_k is the convex hull of n (semi-exposed) proper faces if and only if $k \leqslant 2n$. Thus if $k \geqslant 5$, K_k is prime but the primal faces of K_k each contain only one point. On the other hand, if $k \geqslant 4$, K_k has no proper split faces, so K_k is weakly prime and all its faces are weakly primal.

It is possible to define another stronger concept of primality by omitting the word "semi-exposed" from Definition 5.1. However, our definition permits primality to be described in terms of the ordering of $A(K)$ in the following way (compare [5, 12, 14]). The proof of this result shows that the definition of a primal face is unchanged if "semi-exposed" is replaced by "exposed" in Definition 5.1.

PROPOSITION 5.2. *A closed face F of a compact convex set K is primal if and only if, whenever $a_1, a_2, \ldots, a_n \in A(K)$ have infimum 0 in $A(K)$, then $a_i \mid F = 0$ for some i.*

Proof. Suppose that F is not primal, so that $K = \mathrm{co}(F_1 \cup F_2 \cup \ldots \cup F_n)$ for some semi-exposed faces F_i of K not containing F. Then there exist $a_i \in A(K)^+$ such that $a_i \mid F_i = 0$, $a_i \mid F \neq 0$. Now $a_1, a_2, \ldots, a_n$ have infimum 0 in $A(K)$, so the stated property fails.

Conversely, suppose that F is primal and $a_1, a_2, \ldots, a_n$ have infimum 0 in $A(K)$. Let $F_i = a_i^{-1}(0)$, which is an exposed face of K. Now 0 is the convex lower semi-continuous envelope of the pointwise minimum of $a_1, a_2, \ldots, a_n$, so that for any $x \in \partial_e K$, $\min(a_1(x), \ldots, a_n(x)) = 0$ [7; 1.6.1, 1.6.3]. Thus $x \in F_i$ for some i, so that $K = \mathrm{co}(F_1 \cup \ldots \cup F_n)$ by the Kreĭn-Milman Theorem. Since F is primal, $F \subseteq F_i$, so that $a_i \mid F = 0$, for some i.

For $x \in K$, let F_x be the smallest closed face of K containing x. Then F_x is (weakly) primal if and only if, whenever $n \geqslant 1$ and $F_1, F_2, \ldots, F_n$ are closed (split) semi-exposed faces of K such that $K = \mathrm{co}(F_1 \cup \ldots \cup F_n)$, then $x \in F_1 \cup \ldots \cup F_n$. In these circumstances, we shall say that x is a (*weakly*) *primal* point of K.

PROPOSITION 5.3. *Let K be a compact convex set.*

(1) Each point of $\overline{\partial_e(K)}$ is primal.

(2) Each point of $\overline{\partial_{pr}(K)}$ is weakly primal.

(3) *Each (weakly) primal face of K is contained in a maximal (weakly) primal face.*

(4) *Each maximal weakly primal face is split.*

Proof. (1) Suppose that $K = \mathrm{co}(F_1 \cup \ldots \cup F_n)$ for closed faces F_i. By Milman's Theorem, $\partial_e K \subseteq F_1 \cup \ldots \cup F_n$, so $\overline{\partial_e K} \subseteq F_1 \cup \ldots \cup F_n$.

(2) Suppose that $K = \mathrm{co}(F_1 \cup \ldots \cup F_n)$ for closed split faces F_i. For each i, $\partial_{\mathrm{pr}} K \subseteq F_i \cup F_i'$, where F_i' is the complementary face of F_i. But $F_1' \cap \ldots \cap F_n' = \emptyset$, so $\partial_{\mathrm{pr}} K \subseteq F_1 \cup \ldots \cup F_n$, and hence $\overline{\partial_{\mathrm{pr}} K} \subseteq F_1 \cup \ldots \cup F_n$.

(3) Let $\mathscr{F}$ be a totally ordered family of (weakly) primal faces containing a given (weakly) primal face F_0, and let F be the smallest closed face containing $\cup \mathscr{F}$. Suppose that $K = \mathrm{co}(G_1 \cup \ldots \cup G_n)$ for closed (split) semi-exposed faces G_i not containing F. There exist $F_i \in \mathscr{F}$ such that $F_i \nsubseteq G_i$. Taking F_k to be the largest of $F_1, \ldots, F_n$, this contradicts the fact that F_k is (weakly) primal. Hence F is (weakly) primal. By Zorn's Lemma, F_0 is contained in a maximal (weakly) primal face.

(4) The smallest closed split face containing a weakly primal face is weakly primal.

If A is a unital C^*-algebra and $K = S(A)$, the converse of Proposition 5.3(2) is true, but the converse of (1) holds only if A is an antiliminal extension of an abelian C^*-algebra (see Theorem 3.5, Theorem 5.6 below, and [8, Theorem 3.4]). If K is a Choquet simplex, so that $\partial_{\mathrm{pr}} K = \partial_e K$ and every closed face is split, the converse of (2) may fail, for example if K is a prime simplex in which $\overline{\partial_e K} \neq K$ (see [2; II.7.17]).

Following Briem [11] and Ellis [18], we can now see that the weakly primal faces determine $A(K)$ in the following sense. Since each weakly primal face is contained in a member of the Silov decomposition, this result is an improvement of [17; Theorem 2].

THEOREM 5.4. *Let K be a compact convex set, $X = \overline{\partial_e K}$, f be a continuous real-valued function on X, and suppose that, for each weakly primal face F of K, there exists $a_F \in A(F)$ such that $a_F \mid F \cap X = f \mid F \cap X$. Then there exists $a \in A(K)$ such that $f = a \mid X$.*

Proof. By a result of Briem [11; Theorem 8] and standard convexity theory (see [2; II.4.5], [7; 1.6.9]), it suffices to show firstly that $\int f \,\mathrm{d}\mu = 0$ for all measures μ which are extreme points of the unit ball of the space of boundary affine dependences on K, and secondly that $\int f \,\mathrm{d}\nu = f(x)$ for all probability measures ν on X representing points $x \in X$. It was shown in [18; Lemma 4] that μ is supported by a face F which is weakly prime and hence weakly primal in K. Since μ is a boundary affine dependence on F, and f and a_F coincide on $F \cap X$,

$$\int_K f \, d\mu = \int_{F \cap X} f \, d\mu = \int_F a_F \, d\mu = 0.$$

For $x \in X$, F_x is weakly primal (Proposition 5.3) and any representing measure ν is supported by $F_x \cap X$. Hence

$$\int_X f \, d\nu = \int_{F_x} a_{F_x} \, d\nu = a_{F_x}(x) = f(x).$$

The characterization of $A(K)$ given in Theorem 5.4 is not intrinsically in terms of X, since the conditions assume knowledge of $A(F)$ for primal faces F. However, it does permit characterizations which are intrinsic to larger subsets Y of K.

COROLLARY 5.5. *Let Y be a compact extremal subset of a compact convex set K, and suppose that Y contains each primal face of K. Let f be a continuous real-valued function on Y, and suppose that $f \mid C$ is affine for each convex subset C of Y. Then there exists $a \in A(K)$ such that $f = a \mid Y$.*

Proof. The function $f \mid X$ satisfies the conditions of Theorem 5.4, so there exists $a \in A(K)$ such that $f \mid X = a \mid X$. For any $y \in Y$, the face F_y is contained in Y (see Lemma 3.7), and $f \mid F_y$ and $a \mid F_y$ are continuous affine functions coinciding on $\partial_e F_y$ so that $f(y) = a(y)$.

EXAMPLE. [17; Example 10]. Let K be the state space of the order-unit space $(A, A^+, 1)$, where A is the space of all sequences $\alpha = (\alpha_n)_{n \geqslant 1}$ which converge to a limit α_∞, and

$$A^+ = \left\{ \alpha \in A : \alpha_n \geqslant 0, \ \alpha_\infty \geqslant \frac{1}{2} \alpha_1 \right\}.$$

Now K is a Choquet simplex, with extreme boundary

$$\partial_e K = \{\delta_n : 1 \leqslant n < \infty\} \cup \{2\delta_\infty - \delta_1\}$$

(where δ denotes Dirac measure). A closed face F is (weakly) primal if and only if F is either a singleton or $\text{co}\{\delta_1, 2\delta_\infty - \delta_1\}$; F is (weakly) prime if and only if F is a singleton. The maximal primal (respectively, prime) faces are the faces of the abstract Silov (respectively, Bishop) decomposition [17, 7].

This example shows that Theorem 5.4 cannot be improved either by replacing "weakly primal" by "weakly prime" or by replacing "$F \cap X$" by "$\overline{\partial_e F}$". The example of polygons shows that Theorem 5.4 cannot be improved by replacing "weakly primal" by "primal".

Now let A be a unital C^*-algebra. It is known that if $S(A)$ is weakly prime, then $S(A)$ is prime (and A is a prime C^*-algebra) [12, 13]. We shall now see that similar results hold for weakly primal faces of $S(A)$, and that the notions of primality for ideals and faces (Definitions 3.1, 5.1) correspond under duality, and that all the above-mentioned improvements to Theorem 5.4 can be made.

LEMMA 5.6. *Let $F_1, F_2, \ldots, F_n$ be closed faces of $S(A)$ such that $S(A) = \text{co}(F_1 \cup \ldots \cup F_n)$. Then there exist closed split faces G_i of $S(A)$ such that $G_i \subseteq F_i$ $(1 \leqslant i \leqslant n)$ and $S(A) = \text{co}(G_1 \cup \ldots \cup G_n)$.*

Proof. Let $L_i = (F_i)_\perp$, J_i be the ideal of A generated by L_i, and $G_i = J_i^\perp$. Then G_i is a closed split face contained in F_i. Suppose that $S(A) \neq \text{co}(G_1 \cup \ldots \cup G_n)$. By the Kreĭn-Milman Theorem, there exists $\varphi \in P(A)$ such that $\varphi \notin G_1 \cup \ldots \cup G_n$, so there exist $a_i \in L_i$ and $\xi_i \in H_\varphi$ such that $\pi_\varphi(a_i)\xi_i \neq 0$. The mappings $(\lambda_1, \ldots, \lambda_n) \to \sum_{1\leqslant i\leqslant n} \lambda_i\pi_\varphi(a_j)\xi_i$ $(1 \leqslant j \leqslant n)$ of $\mathbf{C}^n$ into H_φ are non-zero and linear, so $\mathbf{C}^n$ is not the union of their kernels. Hence there exist scalars λ_i $(1 \leqslant i \leqslant n)$ such that $\sum_{1\leqslant i\leqslant n} \lambda_i\pi_\varphi(a_j)\xi_i \neq 0$ $(1 \leqslant j \leqslant n)$. Let $\xi = \|\sum \lambda_i\xi_i\|^{-1} \sum \lambda_i\xi_i$. Then (in the notation of Section 1) $\omega_\xi^\varphi \in P(A)$, but $\omega_\xi^\varphi(a_j^* a_j) \neq 0$, so $\omega_\xi^\varphi \notin F_1 \cup \ldots \cup F_n$. By Milman's Theorem, $\omega_\xi^\varphi \notin \text{co}(F_1 \cup \ldots \cup F_n)$. This is a contradiction.

THEOREM 5.7. *Let A be a unital C^*-algebra, F be a closed face of $S(A)$, $L = F_\perp$ and $I = \{a \in A : aA \subseteq L\}$. The following conditions are equivalent.*

(1) *F is a weakly primal face of $S(A)$.*

(2) *F is a primal face of $S(A)$.*

(3) *I is a primal ideal in A.*

(4) *If $J_1, J_2, \ldots, J_n$ are ideals in A with $J_1J_2 \ldots J_n = \{0\}$, then $J_i \subseteq L$ for some i.*

Proof. (1) $\Leftrightarrow$ (2). This follows easily from Lemma 5.6.

(3) $\Leftrightarrow$ (4). Since I is the largest ideal contained in L, this is clear from Definition 3.1.

(1) $\Leftrightarrow$ (4). This follows easily from the correspondence between closed split faces and ideals, since

$$S(A) = \text{co}(J_1^\perp \cup \ldots \cup J_n^\perp) \Leftrightarrow J_1 \ldots J_n = \{0\}.$$

In view of the correspondence between the self-adjoint part of A and $A(S(A))$, Theorems 5.4 and 5.7 show that if f is a continuous real (or complex) function on $\overline{P(A)}$ and for each primal ideal I in A, there exists $a_I \in A$ such that $f(\varphi) = \varphi(a_I)$ for all $\varphi \in \overline{P(A)} \cap S(A/I)$, then $f(\varphi) = \varphi(a)$ for all $\varphi \in \overline{P(A)}$ for some $a \in A$. The following is a stronger version of this result (see also [18; p. 232] for the case of von Neumann algebras), and it shows that the Bishop decomposition determines $A(S(A))$ amongst Banach subspaces of $C_{\mathbf{R}}(\overline{P(A)})$ (see [7; p. 236]).

THEOREM 5.8. *Let A be a unital C^*-algebra, let f be a uniformly continuous complex-valued function on $P(A)$; and suppose that for each prime ideal I in A, there exists $a_I \in A$ such that $f(\varphi) = \varphi(a_I)$ for all $\varphi \in P(A/I)$. Thus there exists $a \in A$ such that $f(\varphi) = \varphi(a)$ for all $\varphi \in P(A)$.*

Proof. Consider f to be extended by continuity to $\overline{P(A)}$. We shall use the notation established in Section 1.

Let (φ_α) be a net in $P(A)$ which is w*-convergent to $\varphi \in \overline{P(A)}$. There are elements $a_\alpha \in A$ such that $(\varphi_\alpha)_b(a_\alpha) = f((\varphi_\alpha)_b)$ for all $b \in A$ with $\varphi_\alpha(b^*b) > 0$. Furthermore $(\varphi_\alpha)_b \to \varphi_b$, so $\varphi_b \in \overline{P(A)}$, if $\varphi(b^*b) > 0$ (see the proof of Theorem 3.8). Let H_0 be the dense subspace $\pi_\varphi(A)\xi_\varphi$ of H_φ and define $s : H_0 \to \mathbf{C}$ by

$$s(\pi_\varphi(b)\xi_\varphi) = \varphi(b^*b)f(\varphi_b) \quad (\varphi(b^*b) > 0)$$

$$s(0) = 0.$$

Then

$$(*) \qquad s(\eta) = \|\eta\|^2 f(\omega_{\tilde{\eta}}^{\varphi})$$

where $\tilde{\eta} = \eta/\|\eta\|$, for non-zero vectors $\eta \in H_0$. In particular, s is well-defined, and, since f is (norm) continuous and bounded, s is continuous and extends to a continuous function on H_φ given by $(*)$.

Now

$$s(\pi_\varphi(b)\xi_\varphi) = \lim \varphi_\alpha(b^*b)f((\varphi_\alpha)_b) =$$

$$= \lim \varphi_\alpha(b^*a_\alpha b).$$

Thus

$$\sum_{j=0}^{3} i^j s(\pi_\varphi(\lambda_1 b_1 + \lambda_2 b_2 + i^j b_3)\xi_\varphi) =$$

$$= \lim \sum_{j=0}^{3} i^j \varphi_\alpha((\lambda_1 b_1 + \lambda_2 b_2 + i^j b_3)^* a_\alpha(\lambda_1 b_1 + \lambda_2 b_2 + i^j b_3)) =$$

$$= \lim 4\varphi_\alpha(b_3^* a_\alpha(\lambda_1 b_1 + \lambda_2 b_2)) =$$

$$= \sum_{j=0}^{3} i^j(\lambda_1 s(\pi_\varphi(b_1 + i^j b_3)\xi_\varphi) + \lambda_2 s(\pi_\varphi(b_2 + i^j b_3)\xi_\varphi)).$$

Also $s(i^j\eta) = s(\eta)$. Thus the functional $\sigma : H_\varphi \times H_\varphi \to \mathbf{C}$ defined by

$$\sigma(\eta_1, \eta_2) = \frac{1}{4}\sum_{j=0}^{3} i^j s(\eta_1 + i^j\eta_2)$$

is conjugate-symmetric, sesquilinear and continuous. Hence there is a bounded linear operator T on H_φ such that

$$\sigma(\eta_1, \eta_2) = \langle T\eta_1, \eta_2\rangle.$$

The formula $\langle T\eta, \eta\rangle = s(\eta) = f(\omega_\eta^\varphi)$ is valid for unit vectors $\eta \in H_0$, and by continuity for any unit vector $\eta \in H_\varphi$.

Let p be a projection in $\pi_\varphi(A)'$ and η be a unit vector in H_φ. For real t, let

$$\eta_t = p\eta + e^{it}(1-p)\eta.$$

Then $\omega_{\eta_t}^\varphi = \omega_\eta^\varphi$, so $\langle T\eta_t, \eta_t\rangle = \langle T\eta, \eta\rangle$ for all real t. Hence

$$\langle Tp\eta, (1-p)\eta\rangle = 0 = \langle T(1-p)\eta, p\eta\rangle \quad (\eta \in H_\varphi, \|\eta\| = 1)$$

so $Tp = pTp = pT$. Thus $T \in \pi_\varphi(A)''$.

If $\varphi = \sum_{j=1}^{n} \lambda_j\varphi_j$ where $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$, $\varphi_j \in S(A)$, then there exist $x_j' \in \pi_\varphi(A)'$ such that $\varphi_j = \omega^\varphi_{x_j'\xi_\varphi}$ and $\sum_{j=1}^{n} \lambda_j x_j'^* x_j' = 1$. Hence

$$\sum_{j=1}^{n} \lambda_j f(\varphi_j) = \sum_{j=1}^{n} \lambda_j \langle Tx_j'\xi_\varphi, x_j'\xi_\varphi\rangle = \sum_{j=1}^{n} \lambda_j \langle Tx_j'^*x_j'\xi_\varphi, \xi_\varphi\rangle = \langle T\xi_\varphi, \xi_\varphi\rangle = f(\varphi).$$

Thus if ν is any discrete measure on $S(A)$ representing φ, then $\int f d\nu = f(\varphi)$. It follows that the same formula is true for any measure ν on $S(A)$ representing φ.

Let μ be an extreme point of the unit ball of the space of boundary affine dependencies. By [18; Lemma 4], μ is supported by a weakly prime closed split face F. If $P = F_\perp$, then P is a prime ideal in A [14; Lemma 1.1], so that

$$\int f d\mu = \int_F \psi(a_p)\, d\mu(\psi) = 0.$$

As in Theorem 5.4, this suffices to prove the theorem.

COROLLARY 5.9. *Let A be a unital C^*-algebra, let f be a continuous complex-valued function on $\overline{F(A)}$, and suppose that $f \mid C$ is affine for each convex subset C of $\overline{F(A)}$. Then there exists $a \in A$ such that $f(\varphi) = \varphi(a)$ for all $\varphi \in \overline{F(A)}$.*

Proof. Since $S(A/I) \subseteq \overline{F(A)}$ for every prime ideal I (Corollary 3.6), it follows from Theorem 5.8 that there exists $a \in A$ such that $f(\varphi) = \varphi(a)$ for all $\varphi \in \overline{P(A)}$. Since $\overline{F(A)}$ is compact and extremal, it follows as in the proof of Corollary 5.5 that $f(\varphi) = \varphi(a)$ for all $\varphi \in \overline{F(A)}$.

Alternatively, Corollary 5.9 may be proved by applying Corollary 5.5 to the real and imaginary parts of f with $Y = \overline{F(A)}$.

THEOREM 5.10. *Let A and B be unital C^*-algebras and let $\Psi : \overline{F(A)} \to \overline{F(B)}$ be a homeomorphism such that Ψ and Ψ^{-1} are affine on convex subsets of $\overline{F(A)}$ and $\overline{F(B)}$ respectively. Then there is a Jordan isomorphism $\Phi : B \to A$ such that $\Psi = \Phi^* \mid \overline{F(A)}$. Furthermore, Φ is a $*$-isomorphism if and only if $\Psi \mid P(A)$ preserves orientation.*

Proof. Let $b \in B$. Applying Corollary 5.9 to the function $\varphi \to \Psi(\varphi)(b)$ in $C_C(\overline{F(A)})$, we obtain a (necessarily unique) element $a \in A$ such that $\Psi(\varphi)(b) = \varphi(a)$ for all $\varphi \in \overline{F(A)}$. Define $\Phi(b) = a$. It is straightforward to verify that Φ is a unital linear order isomorphism of B onto A (the surjectivity requires an application of Corollary 5.9 to B). Hence Φ is a Jordan isomorphism [21; Corollary 5]. By construction, $\Phi^* \mid \overline{F(A)} = \Psi$. Since $\Phi^*(S(A)) = S(B)$ [10; Theorem 3.2.3], Ψ maps $P(A)$ onto $P(B)$. It follows from [3; Corollary 8.5] that Φ is a $*$-isomorphism if and only if $\Psi \mid P(A)$ preserves orientation.

Alternatively, since Ψ preserves the local affine structure in $\overline{F(A)}$, one may show directly that Ψ maps $P(A)$ onto $P(B)$ (also $\partial_{pr}S(A) = F(A)$ onto $\partial_{pr}S(B) = F(B)$), and that Ψ preserves equivalence and transition probabilities between pure states. If $\Psi \mid P(A)$ preserves orientation then, by a theorem of Shultz [27; Theorem 18], there is a $*$-isomorphism Φ of B onto A such that $\Phi^* \mid P(A) = \Psi \mid P(A)$ and hence $\Phi^* \mid \overline{F(A)} = \Psi$.

In Corollary 5.9 it is not possible to replace $\overline{F(A)}$ by $\overline{P(A)}$ except when this change is trivial, as shown below. The C^*-algebras for which $\overline{F(A)} = \overline{P(A)}$ are described in statement (C) of Section 1.

PROPOSITION 5.11. *Let A be a unital C^*-algebra, and suppose that $\overline{F(A)} \neq \overline{P(A)}$. Then there is a continuous real-valued function f on $\overline{P(A)}$ such that $f \mid C$ is affine, for each convex subset C of $\overline{P(A)}$, and such that there does not exist $a \in A$ with $f(\varphi) = \varphi(a)$ for all $\varphi \in \overline{P(A)}$.*

Proof. By [8; Section 3], A has a non-abelian ideal I with continuous trace. It follows from [32; Theorem 6 and Remark on p. 601] that

$$\overline{P(A)} \subseteq \{\lambda\varphi + (1-\lambda)\psi : 0 \leqslant \lambda \leqslant 1,\ \varphi \in P(I),\ \psi \in S(A/I)\}.$$

Hence any convex subset C of $\overline{\mathrm{P}(A)}$ satisfies

$$C \subseteq \{\lambda\varphi + (1-\lambda)\psi : 0 \leqslant \lambda \leqslant 1,\ \psi \in \mathrm{S}(A/I)\}$$

for some $\varphi \in \mathrm{P}(I)$.

There is an irreducible representation π of I on a Hilbert space H of dimension greater than 1. Let (ξ_1, ξ_2) be an orthonormal pair of vectors in H, and $\eta_\pm = 2^{-1/2}(\xi_1 \pm \xi_2)$. By Kadison's Transitivity Theorem, there exists $a \in I$ such that $\pi(a)\xi_1 = \xi_1$ and $\pi(a)\xi_2 = -\xi_2$. Let $f(\psi) = |\psi(a)|$ $(\psi \in \overline{\mathrm{P}(A)})$. Then f is continuous, and $f \mid C$ is affine. But

$$f(\omega^\pi_{\xi_1}) = 1, \quad f(\omega^\pi_{\xi_2}) = 1, \quad f(\omega^\pi_{\eta_\pm}) = 0,$$

$$\frac{1}{2}(\omega^\pi_{\xi_1} + \omega^\pi_{\xi_2}) = \frac{1}{2}(\omega^\pi_{\eta_+} + \omega^\pi_{\eta_-}),$$

so there is no affine extension of f to $\mathrm{S}(A)$.

We show next how Theorem 5.8 leads to a proof of the Dauns-Hofmann theorem for a unital C^*-algebra A [24; 4.4.8]. Let $g : \mathrm{Prim}(A) \to \mathbf{C}$ be continuous. Let (φ_α) be a net in $\mathrm{P}(A)$ which is w*-convergent to $\varphi \in \overline{\mathrm{P}(A)}$. Let $P_\alpha = \ker \pi_{\varphi_\alpha}$, $I = \ker \pi_\varphi$. Then (P_α) converges in the hull-kernel topology to any $Q \in \mathrm{Prim}(A/I)$. Thus if $f(\psi) = g(\ker \pi_\psi)$ $(\psi \in \mathrm{P}(A))$, $\lim f(\varphi_\alpha)$ exists. Since (φ_α) is an arbitrary convergent net, and $\mathrm{P}(A)$ is relatively compact, it follows that f is uniformly continuous. If J is any prime ideal, g is constant on $\mathrm{Prim}(A/J)$ by Proposition 3.2.(3). Hence there is a scalar λ_J such that $f(\psi) = \lambda_J = \psi(\lambda_J 1)$ for all $\psi \in \mathrm{P}(A/J)$. It follows from Theorem 5.8 that there exists $a \in A$ such that $f(\psi) = \psi(a)$ for all $\psi \in \mathrm{P}(A)$ (in this case the proof of 5.8 may be considerably simplified since f is constant on the supports of the measures ν and μ). Hence $a - g(P)1 \in P$ for all $P \in \mathrm{Prim}(A)$, as required.

Apart from the study of operator algebras, the most important influence in convexity theory in recent years has been the study of function algebras (see [7; Chapter 4], [17], [18]). Let A be a function algebra on a compact Hausdorff space X, S be the state space and K be the complex state space of A, and identify X with a subset of S. Define a closed subset E of X to be *weakly primal* if, whenever $n \geqslant 1$ and $G_1, G_2, \ldots, G_n$ are peak sets for A with $X = G_1 \cup \ldots \cup G_n$, then $E \subseteq G_i$ for some i. Proceeding as in [18; Corollary 2], it can be shown that if F is a weakly primal face of K, then $F \cap X$ is weakly primal in X. Furthermore, if E is a weakly primal subset of X, then E is contained in a weakly primal face of K. But it is not clear that $\mathrm{co}(E \cup (-iE))$ is weakly primal in K even if E is a weakly primal generalised peak set containing more than one point of X. In this setting, it is possible to replace "weakly primal" by "weakly prime" in Theorem 5.4 [18; Theorem 3].

6. STRONGLY PRIMAL FACES

In this section, we introduce a strengthening of the notion of primality of faces, and discuss its possible relevance to C^*-algebras.

DEFINITION 6.1. A closed face F of a compact convex set K is said to be *strongly primal* if, whenever $n \geqslant 1$ and $E_1, E_2, \ldots, E_n$ are compact extremal subsets of K such that $K = \overline{\text{co}}(E_1 \cup \ldots \cup E_n)$, then $\partial_e F \subseteq E_i$ for at least one value of i. A point $x \in K$ is *strongly primal* if F_x is a strongly primal face.

Note that if, in Definition 6.1, "compact extremal subsets" is replaced by "semi-exposed faces" or "closed split faces", then one obtains definitions equivalent to those of primal and weakly primal faces given in Definition 5.1. Clearly any strongly primal face is primal. If K is a Choquet simplex, and E is a compact extremal subset of K, then $\overline{\text{co}}E$ is a closed split face [7; p. 114]. It follows by Milman's Theorem that a weakly prime face of a simplex is strongly primal. An argument similar to the proof of Proposition 5.3 (1) shows that any point of $\overline{\partial_e K}$ is strongly primal.

Let A be a unital C^*-algebra. It has been shown in Theorems 3.5 and 5.6 that a state φ belongs to $\overline{\text{F}(A)}$ if and only if φ is (weakly) primal in $\text{S}(A)$. The following seems plausible:

CONJECTURE. *A strongly primal state of a unital C^*-algebra A belongs to* $\overline{\text{P}(A)}$t

By the remarks above, the converse of this conjecture is true. Although it has not been possible to prove the conjecture, it can be established in any of the following cases.

(1) A is antiliminal (using [8; Proposition 3.1]).

(2) A is primitive (using Theorem 3.11).

(3) A is a von Neumann algebra (using [19; Theorem 4']).

It can also be shown that $\text{S}(A)$ is strongly primal (in itself) if and only if $\overline{\text{P}(A)} = \text{S}(A)$, or, equivalently, A is prime and either antiliminal or one-dimensional (see result (A) in Section 1).

Finally, we observe that if we defined "strongly primal" by replacing "$\partial_e F \subseteq E_i$" by "$F \subseteq E_i$" in Definition 6.1, then the "strongly primal" points of K would form the smallest compact extremal subset containing $\partial_e K$, and any "strongly primal" state of A would belong to $\overline{\text{P}(A)}$ by Theorem 3.8.

REFERENCES

1. AKEMANN, C. A., The general Stone-Weierstrass problem, *J. Functional Analysis*, **4**(1969) 277–294.

2. ALFSEN, E. M., *Compact convex sets and boundary integrals*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

3. ALFSEN, E. M.; HANCHE-OLSEN, H.; SHULTZ, F. W., State spaces of C^*-algebras, *Acta Math.*, **144**(1980), 267–305.

4. ANDERSON, J.; BUNCE, J. W., Stone-Weierstrass theorems for separable C^*-algebras, *J. Operator Theory*, **6**(1981), 363–374.

5. ARCHBOLD, R. J., Prime C^*-algebras and antilattices, *Proc. London Math. Soc. (3)*, **24**(1972), 669–680.

6. ARCHBOLD, R. J., On factorial states of operator algebras, *J. Functional Analysis*, **55**(1984), 25–38.

7. ASIMOV, L.; ELLIS, A. J., *Convexity theory and its applications in functional analysis*, Academic Press, London, 1980.

8. BATTY, C. J. K.; ARCHBOLD, R. J., On factorial states of operator algebras. II, *J. Operator Theory*, **13**(1985), 131–142.

9. BORCEUX, F.; SIMMONS, H.; VAN DEN BOSSCHE, G., A sheaf representation for modules with applications to Gelfand rings, *Proc. London Math. Soc. (3)*, **48**(1984), 230–246.

10. BRATELLI, O.; ROBINSON, D. W., *Operator algebras and quantum statistical mechanics. I*, Springer-Verlag, New York, 1979.

11. BRIEM, E., Extreme orthogonal boundary measures for A(K) and decompositions for compact convex sets, in *Springer Lecture Notes in Mathematics*, **512**(1976), 9–16.

12. CHU, C. H., Antilattices and prime sets, *Math. Scand.*, **31**(1972), 151–165.

13. CHU, C. H., Prime faces in C^*-algebras, *J. London Math. Soc. (2)*, **7**(1973), 175–180.

14. CHU, C. H., On convexity theory and C^*-algebras, *Proc. London Math. Soc. (3)*, **31**(1975), 257–288.

15. DIXMIER, J., *Les C^*-algèbres et leurs représentations*, 2nd ed., Gauthier-Villars, Paris, 1969.

16. DIXMIER, J., *Les algèbres d'operateurs dans l'espace Hilbertien*, 2nd ed., Gauthier-Villars, Paris, 1969.

17. ELLIS, A. J., Central decompositions and the essential set for the space A(K), *Proc. London Math. Soc. (3)*, **26**(1973), 564–576.

18. ELLIS, A. J., Weakly prime compact convex sets and uniform algebras, *Math. Proc. Cambridge Philos. Soc.*, **81**(1977), 225–232.

19. GLIMM, J., A Stone-Weierstrass theorem for C^*-algebras, *Ann. of Math.*, **72**(1960), 216–244.

20. HALPERN, H., Irreducible module homomorphisms of a von Neumann algebra into its centre, *Trans. Amer. Math. Soc.*, **140**(1969), 195–221.

21. KADISON, R. V., A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. of Math.*, **56**(1952), 494–503.

22. KADISON, R. V.; LANCE, E. C.; RINGROSE, J. R., Derivations and automorphisms of operator algebras. II, *J. Functional Analysis*, **1**(1967), 204–221.

23. LONGO, R., Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^*-inclusions, *Invent. Math.*, **76**(1984), 143–156.

24. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

25. POPA, S., Semiregular maximal abelian $*$-subalgebras, and the solution to the factor state Stone-Weierstrass problem, *Invent. Math.*, **76**(1984), 157–161.

26. SAKAI, S., A characterization of type I C^*-algebras, *Bull. Amer. Math. Soc.*, **72**(1966), 508–512.

27. SHULTZ, F. W., Pure states as a dual object for C^*-algebras, *Comm. Math. Phys.*, **82**(1982), 497–509.

28. TAKESAKI, M., *Theory of operator algebras. I*, Springer-Verlag, New York, Heidelberg, Berlin, 1979.

29. TELEMAN, S., Measure-theoretic properties of the Choquet and of the maximal topologies, INCREST preprint, Bucharest, No. 33/1983.

30. Tomiyama, J.; Takesaki, M., Applications of fibre bundles to the certain class of C^*-algebras, *Tohôku Math. J. (2)*, **13**(1961), 498–523.
31. Wils, W., The ideal center of partially ordered vector spaces, *Acta Math.*, **127**(1971), 41–77.
32. Glimm, J., Type I C^*-algebras, *Ann. of Math.*, **73**(1961), 572–612.

R. J. ARCHBOLD
Department of Mathematics,
University of Aberdeen,
Edward Wright Building,
Dunbar Street,
Aberdeen AB9 2TY,
Scotland.

C. J. K. BATTY
Department of Mathematics,
University of Edinburgh,
King's Buildings,
Edinburgh EH9 3JZ,
Scotland.

Received April 6, 1984; revised March 28, 1985.