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MAPPING CONES AND EXACT SEQUENCES IN KK-THEORY

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INTRODUCTION

Let A and B be C^*-algebras and $\varphi : A \to B$ a $*$-homomorphism. The mapping cone for φ is a C^*-algebra C_φ. As the equivalence relation defining Kasparov's KK-functor is homotopy, we get, for all C^*-algebras D, a long exact sequence

$$\ldots \to \mathrm{KK}(D, A(0,1)) \xrightarrow{\varphi_*} \mathrm{KK}(D, B(0,1)) \to \mathrm{KK}(D, C_\varphi) \to$$
$$\to \mathrm{KK}(D, A) \xrightarrow{\varphi_*} \mathrm{KK}(D, B).$$

Assuming moreover that the algebras A and B are separable we also obtain, using Bott periodicity, an exact sequence

$$\mathrm{KK}(B, D) \xrightarrow{\varphi^*} \mathrm{KK}(A, D) \to \mathrm{KK}(C_\varphi, D) \to \mathrm{KK}(B(0,1), D) \xrightarrow{\varphi^*} \mathrm{KK}(A(0,1), D) \to \ldots$$

These two exact sequences are the mapping cone or Puppe exact sequences.

Of course the Puppe sequences can be obtained from the exact sequences associated with an ideal established by Kasparov ([12], § 7; cf. [14] for the $\mathbf{Z}/2$ graded case), since we have a short exact sequence $0 \to B(0,1) \to C_\varphi \to A \to 0$.

On the other hand the exactness of the Puppe sequences is an immediate consequence of the definition of the Kasparov groups and can, as we shall show, in fact be used to give a much simpler proof for the existence of the long exact sequences associated with an ideal.

For this, let $0 \to I \to A \xrightarrow{q} A/I \to 0$ be a short exact sequence (admitting a completely positive cross-section), and let $e : I \to C_q$ be the natural embedding. We construct an inverse for $[e] \in \mathrm{KK}(I, C_q)$, namely the element of $\mathrm{KK}(C_q, I)$ corresponding to the short exact sequence

$$0 \to I(0,1) \to A[0,1) \to C_q \to 0.$$

Therefore in the Puppe sequences associated with q we can replace C_q by I, and obtain the long exact sequences associated with the ideal I.

The fact that the second Puppe sequence is less obvious than the first one, is due to the unsymmetry in A and B, of the equivalence relation that defines the functor $KK(A, B)$, namely homotopy.

We introduce here an equivalence relation, that we call cobordism and which reverses the roles played by A and B. We then prove that (for separable A) cobordism and homotopy coincide.

We illustrate this dual equivalence relation with a "dual Puppe sequence", based on dual mapping cones and dual suspension.

We would like to thank Uffe Haagerup who helped us with a useful remark (2.5.1).

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NOTATIONS

As we deal with $Z/2$-graded C^*-algebras we use the language of Kasparov bimodules (with the notations of [13]) rather than the quasihomomorphism formalism of [7], [8].

Our approach, however, can be put very naturally into the framework of quasihomomorphisms, see Section 5.

If A is a C^*-algebra and X a locally compact space, $A(X)$ denotes the C^*-algebra of A-valued continuous functions on X vanishing at infinity.

All our results hold in the real as well as in the complex case. They also remain true in the equivariant case with respect to a compact group action.

1. EXACTNESS OF THE PUPPE SEQUENCES

Let A, B be graded C^*-algebras and $\varphi : A \to B$ a grading preserving $*$-homomorphism. The cone C_φ is the subalgebra $\{(x,f) \mid \varphi(x) = f(0)\}$ of $A \oplus B[0, 1)$.

Let $p : C_\varphi \to A$ be given by $p(x,f) = x$ and $i : B(0, 1) \to C_\varphi$ be given by $i(f) = (0,f)$.

In this section we prove:

1.1. THEOREM. *Let A, B, D be graded C^*-algebras and $\varphi : A \to B$ a grading preserving $*$-homomorphism. Then the sequences*

$$KK(D, A(0, 1)) \xrightarrow{\varphi_*} KK(D, B(0, 1)) \xrightarrow{i_*} KK(D, C_\varphi) \xrightarrow{p_*} KK(D, A) \xrightarrow{\varphi_*} KK(D, B)$$

and (if A and B are separable)

$$\mathrm{KK}(B, D) \xrightarrow{\varphi^*} \mathrm{KK}(A, D) \xrightarrow{p^*} \mathrm{KK}(C_\varphi, D) \xrightarrow{i^*} \mathrm{KK}(B(0, 1), D) \xrightarrow{\varphi^*} \mathrm{KK}(A(0, 1), D)$$

are exact.

We begin with some observations:

1.2. REMARKS. 1. The cone C_p is isomorphic to the subalgebra $\{(f, g) \mid \varphi(f(0)) = g(0)\}$ of $A[0, 1) \oplus B[0, 1)$.

The map $j : B(0, 1) \to C_p$ given by $j(g) = (0, g)$ is a homotopy equivalence. Indeed let $\psi : C_p \to B(0, 1)$ be defined by

$$\psi(f, g)(t) = \begin{cases} g(2t - 1) & \text{for } 1/2 \leqslant t < 1 \\ \varphi(f(1 - 2t)) & \text{for } 0 < t \leqslant 1/2 . \end{cases}$$

Obviously $\psi \circ j$ is homotopic (in the space of grading preserving $*$-homomorphisms) to the identity of $B(0, 1)$ and $j \circ \psi$ is homotopic to the identity of C_p.

2. The cone C_i is isomorphic to the subalgebra $\{(f, g) \mid \varphi(f(t)) = g(0, t)\}$ of $A(0, 1) \oplus B([0, 1)\times[0, 1) \setminus \{0, 0\})$.

The map $q : C_i \to A(0, 1)$ given by $q(f, g) = f$ is a homotopy equivalence. Indeed let

$$\alpha : [0, 1)\times[0, 1)\setminus\{0, 0\} \to [0, 1]\times(0, 1)$$

be a homeomorphism such that $\alpha(0, t) = (0, t)$, $(t \in (0, 1))$ $\left(\text{for instance } \alpha(t, s) = \left(\frac{2}{\pi} \operatorname{Arctan} \frac{t}{s}, \max(t, s)\right)\right)$.

Using α, C_i becomes isomorphic to the subalgebra $\{(f, g) \mid \varphi(f(t)) = g(0, t)\}$ of $A(0, 1) \oplus B([0, 1]\times(0, 1))$. Let $\omega : A(0, 1) \to C_i$ be defined by $\omega(f) = (f, g)$ $g(s, t) = \varphi(f(s))$. This is obviously a homotopy inverse of q.

Thanks to these remarks we now just have to prove exactness at $\mathrm{KK}(D, A)$ and $\mathrm{KK}(A, D)$.

Proof of Theorem 1.1. Let $(E, F) \in \mathscr{E}(D, A)$. The image $\varphi_*(E, F)$ is the zero element of $\mathrm{KK}(D, B)$ if and only if there exists a homotopy $(\bar{E}, \bar{F}) \in \mathscr{E}(D, B[0, 1])$ with $(\bar{E}_0, \bar{F}_0) = \varphi_*(E, F)$ and $(\bar{E}_1, \bar{F}_1) = (0, 0)$. Then the pair (E, F), $(\bar{E}, \bar{F})$ defines an element of $\mathscr{E}(D, C_\varphi)$. This proves the first assertion.

Define $l : C_\varphi \to B[0, 1)$ by $l(x, f) = f$ and $p_0 : B[0, 1) \to B$ by $p_0(f) = f(0)$. We have $\varphi \circ p = p_0 \circ l$. As the algebra $B[0, 1)$ is contractible $\mathrm{KK}(B[0, 1), D) = 0$ hence $p_0^* = 0$ and $l^* = 0$. Therefore $p^* \circ \varphi^* = (\varphi \circ p)^* = 0$.

Let now $(E, F) \in \mathscr{E}(A, D)$. Assume that $p^*(E, F)$ is the zero element of $\mathrm{KK}(C_\varphi, D)$. Let then $(\bar{E}, \bar{F}) \in \mathscr{E}(C_\varphi, D[0, 1))$ be such that $(\bar{E}_0, \bar{F}_0) = p^*(E, F)$. Let $\tilde{E}$ be the submodule of $\bar{E}$, $\tilde{E} = \{\xi \in \bar{E} \mid \xi(0) = 0\}$. As $\mathscr{K}(\tilde{E})$ is an ideal in $\mathscr{K}(\bar{E})$

we get a map $\mathscr{L}(\bar{E}) \to \mathscr{L}(\hat{E})$. Let $\tilde{F}$ be the image of $\bar{F}$ under this map and let $B(0, 1)$ act in $\tilde{E}$ through the composition $B(0, 1) \xrightarrow{i} C_\varphi \to \mathscr{L}(\bar{E}) \to \mathscr{L}(\hat{E})$. As $p \circ i = 0$, the action of $B(0, 1)$ in $\bar{E}_0 = p^*(E)$ is the zero action. We deduce that $B(0, 1) \cdot \bar{E} \subseteq \tilde{E}$ and that, for $x \in B(0, 1)$ and $y \in \mathscr{K}(\bar{E})$, $xy \in \mathscr{K}(\tilde{E})$. Hence $(\tilde{E}, \tilde{F}) \in \mathscr{E}(B(0, 1), D(0, 1))$. As B is separable the map $\tau_{C(0,1)} : \mathrm{KK}(B, D) \to \mathrm{KK}(B(0, 1), D(0, 1))$ is an isomorphism. Let $\omega \in \mathrm{KK}(B, D)$ be such that $\tau_{C(0,1)}(\omega)$ is the class of $(\tilde{E}, \tilde{F})$.

We want to show that $\varphi^*(\omega)$ is the class of (E, F). But as A is also separable it is enough to show that $\tau_{C(0,1)}(\varphi^*(\omega)) = \tau_{C(0,1)}$ [class of (E, F)]. But $\tau_{C(0,1)}(\varphi^*(\omega)) = \varphi^*(\tau_{C(0,1)}(\omega))$ is the class of $\varphi^*(\tilde{E}, \tilde{F})$.

Let E' be the $D(-1, 1)$ module:

$$E' = \{(\xi, \eta) \mid \xi \in E(-1, 0], \eta \in \bar{E} \text{ such that } \xi(0) = \eta(0)\}.$$

Let $F' \in \mathscr{L}(E')$ be given by $F'(\xi, \eta) = (\xi', \eta')$ with $\xi'(t) = F(\xi(t))$, $\eta' = \bar{F}(\eta)$.

Write $A(-1, 1) = \{(f, g) \mid f \in A(-1, 0],\ g \in A[0, 1), f(0) = g(0)\}$. Let $\overline{\varphi} : A[0, 1) \to C_\varphi$ be given by $\overline{\varphi}(f) = (f(0), \varphi \circ f)$. Let $A(-1, 1)$ act on E' by $(f, g)(\xi, \eta) = (f\xi, \overline{\varphi}(g)\eta)$. Then $(E', F') \in \mathscr{E}(A(-1, 1), D(-1, 1))$. Its restriction to $A(-1, 0)$ is equivalent to $\tau_{C(0,1)}(E, F)$, its restriction to $A(0, 1)$ is equivalent to $\varphi^*(\tilde{E}, \tilde{F})$. Q.E.D.

2. EXCISION

In this section we establish the long exact sequences associated with an ideal ([12], § 7, Theorem 1; [14], Theorem 1.1). This is done using the Puppe exact sequences (Theorem 1.1) and an excision type result (Theorem 2.1).

Let J be a graded ideal in A. Let $j : J \to A$ be the inclusion and $q : A \to A/J$ the quotient map. Let $e : J \to C_q$ be defined by $e(x) = (j(x), 0)$.

2.1. THEOREM. *Assume that A is separable and that the quotient map q admits a completely positive (grading preserving-norm decreasing) cross-section.*
Then the element $e_*(1_J) = e^*(1_{C_q}) \in \mathrm{KK}(J, C_q)$ is invertible.

We will use the following:

2.2. LEMMA. *Let J_1 and J_2 be ideals of the separable C^*-algebra B. Assume that the quotient maps $p_i : B \to B/J_i$ admit completely positive (grading preserving-norm decreasing) cross-sections. Then the map $B \xrightarrow{p} B/J_1 \cap J_2$ admits a completely positive (grading preserving-norm decreasing) cross-section.*

Proof. Let $s_i : B/J_i \to B$ be completely positive cross-section of p_i. Applying Theorem 4 of § 3 of [12] we get multipliers M, N of $J_1 \cap J_2$ such that $MJ_1 \subseteq J_1 \cap J_2$, $NJ_2 \subseteq J_1 \cap J_2$, $M \geqslant 0$, $N \geqslant 0$, $M + N = 1$ and $[M, B] \subseteq J_1 \cap J_2$. For

$x \in B$, put:

$$s(p(x)) = (M^{1/2}s_1 \circ p_1(x)M^{1/2} + N^{1/2}s_2 \circ p_2(x)N^{1/2}, p(x)) \in M(J_1 \cap J_2) \oplus B/J_1 \cap J_2.$$

As

$$s(p(x)) - (x, p(x)) = (M^{1/2}(s_1 \circ p_1(x) - x)M^{1/2} + N^{1/2}(s_2 \circ p_2(x) - x)N^{1/2}, 0) +$$
$$+ (M^{1/2}[x, M^{1/2}] + N^{1/2}[x, N^{1/2}], 0) \in J_1 \cap J_2 \oplus 0$$

we deduce that $s(p(x)) \in B \subseteq M(J_1 \cap J_2) \oplus B/J_1 \cap J_2$ and that s is the desired cross-section. Q.E.D.

We will present the inverse of the element $e_*(1_J) = e^*(1_{C_q})$ in the form of an exact sequence.

Recall that in the graded case the equality $\mathrm{Ext}(A, B)^{-1} = \mathrm{KK}^1(A, B)$ does not hold ([12], p. 569).

However, to an extension $0 \to B \xrightarrow{i} D \xrightarrow{p} A \to 0$ in which the maps i and p preserve the grading and p admits a completely positive (grading preserving, norm decreasing) cross-section, there corresponds an element δ_p of $\mathrm{KK}(A, B \otimes \mathscr{C}_1)$ (cf. [14], § 1) where $\mathscr{C}_1$ is the first Clifford algebra.

Consider the exact sequence

$$0 \to J(0, 1) \to A[0, 1) \xrightarrow{p} C_q \to 0.$$

As $J(0, 1) = J[0, 1) \cap A(0, 1)$, the map p admits a completely positive (grading preserving, norm decreasing) cross-section (Lemma 2.2).

Let $\alpha \in \mathrm{KK}(\mathscr{C}_1(0, 1), \mathbf{C})$ be the Bott inverse element (cf. [12], § 5). Set $u = -\delta_p \otimes_{\mathscr{C}_1(0,1)} \alpha \in \mathrm{KK}(C_q, J)$.

2.3. Lemma. *We have* $e^*(u) = 1_J$.

Proof. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \to & J(0, 1) & \to & A[0, 1) & \xrightarrow{p} & C_q & \to & 0 \\ & & \| & & \uparrow & & \uparrow e & & \\ 0 & \to & J(0, 1) & \to & J[0, 1) & \xrightarrow[p_0]{} & J & \to & 0. \end{array}$$

By [14], Lemma 1.5, we get $e^*\delta_p = \delta_{p_0}$. Consider the exact sequence:

$$0 \to \mathbf{C}(0, 1) \to \mathbf{C}[0, 1) \xrightarrow{q_0} \mathbf{C} \to 0.$$

By construction of δ we have $\delta_{p_0} = \tau_J(\delta_{q_0})$.

But $\delta_{q_0} = -\beta$ where $\beta \in \mathrm{KK}(\mathbf{C}, \mathscr{C}_1(0, 1))$ is the Bott element (cf. [12], § 7).

Moreover $\beta \otimes_{\mathscr{C}_1(0,1)} \alpha = 1_{\mathbf{C}}$ ([12], § 5). Q.E.D.

End of the proof of Theorem 2.1. Consider the exact sequence

$$0 \to C_e \to C_q[0, 1) \xrightarrow{\varphi} A/J[0, 1) \to 0$$

where $\varphi = \pi \circ p_0$ where $p_0 : C_q[0, 1) \to C_q$ is evaluation at 0 and $\pi : C_q \to A/J[0, 1)$ is defined by $\pi(x,f) = f$. As both π and p_0 admit completely positive (grading preserving, norm decreasing) cross-sections so does φ.

From Lemma 2.3 (applied to φ instead of q) we derive that for any separable graded C^*-algebra B the map $\mathrm{KK}(B, C_e) \to \mathrm{KK}(B, C_\varphi)$ is injective.

Now, by Theorem 1.1, $\mathrm{KK}(B, C_\varphi) = 0$. Hence $\mathrm{KK}(B, C_e) = 0$.

Applying again Theorem 1.1, and Bott periodicity we deduce that the map $e_* : \mathrm{KK}(B, J) \to \mathrm{KK}(B, C_q)$ is an isomorphism. The map $\otimes u : \mathrm{KK}(B, C_q) \to \mathrm{KK}(B, J)$, being a left inverse of e_*, is an inverse.

Applying this fact to $B = C_q$ we get $e^*(u) = 1_{C_q}$. Q.E.D.

Combining now Theorems 1.1 and 2.1 we get:

2.4. Corollary. ([12], § 7, Theorem 1; cf. [14], Theorem 1.1). *Let* $0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$ *be a short exact sequence of graded algebras such that* q *admits a completely positive (grading preserving and norm decreasing) cross-section. Then there are long exact sequences*

$$\ldots \xrightarrow{j_*} \mathrm{KK}(D, A(\mathbf{R}^{n+1})) \xrightarrow{q_*} \mathrm{KK}(D, A/J(\mathbf{R}^{n+1})) \xrightarrow{\delta} \mathrm{KK}(D, J(\mathbf{R}^n)) \xrightarrow{j_*}$$

$$\xrightarrow{j_*} \mathrm{KK}(D, A(\mathbf{R}^n)) \to \ldots$$

(D separable)

$$\ldots \xrightarrow{q^*} \mathrm{KK}(A(\mathbf{R}^n), D) \xrightarrow{j^*} \mathrm{KK}(J(\mathbf{R}^n), D) \xrightarrow{\delta} \mathrm{KK}(A/J(\mathbf{R}^{n+1}), D) \xrightarrow{q^*}$$

$$\xrightarrow{q^*} \mathrm{KK}(A(\mathbf{R}^{n+1}), D) \xrightarrow{j^*} \ldots$$

(A, B separable) where δ *is multiplication by* $\delta_q \in \mathrm{KK}^1(A/J, J)$.

Note that under the identification of $\mathrm{KK}^1(A/J, J)$ with $\mathrm{KK}(A/J(\mathbf{R}), J)$, δ_q corresponds to the element of $\mathrm{KK}(A/J(\mathbf{R}), J)$ given by the natural inclusion $A/J(\mathbf{R}) \to C_q \sim J$.

2.5. Remarks. 1. (We are indebted to U. Haagerup for this remark):

Let $s : A/J \to A$ be a completely positive cross-section (not necessarily norm decreasing). Let u_n be an approximate unit in A/J with $0 \leqslant u_n \leqslant 1$. The maps

$\varphi_n : \tilde{A}/J \to \tilde{A}/J$ defined by $\varphi_n(1) = 1$, $\varphi_n(x) = u_n^{1/2} x u_n^{1/2}$ $(x \in A/J)$ admit the completely positive lifting $\psi_n : \tilde{A}/J \to \tilde{A}$ given by $\psi_n(1) = 1$ and $\psi_n(x) = v_n^{-1/2} s(u_n^{1/2} x u_n^{1/2}) v_n^{-1/2}$ $(x \in A/J)$, where $v_n = \sup(1, s(u_n)) \in \tilde{A}$ (φ_n and ψ_n are completely positive by [4], Lemma 3.9).

If $\tilde{A}/J$ is separable, the set of maps admitting a unital lifting being (norm pointwise) closed ([1], Theorem 6), the identity of $\tilde{A}/J$ admits a unital lifting. Hence there exists a completely positive, norm decreasing cross-section $s' : A/J \to A$.

2. In the situation of Theorem 2.1, let $e' : C_j \to A/J(0, 1)$ be defined by $e'(x, f)(t) = q(f(t))$. Another excision result, equivalent to Theorem 2.1, is that the element $e'_*(1_{C_j}) = e'^*(1_{A/J(0,1)})$ of $\mathrm{KK}(C_j, A/J(0, 1))$ is invertible.

Its inverse is given by the exact sequence $0 \to C_j \to A[0, 1) \xrightarrow{p'} A/J \to 0$ where $p'(f) = q(f(0))$.

That these excision results are equivalent comes from the isomorphism of C_e and $C_{e'}$: By Theorem 1.1 e_* (and e^*) is an isomorphism if and only if C_e is K-contractible ([8], Proposition 5.4).

3. The condition on the completely positive cross-section is natural (cf. e.g. [9]). However it is not necessary as for any short exact sequence $0 \to I \to A \to A/I \to 0$ the sequence

$$\ldots \to \mathrm{KK}(D, I[0, 1)) \to \mathrm{KK}(D, A[0, 1)) \to \mathrm{KK}(D, A/I[0, 1)) \to \ldots$$

is exact!

Let us give another example:

Let G be a K-amenable discrete group ([6], Definition 2.2). Let α be an action of G on the C^*-algebra A. We have a short exact sequence:

$$0 \to J \to A \times_\alpha G \xrightarrow{\lambda} A \times_{\alpha, \mathrm{red}} G \to 0 \quad (\text{where } J = \mathrm{Ker}\,\lambda).$$

Let now $0 \to J \to D \xrightarrow{p} D/J \to 0$ be any short exact sequence and let $(E, F) \in \mathscr{E}(B_1, B_2)$. Define $J_{B_i} = \mathrm{Ker}(p \hat{\otimes} \mathrm{id}_{B_i})$. One may then define $\tau_p(E, F) \in \mathscr{E}(J_{B_1}, J_{B_2})$ in the following way:

Let E_J be the Hilbert $D \otimes B_2$ submodule of $D \otimes E$ given by $E_J = \mathrm{Ker}(p \otimes \mathrm{id}_E)$ (thus $E_J = \{\xi \in D \otimes E \mid \langle \xi, \xi \rangle \in J_{B_2}\}$). Note that for all $T \in \mathscr{L}(D \otimes E)$ and $\xi \in E_J$, $T\xi \in E_J$. Hence E_J is a (J_{B_1}, J_{B_2})-bimodule. Let F_J be the restriction of $1_D \hat{\otimes} F$ to E_J.

Note also that for all $x \in J_{B_1}$ and $\xi \in D \otimes E$, $x\xi \in E_J$. Therefore $J_{B_1} \cdot \mathscr{K}(D \otimes E) \subseteq \mathscr{K}(E_J)$. Hence $(E_J, F_J) \in \mathscr{E}(J_{B_1}, J_{B_2})$. We put $(E_J, F_J) = \tau_p(E, F[)$

The homomorphism $\Delta^\alpha : A \times_\alpha G \to C^*(G) \otimes A \times_\alpha G$ of [6], 1.3, induces a homomorphism $\Delta' : J \to J_{C^*(G)}$. Therefore we get a map

$$\mathrm{K}^0(C^*(G)) \xrightarrow{\Delta'^* \circ \tau_\lambda} \mathrm{KK}(J, J).$$

Let t be the class of the trivial representation in $\mathrm{K}^0(C^*(G))$. Obviously $\Delta'^* \circ \tau_\lambda(t) = 1_J \in \mathrm{KK}(J, J)$. Also if $x \in \lambda^*(\mathrm{K}^0(C_{\mathrm{red}}(G)))$ then $\Delta'^* \circ \tau_\lambda(x) = 0$. Therefore, when G is K-amenable, J is K-contractible. Using [6], Theorem 2.1, we get then that the sequences

$$\ldots \to \mathrm{KK}(B, J) \to \mathrm{KK}(B, A \times_\alpha G) \xrightarrow{\lambda_*} \mathrm{KK}(B, A \times_{\alpha,\mathrm{red}} G) \to \ldots$$

and

$$\ldots \to \mathrm{KK}(A \times_{\alpha,\mathrm{red}} G, B) \xrightarrow{\lambda^*} \mathrm{KK}(A \times_\alpha G, B) \to \mathrm{KK}(J, B) \to \ldots$$

are exact.

However λ does not always admit a completely positive cross-section (take for instance $G = \mathbf{F}_2$ and $A = \mathbf{C}$, cf. [5]).

3. COBORDISM

In this section we introduce a new equivalence relation in $\mathscr{E}(A, B)$ where A and B are graded algebras. When A is separable we show that this equivalence relation (called cobordism) coincides with homotopy.

This result is illustrated in Section 4. It admits an interesting interpretation (3.8).

Let (E, F) be a Kasparov (A, B)-bimodule and $p \in \mathscr{L}(E)$ a projection of degree 0 such that $[a, p] = 0$ and $a[p, F] \in \mathscr{K}(E)$ for all a in A. Then the action of A restricts to the submodule pE of E, and (pE, pFp) is a Kasparov (A, B)-bimodule noted $(E, F)_p$.

3.1. DEFINITION. The Kasparov (A, B)-bimodules (E_0, F_0) and (E_1, F_1) are said to be *cobordant* if there exists a triple (E, F, v) such that:

a) (E, F) is a Kasparov (A, B)-bimodule.

b) $v \in \mathscr{L}(E)$ is a partial isometry of degree 0.

c) For all a in A, $[a, v] = 0$ and $a[v, F] \in \mathscr{K}(E)$.

d) $(E, F)_{1-vv^*}$ and (E_0, F_0) are unitarily equivalent.

e) $(E, F)_{1-v^*v}$ and (E_1, F_1) are unitarily equivalent.

3.2. REMARK. The word cobordism is used here for the following reason: One may call a triple (E, F, v), satisfying conditions a), b) and c) of Definition 3.1, a Kasparov (A, B) bimodule with boundary. Its boundary is the formal difference $(E, F)_{1-vv^*} - (E, F)_{1-v^*v}$.

3.3. LEMMA. *Cobordism is an equivalence relation compatible with direct sums.*

Proof. The only thing which is not completely obvious is transitivity. Let (E', F', v'), (E'', F'', v'') be two triples as above. Assume that $(E', F')_{1-v'^*v'}$ and $(E'', F'')_{1-v''v''^*}$ are unitarily equivalent. Let $u \in \mathscr{L}((1 - v'^*v')E', (1 - v''v''^*)E'')$ be a unitary realizing this equivalence.

Let $v \in \mathscr{L}(E' \oplus E'')$, $v = (v' \oplus v'') + u$. Then the triple $(E' \oplus E'', F' \oplus F'', v)$ realizes a cobordism between $(E', F')_{1-v'v'^*}$ and $(E'', F'')_{1-v''^*v''}$. Q.E.D.

3.4. REMARKS. 1. If (E, F) is degenerate (i.e. belongs to $\mathscr{D}(A, B)$) then it is cobordant to $(0, 0)$.

Indeed, let v be an isometry of index -1 in the separable Hilbert space H. Then the triple $(E \otimes H, F \otimes 1, 1 \otimes v)$ defines the desired cobordism.

2. Let $(E, F) \in \mathscr{E}(A, B)$ and let $F' \in \mathscr{L}(E')$ of degree one be such that $A(F - F') \subseteq \mathscr{K}(E)$. Let $v \in \mathscr{L}(E \oplus E)$ be given by the matrix $v = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

The triple $(E \oplus E, F \oplus F', v)$ defines a cobordism between (E, F) and (E, F').

We now compare homotopy and cobordism:

3.5. LEMMA. *Any two cobordant Kasparov bimodules are homotopic.*

Proof. Let (E, F, v) satisfy conditions a), b) and c) of Definition 3.1. Then $E, F)_{vv^*}$ and $(E, F)_{v^*v}$ are unitarily equivalent. Moreover (E, F) is (operator) homotopic to $(E, F)_{vv^*} \oplus (E, F)_{1-vv^*}$ and to $(E, F)_{v^*v} \oplus (E, F)_{1-v^*v}$. As $\mathrm{KK}(A, B)$ is a group, $(E, F)_{1-vv^*}$ and $(E, F)_{1-v^*v}$ are homotopic. Q.E.D.

In fact two cobordant Kasparov bimodules define the same element in the group noted $\widetilde{\mathrm{KK}}(A, B)$ in [13], Definition 2.8.

We now prove the converse of Proposition 3.5 assuming that A is separable.

3.6. LEMMA. *Let A be separable and let $(E, F) \in \mathscr{E}(A, B)$ be operator homotopic to a degenerate element. Then (E, F) is cobordant to $(0, 0)$.*

Proof. Let $v \in \mathscr{L}(H)$ be an isometry of index -1 in the separable Hilbert space H. Let $\mathscr{T} \subseteq \mathscr{L}(H)$ be the C^*-algebra generated by v (the Toeplitz algebra).

Let $\mathscr{K} \subseteq \mathscr{T}$ be the ideal generated by the rank one projection $P = 1 - vv^*$ ($\mathscr{K}$ is equal to $\mathscr{K}(H)$). Assume that (E, F) is operator homotopic to the degenerate (E, F'). Consider the Kasparov $(A \otimes \mathscr{T}, B)$-bimodule $(E \otimes H, F' \otimes 1)$. Its restriction to $(A \otimes \mathscr{K}, B)$ is operator homotopic to $(E \otimes H, F \otimes 1)$. By Lemma 2.4 of [14] there exists $G \in \mathscr{L}(E \otimes H)$ such that $(E \otimes H, G) \in \mathscr{E}(A \otimes \mathscr{T}, B)$ and $(G - F \otimes 1)x \in \mathscr{K}(E \otimes H)$ for all x in $A \otimes \mathscr{K}$.

Restrict the $(A \otimes \mathscr{T}, B)$ bimodule $(E \otimes H, G)$ to $A = A \otimes 1_{\mathscr{T}} \subseteq A \otimes \mathscr{T}$. The triple $(E \otimes H, G, 1 \otimes v)$ defines a cobordism between $(E \otimes PH, (1 \otimes P)G(1 \otimes P))$ and $(0, 0)$. But by Remark 3.4.2 (E, F) and $(E \otimes PH, (1 \otimes P)G(1 \otimes P))$ are cobordant. Q.E.D.

3.7. THEOREM. *If A is separable then two Kasparov (A, B)-bimodules are cobordant if and only if they are homotopic.*

Proof. Let (E_1, F_1) and (E_2, F_2) be homotopic Kasparov (A, B)-bimodules. By Theorem 1, § 6 of [12], see also [13], one can find a Kasparov (A, B)-bimodule (E, F) such that both $(E_1, F_1) \oplus (E, F)$ and $(E_2, F_2) \oplus (E, F)$ are operator homotopic to degenerate elements. Using Lemma 3.6 we then get that both (E_1, F_1) and (E_2, F_2) are cobordant to $(E_1, F_1) \oplus (E_2, F_2) \oplus (E, F)$. Q.E.D.

3.8. REMARK. Theorem 3.7 may be interpreted in the following way:

Let $\mathscr{E}(A, B)$ be the semi-group of classes of Kasparov (A, B)-bimodules, where (E, F) and (E', F') are identified if there exists a unitary $u \in \mathscr{L}(E, E')$ of degree zero, intertwining the action of A, and such that $uFu^* - F' \in \mathscr{K}(E')$.

Let then $\overline{\mathrm{KK}}(A, B)$ be the cancellation semi-group associated with $\overline{\mathscr{E}}(A, B)$. Obviously $\overline{\mathrm{KK}}(A, B)$ is the semi-group of cobordism classes of elements of $\mathscr{E}(A, B)$. In that sense cobordism is the strongest "reasonable" equivalence relation in $\mathscr{E}(A, B)$.

Theorem 3.7 which states that the natural homomorphism $\overline{\mathrm{KK}}(A, B) \to \to \mathrm{KK}(A, B)$ is an isomorphism, means that all "reasonable" equivalence relations on $\mathscr{E}(A, B)$ coincide.

Theorem 3.7 can therefore be considered as a generalization of Lemma 2, § 7 of [12]. It also strengthens this result of Kasparov since, in the relation defining $\overline{\mathscr{E}}(A, B)$ above, the action of A is not allowed to change as in the homology of Definition 3, § 7 in [12] (cf. also [3], Corollary 7.8).

Here is another interpretation of cobordism:

Let $I = \mathbf{C}[0, 1]$ and let $\pi_0, \pi_1 : I \to \mathbf{C}$ be the evaluations at 0 and 1. Note that two Kasparov (A, B)-bimodules (E_0, F_0) and (E_1, F_1) define the same element of $\mathrm{KK}(A, B)$ (i.e. are homotopic) if and only if there exists a Kasparov $(A, B \otimes I)$ bimodule $(\tilde{E}, \tilde{F})$ with $\pi_{i*}(\tilde{E}, \tilde{F})$ unitarily equivalent to (E_i, F_i) $(i = 0, 1)$.

Thus homotopy assigns to A and B unsymmetric roles. Cobordism provides an equivalence relation which reverses the roles played by A and B:

Let $\mathscr{T}$ be the Toeplitz algebra, i.e. the universal C^*-algebra generated by a non-unital isometry v. Let $\mathscr{T}'$ be the C^*-subalgebra of $\mathscr{T} \oplus \mathscr{T}$ generated by $w = v \oplus v^*$. Let $\hat{I}$ denote the kernel of the quotient map $\lambda : \mathscr{T}' \to \mathbf{C}$ given by $\lambda(w) = 1$. Let $j_0, j_1 : \mathbf{C} \to \hat{I}$ be the inclusions given by $j_0(1) = (1 - vv^*) \oplus 0 = 1 - ww^*$ and $j_1(1) = 0 \oplus (1 - vv^*) = 1 - w^*w$. ($\hat{I}$ can also be viewed as the C^*-algebra of the "$ax + b$" group G, the maps j_0 and j_1 corresponding to minimal projections associated with the two square integrable representations of G.)

3.9. DEFINITION. a) The Kasparov (A, B)-bimodules (E_0, F_0) and (E_1, F_1) are said to be *equivalent* if there exist Hilbert B-modules E_0', E_1' and a unitary $u \in \mathscr{L}(E_0 \oplus E_0', E_1 \oplus E_1')$ of degree zero such that, for all a in A, $u(a \oplus 0) = (a \oplus 0)u$ and $(a \oplus 0)(u(F_0 \oplus 0)u^* - (F_1 \oplus 0)) \in \mathscr{K}(E_1 \oplus E_1')$.

b) The Kasparov (A, B)-bimodules (E_0, F_0) and (E_1, F_1) are said to be *cohomotopic* if there exists a Kasparov $(A \otimes \hat{I}, B)$-bimodule (E, F) such that $j_i^*(E, F)$ is equivalent to (E_i, F_i), $i = 0, 1$.

In order to show that homotopy and cohomotopy coincide, we will use the following rather obvious lemma.

3.10. LEMMA. *Let H_1 and H_2 be Hilbert spaces and $v_i \in \mathscr{L}(H_i)$, $(i = 1, 2)$ isometries. Let $u \in \mathscr{L}(H_1, H_2)$ be a partial isometry with $uv_1 = v_2^* u = 0$. Then there exists a $*$-representation*

$$\pi : \mathscr{T}' \to \mathscr{L}(H_1 \oplus H_2) \quad \textit{with } \pi(w) = \begin{pmatrix} v_1^* & 0 \\ u & v_2 \end{pmatrix}.$$

Proof. Let H_2' be the subspace of H_2 generated by $\{v_2^n u\xi \mid \xi \in H_1,\ n \geqslant 0\}$ and let H_2'' be the orthogonal complement of H_2' in H_2. Then, both $H_1 \oplus H_2'$ and $0 \oplus H_2''$ are stable under $\pi(w)$; the restriction of $\pi(w)$ to $0 \oplus H_2''$ is an isometry; its restriction to $H_1 \oplus H_2'$ is a coisometry. Q.E.D.

3.11. PROPOSITION. *If A is separable, two Kasparov (A, B)-bimodules are cohomotopic if and only if they are homotopic.*

Proof. Let (E_0, F_0) and (E_1, F_1) be homotopic Kasparov (A, B)-bimodules. Using Theorem 3.7 we get a cobordism (E, F, v) (resp. (E', F', v')) between $(-(E_0, F_0)) \oplus (E_1, F_1)$ (resp. $(-(E_0, F_0)) \oplus (E_0, F_0)$) and $(0, 0)$. Let $u \in \mathscr{L}(E, E')$ be the identification between the two copies of the bimodule $-E_0$. Let then $\mathscr{T}'$ act on $E \oplus E'$ by the action π given by

$$\pi(w) = \begin{pmatrix} v^* & 0 \\ u & v' \end{pmatrix} \qquad \text{(Lemma 3.10).}$$

Then $(E \oplus E', F \oplus F')$ defines a Kasparov $(A \otimes \mathscr{T}', B)$-bimodule whose restriction to $(A \otimes \hat{I}, B)$ is a cohomotopy between (E_0, F_0) and (E_1, F_1).

In order to prove the converse it is enough to show that $j_0^*(1_{\hat{I}})$ and $j_1^*(1_{\hat{I}})$ are equal in $\mathrm{KK}(\mathbf{C}, \hat{I})$. But $j_i^*(1_{\hat{I}}) = j_{i*}[1] \in \mathrm{K}_0(\hat{I})$; moreover $j_0(1)$ and $j_1(1)$ are stably equivalent projections in $\mathscr{T}' = \tilde{\hat{I}}$. Q.E.D.

In particular cohomotopy is an equivalence relation in $\mathscr{E}(A, B)$. Let $\widehat{\mathrm{KK}}(A, B)$ denote the quotient by this equivalence relation. We have proved:

3.12. THEOREM. *If A is separable, then $\widehat{\mathrm{KK}}(A, B) = \mathrm{KK}(A, B)$.*

3.13. REMARK. When A is not separable, cohomotopy is not an equivalence relation. It is however still symmetric and transitive (cf. proof of Lemma 3.3).

Let $\hat{\mathscr{E}}(A, B)$ be the set of Kasparov (A, B)-bimodules (E, F) such that $(E, F) \oplus \oplus (-E, -F)$ is cobordant to $(0, 0)$. Cohomotopy restricted to $\hat{\mathscr{E}}(A, B)$ is an equivalence relation (cf. proof of Lemma 3.10 b)). Let $\widehat{\mathrm{KK}}(A, B)$ be the set of cohomotopy classes of elements of $\hat{\mathscr{E}}(A, B)$. Then $\widehat{\mathrm{KK}}(A, B)$ is a group and the inclusion $\hat{\mathscr{E}}(A, B) \hookrightarrow \mathscr{E}(A, B)$ induces a homomorphism $\widehat{\mathrm{KK}}(A, B) \to \mathrm{KK}(A, B)$, which is an isomorphism when A is separable.

4. THE DUAL PUPPE SEQUENCE

Let $\varphi : A \to B$ be a (grading-preserving) $*$-homomorphism. In this section we construct a dual cone $\hat{C}_\varphi$ associated with φ and establish a "dual Puppe exact sequence" (Theorem 4.2).

Let $\mathscr{T}$ be the Toeplitz algebra $\mathscr{T} = C^*(v)$ where v is a non unitary isometry. Let $\lambda : \mathscr{T} \to \mathbf{C}$ be the $*$-homomorphism given by $\lambda(v) = 1$. Put $\hat{C} = \mathrm{Ker}\,\lambda$. Let $\mathscr{K}$ be the elementary algebra of the compacts contained in $\hat{C}$ as the ideal generated by $1 - vv^*$. The quotient $\hat{S} = \hat{C}/\mathscr{K}$ is isomorphic to the group C^*-algebra $C^*(\mathbf{R})$ (i.e. to $\mathscr{C}_0(\mathbf{R})$ in the complex case and to the algebra noted $\mathscr{C}_0^{\mathbf{R}}(\mathrm{i}\,\mathbf{R})$ in [8], § 4).

Recall ([8], proof of Theorem 4.4 and Proposition 5.4) that $\hat{C}$ is K-contractible, i.e. $\mathrm{KK}(\hat{C}, \hat{C}) = 0$. This also means that $\mathrm{KK}(A \otimes \hat{C}, B) = 0 = \mathrm{KK}(A, B \otimes \hat{C})$ for all algebras A and B (A separable).

Let $q_0 \colon \hat{C} \to \hat{S}$ be the quotient map and set $P = 1 - vv^* \in \hat{C}$.

4.1. DEFINITION. Let $\varphi : A \to B$ be a grading preserving $*$-homomorphism. The *dual cone* $\hat{C}_\varphi$ is the C^*-subalgebra of $(A \otimes \hat{S}) \oplus (B \otimes \hat{C})$ consisting of all pairs (x, y) such that $\varphi \otimes \mathrm{id}_{\hat{S}}(x) = \mathrm{id}_B \otimes q_0(y)$.

Let $j : B \to \hat{C}_\varphi$ be given by $j(b) = (0, b \otimes P)$ and $q : \hat{C}_\varphi \to A \otimes \hat{S}$ the restriction to $\hat{C}_\varphi$ of the natural projection $(A \otimes \hat{S}) \oplus (B \otimes \hat{C}) \to A \otimes \hat{S}$.

Our result in the "dual Puppe sequence" is:

4.2. THEOREM. *Let A, B be separable graded C^*-algebras and let $\varphi : A \to B$ be a grading preserving $*$-homomorphism. Then the following sequence is exact:*

$$\to \mathrm{KK}(B \otimes \hat{S}, D) \xrightarrow{\varphi^*} \mathrm{KK}(A \otimes \hat{S}, D) \xrightarrow{q^*} \mathrm{KK}(\hat{C}_\varphi, D) \xrightarrow{j^*} \mathrm{KK}(B, D) \xrightarrow{\varphi^*} \mathrm{KK}(A, D)$$

where D is any graded C^*-algebra.

Proof. Let us prove exactness at $\mathrm{KK}(B, D)$.

Let $\hat{C}'_\varphi$ be the subalgebra of $(A \otimes \hat{S}) \oplus (B \otimes \hat{C})$ generated by $(0, b \otimes P)$, $b \in B$ ($P = 1 - vv^* \in \hat{C}$) and $(a \otimes q_0(x), \varphi(a) \otimes x)$, $a \in A$, $x \in \hat{C}$. As $\hat{C}_\varphi$ is the subalgebra of $(A \otimes \hat{S}) \oplus (B \otimes \hat{C})$ generated by $(0, b \otimes k)$, $b \in B$, $k \in \mathscr{K} \subset \hat{C}$ and $(a \otimes q_0(x), \varphi(a) \otimes x)$, $a \in A$, $x \in \hat{C}$, $\hat{C}'_\varphi$ is a full hereditary subalgebra of $\hat{C}_\varphi$. Hence, by [2] and [12], § 4, Theorem 7, the inclusion map from $\hat{C}'_\varphi$ to $\hat{C}_\varphi$ induces an isomorphism in KK-theory. It is therefore enough to prove exactness of the sequence

$$\mathrm{KK}(\hat{C}'_\varphi, D) \xrightarrow{j^*} \mathrm{KK}(B, D) \xrightarrow{\varphi^*} \mathrm{KK}(A, D).$$

Let $i: A \to A \otimes \hat{C}$ be given by $i(a) = a \otimes P$ and $\psi : A \otimes \hat{C} \to \hat{C}'_\varphi$ defined by $\psi(a \otimes x) = (a \otimes q_0(x), \varphi(a) \otimes x)$. We have $j \circ \varphi = \psi \circ i$. But as $\hat{C}$ is K-contractible, we get $(\psi \circ i)^* = 0$, hence $\varphi^* \circ j^* = 0$.

Let $(E, F) \in \mathscr{E}(B, D)$. If $\varphi^*(E, F)$ defines the zero element of $\mathrm{KK}(A, D)$ then by Theorem 3.7 it is cobordant to $(0, 0)$. Let then (E', F', w) be a triple defining this cobordism.

Let $\pi_0 : \mathscr{T} \to \mathscr{L}(E')$ be given by $\pi(v) = w$. Let $\pi_1 : B \to \mathscr{L}(E')$ be the action of B in $(1 - ww^*)E'$ transported from the original action in E by the unitary equivalence of $\varphi^*(E, F)$ with $((1 - ww^*)E', (1 - ww^*)F(1 - ww^*))$.

Then there exists a unique $*$-homomorphism $\pi : \hat{C}'_\varphi \to \mathscr{L}(E')$ satisfying $\pi(0, b \otimes P) = \pi_1(b)$ and $\pi(a \otimes q_0(x), \varphi(a) \otimes x) = a \cdot \pi_0(x)$, $(b \in B, a \in A, x \in \hat{C})$. The pair (E', F') defines then an element of $\mathscr{E}(\hat{C}'_\varphi, D)$ whose restriction to B is cobordant to (E, F).

To prove exactness at the other points, one may, as in § 1, construct homomorphisms $\psi : \hat{C}_j \to A \otimes \hat{S}$ and $\omega : B \otimes \hat{S} \to \hat{C}_q$ which induce isomorphisms in KK-theory. Q.E.D.

4.3. Remarks. 1. This theorem could be used in giving another proof of the six term exact sequence theorem. Let $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$ be a short exact sequence of graded C^*-algebras such that p admits a completely positive cross-section. Let then $\hat{C}'_i$ be the subalgebra of $A \otimes \hat{C}$ generated by $b \otimes x$ and $a \otimes \hat{P}$, $a \in A$, $b \in I$, $x \in \hat{C}$. Let $q : \hat{C}'_i \to A/I$ be the map given by $q(a \otimes P) = p(a)$ and $q(b \otimes x) = 0$. One then shows that q induces isomorphism in KK-theory. (Its inverse is given by the exact sequence

$$0 \to \hat{C}_i \to A \otimes \hat{C} \xrightarrow{p \otimes q_0} A/I \otimes \hat{S} \to 0$$

$\hat{C}'_i$ sitting in $\hat{C}_i$ as a full hereditary subalgebra.)

2. Theorem 4.2 can be deduced from the six term exact sequence theorem as $q : \hat{C}_\varphi \to A \otimes \hat{S}$ is an epimorphism (admitting a completely positive cross-section) and $j(B)$ is a full hereditary subalgebra of $\mathrm{Ker}(q)$. In that way we also get an exact sequence

$$\mathrm{KK}(D, A) \xrightarrow{\varphi_*} \mathrm{KK}(D, B) \xrightarrow{j_*} \mathrm{KK}(D, \hat{C}_\varphi) \xrightarrow{q_*} \mathrm{KK}(D, A \otimes \hat{S}) \xrightarrow{\varphi_*} \mathrm{KK}(D, B \otimes \hat{S}) \to .$$

The cone C_φ is isomorphic to the subalgebra C'_φ of $A \oplus B[-\infty, +\infty)$, $C'_\varphi = \{(x,f) \mid \varphi(x) = f(-\infty)\}$. Let $\mathbf{R}$ act trivially in A and by translations in $B[-\infty, +\infty)$. Then C'_φ is invariant under this action. Call α the induced action of $\mathbf{R}$ in C'_φ.

Consider the exact sequences $0 \to \mathcal{K} \to \hat{C} \to \hat{S} \to 0$ and $0 \to C(-\infty, +\infty)\times_\alpha \mathbf{R} \to C[-\infty, +\infty)\times_\alpha \mathbf{R} \to C^*(\mathbf{R}) \to 0$. They define the same element of $\mathrm{Ext}(C^*(\mathbf{R})) = \mathbf{Z}$. We thus get a commuting diagram

$$\begin{array}{ccc} \hat{C} & \longrightarrow & \hat{S} \\ \downarrow & & \downarrow \\ C[-\infty, +\infty)\times_\alpha \mathbf{R} & \longrightarrow & C^*(\mathbf{R}) \end{array}$$

in which the vertical arrows are isomorphisms (cf. also P. Green, *Pacific J. Math.*, 72(1977), 71–97).

We deduce that $C'_\varphi \times_\alpha \mathbf{R} \cong \hat{C}_\varphi$. Thus Theorem 4.2 is equivalent to Theorem 1.1 (by [10] or [12]).

5. APPENDIX

In Section 2 we have taken the shortest route to the long exact sequence theorem using the full formalism and most of the basic results of Kasparov. It is however possible to give a proof of the equivalence $C_q \sim \mathrm{Ker}\, q$ ($q : A \to B$ a surjective homomorphism with completely positive cross-section) which is elementary in the sense that it uses only the definition of KK_0, the product and a weak form of Bott periodicity, but avoids KK_1, Ext, graded algebras, the Stinespring theorem etc...

The result $C_q \sim \mathrm{Ker}\, q$ may, in this approach, be considered as a "fundamental theorem" of KK-theory since it easily implies not only the exact sequences associated with q, in both variables, but also the homotopy invariance of Ext, full Bott periodicity and can be used to obtain, from q, an element of $\mathrm{KK}(B, \hat{S}\,\mathrm{Ker}\, q)$ or of $\mathrm{KK}(SB, \mathrm{Ker}\, q)$. This means that all basic properties of KK-theory follow from the existence of the product, entirely on the level of KK_0, using rather straightforward homotopy theory and the fact that a certain operator has index 1. In particular, all these properties can be developed naturally on the basis of the approach in [7], thus giving a consistent and complete treatment of KK-theory using quasihomomorphisms.

We outline here how one has to proceed. For the C^*-algebras connected with a point, open interval, half-open interval, closed interval and their duals we use the following notation: $P = \mathbf{C}$, $S = \mathscr{C}_0(\mathbf{R})$, $C = \mathscr{C}_0([0, 1))$, $I = C([0, 1])$, $\hat{P} = \mathcal{K}$, $\hat{S} = C^*(\mathbf{R})$, $\hat{C}$ and $\hat{I}$ as in Sections 4 and 3 (in the real case one uses, of course, the real C^*-algebras). To the exact sequences $0 \to S \to C \to P \to 0$ and $0 \to C \to I \to$

$\to P \to 0$ correspond the dual sequences $0 \to \hat{P} \to C \to S \to 0$ and $0 \to \hat{P} \to \hat{I} \to$ $\to \hat{C} \to 0$. Moreover, we write SA, $\hat{P}A$ etc ... for $S \otimes A$, $\hat{P} \otimes A$ etc ...

1. We say that a short exact sequence $\rho : 0 \to J \to A \to A/J \to 0$ of C^*-algebras is semi-split, if there exists a "complementary" exact sequence $0 \to \hat{P}J \to A^- \to$ $\to A/J \to 0$ such that, for the C^*-algebras of (formal) 2×2-matrices

$$E = \begin{pmatrix} A & J(\hat{P}J) \\ (\hat{P}J)J & A^- \end{pmatrix} \quad J_0 = \begin{pmatrix} J & J(\hat{P}J) \\ (\hat{P}J)J & \hat{P}J \end{pmatrix}$$

(we use here the natural embedding $J \hookrightarrow \hat{P}J$) and for $\pi : E \to E/J_0 \cong A/J \oplus A/J$, there exists a homomorphism $\varphi : A/J \to E$ such that $\pi\varphi(x) = (x, x)$ for $x \in A/J$.

REMARK. From the definition of E one sees that there are KK-equivalences (invertible elements) $E \sim A \oplus A/J$ and $E \sim A^- \oplus A/J$. Combining these, one obtains an invertible element in $\mathrm{KK}(A, A^-)$.

Note that, by Kasparov's generalized Stinespring theorem, ρ is semi-split iff the quotient map $A \to A/J$ admits a completely positive, cross-section. One can easily prove an analogue of Lemma 2.2 above, showing that $0 \to J_1 \cap J_2 \to \mathrm{B} \to$ $\to B/J_1 \cap J_2 \to 0$ is semi-split whenever $0 \to J_i \to B \to B/J_i \to 0$ are so for $i = 1, 2$. With a semi-split $\rho : 0 \to J \to A \to A/J \to 0$ we can, using the cross-section φ, associate a quasihomomorphism $\rho' = (\varphi, \overline{\varphi}) : A/J \to \hat{S}\hat{P}J$ writing $\varphi(x)(t) \equiv$ $\equiv \varphi(x)$, $\overline{\varphi}(x)(t) = F_t \varphi(x) F_t^{-1}$ where $x \in A/J$, $t \in [0, 1]$ and

$$F_t = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i t} \end{pmatrix}$$

(the real algebra $\hat{S}$ is viewed here as the algebra of continuous complex-valued functions f on $(0, 1)$ such that $f(1 - t) = \overline{f(t)}$). Of course, ρ' could depend on the choice of A^- and φ. We will see later that, up to homotopy, this is not the case.

2. One proves, as in Section 1 above, that the Puppe sequence induces an exact sequence in the second variable of KK.

3. The C^*-algebra $S\hat{S}$ can, in the complex case, be described as the algebra of complex-valued continuous functions on $\mathbf{C}$ that vanish at ∞, or, in the real case, as the subalgebra of functions f such that $f(\overline{z}) = \overline{f(z)}$, $z \in \mathbf{C}$. Let $L^2(S\hat{S})$ denote the corresponding Hilbert space of L^2-functions on $\mathbf{C}$, so that $S\hat{S}$ acts by multiplication on $L^2(S\hat{S})$. The Bott element $\beta : P \to S\hat{S}$ and its inverse $\alpha : S\hat{S} \to \hat{P}P$ are quasihomomorphisms that can be described as follows:

To define $\beta = (\varphi, \overline{\varphi})$, it suffices to give the pair of projections $p = \varphi(1)$, $\overline{p} = \overline{\varphi}(1)$.

We take, for $p, \bar{p}$, the 2×2-matrix-valued functions

$$p(z) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \bar{p}(z) = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*$$

with $W = \frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix} z & 1 \\ -1 & \bar{z} \end{pmatrix}$, $z \in \mathbf{C}$ which act by multiplication on $L^2(S\hat{S}) \oplus \oplus L^2(S\hat{S})$.

Remark. It is an immediate consequence of the definitions, that β is exactly the quasihomomorphism obtained as σ' for the semi-split extension $\sigma : 0 \to S \to \to C \to P \to 0$.

We define the quasihomomorphism $\alpha : S\hat{S} \to \hat{P}P$ as the pair $(\mu, \bar{\mu})$ of homomorphisms $S\hat{S} \to \mathscr{L}(L^2(S\hat{S}) \oplus L^2(S\hat{S}))$

$$\mu(f) = \begin{pmatrix} \mu_0(f) & 0 \\ 0 & 0 \end{pmatrix} \quad \bar{\mu}(f) = \hat{W}\begin{pmatrix} \mu_0(f) & 0 \\ 0 & 0 \end{pmatrix}\hat{W}^*$$

with $\hat{W} = \frac{1}{\sqrt{1-\partial\bar{\partial}}}\begin{pmatrix} \bar{\partial} & 1 \\ 1 & \partial \end{pmatrix}$ where $f \in S\hat{S}$, $\partial = \frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}y}$, $\bar{\partial} = \frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}y}$ and $\mu_0(f)$ is multiplication by f, cf. [12], p. 547.

To show that the composition $P \xrightarrow{\beta} S\hat{S} \xrightarrow{\alpha} \hat{P}P$ is equal to id_P in $KK(P, P)$, it suffices to show that the operator

$$\frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} + \frac{1}{\sqrt{1+z\bar{z}}\sqrt{1-\partial\bar{\partial}}}\begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}$$

on $L^2(S\hat{S}) \oplus L^2(S\hat{S})$, has index 1 (which follows from the fact that this operator is "half" of the Euler characteristic operator on the 2-sphere S^2, cf. [11], 2.6).

This is the weak form of Bott periodicity alluded to above (while its full form says that the other composition $S\hat{S} \xrightarrow{\alpha} \hat{P}P \xrightarrow{\beta} S\hat{S}$ gives the identity of $S\hat{S}$, too).

4. Let $0 \to J \to A \xrightarrow{q} A/J \to 0$ be semi-split. We want to show that the inclusion $e : J \to C_q$ is a KK-equivalence.

a) We define the element $u : C_q \to J$ as in Section 2 above: from the extension $\sigma : 0 \to SJ \to CA \to C_q \to 0$, one obtains $\sigma' : C_q \to \hat{S}SJ$ and one sets $u = \sigma'\alpha$. The remark under 3. shows that the composition $J \xrightarrow{e} C_q \xrightarrow{u} J$ gives id_J (the restriction of u to $J \subset C_q$ is equal to the composition $P \to \hat{S}S \to P$ tensored with id_J).

b) One uses a) to show, exactly as in Section 2 above, that $C_e \sim 0$, and in order to deduce that $Se : SJ \to SC_q$ is a KK-equivalence. But then also

$\hat{S}Se : \hat{S}SJ \to \hat{S}SC_q$ is a KK-equivalence. Since, by 3., the KK-groups of $\hat{S}SJ$ and $\hat{S}SC_q$ contain those of J and C_q as direct summands and since $\hat{S}Se$ respects this direct sum decomposition, one sees that e is invertible (with inverse u).

5. a) Combining 2. and 4., one obtains the long exact sequence in the second variable of KK.

b) Applying a) to the exact sequence $0 \to \hat{P} \to \hat{C} \to \hat{S} \to 0$ and using the fact that $\hat{C} \sim 0$ [8], one obtains full Bott periodicity.

c) Using Bott periodicity, one obtains, as in Section 1, exactness of the Puppe sequence in the first variable of KK, and, as a consequence, also the long exact sequence in the first variable.

d) The canonical inclusion map $SA/J \to C_q$ gives, for a semi-split $\rho : 0 \to J \to \to A \xrightarrow{q} A/J \to 0$ an element ρ'' of $\mathrm{KK}(SA/J, C_q) \cong \mathrm{KK}(SA/J, J)$. Under the Bott isomorphism

$$\mathrm{KK}(SA/J, J) \cong \mathrm{KK}(\hat{S}SA/J, \hat{S}J) \cong \mathrm{KK}(A/J, \hat{S}J),$$

this element corresponds to the element ρ' defined in 2..

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