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CARLESON MEASURES AND OPERATORS ON STAR-INVARIANT SUBSPACES

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INTRODUCTION

Let $\mathbf{D} = \{|z| < 1\}$ be the unit disk and let H^p denote the usual classes of functions analytic on $\mathbf{D}$. A function s in H^∞ is called inner if

$$|s(e^{i\theta})| = 1 \quad \text{a.e. } [d\theta].$$

In this paper we study operators on the subspace $K = H^2 \ominus \varphi H^2$ where φ belongs to a certain class of inner functions. Our main tools in this study are the results in [6] in which the *Carleson measures for* K were characterized under the hypothesis that φ satisfy the *connected level set condition*, that is, there exist r_0, $0 < r_0 < 1$, such that $\{|\varphi(z)| < r_0\}$ is connected.

Recall that a measure on the closed disk $\overline{\mathbf{D}}$ which assigns no mass to the singular support of φ is called a Carleson measure for K if there is a constant c such that

$$\int |f|^2 \, d\mu \leqslant c\|f\|_2^2$$

for all $f \in K$. See [6], p. 347. We will rely on Theorems 3.1 and 3.2 of [6], which characterize such measures. Although these theorems are difficult to state at this point we will record the following.

THEOREM 0. *Let φ satisfy the connected level set condition as above. Then the following conditions are equivalent:*

(i) *The measure μ is a Carleson measure for K.*

(ii) *There is a constant c such that*

$$\int \frac{1 - |z|^2}{|1 - \bar{\zeta} z|^2} \, d\mu(\zeta) \leqslant \frac{c}{1 - |\varphi(z)|}$$

for all $z \in \mathbf{D}$.

Note that when μ is supported on the unit circle $\mathbf{T}$, the integral in (ii) is the ordinary Poisson integral of μ.

As examples of inner functions satisfying the connected level set condition we give

$$\varphi_1(z) = \exp\left(\frac{z+1}{z-1}\right)$$

and

$$\varphi_2(z) = \prod_{k=1}^{\infty} \frac{(1-r^k)-z}{1-(1-r^k)z}$$

where $0 < r < 1$.It is true, of course, that "most" inner functions do not satisfy the connected level set condition. Thus it becomes an interesting, perhaps difficult, question as to whether our results extend to all inner functions s.

We now give a somewhat simplified statement of our main results. In all of our theorems, φ denotes an inner function satisfying the connected level set condition.

The main result of Section 1 is the following theorem.

THEOREM. *Let $f \in K$. Then there is a constant c such that*

$$\int_D |f'(z)|^2 \frac{1-|z|}{1-|\varphi(z)|} \, \mathrm{d}x \, \mathrm{d}y \leqslant c\|f\|_2^2 .$$

A weaker inequality was proven by Axler, Chang, and Sarason in [3] and used to study the compactness of certain Hankel operators. We discuss this in greater detail in Section 1.

In Section 2 we give an alternate characterization of Carleson measures for K in case the measure lives on the circle $\mathbf{T}$. Let P denote orthogonal projection onto K. The main result is the following.

THEOREM. *Let $\mathrm{d}\mu = u\mathrm{d}\theta$ where $u \geqslant 0$ and $u \in L^2$. Then the following conditions are equivalent:*

(i) *μ is a Carleson measure for K.*

(ii) *$u = v + \mathrm{Re}(\varphi h)$, where $v \in L^\infty$ and $h \in H^2$.*

(iii) *The operator $T_u(f) \equiv P(uf)$ defined on $K \cap H^\infty$ extends to a bounded operator on K.*

We relate this result to a theorem of Sarason on generalized interpolation in [10]. The same problem has also been discussed by Clark in [4].

In Section 3 we consider perturbations of orthonormal bases of K. Such matters have been discussed by Clark, [5], and Hruščev, Nikolskii and Pavlov,

[8]. We use our results to determine a sufficient condition that the inclusion operator

$$I : K \to L^2(\mathrm{d}\mu)$$

be bounded below if μ is a Carleson measure for K. This problem was solved by Volberg, [11], in case $\mu = w\mathrm{d}\theta$ where $w \in L^\infty$, for all inner functions s. Volberg's characterization, however, does not extend in case μ is simply a measure, as we will show.

We also consider the compactness of the inclusion operator $I : K \to L^2(\mathrm{d}\mu)$. This question was also treated by Volberg in [11] and solved for any inner function s in the case that $\mu = w\mathrm{d}\theta$ where $w \in L^\infty$. This time Volberg's theorem does extend in case μ is a measure and φ satisfies the connected level set condition, as the following theorem shows.

THEOREM. *Let μ be a Carleson measure for K supported on* $\mathbf{T}$. *Then the following conditions are equivalent:*

(i) $I : K \to L^2(\mathrm{d}\mu)$ *is compact.*

(ii) $\lim_{|z| \to 1} \hat{\mu}(z)(1 - |\varphi(z)|) = 0.$

Here, $\hat{\mu}(z)$ denotes the Poisson integral of μ.

We conclude the introduction with some remarks regarding the notation.

The constants c, c_0, c_1, etc., which appear in various theorems and proofs change their values each time they are used in a different context. The symbol $\|f\|_p$ denotes the H^p norm of a function f and the symbol $\|T\|$ denotes the operator norm of an operator T. The measures μ we consider are always positive, even if we do not specifically say so. Finally, if F and E are sets, then $F \setminus E$ denotes their set theoretic difference.

§ 1

We begin this section by establishing some notation and results which we will use throughout the rest of the paper.

Suppose φ is an inner function. As in [6], p. 349 we extend φ to be analytic on $\mathbf{C} \setminus K^*$, where K^* is the reflection of the singular support of φ. For $t > 0$ we define

$$D_t = \{z : \varphi \text{ is analytic at } z \text{ and } |\varphi(z)| < t\}.$$

For $0 < t_1 < t_2$ we define

$$A_{t_1,t_2} = \{z : \varphi \text{ is analytic at } z \text{ and } t_1 < |\varphi(z)| < t_2\}.$$

Suppose φ satisfies the connected level set condition for r_0, where $0 < r_0 < 1$, i.e. D_{r_0} is a connected set. By [6], Corollary 3.1, if $r_0 < r < 1$ and $\delta = r^{-1}$, then D_δ is simply connected. Let $\sigma : \mathbf{D} \to D_\delta$ be a Riemann map of the disk onto D_δ, and let ψ be its inverse.

For the remainder of the paper, φ will denote a fixed inner function satisfying the connected level set condition for r_0, as above. The symbols σ and ψ will be reserved for the univalent functions defined above. We come now to our first result.

LEMMA 1. *With φ, r and ψ as above, if z is in $A_{\frac{1+r}{2}, \frac{3-r}{2}}$ then*

(i) $$|\varphi'(z)| \leqslant r^{-1} \frac{|\psi'(z)|}{1 - |\psi(z)|^2} .$$

and

(ii) $$\frac{|\psi'(z)|}{1 - |\psi(z)|^2} \leqslant \frac{2}{1 - r} |\varphi'(z)|.$$

Proof. To prove (i) define $f : \mathbf{D} \to \mathbf{D}$ by $f(w) = r\varphi(\sigma(w))$. The Schwarz-Pick theorem yields

$$|f'(w)| \leqslant \frac{1 - |f(w)|^2}{1 - |w|^2}.$$

The chain rule now gives (i).

To prove (ii) let z be in $A_{\frac{1+r}{2}, \frac{3-r}{2}}$ and suppose $\varphi(z) = |\varphi(z)|e^{i\theta}$. Let N be the disk centered at $e^{i\theta}$ of radius $1 - r$. It follows from [6], Theorem 1.1, that there is a single valued branch of φ^{-1} defined on N for which $\varphi^{-1}(\varphi(z)) = z$. Define $f : N \to \mathbf{D}$ by $f(w) = \psi(\varphi^{-1}(w))$. Applying the Schwarz-Pick theorem again and observing that $|\varphi(z) - e^{i\theta}| < 1 - r/2$ yields

$$\frac{1 - r}{2} \frac{|\psi'(z)|}{|\varphi'(z)|} \leqslant 1 - |\psi(z)|^2$$

which proves (ii).

LEMMA 2. *Let φ and r be as above. Let z be in $A_{\frac{1+r}{2}, 1}$. Then there is a constant $c = c(r)$, independent of z for which*

$$\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \leqslant c(r)|\varphi'(z)|.$$

Proof. Let $\varphi(z) = |\varphi(z)|e^{i\theta}$ and $\zeta = re^{i\theta}$. Let L be a line tangent to the circle $\{z : |z| = r\}$ passing through ζ and let Ω be the simply connected region bounded by L and the unit circle containing $\varphi(z)$. Let $f : \Omega \to \mathbf{D}$ be a conformal map of Ω onto the disk which takes $\frac{1+r}{2}e^{i\theta}$ to 0. It is not difficult to see that

$$|f'(\varphi(z))| \leqslant M < \infty$$

for a constant M independent of z.

Let φ^{-1} be defined on Ω such that $\varphi^{-1}(\varphi(z)) = z$ and let g be the inverse of f. Define $h : \mathbf{D} \to \mathbf{D}$ by $h(w) = \varphi^{-1}(g(w))$. Applying the Schwarz-Pick theorem we obtain

$$\frac{|g'(w)|}{|\varphi'(z)|} \leqslant \frac{1-|z|^2}{1-|\varphi(z)|^2}.$$

Since $|g'(w)| \geqslant 1/M$, the lemma is proved.

We will need the next lemma, which is an easy consequence of [6], Theorem 3.1. For the remainder of the paper we let $K = H^2 \ominus (\varphi H^2)$, where φ is our fixed inner function. We will also use the symbol $dA(z)$ to denote area measure on the unit disk.

LEMMA 3. *Let φ and r be as above and suppose μ is the measure on $A_{r,1}$ defined by $d\mu = |\varphi'|\,dA$. Then μ is a Carleson measure for K.*

Proof. For each $e^{i\theta}$ construct $\Omega = \Omega_\theta$ as in Lemma 2. Since φ is a covering map on $A_{\frac{1+r}{2},1}$, it follows that $\varphi^{-1}(\Omega_\theta) = \bigcup_{n=1}^{\infty} \Gamma_{n,\theta}$ where $\Gamma_{n,\theta}$ is a connected open set and

$$\varphi : \Gamma_{n,\theta} \to \Omega_\theta$$

is a homeomorphism. Thus

$$\int_{\Gamma_{n,\theta}} |\varphi'|\,d\mu = \int_{\Gamma_{n,\theta}} |\varphi'|^2\,dA = \text{area of } \Omega_\theta \leqslant 2\pi.$$

Theorem 3.2 of [6] implies that μ is a Carleson measure for K. See also the remarks prior to Corollary 5 in Section 3.

We are ready to prove the main result of this section.

THEOREM 1. *Let f be a function in K. Then*

$$\int_D |f'(z)|^2 \frac{1-|z|}{1-|\varphi(z)|}\,dA(z) \leqslant c\|f\|_2^2$$

for a constant c independent of f.

Proof. Since $f \in H^2$ it is enough to show that (see [7], p. 237)

$$\int_{\mathbf{D} \setminus D_r} |f'(z)|^2 \frac{1 - |z|}{1 - |\varphi(z)|} \, dA \leqslant c \|f\|_2^2$$

for some r, $0 < r_0 < r < 1$. By the Schwarz-Pick theorem this will be true if

$$\int_{\mathbf{D} \setminus D_r} |f'(z)|^2 |\varphi'(z)|^{-1} dA \leqslant c \|f\|_2^2$$

for all f in K.

Let $g(w) = f(\sigma(w))\sigma'(w)^{1/2}$. It is shown in [6], p. 358, that since $f \in K$, $g \in H^2$ and $\|g\|_2^2 \leqslant c_1 \|f\|_2^2$. It follows that

$$I = \int_{\mathbf{D}} |g'(w)|^2 (1 - |w|) \, dA(w) \leqslant c_2 \|f\|_2^2.$$

Now calculate that

$$g'(w) = f'(\sigma(w))\sigma'(w)^{3/2} + f(\sigma(w)) \cdot \frac{1}{2} \sigma'(w)^{-1/2} \sigma''(w).$$

Let

$$I_1 = \int_{\psi(\mathbf{D} \setminus D_r)} |f(\sigma(w))|^2 \frac{|\sigma''(w)|^2}{|\sigma'(w)|} (1 - |w|) dA(w).$$

Since σ is univalent it follows from [9], p. 689, that

$$\frac{|\sigma''(w)|}{|\sigma'(w)|} \leqslant \frac{c}{1 - |w|}$$

and therefore

$$I_1 \leqslant c \int_{\psi(\mathbf{D} \setminus D_r)} |f(\sigma(w))|^2 \frac{|\sigma'(w)|}{1 - |w|} \, dA(w).$$

Changing variables yields

$$I_1 \leqslant c \int_{\mathbf{D} \setminus D_r} |f(z)|^2 \frac{|\psi'(z)|}{1 - |\psi(z)|} \, dA(z) \leqslant$$

$$\leqslant c \int_{\mathbf{D} \setminus D_r} |f(z)|^2 |\varphi'(z)| \, dA(z),$$

where we have used Lemma 1. An application of Lemma 3 shows that

$$I_1 \leqslant c\|f\|_2^2.$$

Now define

$$I_2 = \int_{\psi(\mathbf{D}\setminus D_r)} |f'(\sigma(w))|^2 |\sigma'(w)|^3 (1 - |w|)\, \mathrm{d}A(w).$$

Since $I_2 \leqslant 2(|I_1| + |I|)$ it follows that

$$I_2 \leqslant c_3\|f\|_2^2.$$

Changing variables yields

$$I_2 = \int_{\mathbf{D}\setminus D_r} |f'(z)|^2 \frac{1 - |\psi(z)|}{|\psi'(z)|}\, \mathrm{d}A(z) \geqslant$$

$$\geqslant c \int_{\mathbf{D}\setminus D_r} |f'(z)|^2 |\varphi'(z)|^{-1}\, \mathrm{d}A(z),$$

where we have again used Lemma 1. This proves the theorem.

As a consequence of Theorem 1 we see that there is a constant c such that

$$\int_{\{|\varphi| > r\}} |f'(z)|^2 (1 - |z|)\, \mathrm{d}A \leqslant c(1 - r)\|f\|_2^2$$

for $r > r_0$, provided $f \in K$. In [3], Lemma 5, page 292, Axler, Chang and Sarason prove a weaker inequality, which is valid for all inner functions s, that is

$$\int_{\{|s| > r\} \cap \{|z| > 1/2\}} |f'(z)|^2 (1 - |z|)\mathrm{d}A \leqslant c(1 - r)^{\gamma}\|f\|_2^2$$

provided $f \in (sH^2)^{\perp}$.

Here, γ is a constant which is less than 1.

It is not difficult to show that this means that for $f \in (sH^2)^{\perp}$

$$\int_{\mathbf{D}} |f'(z)|^2 \frac{1 - |z|}{(1 - |s(z)|)^p}\, \mathrm{d}A < c\|f\|_2^2$$

for a constant p between 0 and 1. We have not, however, been able to prove this with $p = 1$ for the case of an arbitrary inner function s.

Theorem 1 has a corollary which we pursue next. Let $h \in H^2$. Call h a *multiplier* for K if $fh \in H^2$ for all $f \in K$. The closed graph theorem shows that h is a multiplier if and only if

$$\|fh\|_2^2 \leqslant 2\|f\|_2^2$$

for a constant c independent of f. Let $u = |h|^2$ be defined on the unit circle. Clearly, h is a multiplier for K if and only if

$$u(z) \leqslant \frac{c}{1 - |\varphi(z)|}$$

for some constant c, since φ satisfies the connected level set condition.

It is well known that if h is analytic on $\mathbf{D}$, then $hf \in H^2$ for all $f \in H^2$ if and only if $h \in H^\infty$. Furthermore, if $h \in H^\infty$ then according to [7], Theorem 3.4, p. 240, $d\mu = |h'(z)|^2(1 - |z|)dxdy$ is a Carleson measure for H^2. Theorem 1 allows us to prove the following analogue of the above result.

COROLLARY 1. *Suppose h is a multiplier for K. Then $|h'(z)|^2(1 - |z|)dxdy$ is a Carleson measure for K.*

Proof. Since h is a multiplier for K it follows that

$$|h(z)|^2 \leqslant u(z) \leqslant \frac{c}{1 - |\varphi(z)|}.$$

Let I_1 and I_2 be the integrals

$$I_1 = \int_{\mathbf{D}} |h'f + f'h|^2(1 - |z|)\, dA(z)$$

and

$$I_2 = \int_{\mathbf{D}} |f'(z)h(z)|^2(1 - |z|)\, dA(z).$$

Using the preceding inequality combined with Theorem 1, and the fact that $\|fh\|_2^2 \leqslant \leqslant c\|f\|_2^2$ we see that

$$I_j \leqslant c\|f\|_2^2$$

for $j = 1,2$.

Since

$$\int_{\mathbf{D}} |f(z)h'(z)|^2(1 - |z|)\, dA(z) \leqslant 2I_1 + 2I_2$$

the theorem is proved.

§2

In this section we prove a result related to a theorem of Sarason. We also provide an alternative characterization of when a positive function $u \in L^2$ defines a Carleson measure for $(\varphi H^2)^\perp$ if φ satisfies the connected level set condition.

Suppose s is an arbitrary inner function. Let $P : L^2 \to (sH^2)^\perp$ be orthogonal projection. In [10], Theorem 1, p. 179, Sarason obtained the following result.

THEOREM A. *Let* $g \in H^2$ *and define*

$$T_g(f) = P(gf)$$

where $f \in H^\infty \cap (sH^2)^\perp$. *The* T_g *extends to a bounded operator on* $(sH^2)^\perp$ *if and only if* $g = b + sh$, *where* $b \in H^\infty$. *Furthermore,* b *may be chosen so*

$$\|b\|_\infty = \|T_g\|.$$

REMARK. Theorem A follows from Sarason's theorem as stated in [10] since T_g is in the commutant of $S = T_z$.

Now let $u \in L^2$ be a positive function. We may define T_u on $H^\infty \cap K$ to be

$$T_u(f) = P(uf),$$

where P denotes orthogonal projection onto K. We wish to determine when T_u extends to a bounded operator on K. Let B be the unit ball in K. Since for $f \in H^\infty \cap B$,

$$\|T_u(f)\| = \sup_{g\in B} |\langle T_u(f), g\rangle| = \sup_{g\in B} \left|\int u f \bar{g}\, d\theta\right| \leqslant$$

$$\leqslant \sup_{g\in B} \left[\int |f|^2 u\, d\theta\right]^{1/2} \left[\int |g|^2 u\, d\theta\right]^{1/2}$$

and since

$$\|T_u(f)\| \geqslant \langle T_u f, f\rangle = \int |f|^2 u\, d\theta,$$

it is clear that T_u extends to a bounded operator on K if and only if $u d\theta$ is a Carleson measure for K.

We now prove our main result.

THEOREM 2. *If* u *is a positive function in* L^2 *then the following conditions are equivalent:*

(i) T_u *extends to a bounded operator on* K.

(ii) *The measure* $u d\theta$ *is a Carleson measure for* K.

(iii) *There are functions* $v \in L^\infty$ *and* $h \in H^2$ *such that* $u = \operatorname{Re}(v + \varphi h)$.

Proof. We must show the equivalence of (ii) and (iii). One direction is simple. If $f \in H^\infty \cap K$ then if $u = \mathrm{Re}(v + \varphi h)$

$$\int |f|^2 u \, d\theta = \int |f|^2 \, \mathrm{Re}(v) \, d\theta \leqslant \|v\|_\infty \|f\|_2^2$$

and thus (iii) implies (ii).

Now assume condition (ii) holds. Since φ satisfies the connected level set condition we may apply [6], Theorem 3.1 to deduce that $\nu = u(\sigma(w))|dw|$ defined on $\psi(T)$ is a Carleson measure for $\mathbf{D}$. Furthermore, it follows that if B is the ball in H^2, then there is a constant c such that

$$\sup_{g \in B} \int |g|^2 \, d\nu \leqslant c\|T_u\|$$

since $\|T_u\|$ is essentially the "Carleson constant for K" norm of $u d\theta$.

Let $f \in K$. If $F = f(\sigma)\sigma'$, then $F \in H^1$ and $\|F\|_1 \leqslant c_1\|f\|$; this may be shown using the arguments of [6], page 358. It follows from factorization and changing variables that

$$\int_{\mathbf{T}} |f| u \, d\theta = \int_{\psi(\mathbf{T})} |F| \, d\nu \leqslant c\|T_u\| \, \|F\|_1 \leqslant cc_1\|T_u\| \, \|f\|_1. \tag{2.1}$$

Thus if X denotes the closure in H^1 of K, then $\Lambda_u(f) = \int fu \, d\theta$ defines a bounded linear functional on X. By the Hahn-Banach theorem there is a function $v_1 \in L^\infty$ such that

$$\Lambda_u(f) = \int f v_1 \, d\theta.$$

It follows that $u - v_1 \in L^2 \ominus K$. Since $L^2 \ominus K = \overline{H_0^2} \oplus \varphi H^2$, we have

$$u - v_1 = \overline{g_1 e^{i\theta}} + \varphi g_2.$$

Observe that

$$g_1 e^{i\theta} = u - \overline{v}_1 - \overline{\varphi g_2}$$

and thus

$$T_{g_1 e^{i\theta}} = T_u - T_{\overline{v}_1}.$$

Therefore $T_{g_1 e^{i\theta}}$ extends to a bounded operator on K. Using Theorem A above we see that

$$g_1 e^{i\theta} = b + \varphi g_3,$$

where $b \in H^\infty$ and $g_3 \in H^2$.

Thus

$$u = v_1 + b + \varphi g_3 + \overline{\varphi g_2}$$

and since u is real, the proof is complete.

As a corollary of the proof we can relate $\|T_u\|$ and $\|v\|_\infty$.

COROLLARY 2. *Suppose u is a positive function in L^2 for which T_u extends to a bounded operator on K. Then there is a function $v \in L^\infty$ such that:*

(i) $T_u = T_v$ *and*

(ii) $\|T_u\| \leqslant \|v\|_\infty \leqslant c\|T_u\|$ *where c is an absolute constant.*

Proof. We choose the v_1 of the proof of Theorem 2 so that $\|v_1\|_\infty = \|A_u\|$ and the b of the proof so that $\|b\|_\infty = \|T_u - T_{\bar{v}_1}\|$. Then

$$\|v\|_\infty = \|v_1 + b\|_\infty \leqslant \|v_1\|_\infty + \|b\|_\infty =$$

$$= \|A_u\| + \|T_u - T_{\bar{v}_1}\| \leqslant \|A_u\| + \|T_u\| + \|T_{\bar{v}_1}\| =$$

$$= 2\|A_u\| + \|T_u\| \leqslant c\|T_u\|,$$

where for the last inequality we have used equation (2.1). This completes the proof.

3. PERTURBATION OF ORTHONORMAL BASES

Let s be an arbitrary inner function, $f \in (sH^2)^\perp$ and suppose that for a complex number ζ the mapping

$$\Lambda_\zeta(f) = f(\zeta)$$

defines a bounded linear functional on $(sH^2)^\perp$. This happens, of course, if $|\zeta| < 1$, in which case

$$\Lambda_\zeta(f) = \frac{1}{2\pi}\int_{\mathbf{T}} f(e^{i\theta})\,\frac{1 - s(\zeta)\overline{s(e^{i\theta})}}{1 - \zeta e^{-i\theta}}\,d\theta$$

and

$$K(z, \zeta) = \sqrt{\frac{1-|\zeta|^2}{1-|s(\zeta)|^2}}\,\frac{1-\overline{s(\zeta)}s(z)}{1-\bar{\zeta}z}$$

is the normalization of the reproducing kernel for ζ in $(sH^2)^\perp$.

It can happen that A_ζ is a bounded linear functional on $(sH^2)^\perp$ even if $|\zeta| \geqslant 1$. For the case $|\zeta| = 1$, see [2] for details. We will use the notation $K(z, \zeta)$ to denote the normalized reproducing kernel in $(sH^2)^\perp$ for ζ. In particular, if $|\zeta| = 1$ we have

$$K(z, \zeta) = \frac{1}{|s'(\zeta)|^{1/2}}\,\frac{1-\overline{s(\zeta)}s(z)}{1-\bar{\zeta}z}.$$

Let $\alpha \in T$ and let $E_\alpha = s^{-1}(\alpha)$. Then E_α is a set of Lebesgue measure 0 and the measure ν_α defined by the equation

$$\frac{1-|s(z)|^2}{|\alpha - s(z)|^2} = \frac{1}{2\pi}\int_{\mathbf{T}} \frac{1-|z|^2}{|\eta - z|^2}\, d\nu_\alpha(\eta)$$

lives on E_α. We can now state the following theorem due to Clark, [5], pp. 176—178.

THEOREM B. *Let s be inner and $\alpha \in \mathbf{T}$. Then the following conditions are equivalent:*

(i) *The measure ν_α is purely atomic.*
(ii) *The set $\{K(z, \zeta) : \zeta \in E_\alpha\}$ is an orthonormal basis for $(sH^2)^\perp$.*

Observe that if $\{K(z, \zeta) : \zeta \in E_\alpha\}$ is an orthonormal basis for $(sH^2)^\perp$ then it follows that the map $S : (sH^2)^\perp \to \ell^2$ defined by

$$Sf = \{f(\zeta_k)|s'(\zeta_k)|^{-1/2}\}$$

is one-to-one and onto; here we have written $E_\alpha = \{\zeta_k\}_{k=1}^\infty$. In this case Parseval's equality becomes

$$\|f\|_2^2 = \sum_{k=1}^{\infty} |f(\zeta_k)|^2 |s'(\zeta_\alpha)|^{-1}.$$

These observations are also due to Clark.

We now assume that $s = \varphi$. Since φ satisfies the connected level set condition, the covering map nature of φ makes it clear that for $\alpha \in \mathbf{T}$ E_α is countable and ν_α is purely atomic. If $E_\alpha = \{\zeta_k\}_{k=1}^\infty$ then $\{K(z, \zeta_k) : k = 1, 2, \ldots\}$ is an orthogonal basis for K. Suppose ζ_k' is a small perturbation of ζ_k. It is natural to inquire when $\{K(z, \zeta_k')\}$ is an unconditional basis for K, that is when every $f \in K$ has the unique representation

$$f = \sum a_k K(z, \zeta_k')$$

and when there are constants c_0 and c_1 such that

$$c_0\|f\|_2^2 \leqslant \sum |a_k|^2 \leqslant c_1\|f\|_2^2.$$

This problem has been considered in [5] and [8], where it is mentioned that if ζ'_k is close enough to ζ_k to satisfy

$$\sum_k \|K(\ ;\zeta_k) - K(\ ;\zeta'_k)\|^2 < 1$$

then $\{K(z, \zeta'_k)\}$ is an unconditional basis. With the assumption that φ satisfies the connected level set condition we will show that there is an $\varepsilon_0 > 0$ such that if

$$|\varphi(\zeta_k) - \varphi(\zeta'_k)| < \varepsilon_0 \quad \text{for all } k$$

and that ζ_k and ζ'_k belong to the same component of $\varphi^{-1}(\{w: |w - \alpha| < \varepsilon_0\})$ then $\{K(z, \zeta'_k)\}$ is an unconditional basis for K. (We actually state this condition somewhat differently.) We first prove the following result.

THEOREM 3. *Let $E_\alpha = \{\zeta_k\}_{k=1}^{\infty}$. There is an $\varepsilon_0 > 0$ such that if for each k there is a path (ζ_k, ζ'_k) connecting ζ_k and ζ'_k for which*

$$\int_{(\zeta_k,\, \zeta'_k)} |\varphi'(z)|\, |dz| < \varepsilon_0$$

for all k, then for absolute constants c_0 and c_1,

$$c_0\|f\|_2^2 \leqslant \sum \frac{|f(\zeta'_k)|^2}{|\varphi'(\zeta'_k)|} \leqslant c_1\|f\|_2^2$$

for all $f \in K$.

Proof. Let $\sigma : \mathbf{D} \to D_\delta$ and $\psi : D_\delta \to \mathbf{D}$ be as in Section 1. Recall that if

$$g(w) = f(\sigma(w))\sigma'(w)^{1/2}$$

for $f \in K$, then $g \in H^2$ and

$$\|g\|_2^2 \leqslant c\|f\|_2^2$$

for some constant c.

Let $w_k = \psi(\zeta_k)$. Changing variables in Parseval's equality yields

$$\|f\|_2^2 = \sum \frac{|g(w_k)|^2}{|\sigma'(w_k)|\, |\varphi'(w_k)|}$$

and using Lemma 1 we get constants m and M such that

$$m\|f\|_2^2 \leqslant \sum |g(w_k)|^2(1 - |w_k|) \leqslant M\|f\|_2^2.$$

Since the map $s = \delta^{-1}\varphi(\sigma)$ is inner and satisfies the connected level set condition, it follows from [6], Lemma 2.1, that $\{w_k\}$ is uniformly separated. Moreover, since $S : K \to \ell^2$ is one-to-one and onto, it follows that the map $T : K \to \ell^2$ defined by

$$T(f) = \{g(w_k)\sqrt{1 - |w_k|}\}$$

is also one-to-one and onto.

Since $\{w_k\}$ is uniformly separated, it is not difficult to show that, given $\varepsilon > 0$ there is a τ so small that if for each k

$$\left| \frac{w_k - w_k'}{1 - \overline{w}_k w_k'} \right| < \tau$$

then

$$\sum |g(w_k) - g(w_k')|^2(1 - |w_k|) \leqslant \varepsilon \|g\|_2^2$$

for all $g \in H^2$.

It follows that if $T' : K \to \ell^2$ is defined to be

$$T'f = \{g(w_k')\sqrt{1 - |w_k|}\}$$

then

$$\|T' - T\|^2 \leqslant c \cdot \varepsilon$$

and by taking ε sufficiently small it follows that T' is also one-to-one and onto. Since $\left| \frac{w_k - w_k'}{1 - \overline{w}_k w_k'} \right| < \tau$, it follows that

$$0 < \gamma_1 < \frac{1 - |w_k|}{1 - |w_k'|} < \gamma_2 < \infty$$

for constants γ_1 and γ_2 independent of k. Therefore, $T'' : K \to \ell^2$ defined by

$$T''f = \{g(w_k')\sqrt{1 - |w_k'|}\}$$

is also one-to-one and onto. Now let $\zeta_k' = \sigma(w_k')$. Changing variables we see that

$$T''f = \left\{ \frac{f(\zeta_k')(1 - |\psi(\zeta_k')|)}{\psi'(\zeta_k')^{1/2}} \right\}.$$

Thus, another use of Lemma 1 shows that the operator $S' : K \to \ell^2$ defined by

$$S'f = \left\{ \frac{f(\zeta_k')}{|\varphi'(\zeta_k')|^{1/2}} \right\}$$

is one-to-one and onto. The open mapping theorem now gives constants c_0 and c_1 such that

$$c_0\|f\|_2^2 \leqslant \sum \frac{|f(\zeta_k')|^2}{|\varphi'(\zeta_k')|} \leqslant c_1\|f\|_2^2.$$

To complete the proof we note that the condition

$$\left|\frac{w_k - w_k'}{1 - \bar{w}_k w_k'}\right| < \tau$$

is equivalent to

$$\int_{(w_k, w_k')} \frac{|dw|}{1 - |w|} < \tanh^{-1}(\tau)$$

for some path connecting w_k and w_k'. Changing variables yields the equivalent condition

$$\int_{(\zeta_k, \zeta_k')} \frac{|\psi'(z)|\,|dz|}{1 - |\psi(z)|} < \tanh^{-1}(\tau).$$

A final application of Lemma 1 finishes the argument.

As a corollary of the proof we have the following result.

COROLLARY 3. *Let* $\{\zeta_k'\}$ *satisfy*

$$\int_{(\zeta_k, \zeta_k')} |\varphi'|\,|dz| < \varepsilon_0$$

where $\{\zeta_k\} = E_\alpha$ *and* ε_0 *is as in Theorem* 3. *Then* $S'f = \{f(\zeta_k')|\varphi'(\zeta_k')|^{-1/2}\}$ *is an iso morphism of* K *onto* ℓ^2.

COROLLARY 4. *Let* $\{\zeta_k'\}$ *be as in the above corollary. Then* $\{K(z, \zeta_k')\}_{k=1}^{\infty}$ *is an unconditional basis for* K.

Proof. Let

$$J_k(z) = \frac{1 - \overline{\varphi(\zeta_k')}\varphi(z)}{1 - \bar{\zeta}_k' z}$$

be the unnormalized reproducing kernel for ζ_k' in K. Then

$$\|J_k\|^2 = \begin{cases} \dfrac{1 - |\varphi(\zeta_k')|^2}{1 - |\zeta_k'|^2}, & |\zeta_k'| \neq 1 \\[2ex] |\varphi'(\zeta_k')|, & |\zeta_k'| = 1. \end{cases}$$

Using the restriction on the location of $\{\zeta_k'\}$, if $|\zeta_k'| \leqslant 1$, we see

$$|\varphi'(\zeta_k')| \leqslant \frac{1 - |\varphi(\zeta_k')|^2}{1 - |\zeta_k'|^2} \leqslant c(r)\,|\varphi'(\zeta_k')|$$

where we have used the Schwarz-Pick theorem, and Lemma 2. It is a simple matter to use the equation $\varphi(z)\varphi(1/\bar{z}) = 1$ to show that the above inequality remains valid if $|\zeta_k'| > 1$.

Since $K(z, \zeta_k') = J_k/\|J_k\|$ it follows from Theorem 3 and the above observations that the linear operator $W: K \to \ell^2$ defined by

$$Wf = \{\langle f, K(\ ;\zeta_k')\rangle\}$$

is an isomorphism of K onto ℓ^2. Let

$$e_n = \{\delta_{kn}\}_{k=1}^{\infty}$$

where δ_{kn} is the Kronecker delta. Since W is an isomorphism there are functions $E_n(z)$ in K such that $WE_n = e_n$. Thus

$$\langle E_n, K(\ ;\zeta_k')\rangle = \delta_{kn},$$

that is, $\{E_n\}$ and $\{K(\ ;\zeta_k')\}$ are biorthogonal systems. Since $\{e_n\}$ is an orthogonal basis for ℓ^2 and W is an isomorphism, it follows that $\{E_n\}$ is an unconditional basis for K. Since $\{E_n\}$ and $\{K(\ ;\zeta_k')\}$ are biorthogonal, the proof is complete, see [12] pages 28—29 for exact details.

REMARK. In [1], Ahern and Clark construct an isometry $V: L^2(d\sigma) \to (sH^2)^\perp$ which maps $L^2(d\sigma)$ onto $(sH^2)^\perp$; here $d\sigma$ depends on the inner function s. In the case that $s(z) = \exp\left(\frac{z+1}{z-1}\right)$, the atomic inner function, $d\sigma$ is the Lebesgue measure on [0, 1]. Furthermore if x is a complex number such that

$$\frac{\zeta + 1}{\zeta - 1} = ix$$

then

$$V(e^{ixt}) = cK(z, \zeta)$$

where c is a constant depending on ζ.

Observe that $\exp\left(\frac{z+1}{z-1}\right)$ satisfies the connected level set condition. Thus we have a correspondence between orthonormal bases:

$$\{e^{-2\pi int}\}_{n=-\infty}^{\infty} \sim \{K(z, \zeta_n) : \zeta_n \in E_1\}$$

where $\frac{\zeta_n + 1}{\zeta_n - 1} = -2\pi in$. Any perturbation theorem about one basis yields a theorem about the other basis. Theorem 3 shows therefore that if L is sufficiently small and

$$|2\pi n - x_n| < L$$

for all n, then $\{e^{ix_n t}\}$ is an unconditional basis for $L^2[0, 1]$. This was originally proved by Paley and Wiener. See [12], Chapter 1, pages 42—44 for more details and historical comments.

For other φ's, Theorem 3 combined with the isometry of Ahern and Clark will yield perturbation of basis theorems for particular $L^2(\mathrm{d}\sigma)$ spaces.

We now use Theorem 3 to give a sufficient condition that the inclusion operator $If = f$ which maps K into $L^2(\mathrm{d}\mu)$ be bounded below if μ is a Carleson measure for K. Suppose $E_\alpha = \{\zeta_k\}_{k=1}^{\infty}$ and let ε_0 be as in Theorem 3. For each k let N_k denote the component of $\varphi^{-1}(\{z : |z - \alpha| < \varepsilon_0\})$ containing ζ_k. We observe that the N_k are pairwise disjoint.

Since μ is a measure which lives on $\bigcup_{k=1}^{\infty} N_k$, it follows from [6], Theorem 3.2 that μ is a Carleson measure for K if and only if

$$\int_{N_k} |\varphi'| \,\mathrm{d}\mu < c$$

for a constant c independent of k. The sufficiency of this condition may be seen directly from Theorem 3, for if $f \in K$,

$$\int |f|^2 \,\mathrm{d}\mu = \sum_k \int_{N_k} |f|^2 \,\mathrm{d}\mu =$$

$$= \sum_k \frac{|f(\xi_k)|^2}{|\varphi'(\xi_k)|} \int_{N_k} |\varphi'| \,\mathrm{d}\mu \leqslant c \cdot \sum_k \frac{|f(\xi_k)|^2}{|\varphi'(\xi_k)|}$$

where $\xi_k \in N_k$. The condition that $\xi_k \in N_k$ is just the hypothesis of Theorem 3, that is

$$\int_{(\xi_k,\, \zeta_k)} |\varphi'|\, |\mathrm{d}z| < \varepsilon_0 .$$

Thus

$$\int |f|^2\, \mathrm{d}\mu \leqslant c \cdot c_1 \|f\|_2^2$$

and μ is a Carleson measure for K.

As corollary of this argument we have the following result.

COROLLARY 5. *Let μ be a positive measure which is a Carleson measure for K. Suppose there is a constant $c > 0$ such that*

$$\int_{N_k} |\varphi'|\, \mathrm{d}\mu > c$$

for all k. Then $I : K \to L^2(\mathrm{d}\mu)$ is bounded below.

Proof. Let $f \in K$. Then

$$\int |f|^2\, \mathrm{d}\mu \geqslant \sum_k \int_{N_k} |f|^2\, \mathrm{d}\mu =$$

$$= \sum_k \frac{|f(\xi_k)|^2}{|\varphi'(\xi_k)|} \int_{N_k} |\varphi'| \mathrm{d}\mu \geqslant c \sum \frac{|f(\xi_k)|^2}{|\varphi'(\xi_k)|}$$

where ξ_k is a point in N_k. An application of Theorem 3 shows that

$$\|If\|^2 = \int |f|^2 \mathrm{d}\mu \geqslant c \|f\|_2^2$$

and the proof is complete.

In [11] Volberg proved the following theorem.

THEOREM C. *Let s be inner and $w \geqslant 0$ be in L^∞. Then the following conditions are equivalent:*

(i) *$I : H^2 \ominus sH^2 \to L^2(w\mathrm{d}\theta)$ is bounded below.*

(ii) $\inf_{z \in \mathbf{D}} [w(z) + |s(z)|] > 0.$

We give an example which shows that this result does not hold if we replace w by μ, a Carleson measure for K with support on $\mathbf{T}$.

For any $\alpha \in \mathbf{T}$ let $\mu_1 = \nu_\alpha$, i.e.

$$\mu_1 = \sum_{k=1}^{\infty} \frac{1}{|\varphi'(\zeta_k)|} \delta_{\zeta_k},$$

where $E_\alpha = \{\zeta_k\}_{k=1}^{\infty}$. It is easy to see that if $\hat{\mu}_1$ is the Poisson integral of μ_1,

$$\hat{\mu}_1(z) + |\varphi(z)| = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} + |\varphi(z)|$$

and

$$\inf_{z \in \mathbf{D}} [\hat{\mu}_1(z) + |\varphi(z)|] > 0.$$

Let $\mu = \mu_1 - \dfrac{1}{|\varphi'(\zeta_1)|} \delta_{\zeta_1}$. Since μ_1 lives on D_δ and since

$$\hat{\mu}_1(z) = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} \leqslant \frac{2}{1 - |\varphi(z)|},$$

μ_1 is a Carleson measure for K, by Theorem 0 of the introduction. Thus μ is also a Carleson measure for K. It is simple to verify that the condition

$$\inf_{z \in \mathbf{D}} [\hat{\mu}(z) + |\varphi(z)|] > 0$$

still holds. However, if

$$f(z) = K(z, \zeta_1)$$

then $f \in K$, $\|f\|_2^2 = 1$, but

$$\int |f|^2 \, \mathrm{d}\mu = 0.$$

Thus condition (ii) in Volberg's theorem is no longer *sufficient* to imply condition (i), in case w is replaced by a measure.

We remark that condition (ii) is always a *necessary* consequence of condition (i), even if w is replaced by a Carleson measure for $(sH^2)^\perp$.

Suppose μ is a Carleson measure for K. We next consider the question of when the inclusion operator $I : K \to L^2(\mathrm{d}\mu)$ is compact.

In case $\mu = w\mathrm{d}\theta$ for a positive function $w \in L^\infty$ this problem has been solved by Volberg, [11] for arbitrary inner functions s. His result is the following.

THEOREM D. *The following conditions are equivalent:*

(i) $I : (sH^2)^\perp \to L^2(w\mathrm{d}\theta)$ *is compact.*

(ii) $\lim_{|z|\to 1} w(z)(1 - |s(z)|) = 0.$

Here, $w(z)$ is the harmonic extension of w to $\mathbf{D}$ given by the Poisson integral.

We will be able to extend Volberg's theorem to arbitrary Carleson measures for K supported on $\mathbf{T}$. Of course we are also using the additional assumption that φ satisfies the connected level set condition. Our first result is in terms of the behavior of μ on "preimages" of the circle $\mathbf{T}$.

Let $\varphi^{-1}(\mathbf{T})=\bigcup_{n=1}^{\infty} I_n$ where each I_n is a connected half-open arc and $\varphi : I_n \to \mathbf{T}$ is a homeomorphism.

THEOREM 4. *Let μ be a Carleson measure for K supported on the circle $\mathbf{T}$. Then the following conditions are equivalent:*

(i) $I : K \to L^2(\mathrm{d}\mu)$ *is compact.*

(ii) $\lim_{n\to\infty} \int_{I_n} |\varphi'|\, \mathrm{d}\mu = 0.$

Proof. We first show that (ii) implies (i). It is enough to do this under the assumption that μ lives on a union of arcs $\bigcup (\zeta_k, \xi_k)$ where $\zeta_k \in E_\alpha$, $\xi_k \in E_\beta$ and

$$\int_{(\zeta_k, \xi_k)} |\varphi'|\, \mathrm{d}\theta < \varepsilon_0,$$

since any μ has a decomposition as the sum of finitely many such measures.

Let μ_N be the restriction of μ to $\bigcup_{k=1}^{N} (\zeta_k, \xi_k)$. Observe that the support of μ_N is contained in D_δ and that functions in K belong to $E^2(D_\delta)$; see [6], Lemma 3.1 for details. It is therefore easy to see that the inclusion operator $I_N : K \to L^2\,(\mathrm{d}\mu_N)$ is compact.

Furthermore,

$$\int |f|^2(\mathrm{d}\mu - \mathrm{d}\mu_N) = \sum_{k>N} \int_{(\zeta_k, \xi_k)} |f|^2\, \mathrm{d}\mu \leqslant \sum_{k>N} \frac{|f(z_k)|^2}{|\varphi'(z_k)|} \int_{(\zeta_k, \xi_k)} |\varphi'|\mathrm{d}\mu,$$

where z_k is some point in (ζ_k, ξ_k). Using Theorem 3 we see

$$\|I - I_N\|^2 \leqslant c \cdot \sup_{k>N} \int_{(\zeta_k, \xi_k)} |\varphi'|\, \mathrm{d}\mu$$

and thus I is the uniform limit of compact operators. This establishes that (ii) implies (i).

We now show that (i) implies (ii). Restrict z to be on the level set $\{|\varphi(z)| = r\}$ where r is as in Section 1. If $I : K \to L^2(d\mu)$ is compact, it follows that

$$\lim_{|z|\to 1} \|I(K(\ ;z))\| = 0.$$

Condition (ii) now follows from Lemma 3.2 of [6].

Now let $\hat{\mu}(z)$ denote the Poisson integral of μ.

COROLLARY 6. *The following conditions are equivalent:*

(i) $I : K \to L^2(d\mu)$ *is compact.*

(ii) $\lim_{|z|\to 1} \hat{\mu}(z)(1 - |\varphi(z)|) = 0.$

Proof. We must show condition (ii) of the corollary is equivalent to condition (ii) of the theorem.

First observe that condition (ii) above is an easy necessary condition for compactness as shown by Volberg in [11], page 475, even in the case where $\hat{\mu}$ is the Poisson integral of a measure.

We need only show then that condition (ii) above implies condition (ii) of Theorem 4. For this, again restrict z to the level set $\{|\varphi(z)| = r\}$. Then it follows that

$$\lim_{|z|\to 1} \hat{\mu}(z) = 0.$$

Condition (ii) of Theorem 4 now follows exactly as in the proof of Theorem 3.2 of [6] that (ii) implies (iii) b, on page 362.

This completes the argument.

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