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DOUBLY SHIFT-INVARIANT SPACES IN H^2

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Dedicated to Paul Halmos, in honor of his 70th year

1. INTRODUCTION

In the theory of square-summable power series developed by L. de Branges and J. Rovnyak [4], there arises a class of Hilbert spaces that are contained contractively in the Hardy space H^2 of the unit disk and are invariant under both the unilateral shift operator, S (the operator on H^2 defined by $(Sf)(z) = zf(z)$), and its adjoint, S^*. One such space is associated with each function b lying in the unit ball of H^∞ that is not an extreme point of the unit ball. The space associated with b is denoted by $\mathscr{H}(b)$ and, as a vector space, equals the range of the operator $(1 - T_b T_{\bar{b}})^{1/2}$ (where, in general, T_u denotes the Toeplitz operator on H^2 with symbol u). The norm in $\mathscr{H}(b)$ is denoted by $\| \ \|_b$ and is defined by the relation $\|(1 - T_b T_{\bar{b}})^{1/2} f\|_b = \|f\|_2$ (where $\| \ \|_2$ denotes the usual norm in H^2).

The spaces $\mathscr{H}(b)$ have a rich and fascinating structure whose exploration is the aim of the present paper. For someone interested in concrete operator theory, it is natural to seek to understand the two operators naturally associated with $\mathscr{H}(b)$, the operators $X = S^*|\mathscr{H}(b)$ and $Y = S|\mathscr{H}(b)$. The main theme in what follows will be the interplay between properties of X and Y, on the one hand, and, on the other hand, properties of the function b and of a related function, a, the outer function whose modulus on the unit circle equals $(1 - |b|^2)^{1/2}$ (normalized to be positive at the origin).

This paper is a sequel to [12], a crucial result from which will be stated and further developed in Section 2. Let it be mentioned here that both [4] and [12] contain proofs, from different viewpoints, of the invariance of $\mathscr{H}(b)$ under S and S^*. (The space $\mathscr{H}(b)$ can of course just as well be defined as above when b is an extreme point of the unit ball of H^∞; in that case $\mathscr{H}(b)$ is S^*-invariant but not S-invariant. Versions of these spaces exist also in vector-valued H^2 spaces and have been studied by de Branges and Rovnyak [5], more recently by de Branges alone [3], and most recently by J. A. Ball and T. L. Kriete [2]. Some of the results from [12] are special cases of results to be found in the latter paper.)

Section 3 is along with Section 2 of a preliminary nature. It contains a few properties of the spaces $\mathscr{H}(b)$ that lie fairly near the surface. (More general versions of some of these properties appear in [2].) The main results are in Sections 4—8. Section 4 explores the inclusion $H^\infty \subset \mathscr{H}(b)$, Section 5 the conditions under which X is similar to S^*, and Section 6 the conditions under which Y is similar to S. In Section 7 it is shown that the invariant subspaces of X are the intersections with $\mathscr{H}(b)$ of the invariant subspaces of S^*. The lattice of invariant subspaces of X is thus isomorphic to the complement of the Beurling lattice (the lattice of inner functions). The situation with regard to the invariant subspaces of Y is more complicated, however. That is illustrated in Section 8, where the invariant subspaces of Y are classified for the special case $b(z) = (1 + z)/2$. The concluding Section 9 contains an example that establishes the independence of two conditions from the main theorem of Section 6.

NOTATIONS AND CONVENTIONS. Throughout the paper, b, a, X and Y will have the meanings assigned above. To avoid a trivial exceptional case, it will be assumed that b is not a constant function.

The open unit disk will be denoted by D and the unit circle by ∂D.

The inner product in H^2 will be denoted by $\langle\,,\rangle$ and that in $\mathscr{H}(b)$ by $\langle\,,\rangle_b$.

The kernel function in H^2 for the functional of evaluation at the point w of D will be denoted by k_w ($k_w(z) = (1 - \bar{w}z)^{-1}$); it is shown in [12] that these functions belong to $\mathscr{H}(b)$.

With each function u in L^∞ of ∂D, we associate a Hilbert space $\mathscr{M}(u)$ that is contained boundedly in H^2. As a vector space, $\mathscr{M}(u)$ equals the range of the operator T_u; one defines its norm by setting $\|T_u f\|_{\mathscr{M}(u)} = \|f\|_2$ provided f is orthogonal to $\ker T_u$. The cases of interest below are $u = \bar{a}$ and $u = a$, in both of which $\ker T_u$ is trivial. (The space $\mathscr{H}(b)$ is what de Branges calls the complementary space of $\mathscr{M}(b)$ [3].)

If two of the Hilbert spaces we are considering, for example $\mathscr{M}(u)$ and $\mathscr{M}(v)$, are equal as vector spaces, we shall write $\mathscr{M}(u) = \mathscr{M}(v)$, even when the Hilbert space structures on the two spaces differ. If $\mathscr{M}(u) = \mathscr{M}(v)$ then, because both spaces are continuously and injectively embedded in H^2, the closed graph theorem implies that their norms are equivalent, that is, the identity map of either onto the other is bounded.

2. PRELIMINARY LEMMAS

For $0 < r < 1$ let a_r denote the outer function with modulus $(1 - r^2|b|^2)^{1/2}$ on ∂D and a positive value at the origin. In [12] the equality

$$(1) \qquad (1 - r^2 T_b T_{\bar{b}})^{-1} = 1 + T_{rb/a_r} T_{r\bar{b}/\bar{a}_r}$$

is derived and used to obtain the following result.

LEMMA 1. *The H^2 function f belongs to $\mathscr{H}(b)$ if and only if* $\lim_{r\to 1} \|T_{\bar{b}/\bar{a}_r} f\|_2 < \infty$. *If f_1 and f_2 are two functions in $\mathscr{H}(b)$, then*

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle + \lim_{r\to 1} \langle T_{\bar{b}/\bar{a}_r} f_1, T_{\bar{b}/\bar{a}_r} f_2\rangle.$$

The following consequence will be used extensively.

LEMMA 2. *The H^2 function f belongs to $\mathscr{H}(b)$ if and only if $T_{\bar{b}} f$ is in $T_{\bar{a}} H^2$ If f_1 and f_2 are functions in $\mathscr{H}(b)$ and $T_{\bar{b}} f_j = T_{\bar{a}} g_j$ $(j = 1, 2)$, then*

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle + \langle g_1, g_2\rangle.$$

In fact, if $T_{\bar{b}} f = T_{\bar{a}} g$, one easily sees that $T_{\bar{b}/\bar{a}_r} f = T_{\bar{a}/\bar{a}_r} g$, so that

$$\|T_{\bar{b}/\bar{a}_r} f\|_2 \leqslant \|a/a_r\|_\infty \|g\|_2 \leqslant \|g\|_2 .$$

In virtue of Lemma 1, that implies f is in $\mathscr{H}(b)$, establishing one direction in the first assertion of Lemma 2.

To establish the second assertion of Lemma 2 note that, as r tends to 1, the functions a/a_r tend pointwise in D to the constant function 1, so they tend to 1 in the weak-star topology of H^∞, and therefore also in the weak-star topology of L^∞. Therefore $\bar{a}/\bar{a}_r \to 1$ in the weak-star topology of L^∞, which implies $T_{\bar{a}/\bar{a}_r} \to 1$ in the weak operator topology. Thus, under the assumption $T_{\bar{b}} f = T_{\bar{a}} g$, we can conclude that $T_{\bar{b}/\bar{a}_r} f \to g$ weakly in H^2 as $r \to 1$. But because $\|T_{\bar{b}/\bar{a}_r} f\|_2 \leqslant \|g\|_2$, the preceding weak convergence is actually norm convergence. The second assertion of Lemma 2 follows immediately from this in conjunction with the second assertion of Lemma 1.

To establish the other direction in the first assertion of Lemma 2, assume f is in $\mathscr{H}(b)$. We have

$$T_{\bar{a}} T_{\bar{b}/\bar{a}_r} f = T_{\bar{a}/\bar{a}_r} T_{\bar{b}} f.$$

The functions $T_{\bar{b}/\bar{a}_r} f$ remain bounded in H^2-norm as $r \to 1$, so they have a weak cluster point, say g. Because $T_{\bar{a}/\bar{a}_r} \to 1$ in the weak operator topology, the equality $T_{\bar{b}} f = T_{\bar{a}} g$ follows, and the proof of Lemma 2 is complete.

It is worth mentioning an alternative proof of Lemma 2. Let B denote the 2-by-1 matrix inner function $\begin{pmatrix} b \\ a \end{pmatrix}$, and let $\mathscr{H}(B)$ denote the orthogonal complement of BH^2 in $H^2 \oplus H^2$. It is shown in [12] that the projection $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $H^2 \oplus H^2$ defines an isometry of $\mathscr{H}(B)$ onto $\mathscr{H}(b)$, and, moreover, that $\mathscr{H}(B)$ is the range of the projection

$$E = \begin{pmatrix} 1 - T_b T_{\bar b} & -T_b T_{\bar a} \\ -T_a T_{\bar b} & 1 - T_a T_{\bar a} \end{pmatrix}.$$

In particular, if f is in $\mathscr{H}(b)$, then there is a function g in H^2 such that $\begin{pmatrix} f \\ -g \end{pmatrix}$ is in $\mathscr{H}(B)$. The latter function is then sent into itself by the projection E, which implies

$$f = f - T_b T_{\bar b} f + T_b T_{\bar a} g,$$

in other words, $T_{\bar b} f = T_{\bar a} g$. This establishes one direction in the first assertion of Lemma 2. A straightforward reversal of the preceding reasoning yields the other direction in the first assertion. The second assertion follows because P_+ defines an isometry of $\mathscr{H}(B)$ onto $\mathscr{H}(b)$.

It should be noted that, if f is in $\mathscr{H}(b)$, then the function g satisfying $T_{\bar b} f = T_{\bar a} g$ is unique, because the operator $T_{\bar a}$ has a trivial kernel (the function a being an outer function).

3. SIMPLE INCLUSIONS

This section contains three simple lemmas about inclusions in $\mathscr{H}(b)$, followed by some remarks about multipliers of $\mathscr{H}(b)$.

LEMMA 3. *The space $\mathscr{M}(\bar a)$ is contained contractively in $\mathscr{H}(b)$.*

This assertion is equivalent to the assertion that there is a factorization $T_{\bar a} = (1 - T_b T_{\bar b})^{1/2} R$ with $\|R\| \leqslant 1$, which, by a well-known criterion of R. G. Douglas, [8], is equivalent to the inequality $T_{\bar a} T_a \leqslant 1 - T_b T_{\bar b}$. The preceding inequality holds because $T_b T_{\bar b} \leqslant T_{\bar b} T_b$ (the operator T_b being subnormal, and therefore hyponormal) and $T_{\bar a} T_a + T_{\bar b} T_b = T_{|a|^2} + T_{|b|^2} = 1$. (It is equally simple to deduce Lemma 3 from Lemma 2.)

LEMMA 4. *The space $\mathscr{M}(a)$ is contained contractively in $\mathscr{M}(\bar a)$ (and therefore is also contained contractively in $\mathscr{H}(b)$).*

Because of the criterion in [8], this follows from the inequality $T_a T_{\bar a} \leqslant T_{\bar a} T_a$, that is, from the hyponormality of T_a.

LEMMA 5. *The operator T_b maps $\mathscr{M}(\bar a)$ contractively into $\mathscr{H}(b)$.*

Using the criterion in [8], once more, we can reduce this assertion to the inequality $T_b T_{\bar a} T_a T_{\bar b} \leqslant 1 - T_b T_{\bar b}$. Because $T_{\bar a} T_a = 1 - T_{\bar b} T_b$, the difference between the right and left sides of the desired inequality equals $1 - 2T_b T_{\bar b} + (T_b T_{\bar b})^2$, which equals $(1 - T_b T_{\bar b})^2$. The inequality therefore holds.

A function u in H^∞ will be called a multiplier of $\mathcal{H}(b)$ if $u\mathcal{H}(b) \subset \mathcal{H}(b)$. It follows by the closed graph theorem that a multiplier of $\mathcal{H}(b)$ induces a bounded operator on $\mathcal{H}(b)$. These are in fact precisely the bounded operators on $\mathcal{H}(b)$ that commute with the operator Y (the operator of multiplication by z). The standard reasoning [13] that an operator in the commutant of Y is induced by a multiplier goes as follows. For w in D, let k_w^b denote the kernel function in $\mathcal{H}(b)$ for the functional of evaluation at w. (Explicitly, $k_w^b = (1 - \overline{b(w)b})k_w$ [12].) Let A be an operator on $\mathcal{H}(b)$ that commutes with Y. Since k_w^b is an eigenvector of Y^* of unit multiplicity (with eigenvalue $\bar{w}$), it is also an eigenvector of A^*. Let $u(w)$ be the complex conjugate of the eigenvalue of k_w^b as an eigenvector of A^*. Obviously $|u(w)| \leqslant \|A\|$. Moreover

$$u(w) = \langle 1, \overline{u(w)}k_w^b\rangle_b = \langle A1, k_w^b\rangle_b,$$

so $u = A1$, and it is now clear that u is in H^∞. Finally, for f in $\mathcal{H}(b)$,

$$(Af)(w) = \langle f, A^*k_w^b\rangle_b = u(w)\langle f, k_w^b\rangle_b = u(w)f(w),$$

so A is the operator of multiplication by u.

In certain cases, although not in general, every function in H^∞ is a multiplier of $\mathcal{H}(b)$; that obviously happens, for example, if $\|b\|_\infty < 1$ (so that $\mathcal{H}(b) = H^2$). Necessary and sufficient conditions for every function in H^∞ to be a multiplier of $\mathcal{H}(b)$ are given in Section 6. Here, for purposes of orientation, it will be shown that, for any b, every function that is holomorphic in a neighborhood of $\bar{D}$ is a multiplier of $\mathcal{H}(b)$ (and so, in particular, is an element of $\mathcal{H}(b)$).

The proof depends upon the equality $Y = X^* + (b \otimes S^*b)$ from [4]. (An alternative proof is presented in [12]. By $b \otimes S^*b$ is meant, as is standard, the operator on $\mathcal{H}(b)$ that sends the function f to $\langle f, S^*b\rangle_b b$. The proof that b belongs to $\mathcal{H}(b)$ can be found in [12]; this inclusion depends on the assumption that b is not an extreme point of the unit ball of H^∞.) Since X is a contraction, it follows that the essential spectrum of Y is contained in $\bar{D}$. Therefore, if $|w| > 1$, then $Y - w$ is a Fredholm operator of index 0, so it must be invertible, because Y obviously has no eigenvectors. Consequently, the spectrum of Y is contained in $\bar{D}$ — in fact, it clearly equals $\bar{D}$. Hence, if the function u is holomorphic in a neighborhood of $\bar{D}$, then $u(Y)$ is defined by the standard holomorphic functional calculus, and one easily verifies that $u(Y)$ is the operator on $\mathcal{H}(b)$ of multiplication by u, so that u is a multiplier of $\mathcal{H}(b)$.

4. CONDITIONS FOR $\mathscr{H}(b)$ TO CONTAIN H^∞

THEOREM 1. *The following conditions are equivalent.*

(i) $H^\infty \subset \mathscr{H}(b)$.

(ii) $\sup_{n\geqslant 0} \|z^n\|_b < \infty$.

(iii) *b/a is in H^2.*

(iv) *$(1 - |b|^2)^{-1}$ is integrable on ∂D.*

That (i) implies (ii) follows in standard fashion from the closed graph theorem. Because a is an outer function, (iii) is equivalent to the integrability on ∂D of $|b|^2|a|^{-2}$, which, in turn, is equivalent to (iv) because $(1 - |b|^2)^{-1} = 1 + |b|^2|a|^{-2}$ on ∂D. Similarly, (iv) is equivalent to the condition that $1/a$ belong to H^2, which implies (i) by Lemma 4. We can complete the proof of Theorem 1 by showing that (ii) implies (iii), which is an immediate consequence of the following lemma (applied to the function $f = 1$).

LEMMA 6. *If f is in $\mathscr{H}(b)$, then $\sup_{n\geqslant 0} \|Y^n f\|_b < \infty$ if and only if $T_b f$ is in $T_a H^2$.*

To prove the lemma, suppose first that $\sup_{n\geqslant 0} \|Y^n f\|_b = C < \infty$. By Lemma 1, then,

$$\lim_{r\to 1} \|T_{\bar b/\bar a_r} z^n f\|_2 \leqslant C \quad (n = 0, 1, \ldots).$$

Moreover, the equality (1) implies that, for any function g in H^2, the numbers $r\|T_{\bar b/\bar a_r} g\|_2$ increase with r. Hence, for all n and r we have

$$r\|T_{\bar b/\bar a_r} z^n f\|_2 \leqslant C,$$

in other words,

$$r\left\| P\left(\frac{z^n \bar b f}{\bar a_r}\right)\right\|_2 \leqslant C,$$

where P denotes the orthogonal projection of L^2 onto H^2. As $n \to \infty$, the left side of the preceding inequality tends to the L^2-norm of $r\bar b f/\bar a_r$. Thus $\|bf/a_r\|_2 \leqslant C/r$. Now letting r tend to 1 in the last inequality, we can conclude that bf/a is in H^2. To say bf/a is in H^2 is the same as to say $T_b f$ is in $T_a H^2$, so one direction of Lemma 6 is established.

We shall use Lemma 2 to establish the other direction. Suppose there is a function g in H^2 such that $T_b f = T_a g$. For $n \geqslant 0$,

$$T_{\bar b} Y^n f = T_{\bar b/b} T_b S^n f = T_{\bar b/b} T_a S^n g = T_{\bar a} T_{a\bar b/b\bar a} S^n g.$$

Therefore, by Lemma 2,

$$\|Y^n f\|_b^2 = \|S^n f\|_2^2 + \|T_{a\bar{b}/b\bar{a}} S^n g\|_2^2 \leqslant \|f\|_2^2 + \|g\|_2^2,$$

and the proof of Lemma 6 is complete.

Theorem 1 can be made a bit more precise. Let $\sum_0^\infty c_j z^j$ denote the power series at the origin for the function b/a.

PROPOSITION. $\|z^n\|_b^2 = 1 + \sum_0^n |c_j|^2 \quad (n = 0, 1, \ldots)$.

The implication (ii) ⇒ (iii) of Theorem 1 is an obvious corollary.

The case $n = 0$ of the proposition is given in [12]. Here is the proof of the induction step that yields the general case. A calculation is needed.

LEMMA 7. $\langle S^*b, z^n\rangle_b = a(0)c_{n+1} \quad (n = 0, 1, \ldots)$.

We shall deduce this from Lemma 1. We have

$$r^2 T_{\bar{b}/\bar{a}_r} S^*b = P\left(\frac{\bar{z} r^2 |b|^2}{\bar{a}_r}\right) = P\left(\frac{\bar{z}(1 - |a_r|^2)}{\bar{a}_r}\right) =$$

$$= - P(\bar{z} a_r) = - \bar{z}(a_r - a_r(0)).$$

Therefore

$$T_{rb/a_r} T_{r\bar{b}/\bar{a}_r} S^*b = - \bar{z}\left(b - \frac{a_r(0) b}{a_r}\right) = - S^*b + a_r(0) S^*(b/a_r).$$

This in conjunction with Lemma 1 gives

$$\langle S^*b, z^n\rangle_b = \langle S^*b, z^n\rangle + \lim_{r\to 1} \langle T_{rb/a_r} T_{r\bar{b}/\bar{a}_r} S^*b, z^n\rangle =$$

$$= \langle S^*b, z^n\rangle + \lim_{r\to 1} [-\langle S^*b, z^n\rangle + a_r(0) \langle S^*(b/a_r), z^n\rangle] =$$

$$= \lim_{r\to 1} a_r(0) \langle b/a_r, z^{n+1}\rangle = a(0) c_{n+1},$$

as desired.

To establish the proposition we use the formula $Y^* = X + (S^*b \otimes b)$ (the adjoint of which is mentioned in Section 3) to get

$$Y^*Y = [X + (S^*b \otimes b)]Y = XY + (S^*b \otimes Y^*b) =$$

$$= 1 + (S^*b \otimes S^*b) + \|b\|_b^2 (S^*b \otimes S^*b).$$

As $\|b\|_b^2 = |a(0)|^{-2} - 1$ [12], we thus have

$$Y^*Y = 1 + |a(0)|^{-2}(S^*b \otimes S^*b).$$

Consequently

$$\|z^{n+1}\|_b^2 = \langle Y^*Yz^n, z^n\rangle_b =$$

$$= \|z^n\|_b^2 + |a(0)|^{-2}\langle (S^*b \otimes S^*b)z^n, z^n\rangle_b =$$

$$= \|z^n\|_b^2 + |a(0)|^{-2}|\langle S^*b, z^n\rangle_b|^2 = \|z^n\|_b^2 + |c_{n+1}|^2,$$

the final equality following by Lemma 7. The proposition now follows.

5. CONDITIONS FOR X TO BE SIMILAR TO S^*

The functions a and b will be said to form a corona pair if they satisfy the hypotheses of L. Carleson's corona theorem [10], that is, if $|a|^2 + |b|^2$ is bounded away from 0 in D. Because $|a|^2 + |b|^2 = 1$ on ∂D, one might guess for an instant that a and b would form a corona pair automatically in the case where b is an outer function. That is not true, however. To obtain a simple counterexample, let b be the outer function whose modulus on ∂D is given by $|b(e^{i\theta})|^2 = (2\pi)^{-1}\theta$ ($0 < \theta < 2\pi$). Then both a and b have 0 as a one-sided limit along ∂D at the point 1, so, by a theorem of Lindelöf [10, p. 92], they both have 0 as a radial limit at the same point.

THEOREM 2. *The following conditions are equivalent.*
(i) *X is similar to S^*.*
(ii) *$\mathscr{H}(b) = \mathscr{M}(\bar{a})$.*
(iii) *b is a multiplier of $\mathscr{H}(b)$.*
(iv) *a and b form a corona pair.*

That (ii) implies (iii) follows immediately from Lemma 5. If (ii) holds then the norms in $\mathscr{H}(b)$ and $\mathscr{M}(\bar{a})$ are equivalent, which implies (i) because the operator $S^* \mid \mathscr{M}(\bar{a})$ is obviously unitarily equivalent to S^* (the unitary equivalence being implemented by $T_{\bar{a}}$). We shall complete the proof of Theorem 2 by establishing the implications (iii) $\Rightarrow$ (ii), (iv) $\Leftrightarrow$ (ii), and (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (ii). Suppose f is a function in H^2 such that bf is in $\mathscr{H}(b)$, and let g be the function in H^2 satisfying $T_{\bar{b}}bf = T_{\bar{a}}g$. Since

$$T_{\bar{b}}bf = T_{|b|^2}f = f - T_{|a|^2}f = f - T_{\bar{a}}af,$$

it follows that $f = T_{\bar{a}}(g + af)$, so f is in $\mathscr{M}(\bar{a})$. Hence, if b is a multiplier of $\mathscr{H}(b)$, then $\mathscr{H}(b) \subset \mathscr{M}(\bar{a})$, which, together with Lemma 3, implies $\mathscr{H}(b) = \mathscr{M}(\bar{a})$.

The preceding argument, incidentally, establishes a converse to Lemma 5: If f is in H^2, then $T_b f$ lies in $\mathscr{H}(b)$ only if f is in $\mathscr{M}(\bar{a})$.

(ii) $\Rightarrow$ (iv). Assume $\mathscr{H}(b) = \mathscr{M}(\bar{a})$. Then the norms in $\mathscr{H}(b)$ and $\mathscr{M}(\bar{a})$ are equivalent. Hence, there is a positive constant c such that

$$\|k_w\|_b \geqslant c\|k_w\|_{\mathscr{M}(\bar{a})} \quad (|w| < 1). \tag{2}$$

Since k_w is an eigenvector of $T_{\bar{a}}$ with eigenvalue $\overline{a(w)}$, we have

$$\|k_w\|^2_{\mathscr{M}(\bar{a})} = |a(w)|^{-2}\|k_w\|_2^2 = |a(w)|^{-2}(1 - |w|^2)^{-1},$$

while it is shown in [12] that

$$\|k_w\|_b^2 = (|a(w)|^2 + |b(w)|^2)|a(w)|^{-2}(1 - |w|^2)^{-1}.$$

The last two equalities together with (2) give $|a(w)|^2 + |b(w)|^2 \geqslant c^2$, which means a and b form a corona pair.

(iv) $\Rightarrow$ (ii). Assume a and b form a corona pair. Then, by the corona theorem, there are functions u and v in H^∞ such that $au + bv = 1$. To show $\mathscr{H}(b) = \mathscr{M}(\bar{a})$ it will suffice, because of Lemma 3, to establish the inclusion $\mathscr{H}(b) \subset \mathscr{M}(\bar{a})$. Let f be any function in $\mathscr{H}(b)$, and let g be the function in H^2 satisfying $T_{\bar{b}} f = T_{\bar{a}} g$. We have

$$f = T_{\bar{a}} T_{\bar{u}} f + T_{\bar{v}} T_{\bar{b}} f = T_{\bar{a}} T_{\bar{u}} f + T_{\bar{v}} T_{\bar{a}} g = T_{\bar{a}}(T_{\bar{u}} f + T_{\bar{v}} g),$$

so f is in $\mathscr{M}(\bar{a})$, as desired.

(i) $\Rightarrow$ (ii). Assume X is similar to S^*, say $X = AS^*A^{-1}$, where A is an invertible linear map of H^2 onto $\mathscr{H}(b)$. Since $\mathscr{H}(b)$ is contained contractively in H^2, we can regard A as a bounded linear map of H^2 into itself, and, when so interpreted, A becomes an operator in the commutant of S^*. Therefore $A = T_{\bar{u}}$ for some function u in H^∞, and we have $\mathscr{H}(b) = \mathscr{M}(\bar{u})$. This implies [8] that $T_{\bar{u}} T_u \leqslant C(1 - T_b T_{\bar{b}})$ for some positive constant C, and in fact, replacing u by a scalar multiple of itself, we can assume without loss of generality that $C = 1$. As $T_u T_{\bar{u}} \leqslant T_{\bar{u}} T_u$, we have, a fortiori, $T_u T_{\bar{u}} + T_b T_{\bar{b}} \leqslant 1$. Hence, for w in D,

$$\|T_{\bar{u}} k_w\|_2^2 + \|T_{\bar{b}} k_w\|_2^2 \leqslant \|k_w\|_2^2,$$

in other words,

$$|u(w)|^2\|k_w\|_2^2 + |b(w)|^2\|k_w\|_2^2 \leqslant \|k_w\|_2^2.$$

The inequality $|u|^2 + |b|^2 \leqslant 1$ therefore holds in D, so it holds also on ∂D, which means $|u| \leqslant |a|$ on ∂D. Since a is an outer function, the function $v = u/a$ is thus in H^∞, and we have the operator factorization $T_{\bar{u}} = T_{\bar{a}} T_{\bar{v}}$. Hence $\mathscr{M}(\bar{u}) \subset \mathscr{M}(\bar{a})$, and it follows that $\mathscr{H}(b) \subset \mathscr{M}(\bar{a})$, which together with Lemma 3 gives $\mathscr{H}(b) = \mathscr{M}(\bar{a})$, completing the proof of Theorem 2.

6. CONDITIONS FOR Y TO BE SIMILAR TO S

THEOREM 3. *The following conditions are equivalent.*

(i) *Y is similar to S.*

(ii) *Y is polynomially bounded.*

(iii) *Y is power bounded.*

(iv) *Every function in H^∞ is a multiplier of $\mathscr{H}(b)$.*

(v) *$\mathscr{H}(b) = \mathscr{M}(a)$.*

(vi) *a and b form a corona pair, and the operator $T_{a/\bar{a}}$ is invertible.*

Some comments on condition (vi) are in order prior to the proof. The invertibility of $T_{a/\bar{a}}$ can be characterized in several ways, thanks to the work of A. Devinatz, H. Widom, H. Helson and G. Szegö, and R. Hunt, B. Muckenhoupt and R. Wheeden. These matters are discussed in [11], where references and some of the proofs can be found; the facts are as follows. A criterion due to Devinatz and Widom states that $T_{a/\bar{a}}$ is invertible if and only if $|a|^2$ satisfies what is called the Helson-Szegö condition, which is that $|a|^2$ be writable (on ∂D) as $\exp(x+\tilde{y})$, where x and y are real functions in L^∞, $\tilde{y}$ denotes the conjugate function of y, and $\|y\|_\infty < \pi/2$. By a theorem of Helson and Szegö, $|a|^2$ satisfies their condition if and only if P, the orthogonal projection of L^2 onto H^2, defines a bounded operator in the weighted space $L^2(|a|^2)$. Finally, the theorem of Hunt, Muckenhoupt and Wheeden states that the latter condition holds if and only if $|a|^2$ satisfies Muckenhoupt's condition (A_2). (The reader unfamiliar with (A_2) will find a thorough discussion in [10] which also contains detailed proofs of the second and third of the three equivalences just stated. Proofs of the first two equivalences are in [11].)

The two parts of condition (vi) are independent. That $T_{a/\bar{a}}$ can be noninvertible when a and b form a corona pair is shown by the example $b(z)=(1+z)/2$; in that case $a(z)=(1-z)/2$, and $T_{a/\bar{a}} = -S$. On the other hand, given b such that $T_{a/\bar{a}}$ is invertible, if we replace b by its product with an inner function we produce no alteration in a; one can clearly create by such replacements examples where $T_{a/\bar{a}}$ is invertible yet a and b do not form a corona pair. In none of these examples, however, is b an outer function. An example in which b is an outer function is constructed in Section 9.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Theorem 3 are trivial. Also, it is very easy to see that (ii) and (iv) are equivalent. In fact, (ii) amounts to the existence of a positive constant C such that $\|pf\|_b \leqslant C\|p\|_\infty\|f\|_b$ for every polynomial p and every function f in $\mathscr{H}(b)$. Suppose such a C exists, and let u be any function in H^∞. Let (p_n) be the sequence of Féjer means of the power series of u. Then, for any f in $\mathscr{H}(b)$, the sequence (p_nf) is bounded in $\mathscr{H}(b)$-norm and converges pointwise in D to uf, from which one easily deduces that uf is in $\mathscr{H}(b)$. This shows that (ii) implies (iv). In the other direction, if (iv) holds then we have a natural map of H^∞

into the space of operators on $\mathscr{H}(b)$, namely, the map that sends each function in H^∞ to the multiplication operator it induces. From the remarks in Section 3 it is clear that the inverse of that map is a contraction and that its range, being the commutant of Y, is norm closed. By the open mapping theorem, the map itself must be bounded, which means (ii) holds.

The implication (v) $\Rightarrow$ (i) is also trivial, because $S \mid \mathscr{M}(a)$ is obviously unitarily equivalent to S, and if $\mathscr{H}(b) = \mathscr{M}(a)$ then the norms in $\mathscr{H}(b)$ and $\mathscr{M}(a)$ are equivalent.

The implication (vi) $\Rightarrow$ (v) is easy. In fact, if a and b form a corona pair then $\mathscr{H}(b) = \mathscr{M}(\bar{a})$, by Theorem 2. If also $T_{a/\bar{a}}$ is invertible, then, because $T_a = T_{\bar{a}} T_{a/\bar{a}}$, the operators T_a and $T_{\bar{a}}$ have the same range, so that (v) holds.

We shall complete the proof of Theorem 3 by showing that (iii) implies (vi), which is simply done on the basis of Lemma 6. Assume Y is power bounded. If f is any function in $\mathscr{H}(b)$ then, by Lemma 6, the function bf/a is in H^2. Since $|bf/a|^2 + |f|^2 = |f/a|^2$, it follows that f/a is in H^2, so we can conclude that $\mathscr{H}(b) \subset \mathscr{M}(a)$. The opposite inclusion is given by Lemma 4, and thus $\mathscr{H}(b) = \mathscr{M}(a)$. Since $\mathscr{M}(\bar{a})$ lies between $\mathscr{H}(b)$ and $\mathscr{M}(a)$ (Lemmas 3 and 4), we also have $\mathscr{H}(b) = \mathscr{M}(\bar{a})$, so a and b form a corona pair by Theorem 2. Furthermore, in view of the factorization $T_a = T_{\bar{a}} T_{a/\bar{a}}$, the equality of $\mathscr{M}(a)$ and $\mathscr{M}(\bar{a})$ implies $T_{a/\bar{a}}$ is surjective (since $T_{\bar{a}}$ is one-to-one). The same factorization shows $T_{a/\bar{a}}$ is one-to-one (since T_a is one-to-one). Hence $T_{a/\bar{a}}$ is invertible, and the proof of Theorem 3 is complete.

Theorem 3 has a close analogue for the spaces $\mathscr{M}(\bar{u})$ with u in H^∞. One looses no generality in taking u to be an outer function, for if it is not (and is not identically 0), one can replace it by its outer factor without altering $\mathscr{M}(\bar{u})$.

THEOREM 4. *If u is an outer function in H^∞, then the space $\mathscr{M}(\bar{u})$ is invariant under S, and the following conditions are equivalent for the operator $Z = S \mid \mathscr{M}(\bar{u})$.*

(i) *Z is similar to S.*

(ii) *Z is polynomially bounded.*

(iii) *Z is power bounded.*

(iv) *Every function in H^∞ is a multiplier of $\mathscr{M}(\bar{u})$.*

(v) *$\mathscr{M}(\bar{u}) = \mathscr{M}(u)$.*

(vi) *$T_{u/\bar{u}}$ is invertible.*

The S-invariance of $\mathscr{M}(\bar{u})$ holds because $T_{\bar{u}} S - S T_{\bar{u}}$ is a rank-one operator whose range is spanned by the constant function 1, which belongs to $\mathscr{M}(\bar{u})$ (as it is an eigenvector of $T_{\bar{u}}$). All of the arguments used in the proof of Theorem 3 carry over to Theorem 4 except for the proof that (iii) implies (vi), which implication can be established as follows.

Saying that Z is power bounded is the same as saying that there exist factorizations $S^n T_{\bar{u}} = T_{\bar{u}} R_n$, $n = 0, 1, \ldots$, with $\sup_{n \geq 0} \|R_n\| = C < \infty$. By Douglas's criterion [8], the last condition is equivalent to the inequalities

$$(S^n T_{\bar u})(S^n T_{\bar u})^* \leqslant C^2 T_{\bar u} T_u \quad (n = 0, 1, \ldots).$$

Hence, assuming Z is power bounded, we have $\|uS^{*n}f\|_2 \leqslant C\|uf\|_2$ $(n = 0, 1, \ldots)$ for all f in H^2. Let q be any trigonometric polynomial, and choose n so that $z^n q$ is in H^2. The preceding inequality applied to the function $f = z^n q$ gives

$$\int_{-\pi}^{\pi} |Pq|^2 |u|^2 \, d\theta \leqslant C^2 \int_{-\pi}^{\pi} |q|^2 |u|^2 \, d\theta.$$

We can conclude that P is bounded in $L^2(|u|^2)$, so $T_{u/\bar u}$ is invertible by the theorems of Devinatz-Widom and Helson-Szegö. The proof of Theorem 4 is complete.

7. THE INVARIANT SUBSPACES OF X

THEOREM 5. *The invariant subspaces of X are the intersections with $\mathscr{H}(b)$ of the invariant subspaces of S^*.*

By the theorem of A. Beurling [10], the invariant subspaces of S^* are the subspaces $\mathscr{H}(u)$ $(= H^2 \ominus uH^2)$ with u an inner function. The intersections of those subspaces with $\mathscr{H}(b)$ are obviously invariant subspaces of X; it remains to show that they are the only ones. A few remarks about the functional calculus for X are needed.

From Lemma 2 it follows immediately that, for any function u in H^∞, the space $\mathscr{H}(b)$ is invariant under $T_{\bar u}$ and the restriction of $T_{\bar u}$ to $\mathscr{H}(b)$ is a bounded operator of norm at most $\|u\|_\infty$. If u is a polynomial then $T_{\bar u}|\mathscr{H}(b) = u^*(X)$, where u^* is defined by $u^*(z) = \overline{u(\bar z)}$. If u is any function in H^∞ and (p_n) is the sequence of Féjer means of its power series, then, for f in $\mathscr{H}(b)$, the sequence $(T_{\bar p_n} f)$ converges pointwise in D to $T_{\bar u} f$ and is bounded in the norm of $\mathscr{H}(b)$, from which one easily sees that $T_{\bar p_n} f \to T_{\bar u} f$ weakly in $\mathscr{H}(b)$. Thus, $T_{\bar u} \mid \mathscr{H}(b)$ lies in the weakly closed operator algebra generated by X, so every invariant subspace of X is also invariant under $T_{\bar u}$. (Since X is a completely nonunitary contraction [12], the H^∞ functional calculus of B. Sz.-Nagy and C. Foiaş [14] applies to it. The preceding remarks show that, for u in H^∞, $u(X) = T_{\bar u *}|\mathscr{H}(b)$.)

The following lemma contains the key step in the proof of Theorem 5.

LEMMA 7. *If $\mathscr{J}$ is an invariant subspace of X, then $T_{\bar a}\mathscr{J}$ is dense in $\mathscr{J}$.*

In fact, suppose the function f in $\mathscr{J}$ is orthogonal to $T_{\bar a}\mathscr{J}$, and let g be the function in H^2 satisfying $T_{\bar b} f = T_{\bar a} g$. The function f is then orthogonal to $S^{*n} T_{\bar a} f$ for any nonnegative integer n. Since $T_{\bar{b}}S^{*n}T_{\bar{a}}f=T_{\bar{a}}S^{*n}T_{\bar{a}}g$, it follows by Lemma 2 that

$$0=\langle f, S^{*n}T_{\bar{a}}f\rangle_b=\langle f, S^{*n}T_{\bar{a}}f\rangle+\langle g, S^{*n}T_{\bar{a}}g\rangle=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}a(|f|^2+|g|^2)\,d\theta \qquad (n=0, 1, \ldots).$$

Consequently, the function $a(|f|^2+|g|^2)$ belongs to the space H_0^1 (that is, it belongs to H^1 and vanishes at the origin). Since a is an outer function, the function $|f|^2+ |g|^2$ therefore also belongs to H_0^1. Because the only nonnegative function in H_0^1 is 0, we can conclude that $f=0$, concluding the proof of the lemma.

Lemma 7 has a couple of corollaries that will be mentioned before it is used to complete the proof of Theorem 5.

COROLLARY 1. *The polynomials are dense in* $\mathscr{H}(b)$.

In fact, it follows from Lemma 7 that $\mathscr{M}(\bar{a})$ is dense in $\mathscr{H}(b)$. The functions $T_{\bar{a}}z^n$ $(n=0, 1, \ldots)$ obviously span $\mathscr{M}(\bar{a})$, so their linear combinations are dense in $\mathscr{M}(\bar{a})$ relative to the norm of $\mathscr{H}(b)$. These linear combinations are therefore dense in $\mathscr{H}(b)$. Since $T_{\bar{a}}z^n$ is a polynomial, the corollary is established.

COROLLARY 2. *The operators commuting with* X *are the operators* $T_{\bar{u}} \mid \mathscr{H}(b)$ *with* u *in* H^∞.

Suppose the operator A on $\mathscr{H}(b)$ commutes with X. For $|w|<1$, the function k_w is an eigenvector of X of unit multiplicity, so it is an eigenvector of A; let the corresponding eigenvalue be denoted by $\overline{u(w)}$. The function k_w $(=(1-\bar{w}Y)^{-1}1)$ is an antiholomorphic function of w, implying that u is a holomorphic function of w. Obviously $\|u\|_\infty \leqslant \|A\|$. The operators A and $T_{\bar{u}}|\mathscr{H}(b)$ thus coincide on k_w for every w; we shall be able to conclude that $A=T_{\bar{u}}|\mathscr{H}(b)$ once we know that the functions k_w span $\mathscr{H}(b)$. To establish the last fact it will suffice, because of Corollary 1, to show that every polynomial lies in the span of the functions k_w. That can be seen, for example, on the basis of the holomorphic functional calculus; if p is a polynomial, then

$$p=p(Y)1=\frac{1}{2\pi i}\int_{|w|=2} p(w)[(w-Y)^{-1}1]\,dw.$$

The integral exists as an ordinary $\mathscr{H}(b)$-valued Riemann integral, and its Riemann sums are linear combinations of the functions k_w. The proof of Corollary 2 is complete.

Returning now to the proof of Theorem 5, suppose $\mathscr{J}$ is an invariant subspace of X, and let $\mathscr{K}$ be the closure of $\mathscr{J}$ in H^2. Then $\mathscr{K}$ is an invariant subspace of S^*. If f is in $\mathscr{K}$, there is a sequence (f_n) in $\mathscr{J}$ converging to f in the norm of H^2. Then $T_{\bar{a}} f_n \to T_{\bar{a}} f$ in the norm of $\mathscr{M}(\bar{a})$ and so also in the norm of $\mathscr{H}(b)$ (Lemma 3). As $T_{\bar{a}} f_n$ belongs to $\mathscr{J}$ for each n, the function $T_{\bar{a}} f$ thus also belongs to $\mathscr{J}$. Hence $T_{\bar{a}} \mathscr{K} \subset \mathscr{J}$. By Lemma 7, $T_{\bar{a}}(\mathscr{K} \cap \mathscr{H}(b))$ is dense in $\mathscr{K} \cap \mathscr{H}(b)$, so $\mathscr{K} \cap \mathscr{H}(b) \subset J$. The opposite inclusion being trivial, we can conclude that $\mathscr{J} = \mathscr{K} \cap \mathscr{H}(b)$, completing the proof of Theorem 5.

According to the theory of Sz.-Nagy and Foiaş [14], each invariant subspace of X determines a factorization of the characteristic operator function of X. It would be interesting to see those factorizations displayed concretely.

In connection with the first corollary of Lemma 7 the question arises whether the polynomials are dense in the space of multipliers of $\mathscr{H}(b)$ relative to the weak operator topology. The author does not know the answer. A related question is whether, for f in $\mathscr{H}(b)$, the functions $f_r(z) = f(rz)$ $(0 < r < 1)$ remain bounded in the norm of $\mathscr{H}(b)$.

8. THE CASE $b(z) = (1 + z)/2$

If u is an inner function, then $\mathscr{H}(b) \cap \mathscr{M}(u)$ is obviously an invariant subspace of the operator Y. If the conditions of Theorem 3 hold, Beurling's theorem implies every nontrivial invariant subspace of Y has the preceding form. In this section a simple case will be examined where the conditions of Theorem 3 fail. The invariant subspace lattice of Y will be determined and seen to be more complicated than Beurling's lattice.

We take $b(z) = (1 + z)/2$, which gives $a(z) = (1 - z)/2$. The functions a and b then form a corona pair, so $\mathscr{H}(b) = \mathscr{M}(\bar{a})$ by Theorem 2. The operator Y is similar to the restriction of S to $\mathscr{M}(\bar{a})$, enabling us to work instead with the latter operator. It is actually slightly more convenient (and obviously permissible) to work in $\mathscr{M}(2\bar{a})$ rather in $\mathscr{M}(\bar{a})$. We let e denote the function $2a$, that is, $e(z) = 1 - z$, and we denote the norm and inner product in the space $\mathscr{M}(\bar{e})$ by $\| \ \|_{\bar{e}}$ and $\langle \ , \ \rangle_{\bar{e}}$. The operator $S \mid \mathscr{M}(\bar{e})$ will be denoted by Z.

The following lemma describes the functions in $\mathscr{M}(\bar{e})$.

LEMMA 8. *The H^2 function f belongs to $\mathscr{M}(\bar{e})$ if and only if it can be written as $f = (S - 1)g + c$ where g is in H^2 and c is a constant. If f_1 and f_2 are two functions in $\mathscr{M}(\bar{e})$ and $f_j = (S - 1)g_j + c_j$ $(j = 1, 2)$, then*

$$\langle f_1, f_2 \rangle_{\bar{e}} = \langle g_1, g_2 \rangle + c_1 \bar{c}_2. \tag{3}$$

To prove the lemma one needs only to note that $T_{\bar{e}} = 1 - S^*$ and then to use the easily established identity $1-S^*=(S-1)S^*+(1 \otimes 1)$. If $f = (1 - S^*)h$ is a typical function in $\mathscr{M}(\bar{e})$, then the function g and constant c of the lemma are given by $g = S^*h$, $c = h(0)$. Conversely, if g and c are given, h is determined by the equality $h = Sg + c$. If $f_j = (1 - S^*)h_j$ $(j = 1, 2)$ are two functions in $\mathscr{M}(\bar{e})$ then, by definition, $\langle f_1, f_2\rangle_{\bar{e}} = \langle h_1, h_2\rangle$, which coincides with the right side of (3) when g_1, c_1, g_2, c_2 are defined in terms of h_1 and h_2 as indicated above.

The first statement in the lemma says that the functions in $\mathscr{M}(\bar{e})$ are the functions in H^2 that are divisible by the function $z - 1$, or that differ by a constant from such a function. In virtue of the estimate $|g(z)| = o((1 - |z|)^{-1})$ $(|z| \to 1)$, valid for any function g in H^2, it follows that every function f in $\mathscr{M}(\bar{e})$ has a radial limit at the point 1, denoted hereafter by $f(1)$. The radial limit $f(1)$ is the constant c of Lemma 8. The functional $f \to f(1)$ on $\mathscr{M}(\bar{e})$ is the bounded linear functional induced by the constant function 1. The kernel of that linear functional is $\mathscr{M}(e)$, which is thus a closed subspace of $\mathscr{M}(\bar{e})$ of codimension 1. By Lemma 8, the norm in $\mathscr{M}(e)$ is the norm it acquires as a subspace of $\mathscr{M}(\bar{e})$. The subspace $\mathscr{M}(e)$ is obviously an invariant subspace of Y.

Every function in H^∞ obviously is a multiplier of $\mathscr{M}(e)$. As $\mathscr{M}(\bar{e})$ is spanned by $\mathscr{M}(e)$ and the constant function 1, a function in H^∞ is a multiplier of $\mathscr{M}(\bar{e})$ if and only if it belongs to $\mathscr{M}(\bar{e})$. The multipliers of $\mathscr{M}(\bar{e})$ are thus described, in a sense, by Lemma 8. For inner functions more precise information is available.

THEOREM 6. *Let u be the inner function with zero sequence (z_j) and singular measure μ. The following conditions are equivalent.*

(i) *u is in $\mathscr{M}(\bar{e})$.*

(ii) *$1 - |u(r)|^2 = O(1 - r^2)$ as $r \to 1-$.*

(iii) *u has an angular derivative at the point 1.*

(iv) $$\sum \frac{1 - |z_j|^2}{|z_j - 1|^2} + \int \frac{d\mu(z)}{|z - 1|^2} < \infty.$$

The equivalence of (ii) and (iii) is a special case of a well-known theorem of C. Carathéodory [6], [7]; it will not play a role below but is mentioned for the sake of completeness. From condition (iv) one sees that an inner multiplier of $\mathscr{M}(\bar{e})$ can have a singularity at the point 1. (Every inner function that is regular at 1 is obviously a multiplier.) The equivalence of (ii) and (iv) can be extracted from a paper of P. R. Ahern and D. N. Clark [1]; the case of a Blaschke product goes back to O. Frostman [9] and Carathéodory [7, pp. 31 ff.]. For completeness a simple proof of the equivalence of (ii) and (iv) will be presented here. Before that, however, the equivalence of (i) and (ii) will be established.

Suppose first that the inner function u belongs to $\mathcal{M}(\bar{e})$. Since the function $(u - u(1))/(z - 1)$ belongs to H^2, while $|u| = 1$ on ∂D, we must have $|u(1)| = 1$, and clearly we loose no generality if we assume $u(1) = 1$. For $0 < r < 1$ we have

$$\infty > \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{u-1}{e^{i\theta}-1}\right|^2 d\theta \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{u-1}{e^{i\theta}-1/r}\right|^2 d\theta =$$

$$= \frac{r}{\pi(1-r^2)}\int_{-\pi}^{\pi}(1 - \operatorname{Re} u)\,\frac{1-r^2}{|e^{i\theta}-r|^2}\,d\theta =$$

$$= \frac{2r}{1-r^2}(1 - \operatorname{Re} u(r)) = \frac{r}{1-r^2}(|1-u(r)|^2 + 1 - |u(r)|^2).$$

The estimate $1 - |u(r)|^2 = O(1 - r^2)$ $(r \to 1-)$ follows immediately from the last inequality, so the implication (i) $\Rightarrow$ (ii) is established.

To establish the reverse implication, assume that u satisfies (ii). Let v be the inner function uu^* (where $u^*(z) = \overline{u(\bar{z})}$). Condition (ii) for u implies that v has 1 as a radial limit at the point 1, and $1 - v(r) = O(1 - r^2)$ as $r \to 1-$. If we replace u by v in the string of equalities above, we get, for $0 < r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{v-1}{e^{i\theta}-1/r}\right|^2 d\theta = \frac{2r(1-v(r))}{1-r^2}.$$

The left side is therefore $O(1)$ as $r \to 1-$, so applying the monotone convergence theorem, we can conclude that the function $(v - 1)/(z - 1)$ is in H^2 and hence (by Lemma 8) that v is in $\mathcal{M}(\bar{e})$. As $u = T_{\bar{u}^*}v$, the function u is also in $\mathcal{M}(\bar{e})$, and the implication (ii) $\Rightarrow$ (i) is established.

To establish the equivalence of (ii) and (iv), fix a Stolz angle G with vertex at the point 1, symmetric with respect to the radius to that point. If the zero sequence (z_j) clusters at the point 1 from within G, conditions (ii) and (iv) both clearly fail. We can assume therefore that (z_j) does not cluster at 1 from within G. The inner factor of u whose zeros are the points z_j lying inside G and whose singular measure is the portion of μ carried by G clearly satisfies both (ii) and (iv), so u will satisfy one of these conditions or the other if and only if its complementary inner factor does. We can therefore assume without loss of generality that every z_j lies outside of G and that μ is carried by the complement of G.

Let ν be the measure $\mu + \sum(1 - |z_j|^2)\delta_{z_j}$. Condition (iv) can then be rewritten as

(4)
$$\int \frac{d\nu(z)}{|z - 1|^2} < \infty.$$

Condition (ii) holds if and only if $\log \frac{1}{|u(r)|}$ is $O(1 - r^2)$ as $r \to 1-$. We have

$$\log \frac{1}{|u(r)|} = \sum \log \left| \frac{1 - \bar{z}_j r}{z_j - r} \right| + \int \frac{1 - r^2}{|z - r|^2} d\mu(z).$$

The quantities $\left| \frac{1 - \bar{z}_j r}{z_j - r} \right|$ remain uniformly bounded because the points z_j are outside of G. Therefore, the ratio

$$\log \left| \frac{1 - \bar{z}_j r}{z_j - r} \right| \Big/ \left(\left| \frac{1 - \bar{z}_j r}{z_j - r} \right|^2 - 1 \right)$$

remains bounded away from 0 (and of course is less than 1). Because

$$\left| \frac{1 - \bar{z}_j r}{z_j - r} \right|^2 - 1 = \frac{(1 - r^2)(1 - |z_j|^2)}{|z_j - r|^2},$$

we see that $\log \frac{1}{|u(r)|}$ is comparable to

$$\sum \frac{(1 - r^2)(1 - |z_j|^2)}{|z_j - r|^2} + \int \frac{1 - r^2}{|z - r|^2}$$

in the sense that the ratio of $\log \frac{1}{|u(r)|}$ to the preceding quantity is bounded (by 1) and bounded away from 0 ($0 < r < 1$). Hence condition (ii) is equivalent to the condition

(5)
$$\int \frac{d\nu(z)}{|z - r|^2} = O(1) \quad (r \to 1-).$$

It is elementary that (4) and (5) are equivalent. That (4) implies (5) follows by Fatou's lemma, and the opposite implication holds because the ratio $|z - 1|/|z - r|$, for z in the support of ν and $0 < r < 1$, does not exceed the cosecant of half the aperature of G. The proof of Theorem 6 is complete.

A passing REMARK. By using the reasoning employed above to establish the equivalence of (i) and (ii), together with Carathéodory's theorem, one can show that, if the inner function u belongs to $\mathcal{M}(\bar{e})$, then $\|u\|_{\bar{e}}^2 = 1 + |u'(1)|$, where $u'(1)$ denotes the angular derivative of u at 1. The proof will be omitted as the result is not needed here.

THEOREM 7. *The invariant subspaces of* Z, *besides* $\{0\}$, *are the subspaces* $\mathcal{M}(\bar{e}) \cap \mathcal{M}(u)$ *and* $\mathcal{M}(e) \cap \mathcal{M}(u)$ *with* u *an inner function.*

If the inner function u is not in $\mathcal{M}(\bar{e})$ then $\mathcal{M}(\bar{e}) \cap \mathcal{M}(u)$ and $\mathcal{M}(e) \cap \mathcal{M}(u)$ coincide, but in the contrary case they are different. Thus, an inner function is associated with each nontrivial invariant subspace of Z, but certain inner functions, those in $\mathcal{M}(\bar{e})$, are associated with two different invariant subspaces.

That the subspaces mentioned in the theorem are invariant under Z is obvious. It remains to show that there are no others (besides $\{0\}$).

The operator $Z|\mathcal{M}(e)$ is unitarily equivalent to S, the unitary equivalence being implemented by T_e. This together with Beurling's theorem implies that the Z-invariant subspaces contained in $\mathcal{M}(e)$, other than $\{0\}$, are the subspaces $\mathcal{M}(e) \cap \mathcal{M}(u)$ with u an inner function. It remains to treat invariant subspaces of Z not contained in $\mathcal{M}(e)$.

Let $\mathcal{I}$ be a Z-invariant subspace not contained in $\mathcal{M}(e)$, and suppose that the greatest common inner divisor of the functions in $\mathcal{I}$ is 1. Then 1 is also the greatest common inner divisor of the functions in $\mathcal{I} \cap \mathcal{M}(e)$ (since $(Z-1)\mathcal{I} \subset \mathcal{I}$), and therefore $\mathcal{I} \supset \mathcal{M}(e)$ (by what was noted in the preceding paragraph). Since $\mathcal{I}$ contains a function that does not vanish at the point 1, it thus contains the constant function 1, and we can conclude that $\mathcal{I} = \mathcal{M}(\bar{e})$.

To finish the proof of the theorem, let $\mathcal{I}$ be any Z-invariant subspace not contained in $\mathcal{M}(e)$, and let u be the greatest common inner divisor of the functions in $\mathcal{I}$. Choose a function f in $\mathcal{I}$ with $f(1) \neq 0$. Then $f = ug$ where $g\ (= T_{\bar{u}}f)$ is in $\mathcal{M}(\bar{e})$. Obviously $g(1) \neq 0$, so u has a (nonzero) radial limit $u(1)$ at the point 1. We have

$$\frac{f - f(1)}{z-1} = \frac{u(g - g(1))}{z-1} + \frac{g(1)\,(u - u(1))}{z-1}.$$

The term on the left and the first term on the right belong to H^2, and therefore so does the second term on the right, which means that u is in $\mathcal{M}(\bar{e})$. Let $\mathcal{K} = T_{\bar{u}}\mathcal{I}$. Then $\mathcal{K}$ is contained in $\mathcal{M}(\bar{e})$, and its closure in $\mathcal{M}(\bar{e})$ is a Z-invariant subspace. By what was established in the preceding paragraph, the closure of $\mathcal{K}$ equals $\mathcal{M}(\bar{e})$. Since u is a multiplier of $\mathcal{M}(\bar{e})$, it follows that $u\mathcal{K} = \mathcal{I}$ contains $u\mathcal{M}(\bar{e}) = \mathcal{M}(\bar{e}) \cap \mathcal{M}(u)$. The opposite inclusion being trivial, we can conclude that $\mathcal{I} = \mathcal{M}(\bar{e}) \cap \mathcal{M}(u)$, completing the proof of Theorem 7.

The equality $S^*S - SS^* = 1 \otimes 1$ together with a little juggling enables one to show that $ZT_{\bar{e}} = T_{\bar{e}}(S + (1 \otimes 1))$, which tells us that the operator Z is unitarily equivalent to the operator $S + (1 \otimes 1)$. Theorem 7 thus raises the question of describing the invariant subspace lattice of $S + c(1 \otimes 1)$, where c is a complex number other than 1. The case $|c| = 1$ is really the same as the case $c = 1$, because if $|c| = 1$ we can write $S + c(1 \otimes 1) = c(\bar{c}S + (1 \otimes 1))$, and $\bar{c}S$ is unitarily equivalent to S.

The case $|c| < 1$ is very simple, because in that case $S + c(1 \otimes 1)$ is similar to S; in fact, one can easily check that

$$S + c(1 \otimes 1) = (1 - cS^*)^{-1}S(1 - cS^*).$$

Thus, when $|c| < 1$, the invariant subspace lattice of $S + c(1 \otimes 1)$ is isomorphic to Beurling's lattice.

The case $|c| > 1$ is only slightly more complicated. In that case one can easily check that c is an eigenvalue of $S + c(1 \otimes 1)$, the corresponding eigenspace being spanned by the function $(c - z)^{-1}$. Let $\mathcal{N}_c$ denote the eigenspace. Then H^2 is the (nonorthogonal) direct sum of $\mathcal{N}_c$ and H_0^2, both of which are invariant under $S + c(1 \otimes 1)$. The restriction of $S + c(1 \otimes 1)$ to H_0^2 is the same as the restriction of S, so the invariant subspaces of $S + c(1 \otimes 1)$ contained in H_0^2 are described by Beurling's theorem. An invariant subspace not contained in H_0^2 is the vector sum of $\mathcal{N}_c$ and its intersection with H_0^2. The invariant subspace lattice of $S + c(1 \otimes 1)$, when $|c| > 1$, is thus isomorphic to the direct sum of Beurling's lattice with the lattice $\{0, 1\}$. Each inner function gives rise to two invariant subspaces.

From Theorem 7 we see that, in some sense, the invariant subspace lattice of $S + c(1 \otimes 1)$ for $|c| = 1$ is intermediate between the lattices for $|c| < 1$ and $|c| > 1$.

9. AN EXAMPLE

An example will now be constructed to show that it is possible for b to be an outer function and $|a|^2$ to satisfy the condition (A_2) (implying the invertibility of $T_{a/\bar{a}}$) even though a and b fail to form a corona pair. The example is needed to show that Theorem 3 cannot be strengthened in one plausible direction. The simple example in Section 5 of an outer function b such that a and b do not form a corona pair does not suffice here, for if $|a|^2$ satisfies (A_2), then $\log |a|$ must belong to BMO, the space of functions of bounded mean oscillation on ∂D [10], [11], a requirement that is not met by the example in Section 5.

The following notations will be used. The Poisson kernel for the point w in D will be denoted by P_w. The value at w of the Poisson integral of the integrable function x will be denoted by $x(w)$ or by $P_w(x)$. We shall employ the standard seminorm $\| \ \|_*$ in the space BMO [10], [11]: if x is in BMO, then $\|x\|_*$ is the supremum of the mean oscillations of x over the subarcs of ∂D. The seminorm $\| \ \|_*$ fails to be a norm because it annihilates the constant functions, but we shall nevertheless refer to it as the $*$-norm. The usual norm in BMO (or rather, one of the usual norms) is defined by $\|x\|_{\mathrm{BMO}} = \|x\|_* + |x(0)|$. The letter C will stand for a generic absolute constant, possibly varying in magnitude from one occurence to the next.

The construction is ever so slightly delicate for the following reason. The condition (A_2) on $|a|^2$ implies the inequality

$$|a(w)|^2 \geqslant \text{const.}\, P_w(|a|^2) \qquad (|w| < 1)$$

with a constant that is independent of w (and is in fact equivalent to that condition together with its analogue for a^{-1} [11, p. 83]). Thus, if b is an outer function, and if $|a|^2$ and $|b|^2$ both satisfy (A_2), then a and b will automatically form a corona pair (since $P_w(|a|^2) + P_w(|b|^2) = 1$ for all w). In the desired example, therefore, (A_2) must fail for $1 - |a|^2$ while holding for $|a|^2$, entailing a mild schizophrenia on the part of the function a. Roughly speaking, such behavior is possible because, while the (A_2) condition for $|a|^2$ imposes severe restrictions on $|a|^2$ where it is small, it does not impose comparable restrictions on $|a|^2$ where it is close to 1.

The following lemma contains the main step in the construction.

LEMMA 9. *There exists a pair x, y of nonnegative functions on ∂D with the following properties:*

(i) *x is in* BMO *with $\|x\|_*$ less than any preassigned positive number;*

(ii) *y is integrable;*

(iii) $\min\{x(w), y(w)\} \leqslant 1$ *for each w in ∂D;*

(iv) *there is a sequence (w_n) in D with $|w_n| \to 1$ such that $x(w_n) \to \infty$ and $y(w_n) \to \infty$.*

Once this lemma is established, we can obtain the desired functions a and b as follows. It is known [10], [11] that a nonnegative function on ∂D will satisfy (A_2) if its logarithm is in BMO and has a sufficiently small $*$-norm. We can therefore suppose that we have selected functions x and y as in Lemma 9 such that e^{-2x} satisfies (A_2). Let u be the outer function with modulus e^{-x} and v the outer function with modulus e^{-y}. Condition (iv) implies that $u(w_n) \to 0$ and $v(w_n) \to 0$, while (iii) implies that $|u|^2 + |v|^2$ is bounded away from 0 on ∂D. Let s be the outer function with modulus $(|u|^2 + |v|^2)^{-1/2}$. The functions $a = su$ and $b = sv$ then meet all of our requirements.

The proof of the lemma involves certain auxiliary functions. The function x will be constructed first. For $0 < t < 1$, let I_t denote the closed subarc of ∂D centered at the point 1 of arclength $2t^{2/3}$. Let x_t denote the function on ∂D whose value at $e^{i\theta}$ is $\log(t^{2/3}/|\theta|)$ for $|\theta| \leqslant t^{2/3}$ and 0 for $t^{2/3} \leqslant |\theta| \leqslant \pi$.

SUBLEMMA 1. *$\|x_t\|_* \leqslant C \quad (0 < t < 1)$.*

In fact, the function x_t is the maximum of 0 and the function $\log(t^{2/3}/|\theta|)$ $(0 < |\theta| \leqslant \pi)$, so, as is easily seen, its $*$-norm is at most twice the $*$-norm of the latter function. That function is the sum of the function $\log(1/|\theta|)$ $(0 < |\theta| \leqslant \pi)$ and a constant, so its $*$-norm is the same as that of the function $\log(1/|\theta|)$ (which is well known to be finite).

SUBLEMMA 2. *If $r = 1 - t$, then $P_r \geqslant Ct^{-1/3}$ on I_t.*

This follows immediately from the expression

$$P_r(\theta) = \frac{1 - r^2}{(1 - r)^2 + 4r\sin^2\dfrac{\theta}{2}}\,.$$

SUBLEMMA 3. *If $r = 1 - t$, then $P_r(x_t) \geqslant C\log(1/t)$.*

As $P_r(\theta) \geqslant \dfrac{C}{1 - r}$ for $|\theta| < 1 - r$, we have

$$P_r(x_t) \geqslant \frac{C}{1 - r}\int_0^t \log(t^{2/3}/\theta)\,\mathrm{d}\theta =$$

$$= \frac{C}{1 - r}(t\log(1/t^{1/3}) + t) \geqslant C\log(1/t).$$

To construct the function x, we choose a sequence $(\mathrm{e}^{\mathrm{i}\theta_n})$ of points of ∂D and two sequences (c_n) and (t_n) of positive numbers, with $t_n < 1$ for all n, such that the following conditions are satisfied:

(A) The arcs $I_n = \mathrm{e}^{\mathrm{i}\theta_n} I_{t_n}$ are mutually disjoint.

(B) $\sum c_n < \infty$.

(C) $c_n \log(1/t_n) \to \infty$.

(D) $\sum n t_n^{1/3} < \infty$.

One can obviously do this by choosing $(\mathrm{e}^{\mathrm{i}\theta_n})$ first, then (c_n), and then (t_n), and at the same time make $\sum c_n$ as small as desired. For each n, let $w_n = (1 - t_n)\mathrm{e}^{\mathrm{i}\theta_n}$ and $x_n(\mathrm{e}^{\mathrm{i}\theta}) = c_n x_{t_n}(\mathrm{e}^{\mathrm{i}(\theta - \theta_n)})$. Then $\|x_n\|_* \leqslant Cc_n$ by Sublemma 1. Since x_n vanishes on at least half of ∂D, one easily checks that $\|x_n\|_{\mathrm{BMO}} \leqslant 2\|x_n\|_*$, and therefore the series $\sum x_n$ converges in the norm of BMO. Defining $x = \sum x_n$, we have $\|x\|_* \leqslant$ $\leqslant C\sum c_n$, so we can attain condition (i) of Lemma 9 by choosing $\sum c_n$ sufficiently small. Condition (iv) holds for x because of Sublemma 3 and condition (C).

To construct the function y, let E_n denote the subset of I_n where $x_n < 1$. The set E_n has positive measure because x_n is continuous (except at the center of I_n) and vanishes at the endpoints of I_n. Let y_n be that multiple of the characteristic function of E_n satisfying $P_{w_n}(y_n) = n$, and let $y = \sum y_n$. The integrability of y follows from Sublemma 2 and condition (D):

$$\int_{-\pi}^{\pi} y \, d\theta = \sum \int_{I_n} y_n \, d\theta \leqslant$$

$$\leqslant C \sum t_n^{1/3} \int_{-\pi}^{\pi} y_n P_{w_n} \, d\theta = C \sum n t_n^{1/3} < \infty.$$

Condition (iv) for y and condition (iii) being immediate from the construction, the proof of Lemma 9 is complete.

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