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DUAL ALGEBRAIC STRUCTURES ON OPERATOR ALGEBRAS RELATED TO FREE PRODUCTS

DAN VOICULESCU

In [3] L. G. Brown considered C^*-algebras $\mathscr{U}_{\mathrm{nc}}(n)$, $\mathscr{G}r_{\mathrm{nc}}(n)$ corresponding to the unitary group and to the grassmanian. On the other hand in [12] we introduced operations for probability measures on $\mathbf{R}$ and $\mathbf{T}$ (studied in [13], [14]) related to reduced free products, which are similar to convolution on these groups. The motivation for the present note is to show that from the general point of view of categories these analogies can be given a precise meaning in the framework of dual algebraic structures related to free products and to present further examples. Thus $\mathscr{U}_{\mathrm{nc}}(n)$ is a group, in a dual sense, which acts on $\mathscr{G}r_{\mathrm{nc}}(n)$ and the operators in [12] are precisely the convolution on the state spaces of two operator algebras which are dual groups related to $\mathbf{R}$ and $\mathbf{T} = \mathscr{U}(1)$. Also, with each operator algebra with a dual group structure there is associated a group (for $\mathscr{U}_{\mathrm{nc}}(n)$ it is $\mathscr{U}(n)$) and a dual action on some operator algebra gives rise also to an action of the associated group on that operator algebra (see 2.5). In fact the action of $\mathscr{U}(n, 1)$ on the Cuntz-algebra $\mathscr{O}_n$ constructed in [12] is due to the existence of such a dual action of a certain dual group $\mathscr{U}_{\mathrm{nc}}(n, 1)$. The algebras we consider are operator algebras, however it is clear that the dual algebraic structures make sense also in a purely algebraic context and the examples in Section 5 have analogs in the purely algebraic context.

In order to deal also with "non-compact" dual groups we shall have to work with pro-C^*-algebras. In the case of a countable family of semi-norms these are essentially the same thing as Arveson's σ-C^*-algebras [1]. The necessary facts about pro-C^*-algebras are given in Section 1.

The generalities about dual algebraic structures (in the usual sense of category theory) are in Section 2.

The convolution on state spaces is defined in Section 3.

In Section 4 a generalization of Kuiper's theorem is proven for representations of dual groups.

Section 5 is a list of examples of dual groups and of dual actions of these, analogous to certain Lie groups and homogeneous spaces. In particular we exhibit a generalization of the C^*-algebras of Cuntz [6] and of the $\mathcal{U}(n, 1)$ action on these algebras [12].

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1. PRO-C^*-ALGEBRAS

1.1. In view of the examples in Section 5, the limitation of our considerations to complex C^*-algebras would be artificial. Hence, as in [9], the C^*-algebras will be either real, complex or "real" (i.e. complex with an antilinear involutive automorphism) (see § 1 of [9] and [10]). Accordingly the representations of these C^*-algebras will be on Hilbert spaces which are real, complex and respectively complex with an antiunitary involution. By Λ we shall denote the C^*-algebra $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{C}$ with complex conjugation corresponding to each of the three cases considered. For real C^*-algebras the states will be states of their complexification which are real. Similarly for "real" C^*-algebras the states will be assumed to be real on the real subalgebra.

1.2. A pro-C^*-algebra is a C^*-algebra $(A, \| \ \|)$ endowed with a family of C^*-seminorms $(\| \ \|_\alpha)_{\alpha \in I}$ indexed by some directed set I so that $\alpha \leqslant \beta \Rightarrow \|x\|_\alpha \leqslant \leqslant \|x\|_\beta$, $\sup_{\alpha \in I} \|x\|_\alpha = \|x\|$ and moreover:

$$A_1 = \varprojlim A_{\alpha 1}$$

where the subscript 1 indicates the unit-ball and A_α is the quotient of A by the ideal annihilated by $\| \ \|_\alpha$.

Note that given a directed set I and epimorphisms of C^*-algebras $\varphi_{\alpha\beta} : A_\beta \to A_\alpha$ when $\alpha \leqslant \beta$, the inverse limit C^*-algebra is naturally equiped with a pro-C^*-algebra structure.

1.3. EXAMPLE. X a locally compact space, $A = C_b(X)$ the bounded continuous functions on X and the C^*-seminorms $\|f\|_K = \sup_{t \in K} | f(t) |$, where K runs over the compact subsets of X.

1.4. If $(A, (\| \ \|_\alpha)_{\alpha \in I})$ and $(B, \| \ \|_\beta)_{\beta \in J})$ are pro-C^*-algebras then a morphism is a $*$-homomorphism (intertwining the involutive antilinear $*$-automorphisms in the "real" case) $\varphi : A \to B$ such that for every $\beta \in J$ there is $\alpha \in I$ so that $\|\varphi(x)\|_\beta \leqslant \|x\|_\alpha$. Note, that this implies $(A, (\| \ \|_\alpha)_{\alpha \in I})$ and $(A, (\| \ \|_\beta)_{\beta \in J})$ are isomorphic pro-C^*-algebras if for every $\alpha \in I$ there is $\beta \in J$ so that $\|x\|_\alpha \leqslant \|x\|_\beta$

and vice-versa for every $\beta \in J$ there is $\alpha \in I$ with $\|x\|_\beta \leqslant \|x\|_\alpha$. Also, this definition of morphisms implies that a representation on a Hilbert space of $(A, (\| \ \|_\alpha)_{\alpha \in I})$ actually factors through some A_α.

A state on a pro-C^*-algebra A will be always assumed to be continuous with respect to one of the seminorms. This ensures that the Gelfand-Naimark-Segal construction yields a representation of A.

1.5. A pro-C^*-algebra will be called separable if it is isomorphic to a pro-C^*-algebra having a countable family of seminorms and such that each of the quotients A_α is separable. A separable pro-C^*-algebra is always isomorphic to one with the seminorms indexed by $\mathbf{N}$.

1.6. Let $(A, (\| \ \|_\alpha)_{\alpha \in I})$ be a pro-C^*-algebra and B a C^*-algebra. Then $\mathrm{Hom}(A, B)$ is the direct limit $\varinjlim \mathrm{Hom}(A_\alpha, B)$. Endowing each $\mathrm{Hom}(A_\alpha, B)$ with the topology of point-norm convergence and the direct limit with the direct limit topology yields a topology on $\mathrm{Hom}(A, B)$, which depends only on the isomorphism class of A. This implies that if $y \in \varprojlim A_\alpha$, the inverse limit being in the category of sets, the $\mathrm{Hom}(A, B) \ni \pi \to \pi(y) \in B$ is a continuous map. If X is compact and $F : X \to \mathrm{Hom}(A, B)$ is continuous, then $\sup_{x \in X} \| F(x)(y) \| < \infty$ for $y \in \varprojlim A_\alpha$ which, in turn, if I is countable, easily yields that $F(X) \subset \mathrm{Hom}(A_\alpha, B)$ for some $\alpha \in I$.

1.7. If φ is a morphism of pro-C^*-algebras as in 1.4, then $\mathrm{Ker}\,\varphi$ together with the restriction of the seminorms $(\| \ \|_\alpha)_{\alpha \in I}$ to $\mathrm{Ker}\,\varphi$ is a pro-C^*-algebra as can be easily checked.

Let $\mathscr{I} \subset A$ be a closed two-sided ideal of A which is pro-C^* with respect to the restriction of the seminorms $(\| \ \|_\alpha)_{\alpha \in I}$ of A. Then $A_\alpha / \mathscr{I}_\alpha$ is a quotient of $A/\mathscr{I}$ and let $\| \ \|'_\alpha$ be the corresponding C^*-seminorm on $A/\mathscr{I}$. Assuming moreover that the set I is countable (in which case there is an isomorphic pro-C^*-algebra with seminorms indexed by $\mathbf{N}$) one can easily show that $(A/\mathscr{I}, (\| \ \|'_\alpha)_{\alpha \in I}$ is a pro-C^*--algebra.

1.8. We pass now to free products (see [2]). *Throughout this paper we shall consider only free products of unital C^*-algebras and the free product will be assumed to be with amalgamation over A. Thus we shall write $A * B$ for $A *_A B$.*

The definition of the free product extends in an obvious way to real and "real" C^*-algebras. Remark also, that the free product of "real" C^*-algebras, after forgetting the "real" structure does not coincide with the free product of the underlying complex C^*-algebras, since the free product for "real" C^*-algebras is universal with respect to morphisms which intertwine the involutions.

1.9. For unital pro-C^*-algebras $(A, (\| \ \|_\alpha)_{\alpha\in I})$ and $(B, (\| \ \|_\beta)_{\beta\in J})$ we define their free product as follows: the $A_\alpha * B_\beta$ form a projective system and we define $(A \mathbin{\overline{*}} B)_1 = \varprojlim (A_\alpha * B_\beta)_1$ and consider on $A \mathbin{\overline{*}} B$ the C^*-seminorms $\| \ \|_{(\alpha,\beta)}$ corresponding to the quotients $A_\alpha * B_\beta$. Note that $A \mathbin{\overline{*}} B$ and $A * B$ are not equal in general.

2. DUAL ALGEBRAIC STRUCTURES

The algebraic structures, to which we refer, are algebraic structures in a category as defined for instance in Chapter IV, § 1 of [4], the category of unital pro-C^*-algebras with unit-preserving morphisms. This section briefly recalls, for the readers convenience, the generalities about algebraic structures as they appear in the context of this particular category. Direct products exist in this dual category and are precisely the free products of pro-C^*-algebras (with amalgamation over Λ) and Λ is a final object. Hence, roughly speaking, the recipe for defining a dual algebraic structure on a pro-C^*-algebra is to define an n-ary dual operation as a unital morphism $\mu : A \to \underbrace{A * \ldots * A}_{n\text{-times}}$ and a null-ary operation as a unital morphism $\chi : A \to \Lambda$ and to require that the diagrams defining the algebraic structure (with inverted arrows) be commutative.

2.1. A dual group is a unital pro-C^*-algebra A together with unital morphisms $\mu : A \to A * A$, $j : A \to A$, $\chi : A \to \Lambda$ so that:

$$(\mu * \mathrm{id}_A)\mu = (\mathrm{id}_A * \mu)\mu$$

$$j^2 = \mathrm{id}_A$$

$$\delta_A(j * \mathrm{id}_A) = \varepsilon_A \chi$$

$$\delta_A(\mathrm{id}_A * j) = \varepsilon_A \chi$$

$$(\chi * \mathrm{id}_A)\mu = \mathrm{id}_A$$

$$(\mathrm{id}_A * \chi)\mu = \mathrm{id}_A$$

where $\varepsilon_A : \Lambda \to A$ is the unique (unital) morphism and $\delta_A : A * A \to A$ is the natural morphism mapping both A's in the free product identically into A.

2.2. A dual action of the dual group (A, μ, j, χ) on the pro-C^*-algebra M is given by a unital morphism $\sigma : M \to A \ast M$ such that

$$(\mu \ast \mathrm{id}_M)\sigma = (\mathrm{id}_A \ast \sigma)\sigma$$

$$(\chi \ast \mathrm{id}_M)\sigma = \mathrm{id}_M.$$

2.3. Under the assumption in 2.2, if B is a unital pro-C^*-algebra then $\mathrm{Hom}(A, B)$ is a group the operation being $\pi_1[\mu]\pi_2 = \delta_B(\pi_1 \ast \pi_2)\mu$, the inverse of π is πj, and the unit $\varepsilon_B\chi$. The group $\mathrm{Hom}(A, B)$ acts on $\mathrm{Hom}(M, B)$, i.e. for each $\pi \in \mathrm{Hom}(A, B)$ there is a map $\sigma(\pi) : \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, B)$ defined by $\sigma(\pi)(\rho) = \delta_B(\pi \ast \rho)\sigma$ and $\sigma(\pi_1)\sigma(\pi_2) = \sigma(\pi_1[\mu]\pi_2)$, $\sigma(\varepsilon_B\chi)\rho = \rho$.

2.4. If $\pi \in \mathrm{Hom}(A, M)$ we define $\beta(\pi) \in \mathrm{Hom}(M, M)$ by $\beta(\pi) = \sigma(\pi)(\mathrm{id}_M)$. We have $\beta(\pi_1)\beta(\pi_2) = \beta((\beta(\pi_1)\pi_2)[\mu]\pi_1)$.

2.5. In particular there is a homomorphism $\gamma : \mathrm{Hom}(A, A) \to \mathrm{Aut}(M)$ defined by

$$\gamma(\xi) = \beta(\varepsilon_M \xi j).$$

Indeed, we have

$$\gamma(\xi_1)\gamma(\xi_2) = \beta((\beta(\varepsilon_M\xi_1 j)\varepsilon_M\xi_2 j)[\mu]\varepsilon_M\xi_1 j) =$$

$$= \beta(\varepsilon_M\xi_2 j[\mu]\varepsilon_M\xi_1 j) = \beta(\varepsilon_M(\xi_2 j[\mu]\xi_1 j)) =$$

$$= \beta(\varepsilon_M(\xi_1[\mu]\xi_2)j) = \gamma(\xi_1[\mu]\xi_2),$$

and also

$$\gamma(\chi)m = \beta(\varepsilon_M\chi j)m = \sigma(\varepsilon_M\chi j)(\mathrm{id}_M)m =$$

$$= \delta_M((\varepsilon_M\chi j) \ast \mathrm{id}_M)\sigma m =$$

$$= \delta_M(\varepsilon_M \ast \mathrm{id}_M)(\chi \ast \mathrm{id}_M)\sigma m = m$$

where we have used the fact that $\chi j = \chi$.

This should be kept in mind for the examples we shall consider: *a dual action of the dual group A on M gives an action of the group* $\mathrm{Hom}(A, A)$ *on* M.

3. CONVOLUTION ON STATE SPACE

Let A and B be unital pro-C^*-algebras and let φ and ψ be states of A, and respectively B. The definition of the free product state $\varphi \ast \psi$ (see [2], [12]) extends also to the real and "real" cases. If A is a dual group (actually dual semigroup would suffice) then there is an associated binary operation on the state space $E(A)$ defined by

$$\varphi_1 \circledast \varphi_2 = (\varphi_1 * \varphi_2) \circ \mu.$$

Moreover, the dual nullary operation $\chi : A \to A$ is a neutral element for the operation $\circledast$ and in case A is a group the map $\varphi \to \varphi \circ j$ is an antiautomorphism of $E(A)$, i.e.:

$$(\varphi_1 \circ j) \circledast (\varphi_2 \circ j) = (\varphi_2 \circledast \varphi_1) \circ j.$$

Since the free product of two trace-states is again a trace state, it follows that the traces form a subsemigroup of $E(A)$.

4. A GENERALIZATION OF KUIPER'S THEOREM

4.1. Let X be a compact space, A a separable pro-C^*-algebra and B a C^*-algebra. Let Map(X, Hom(A, B)) denote the space of continuous maps from X to Hom(A, B) topologized as in 1.6. In view of 1.6, Map(X, Hom(A, B)) identifies with the direct limit of Map(X, Hom(A_α, B)) which in turn identify with Hom(A_α, $C(X, B)$) the direct limit being Hom(A, $C(X, B)$). We shall endow Map(X, Hom(A, B)) with the topology obtained from the identification with Hom(A, $C(X, B)$).

4.2. THEOREM. *Let X be a separable compact space, H a separable infinite-dimensional Hilbert space and A a separable pro-C^*-algebra which is a dual group. Then the space* Map(X, Hom(A, $L(H)$)) *is connected.*

Proof. Let $F \in$ Map(X, Hom(A, $L(H)$)). We shall prove that the connected component of F contains the constant map equal to $\chi \otimes I_H$, where χ is the nullary operation of A.

Let $\Omega = \overline{\bigcup_{x \in X} F(x)(A)} \subset L(H)$ and let D be the C^*-algebra generated by Ω. In view of our assumptions D is separable and there is a continuous map $\tilde{F} : X \to \mathrm{Hom}(A, D)$ such that $F(x) = \iota \circ \tilde{F}(x)$ where $\iota : D \to L(H)$ is the inclusion.

We shall use several times the fact that the connected component of F and of a limit of maps of the form UFU^* where U is a constant unitary on H (real in the "real" case) are the same.

Using the non-commutative Weyl-von Neumann type theorem of [11] (for the adaptation to the real and "real" case see [9]) we may replace H by $H \oplus H \oplus \dots$ and F by Φ where

$$\Phi(x) = F(x) \oplus G(x) \oplus G(x) \oplus \dots$$

with $G(x) = \rho \circ \tilde{F}(x)$, ρ being a representation of D on H.

Let $\alpha(t)$ be the inner automorphism of $L(H \oplus H)$ corresponding to conjugation with

$$U(t) = \begin{pmatrix} \cos \frac{\pi t}{2} I_H & -\sin \frac{\pi t}{2} I_H \\ \sin \frac{\pi t}{2} I_H & \cos \frac{\pi t}{2} I_H \end{pmatrix}.$$

We consider

$$\Psi_1(x, t) = F(x) \oplus E_1(x, t) \oplus E_1(x, t) \oplus \dots$$

where $E_1 \in \mathrm{Map}(X \times [0,1], \mathrm{Hom}(A, L(H \oplus H))$ is defined by

$$E_1(x, t) = (G(x) + \chi I_H)[\mu](\alpha(t) \circ ((\chi \otimes I_H) + G(x)).$$

Then Ψ_1 connects $\Phi(x) = \Psi_1(x, 0)$ with $\Psi_1(x, 1)$ which after permuting the summands in $H \oplus H \oplus \dots$ can be written as

$$K(x) \oplus (\chi \otimes I_H) \oplus (\chi \otimes I_H) \oplus \dots$$

where

$$K(x) = F(x) \oplus (G(x)[\mu]G(x)) \oplus (G(x)[\mu]G(x)) \oplus \dots .$$

After identification of $H \oplus H \oplus \dots$ with H, we may assume $K(x)$ is a representation of A on H.

Consider further

$$\Psi_2(x, t) = K(x) \oplus E_2(x, t) \oplus E_2(x, t) \oplus \dots$$

$$\Psi_3(x, t) = E_2(x, 1 - t) \oplus E_2(x, 1 - t) \oplus \dots$$

where

$$E_2(x, t) = (K(x) \oplus (\chi \otimes I_H))[\mu](\alpha(t) \circ ((K(x) \circ j) \oplus (\chi \otimes I_H))).$$

We have

$$\Psi_2(x, 0) = \Psi_1(x, 1)$$

$$\Psi_2(x, 1) = K(x) \oplus (K(x) \oplus (K(x) \circ j)) \oplus (K(x) \oplus (K(x) \circ j)) \oplus \dots$$

$$\Psi_3(x, 0) = (K(x) \oplus (K(x) \circ j)) \oplus (K(x) \oplus (K(x) \circ j)) \oplus \dots$$

$$\Psi_3(x, 1) = (\chi \otimes I_H) \oplus (\chi \otimes I_H) \oplus \dots .$$

Since $\Psi_2(x, 1)$ and $\Psi_3(x, 0)$ can be identified after a permutation, we can connect F to

$$(\chi \otimes I_H) \oplus (\chi \otimes I_H) \oplus \dots$$

which is equivalent to $\chi \otimes I_H$. Q.E.D.

4.3. As we shall see in Section 5, $C(\mathbf{T})$ has a dual group structure and $\mathrm{Hom}(C(\mathbf{T}), L(H))$ being homeomorphic to the unitary group $\mathscr{U}(H)$, the preceding theorem implies that $\mathrm{Map}(X, \mathscr{U}(H))$ is connected, which is the essence of Kuiper's theorem.

5. EXAMPLES

In this section we shall discuss examples of dual groups (A, μ, j, χ) and of dual actions on pro-C^*-algebras M, $\sigma : M \to A \mathbin{\overline{*}} M$.

The natural injections of B into $B \mathbin{\overline{*}} B$ will be denoted by i_1, i_2.

For "real" algebras the antilinear automorphism will be denoted by $a \to \bar{a}$.

The pro-C^*-algebras A will arise as inverse limits of C^*-algebras $A_1 \leftarrow A_2 \leftarrow \leftarrow A_3 \leftarrow \dots$.

5.1. THE DUAL GROUPS $\mathbf{R}_{\mathbf{C},\mathrm{nc}}$, $\mathbf{R}_{\mathbf{R},\mathrm{nc}}$. We take $A_n = C([-n, n])$ to be the continuous functions on $[-n, n]$, complex-valued in the first case and real-valued in the second case. Let T_n be the identical function in $C([-n, n])$ and let $\mu_n : A_{2n} \to \to A_n * A_n$ be given by $\mu_n(T_{2n}) = i_1(T_n) + i_2(T_n)$. μ is the inverse limit of the μ_n's, j the inverse limit of j_n's, $j_n(T_n) = -T_n$ and χ the inverse limit of χ_n's, $\chi_n(f) = f(0)$.

The groups $\mathrm{Hom}(\mathbf{R}_{\mathbf{C},\mathrm{nc}}, A)$, $\mathrm{Hom}(\mathbf{R}_{\mathbf{R},\mathrm{nc}}, A)$ are isomorphic to $\mathbf{R}$.

The convolution on the state space of $\mathbf{R}_{\mathbf{C},\mathrm{nc}}$ is the operation on compactly supported probability measures on $\mathbf{R}$, which we introduced in [12] and studied in [13].

5.2. THE DUAL GROUPS CORRESPONDING TO CERTAIN MATRIX GROUPS. All these dual groups will arise as inverse limits of A_n's generated by $(X_{i,j;n})_{1 \leqslant i,j \leqslant d}$ satisfying

$$n^{-1} I_m \leqslant X_n^* X_n \leqslant n I_m$$

$$n^{-1} I_m \leqslant X_n X_n^* \leqslant n I_m$$

where $X_n = \sum_{1 \leqslant i, j \leqslant d} X_{i,j;n} \otimes e_{ij} \in A_n \otimes \mathcal{M}_d$ together with additional relations which will be specified in each case. The dual operations μ and j arise from $\mu_n : A_{n^2} \to A_n \ast A_n$, $j_n : A_n \to A_n$ where

$$\mu_n(X_{i,j;n^2}) = \sum_{1 \leqslant k \leqslant d} i_1(X_{i,k;n}) i_2(X_{k,j;n})$$

$$\sum_{1 \leqslant i, j \leqslant d} j_n(X_{i,j;n}) \otimes e_{ij} = X_n^{-1}.$$

The nullary operation χ is the inverse limit of $\chi_n : A_n \to A$ where $\chi_n(X_{i,j;n}) = \delta_{ij}$.

5.3. The examples in 5.4—5.10 will correspond to the matrix groups listed in § 4, Chapter IX of [7]. For these dual groups it will be convenient to recall some notations from [7] which we shall use.

I_m will denote the unit matrix of order m in a given algebra of the form $B \otimes \mathcal{M}_m$ and

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$$

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

$$K_{p,q} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}.$$

If B is "real", for a matrix $Y \in B \otimes \mathcal{M}_m$ we shall write $\overline{Y}$ for the matrix with entries $(\overline{Y})_{ij} = \overline{Y_{ij}}$.

5.4. THE DUAL GROUPS $\mathcal{GL}_{\mathrm{nc}}(d, \mathbf{C})$, $\mathcal{GL}_{\mathrm{nc}}(d, \mathbf{R})$. The algebras are complex and respectively real and the generators satisfy only the relations in 5.2.

We have $\mathrm{Hom}(\mathcal{GL}_{\mathrm{nc}}(d, \mathbf{C}), A) = \mathcal{GL}(d, \mathbf{C})$ and $\mathrm{Hom}(\mathcal{GL}_{\mathrm{nc}}(d, \mathbf{R}), A) = \mathcal{GL}(d, \mathbf{R})$.

5.5. THE DUAL GROUPS $\mathcal{U}^*_{\mathrm{nc}}(2m)$. We have $d = 2m$, the algebras are "real" and we have the additional relations

$$X_{k,l;n} = X_{m+k,m+l;n}$$

$$X_{m+k,l;n} = -X_{k,m+l;n}$$

where $1 \leqslant k, l \leqslant m$.

5.6. THE DUAL GROUPS $\mathscr{U}_{\mathrm{nc}}(p, q), \mathscr{O}_{\mathrm{nc}}(p, q)$. The algebras are complex and respectively real, $d = p + q$ and we required that

$$X_n^* I_{p,q} X_n = I_{p,q}.$$

We have $\mathrm{Hom}(\mathscr{U}_{\mathrm{nc}}(p, q), A) = \mathscr{U}(p, q)$, $\mathrm{Hom}(\mathscr{O}_{\mathrm{nc}}(p, q), A) = \mathscr{O}(p, q)$.

5.7. THE DUAL GROUP $\mathscr{O}_{\mathrm{nc}}(m, \mathbf{C})$. The algebras are "real" and we require that

$$\overline{X_n^*} X_n = I_n.$$

We have $\mathrm{Hom}(\mathscr{O}_{\mathrm{nc}}(m, \mathbf{C}), A) = \mathscr{O}(m, \mathbf{C})$.

5.8. THE DUAL GROUP $\mathscr{O}_{\mathrm{nc}}^*(2m)$. The algebras are "real", $d = 2m$ and we require that

$$\overline{X_n^*} X_n = I_d, \quad J_m X_n J_m = -\overline{X_n}.$$

We have $\mathrm{Hom}(\mathscr{O}_{\mathrm{nc}}^*(2m), A) = \mathscr{O}^*(2m)$.

5.9. THE DUAL GROUP $\mathscr{S}p_{\mathrm{nc}}(m, \mathbf{C})$. The algebras are "real" $d = 2m$ and we require that

$$\overline{X_n^*} J_m X_n = J_m.$$

We have $\mathrm{Hom}(\mathscr{S}p_{\mathrm{nc}}(m, \mathbf{C}), A) = \mathscr{S}p(m, \mathbf{C})$.

5.10. THE DUAL GROUP $\mathscr{S}p_{\mathrm{nc}}(m, \mathbf{R})$. The algebras are real, $d = 2m$ and we require that

$$X_n^* J_m X_n = J_m.$$

We have $\mathrm{Hom}(\mathscr{S}p_{\mathrm{nc}}(m, \mathbf{R}), A) = \mathscr{S}p(m, \mathbf{R})$.

5.11. THE DUAL GROUP $\mathscr{S}p_{\mathrm{nc}}(p, q)$. The algebras are "real" $d = 2(p + q)$ and we require that

$$\overline{X_n^*} J_{p+q} X_n = J_{p+q}$$

$$X_n^* K_{p,q} X_n = K_{p,q}.$$

We have $\mathrm{Hom}(\mathscr{S}p_{\mathrm{nc}}(p, q), A) = \mathscr{S}p(p, q)$.

We pass now to examples of dual actions of dual groups.

5.12. THE DUAL ACTION OF $\mathscr{U}_{\mathrm{nc}}(p, q)$ ON THE ALGEBRA $A_{p,q}$. The algebra $A_{p,q}$ is the complex C^*-algebra defined by the generators S_{ij} ($1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$) and the relations

$$S^* S = I_p, \quad S S^* = I_q$$

where S is the $p \times q$ matrix with entries S_{ij}.

The dual action $\sigma : A_{p,q} \to \mathscr{U}_{nc}(p, q) \overline{*} A_{p,q}$ is given as the inverse limit of $\sigma_n : A_{p,q} \to A_n * A_{p,q}$ (recall $\mathscr{U}_{nc}(p,q)$ is the inverse limit of A_n's) where σ_n is defined by the matrix equality

$$\sigma_n(S) = (\alpha_n + S\gamma_n)^{-1}(\beta_n + S\delta_n)$$

where we have written

$$X_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n$ are $p \times p$ and respectively $p \times q$, $q \times p$ and $q \times q$ matrices and we have identified A_n with the corresponding subalgebra of $A_n \overline{*} A_{pq}$.

Once we have shown that σ_n is a well defined homomorphism, the fact that σ is a dual action is an easy algebraic exercise which we have left to the reader.

To check that σ_n is well defined, amounts, in view of the definition of the algebras, to the following: given X_{kl} $(1 \leqslant k, l \leqslant p + q)$, S_{ij} $(1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q)$ bounded operators on some Hilbert space H such that $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ the matrix with entries X_{kl} has bounded inverse (actually of norm $\leqslant \sqrt{n}$), $X^* I_{p,q} X = I_{p,q}$ and S satisfies $S^*S = I_p$, $SS^* = I_q$, one must show that $T = (\alpha + S\gamma)^{-1}(\beta + S\delta)$ satisfies

$$T^*T = I_p, \quad TT^* = I_q.$$

Note that H may be assumed to be separable since we are dealing with finitely many operators and also infinite dimensional, since we may take an infinite orthogonal sum.

First we must prove that $\alpha + S\gamma$ is invertible. Since $X^* I_{p,q} X = I_{p,q}$ we have $\alpha^*\alpha - \gamma^*\gamma = I_p$ and hence for $\xi \in H \otimes \mathbf{C}^p$ we have

$$\|(\alpha + S\gamma)\xi\| \geqslant (\|\alpha\xi\|^2 - \|\gamma\xi\|^2)(\|\alpha\| + \|\gamma\|)^{-1}\|\xi\|^{-1} = \|\xi\|(\|\alpha\| + \|\gamma\|)^{-1}$$

so that $\alpha + S\gamma$ has a bounded left inverse. By the generalization of Kuiper's theorem in Section 4 the set of $X_{k,l}$ such that X, X^{-1} are bounded and $X^* I_{p,q} X = I_{p,q}$ is connected, hence $\alpha + S\gamma$ is left invertible and is in the same connected component of left invertible elements as $I + S \cdot 0 = I$ and hence invertible.

Putting $\Sigma = (I_p, S)$, we have $\Sigma I_{p,q} \Sigma^* = 0$ so that

$$(I, T) I_{p,q} (I, T)^* = (\alpha + S\gamma)^{-1} \Sigma X I_{p,q} X^* \Sigma^* (\alpha + S\gamma)^{*-1} =$$

$$= (\alpha + S\gamma)^{-1} \Sigma I_{p,q} \Sigma^* (\alpha + S\gamma)^{*-1} = 0$$

which gives $I = TT^*$. To see that $T^*T = I$ we use again the connectedness argument based on the generalization of Kuiper's theorem: $T : H \otimes \mathbf{C}^p \to H \otimes \mathbf{C}^q$ is an isometry which has the same index as the isometry obtained when $X = I$, i.e. S which has index 0. Thus we also have $TT^* = I$.

The algebras $A_{p,q}$ are a generalization of the C^*-algebras of Cuntz [5] and the dual $\mathscr{U}_{\mathrm{nc}}(p, q)$ action in particular implies the $\mathscr{U}(n, 1)$ action on the Cuntz-algebras ([12], see also [5]).

5.13. THE DUAL ACTION OF $\mathscr{O}^*_{\mathrm{nc}}(2m)$ ON THE ALGEBRA D_m. The algebra D_m is a "real" C^*-algebra defined by the generators S_{ij} $(1 \leqslant i, j \leqslant m)$ and the relations

$$S^*S = SS^* = I_m, \quad S^* = -\overline{S}$$

where S is the $m \times m$ matrix with entries S_{ij}. The dual action $\sigma : D_m \to \mathscr{O}^*_{\mathrm{nc}}(2m) \overline{*} D_m$ is defined as the inverse limit of $\sigma_n : D_m \to A_n * D_m$ (recall $\mathscr{O}_{\mathrm{nc}}(2m)$ is the inverse limit of A_n's) where σ_n is defined by the matrix equality $\sigma_n(S) = (\alpha_n + S\gamma_n)^{-1} \cdot (\beta_n + S\delta_n)$ where we have written

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} = \Omega^*_m X_n \Omega_m, \quad \Omega_m = 2^{-1/2} \begin{pmatrix} I_m & \mathrm{i}I_m \\ \mathrm{i}I_m & I_m \end{pmatrix}.$$

Note that for $Y_n = \Omega^*_m X_n \Omega_m$ we have $Y^*_n I_{m,m} Y^*_n = I_{m,m}$ and $\alpha_n = \overline{\delta}_n$, $\beta_n = -\overline{\gamma}_n$.

That $\sigma_n(S)$ is unitary follows from 5.11.

Moreover we have $\alpha_n \alpha^*_n - \beta_n \beta^*_n = I_n$, $\alpha_n \overline{\beta}^*_n + \beta_n \overline{\alpha}^*_n = 0$ so that

$$(S^*\alpha_n - \overline{\beta}_n)(\alpha^*_n - \overline{\beta}_n S^*) = (S^*\beta_n + \overline{\alpha}_n)(\beta^*_n + \overline{\alpha}^*_n S^*)$$

and hence

$$(\sigma_n(S))^* = (\beta^*_n + \overline{\alpha}^*_n S)(\alpha^*_n - \overline{\beta}^*_n S)^{-1} =$$

$$= (S^*\beta_n + \overline{\alpha}^*_n)^{-1}(S^*\alpha_n - \overline{\beta}_n) = -(\overline{\alpha}_n - \overline{S}\beta_n)^{-1}(\overline{S}\alpha_n + \overline{\beta}_n) = -\overline{\sigma_n(S)}$$

which shows that σ_n is a well-defined homomorphism.

5.14. THE DUAL ACTION OF THE COMPLEXIFICATION OF $\mathscr{S}p_{\mathrm{nc}}(m, \mathbf{R})$ ON THE ALGEBRA C_m. The complexification of $\mathscr{S}p_{\mathrm{nc}}(m, \mathbf{R})$ is a "real" pro-C^*-algebra and the defining relations can be written

$$X^*_n J_m X_n = J_m, \quad \overline{X}_n = X_n.$$

The C^*-algebra C_m is a "real" C^*-algebra defined by the generators S_{ij} $(1 \leqslant i, j \leqslant m)$ and the relations

$$SS^* = S^*S = I_m, \quad S^* = \overline{S}$$

where S is the $m \times m$ matrix with entries S_{ij}. The dual action σ is the inverse limit of the $\sigma_n : C_m \to A_n \ast C_m$ where σ_n is defined by $\sigma_n(S) = (\alpha_n + S\gamma_n)^{-1}(\beta_n + S\delta_n)$ where we have written

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} = \Omega_m^* X_n \Omega_m, \quad \Omega_m = 2^{-1/2}\begin{pmatrix} I_m & iI_m \\ iI_m & I_m \end{pmatrix}.$$

Note that for $Y_n = \Omega_m^* X_n \Omega_m$ we have $Y_n^* I_{m,m} Y_n = I_{m,m}$ and $\alpha_n = \overline{\delta}_n$, $\beta_n = \overline{\gamma}_n$. The unitarity of $\sigma_n(S)$ follows from 5.11.

Moreover we have $\alpha_n\alpha_n^* - \beta_n\beta_n^* = I_m$, $\overline{\beta}_n\alpha_n^* = \overline{\alpha}_n\beta_n^*$ which implies

$$(S^*\alpha_n + \overline{\beta}_n)(\alpha_n^* + \overline{\beta}_n^* S^*) = (S^*\beta_n + \overline{\alpha}_n)(\beta_n^* + \overline{\alpha}_n^* S^*)$$

and hence

$$(\sigma_n(S))^* = (\beta_n^* + \overline{\alpha}_n^* S^*)(\alpha_n^* + \overline{\beta}_n^* S^*)^{-1} = (S^*\beta_n + \overline{\alpha}_n)^{-1}(S^*\alpha_n + \overline{\beta}_n) =$$

$$= (\overline{S}\beta_n + \overline{\alpha}_n)^{-1}(\overline{S}\alpha_n + \overline{\beta}_n) = \overline{\sigma_n(S)}$$

which shows that σ_n is a well defined homomorphism.

5.15. The dual actions defined in 5.12, 5.13, 5.14 can be in fact extended to bigger algebras, namely to the algebras where the condition that S be unitary is replaced by the condition that S be a contraction. The proofs are along similar lines.

5.16. FURTHER EXAMPLES. In the examples 5.4—5.11 the groups Hom(A, A) were reductive, but there are many examples where Hom(A, A) is not reductive. For instance there are dual groups for which Hom(A, A) is isomorphic to various solvable groups of upper triangular matrices. Also, taking free products yields further examples.

There are also many examples of dual actions. We should especially draw attention to the natural dual action of $\mathcal{GL}_{\mathrm{nc}}(m, \mathbf{C})$ (and of its various "dual subgroups") on the non-commutative grassmanian considered in [3] or on similarly defined non-commutative flag manifolds (i.e. C^*-algebras generated by $P_{ij}^{(k)}$ $1 \leqslant i, j \leqslant m$, $1 \leqslant k \leqslant n$ such that $P^{(k)} = (P_{ij}^{(k)})_{1 \leqslant i,j \leqslant n}$ are self-adjoint idempotents and $P^{(1)} \geqslant P^{(2)} \geqslant \ldots \geqslant P^{(n)}$).

5.17. An extension of the considerations in the present paper to free products with amalgamation over some fixed algebra has been indicated in [15].

REFERENCES

1. ARVESON, W. B., The harmonic analysis of automorphism groups, in *Operator Algebras and Applications* (ed. R. V. Kadison), Proc. Symp. Pure Math., **38**, Amer. Math. Soc., Providence, 1981.
2. AVITZOUR, D., Free products of C^*-algebras, *Trans. Amer. Math. Soc.*, **271**(1982), 423--425.
3. BROWN, L. G., Ext of certain free product C^*-algebras, *J. Operator Theory*, **6**(1981), 135--141.
4. BUCUR, I.; DELEANU, A., *Introduction to the theory of categories and functiors*, J. Wiley, New York, 1968.
5. CAREY A. L.; EVANS, D. E., On an automorphic action of $\mathcal{U}(n, 1)$ on $\mathcal{O}_n$, preprint.
6. CUNTZ, J., Simple C^*-algebras generated by isometries, *Comm. Math. Phys.*, **57**(1977), 173--185.
7. HELGASON, S., *Differential geometry, Lie groups and symmetric spaces*, Academic Press, 1978.
8. HUA, L. K., *Harmonic analysis of functions of several complex variables in the classical domains*, AMS Translations of Mathematical Monographs, vol. 6, 1963; revised 1979.
9. KASPAROV, G. G., Hilbert C^*-modules: theorems of Stinespring and Voiculescu, *J. Operator Theory*, **4**(1980), 133--150.
10. PALMER, T. W., Real C^*-algebras, *Pacific J. Math.*, **35**(1970), 195--204.
11. VOICULESCU, D., A non-commutative Weyl-von Neumann theorem, *Rev. Roumaine Math. Pures Appl.*, **21**(1976), 97--115.
12. VOICULESCU, D., Symmetries of some reduced free product C^*-algebras, in *Operator Algebras and their connection with Topology and Ergodic Theory*, Springer Lecture Notes, vol. **1132**, pp. 556--588.
13. VOICULESCU, D., Addition of certain non-commuting random variables, INCREST preprint No. 64/1984.
14. VOICULESCU, D., Multiplication of certain non-commuting random variables, INCREST preprint No. 76/1985.
15. VOICULESCU, D., Operations on certain non-commuting operator-valued random variables, INCREST preprint No. 42/1986.

DAN VOICULESCU
Department of Mathematics, INCREST,
Bdul Păcii 220, 79622 Bucharest,
Romania.

Present address:
Department of Mathematics,
University of California,
Berkeley, CA 94720,
U.S.A.

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1. INOUE, A., Locally C^*-algebras, *Mem. Fac. Sci. Kyushu Univ. Ser. A*, **25**(1971), 197--235.
2. FRAGOULOPOLOU, M., Space of representations and enveloping L.M.C.*-algebras, *Pacific J. Math.*, **95**(1981), 61--73.
3. FRAGOULOPOLOU, M., Representations of tensor product L.M.C.*-algebras, *Manuscripta Math.*, **42**(1983), 115--182.
4. FRAGOULOPOLOU, M., *J. Math. Anal. Appl.*, **108**(1985), 422--429.
5. PHILLIPS, N.C., Work in progress.
6. SCHMÜDGEN, K., Über L.M.C.*-algebren, *Math. Nachr.*, **68**(1975), 167--182.