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TIME PROJECTIONS IN A VON NEUMANN ALGEBRA

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INTRODUCTION

The discussion of stopping for L^2-processes described in [3] is tied quite firmly to the context of the Clifford Filtration employing the Itô-Clifford integral in its definitions and (hence) proofs. However the results of [3] indicate an alternative approach to stopping and it is this that we discuss below. While this alternative approach is not restricted to the Clifford Filtration and does not employ a theory of Stochastic Integration the connection between stopping and Stochastic Integration remains. In the last section we return to the Clifford Filtration and employ this connection to characterize some stopped processes.

1. NOTATIONS AND PRELIMINARIES

Let $(\mathscr{A}_\alpha)$, $\alpha \in \mathbf{R}^+$ be an increasing, right continuous family of (finite) von Neumann algebras acting in the Hilbert space $\mathscr{H}$. Let $\mathscr{A}_\infty$ be the von Neumann algebra generated by $\bigcup_\alpha \mathscr{A}_\alpha$. We assume that $\mathscr{A}_\infty$ is finite and that φ is a faithful normal finite trace on $\mathscr{A}_\infty$ with $\varphi(I) = 1$, where I denotes the identity operator on $\mathscr{H}$. For each $\alpha \in \mathbf{R}^+$, let $M_\alpha : \mathscr{A}_\infty \to \mathscr{A}_\alpha$ denote the conditional expectation and for each fixed $p \in [1, \infty)$, $L^p(\mathscr{A}_\alpha)$ the non-commutative Lebesgue space consisting of (possibly unbounded) operators, X, on $\mathscr{H}$, with dense domain, affiliated to $\mathscr{A}_\alpha$ for which $\varphi(|X|^p)^{1/p} = \|X\|_p < \infty$ (see [9] for details). The expectations, M_α, extend to self-adjoint idempotents on the Hilbert space $L^2(\mathscr{A}_\infty)$ with range $L^2(\mathscr{A}_\alpha)$. The family $(L^2(\mathscr{A}_\alpha))$ is an increasing (strongly) right continuous filtration of subspaces of $L^2(\mathscr{A}_\infty)$ which is the closure of their union [2]. An L^p-process is a family (X_α) with $X_\alpha \in L^p(\mathscr{A}_\alpha)$. In fact this last definition is not quite complete because we shall allow our processes to assume a value at time $\alpha = \infty$. Often this value arises quite naturally, for example an L^2-bounded martingale $X = (X_\alpha)$ has an L^2 limit $X_\infty = \lim_{\alpha\to\infty} X_\alpha$ such that $M_\alpha(X_\infty) = X_\alpha$. However some processes (e.g. $Y_\alpha = \alpha I$) have no natural candidate for the value at infinity and in this case we usually assign the value 0. So an L^p-process is a family (X_α) with $X_\alpha \in L^p(\mathscr{A}_\alpha)$

and some stipulation about what happens at infinity. We will describe this by phrases like "the process $X = (X_\alpha)$ *closed* by X_∞".

We will need to consider integration of vector valued functions by a real measure (usually Lebesgue measure). For this we employ the theory described in Chapter III of [6]. In this connection we will also have to consider "Riemann sums" of vector integrals on the interval $[a, b]$, $0 \leqslant a \leqslant b \leqslant \infty$. So let $\mathscr{P}[a, b]$ denote partitions of $[a, b]$. We note for future reference that families indexed by the elements of $\mathscr{P}[a, b]$ will be thought of as nets.

2. STOPPING REFORMULATED

Let $\mathscr{B} = (\Omega, \Sigma, P, (\Sigma_\alpha), \mathbf{R}^+)$ be a stochastic base [8]. A map $\tau : \Omega \to \mathbf{R}^+$ is a stopping time if $\{\tau \leqslant \alpha\} \in \Sigma_\alpha$ for each $\alpha \in \mathbf{R}^+$. In the presence of right continuity of the filtration (Σ_α) this is equivalent to requiring $\{\tau < \alpha\} \in \Sigma_\alpha$ for $\alpha \in \mathbf{R}^+$. In [3] stopping was reformulated using the sets $\{\tau \leqslant \alpha\}$ and the formalism so obtained abstracted to a non-commutative context. While it makes no difference to the formalism obtained it is better to reformulate in terms of the sets $\{\tau < \alpha\}$ for our purpose since we allow our times to take the value ∞. Bearing this in mind with the reformulation given in [3] we have

2.1. Definitions. (i) A (stopping) time adapted to the filtration of von Neumann algebras $(\mathscr{A}_\alpha)$ is a map $\tau : [0, \infty] \to \mathscr{A}_\infty^{\text{Proj}}$ with $\tau(0) = 0$, $\tau(\infty) = I$, $\tau(\alpha) \in \mathscr{A}_\alpha^{\text{Proj}}$ for $\alpha \in (0, \infty)$ and if $\alpha_1 \leqslant \alpha_2$, $\tau(\alpha_1) \leqslant \tau(\alpha_2)$. In other words a time is an increasing projection valued process starting at 0 and finishing (at infinity) at I. We will often write P_α, Q_α, R_α to denote $\tau(\alpha)$ and sometimes $\tau = (P_\alpha)$.

(ii) Let $p \in [1, \infty)$ be fixed and $X = (X_\alpha)$ be an L^p-process closed by X_∞. Let $\tau = (P_\alpha)$ be a time. Let $\theta \in \mathscr{P}[0, \infty]$, say $\theta = \{\alpha_i\}$, then $X_{\tau(\theta)} \overset{\text{def}}{=} \sum_\theta X_{\alpha_i} \Delta P_{\alpha_i}$ where $\Delta P_{\alpha_i} = P_{\alpha_i} - P_{\alpha_{i-1}}$.

(iii) With the notations of (ii); note that $X_{\tau(\theta)} \in L^p(\mathscr{A}_\infty)$ and $\theta \mapsto X_{\tau(\theta)}$ is a net, $\mathscr{P}[0, \infty]$ being ordered by inclusion. If $\lim_\theta X_{\tau(\theta)}$ exists in $\| \;\|_p$ then we denote this limit by X_τ and call it the *stopped operator* (associated with X and τ).

(iv) If $\sigma = (\varphi_\alpha)$, $\tau = (P_\alpha)$ are times then we define

$$\sigma \leqslant \tau \Leftrightarrow P_\alpha \leqslant \varphi_\alpha \quad \forall \alpha \in \mathbf{R}^+.$$

(v) Let $\alpha \in \mathbf{R}^+$ we will identify this 'time' with the process (P_β) where

$$P_\beta = \begin{cases} 0 & \text{if } \beta \leqslant \alpha \\ I & \text{if } \beta > \alpha \end{cases}.$$

(vi) A stopping time $\tau = (P_\alpha)$ is bounded if $\exists s \in \mathbf{R}^+ : P_\alpha = I, \forall \alpha \geqslant s$.

2.2. REMARKS. In (ii) above one could argue that ${}_{\tau(\theta)}X = \sum_i \Delta P_{\alpha_i} X_{\alpha_i}$ is just as good a definition as $X_{\tau(\theta)}$. Note however that if $X^* = (X_\alpha^*)$ then $({}_{\tau(\theta)}X^*) = X^*_{\tau(\theta)}$. Moreover the involution is very well behaved on a finite von Neumann algebra so it seems likely that we could convert most of our results about $X_{\tau(\theta)}$ to results about ${}_{\tau(\theta)}X$ and conversely. But there is yet another perfectly reasonable definition from the point of view of generalisation, that is

$$X(\tau(\theta)) = \sum_i \Delta P_{\alpha_i} X_{\alpha_i} \Delta P_{\alpha_i}.$$

This has many nice features in that the operation is positivity and adjoint preserving and, anticipating § 3, the map

$$x \mapsto \sum_i \Delta P_{\alpha_i} M_{\alpha_i}(X) \Delta P_{\alpha_i}$$

is a *very* nice map on L^2 (an expectation). However this 'bilateral' stopping involves a restriction on both the domain *and* the range of the operators involved whereas the form we have chosen restricts just the domain, which is in keeping with the commutative case.

In (ii) the value of $X_{\tau(\theta)}$ depends in principle upon the choice of X_∞ that closes X but as we remarked above there is often a natural choice for X_∞.

The existence of $\lim_\theta X_{\tau(\theta)}$ in the appropriate p-norm is by no means assured although we shall see that for various L^1- and L^2-processes it is guaranteed.

It is not difficult to show that if $X = (X_\alpha)$ is an L^p-right continuous L^p-process and $\alpha \in \mathbf{R}^+$ and $\tau = (P_\beta)$ is the process defined in 2.1(v) then $\lim_{\theta \in \mathscr{P}[0,\infty]} X_{\tau(\theta)} = X_\alpha$. However this does not follow if there is not 'enough' right continuity present. So our reformulation of stopping should be regarded as a reformulation of stopping for stochastic processes with right continuous paths applicable only to those non-commutative processes which have some kind of right continuity, e.g. in the strong operator topology or in L^1 norm.

To our knowledge the first investigation of stopping in a non-commutative context was undertaken by R. L. Hudson (perhaps as early as 1973). This work led eventually to the publication [7]. More recently D. B. Appelbaum has proved the 'fermionic' version of Hudson's original result in [1].

2.3. LEMMA. *The set T of stopping times is partially ordered by the relation defined in* 2.1 (iv). *Under this relation the set of times form a complete lattice.*

Proof. The times $\tau(\alpha) \equiv 0$ and $\sigma(\alpha) \equiv I$ for $\alpha \in (0, \infty)$ are respectively the greatest and least times with respect to $\leqslant$. Let $S \neq \emptyset$ be a subset of T. Fix

$\alpha \in [0,\infty]$ and set $P_\alpha = \inf_{\tau \in S} \tau(\alpha)$ then P_α is a projection in $\mathscr{A}_\alpha$ for the projection lattice of $\mathscr{A}_\alpha$ is complete. If $\alpha \leqslant \beta$ then as $\tau(\alpha) \leqslant \tau(\beta)$ for any $\tau \in S$ it follows that $P_\alpha \leqslant P_\beta$. The family $\sigma = (P_\alpha)$ is a time and clearly $\sigma \geqslant \tau$, $\forall \tau \in S$. If ρ is a time, $\rho = (R_\alpha)$ and ρ is an upper bound for S, then $R_\alpha \leqslant \tau(\alpha)$ $\forall \tau \in S$ so $R_\alpha \leqslant \inf_{\tau \in S} \tau(\alpha) = P_\alpha$. So every subset of T has a supremum constructed as above. A dual argument shows that $\varphi_\alpha = \sup_{\tau \in S} \tau(\alpha)$ defines the infimum of S.

2.4. REMARK. We will use the symbols $\sigma \vee \tau$ and $\sigma \wedge \tau$ to denote the supremum and infimum respectively of times σ, τ.

Let us now consider what the definition of stopped operator yields when $\mathscr{A}_\infty$ is commutative and all of the operators may be considered to be (multiplication by) measurable functions. Indeed let (X_α) be an $L^1(\Omega, \Sigma, \mathbf{P})$ martingale adapted to the stochastic base $\mathscr{B}$ and closed by $X_\infty \in L^1(\Omega, \Sigma, \mathbf{P})$. Let τ be a stopping time on the base $\mathscr{B}$. To assist the discussion we will use the following notation: upper case letters will denote elements of $L^1(\Omega, \Sigma, \mathbf{P})$ and lower case letters elements of $\mathscr{L}^1(\Omega, \Sigma, \mathbf{P})$, the Σ-measurable, $\mathbf{P}$-integrable, $\mathbf{C}$-valued functions on Ω. If $y \in \mathscr{L}^1(\Omega, \Sigma, \mathbf{P})$, then Y denotes its class in $L^1(\Omega, \Sigma, \mathbf{P})$. If $Y \in L^1(\Omega, \Sigma, \mathbf{P})$, then y denotes a representative of Y. Let $p_\alpha = \chi_{\{\tau < \alpha\}}$ and $x = (x_\alpha)$ the pathwise right-continuous modification of (X_α). It is known that x is unique up to indistinguishability of stochastic processes, is a martingale and $\lim_{\alpha \to \infty} x_\alpha = x_\infty$ almost surely and in L^1 norm [8]. If $\theta \in \mathscr{P}[0,\infty]$ then $\sum_\theta X_{\alpha_i} \Delta P_{\alpha_i}$ is the class of $\sum_\theta x_{\alpha_i} \Delta p_{\alpha_i}$. But $\sum_\theta x_{\alpha_i} \Delta p_{\alpha_i} = x_{\tau(\theta)}$ where $\tau(\theta)$ is the stopping time $\sum_\theta \alpha_i \chi_{\{\alpha_{i-1} \leqslant \tau < \alpha_i\}}$. So $X_{\tau(\theta)}$ as defined in 2.1 (ii) is indeed the class of $x_{\tau(\theta)}$ (defined in the usual way) in L^1. Now x is right-continuous and so $x_{\tau(\theta)} \to x_\tau$ $\mathbf{P}$-almost surely as θ is refined. In addition the family $\{x_{\tau(\theta)} : \theta \in \mathscr{P}[0,\infty]\}$ is uniformly integrable because $x_{\tau(\theta)} = E(x_\infty \mid \Sigma_{\tau(\theta)})$ and $\mathbf{P}$ is a finite measure so all the conditions of Theorem 6, Chapter III of [6] are met bar countability and total order of the net. But, clearly, we can choose an increasing sequence $(\theta_n) \subset \mathscr{P}[0,\infty]$ with $\tau(\theta_n) \downarrow \tau$ pointwise. We then have $X_{\tau(\theta_n)} \to X_\tau$ in L^1 by the theorem last quoted. But for any $\theta \in \mathscr{P}[0,\infty]$ $\tau(\theta) \geqslant \tau$ and hence if $\theta \supseteq \theta_n$ for some n we have (using 5.4 and 5.5 of [8])

$$\|X_\tau - X_{\tau(\theta)}\|_1 = \|E(X_\infty \mid \Sigma_\tau) - E(X_\infty \mid \Sigma_{\tau(\theta)})\|_1 =$$

$$= \|E(E(X_\infty \mid \Sigma_\tau) - E(X_\infty \mid \Sigma_{\tau(\theta_n)}) \mid \Sigma_{\tau(\theta)})\|_1 \leqslant$$

$$\leqslant \|E(X_\infty \mid \Sigma_\tau) - E(X_\infty \mid \Sigma_{\tau(\theta_n)})\|_1 = \|X_\tau - X_{\tau(\theta_n)}\|_1 .$$

This shows that for a uniformly integrable $L^1(\Omega, \Sigma, \mathbf{P})$ martingale the definition of stopping in 2.1 (iii) will agree with that obtained via the usual definition.

2.5. THEOREM. *Let $X = (X_\alpha)$ be a weakly relatively compact L^1 martingale and $\tau = (P_\alpha)$ a stopping time. Suppose further that X_τ exists. Then for each $\alpha \in \mathbf{R}^+$, $X_{\tau\wedge\alpha}$ exists and the process $X^\tau = (X_{\tau\wedge\alpha})$ is a martingale.*

Proof. Since X is weakly relatively compact, there is $X_\infty \in L^1$ such that $L^1\text{-}\lim_{\alpha\to\infty} X_\alpha = X_\infty$ and $X_\alpha = M_\alpha(X_\infty)$. Consider $\theta \in \mathscr{P}[0,\infty]$ and let $\alpha \in \mathbf{R}^+$ be fixed. It is no loss of generality to assume $\alpha \in \theta$ for what follows. Suppose for argument's sake $\theta = \{\alpha_i\}$ and $\alpha = \alpha_r$. Now $X_{\tau\wedge\alpha(\theta)} = \sum_{i\leqslant r} X_{\alpha_i}\Delta P_{\alpha_i} + X_\alpha(I - P_\alpha)$ by definition but also,

$$M_\alpha(X_{\tau(\theta)}) = \sum_{i\leqslant r} X_{\alpha_i}\Delta P_{\alpha_i} + M_\alpha(\sum_{i>r} X_{\alpha_i}\Delta P_{\alpha_i}) =$$

$$= \sum_{i\leqslant r} X_{\alpha_i}\Delta P_{\alpha_i} + M_\alpha(\sum_{i>r} M_{\alpha_i}(X_\infty\Delta P_{\alpha_i})) =$$

$$= \sum_{i\leqslant r} X_{\alpha_i}\Delta P_{\alpha_i} + \sum_{i>r} M_\alpha(X_\infty\Delta P_{\alpha_i}) = X_{\tau\wedge\alpha(\theta)}.$$

Noting that M_α is L^1-continuous, we have that $L^1\text{-}\lim_\theta X_{\tau\wedge\alpha(\theta)} = L^1\text{-}\lim_\theta M_\alpha(X_{\tau(\theta)}) = M_\alpha(X_\tau)$. So $X_{\tau\wedge\alpha}$ exists and is equal to $M_\alpha(X_\tau)$. It is now clear that $(X_{\tau\wedge\alpha})$ is a martingale.

3. STOPPING FOR L^2-MARTINGALES

Let $X = (X_\alpha)$ be an L^2-martingale adapted to the filtration $(\mathscr{A}_\alpha)$ and suppose (X_α) is a bounded subset of L^2. Then there is $X_\infty \in L^2(\mathscr{A}_\infty)$ such that $X_\alpha = M_\alpha(X_\infty)$, $\lim_{\alpha\to\infty} M_\alpha(X_\infty) = X_\infty$ and (X_α) is L^2-right-continuous. If $\theta \in \mathscr{P}[0,\infty]$ and $\tau = (P_\alpha)$ is a stopping time, then $X_{\tau(\theta)} = \sum_\theta X_{\alpha_i}\Delta P_{\alpha_i} = \sum_\theta M_{\alpha_i}(X_\infty)\Delta P_{\alpha_i} = M_{\tau(\theta)}(X_\infty)$ say, where $M_{\tau(\theta)}(\cdot) = \sum_\theta M_{\alpha_i}(\cdot)\Delta P_{\alpha_i}$. This observation is the basis for the alternative approach to stopping we alluded to in the introduction. We proceed by investigating the properties of the functions $M_{\tau(\theta)}$.

3.1. DEFINITION. Let $\theta \in \mathscr{P}[a, b]$, $0 \leqslant a \leqslant b \leqslant \infty$ and $\tau = (P_\alpha)$ a stopping time. Then we write $M_{\tau(\theta)}$ for the (bounded) linear map

$$L^2(\mathscr{A}_\infty) \ni X \mapsto \sum_\theta M_{\alpha_i}(X)\Delta P_{\alpha_i} \in L^2(\mathscr{A}_\infty).$$

3.2. PROPOSITION. *Let τ be a stopping time.*

(i) $\forall\, \theta \in \mathscr{P}[a, b]$, *$M_{\tau(\theta)}$ is a self-adjoint projection on $L^2(\mathscr{A}_\infty)$.*

(ii) *If $\theta_1, \theta_2 \in \mathscr{P}[a, b]$ and $\theta_1 \subseteq \theta_2$ then $M_{\tau(\theta_1)} \geqslant M_{\tau(\theta_2)}$.*

(iii) *If $\sigma \leqslant \tau$ and $\theta \in \mathscr{P}[0,\infty]$ then $M_{\sigma(\theta)} \leqslant M_{\tau(\theta)}$.*

Proof. (i)

$$\sum_i M_{\alpha_i}(\sum_j M_{\alpha_j}(X)\Delta P_{\alpha_j})\Delta P_{\alpha_i} = \sum_i M_{\alpha_i}(\sum_j M_{\alpha_j}(X)\Delta P_{\alpha_j}\Delta P_{\alpha_i}) =$$

$$= \sum_i M_{\alpha_i}(M_{\alpha_i}(X)\Delta P_{\alpha_i}) = \sum_i M_{\alpha_i}(X)\Delta P_{\alpha_i}$$

using adaptedness, orthogonality of the ΔP_{α_i}'s and properties of the conditional expectation.

$$\varphi(Y^* M_{\tau(\theta)}(X)) = \sum_\theta \varphi(Y^* M_{\alpha_i}(X)\Delta P_{\alpha_i}) = \sum_\theta \varphi(\Delta P_{\alpha_i} M_{\alpha_i}(Y)^* X) =$$

$$= \varphi((\sum_\theta \Delta P_{\alpha_i} M_{\alpha_i}(Y)^*)X) = \varphi(X(\sum_\theta M_{\alpha_i}(Y)\Delta P_{\alpha_i})^*) = \varphi(X M_{\tau(\theta)}(Y)^*)$$

so $M_{\tau(\theta)}$ is self-adjoint.

(ii) It is enough to consider the case $\theta_2 = \theta_1 \cup \{\beta\}$ where $\beta \in \mathbf{R}\setminus\theta_1$. We will suppose $\alpha_{r-1} < \beta < \alpha_r$ for some r. Let $X \in L^2(\mathscr{A}_\infty)$.

$$M_{\tau(\theta_2)} \cdot M_{\tau(\theta_1)}(X) = \sum_{j=1}^{r-1} M_{\alpha_j}\left(\sum_{i=1}^{n} M_{\alpha_i}(X)\Delta P_{\alpha_i}\right)\Delta P_{\alpha_j} +$$

$$+ M_\beta\left(\sum_{i=1}^{n} M_{\alpha_i}(X)\Delta P_{\alpha_i}\right)(P_\beta - P_{\alpha_{r-1}}) + M_{\alpha_r}\left(\sum_{i=1}^{n} M_{\alpha_i}(X)\Delta P_{\alpha_i}\right)(P_{\alpha_r} - P_\beta) +$$

$$+ \sum_{j=r+1}^{n} M_{\alpha_j}\left(\sum_{i=1}^{n} M_{\alpha_i}(X)\Delta P_{\alpha_i}\right)\Delta P_{\alpha_j}.$$

Now use adaptedness and orthogonality of ΔP_{α_j}'s in each of these four terms (as in the first part of (i)). Noting that $M_\beta \cdot M_{\alpha_r} = M_\beta$, one obtains the sum for $M_{\tau(\theta_2)}(X)$.

(iii) Let $\sigma = (\varphi_\alpha)$ and $\tau = (P_\alpha)$ and $\theta \in \mathscr{P}[0, \infty]$, $\theta = \{\alpha_i\}_{i=1}^n$. Write $E_i = \Delta\varphi_{\alpha_i}$ and $F_i = \Delta P_{\alpha_i}$ and (for consistency) $M_i = M_{\alpha_i}$. The condition $\sigma \leqslant \tau$ means that $\forall k$, $\sum_{i=1}^{k} F_i \leqslant \sum_{i=1}^{k} E_i$ and $\sum_\theta E_i = \sum_\theta F_i = I$. Now

$$M_{\sigma(\theta)} \cdot M_{\tau(\theta)}(X) = \sum_i M_i(\sum_j M_j(X)F_j)E_i$$

so consider $M_i(\sum_j M_j(X)F_j)E_i$ for a fixed i. If $j < i$, then $\sum_{j<i} F_j \leqslant \sum_{j<i} E_j \perp E_i$ and E_i is adapted, so

$$M_i(\sum_j M_j(X)F_j)E_i = M_i(\sum_{j\geqslant i} M_j(X)F_j)E_i.$$

Now

$$M_i(\sum_{j\geqslant i} M_j(X)F_j)E_i = M_i(\sum_{j\geqslant i} M_j(XF_j))E_i =$$

$$= \sum_{j\geqslant i} M_i(XF_j)E_i = M_i(\sum_{j\geqslant i} XF_j)E_i = M_i(X(I - \sum_{j<i} F_j))E_i$$

and as we have already noted $\sum_{j<i} F_j \perp E_i$ so the last term is just $M_i(X)E_i$ (using adaptedness of E_i). Now sum on i.

REMARK. 3.2(iii) can be extended to an interval $[a, b]$ for σ, τ such that $\sigma(a) = \tau(a) = 0$ and $\sigma(b) = \tau(b) = I$. We will use this later.

The second result shows that for fixed a, b the family $M_{\tau(\theta)}$, $\theta \in \mathscr{P}[a, b]$, form a decreasing net of projections on $L^2(\mathscr{A}_\infty)$. It follows that their infimum exists. We recognise this in

3.3. DEFINITION. Let $0 \leqslant a \leqslant b \leqslant \infty$ and $\tau = (P_\alpha)$ be a stopping time. For $Y \in L^2(\mathscr{A}_\infty)$ we write $\int_a^b M_\alpha(Y)\mathrm{d}P_\alpha$ to denote $(\inf_{\theta \in \mathscr{P}[a, b]} M_{\tau(\theta)})(Y)$. If $a = 0$ and $b = \infty$ we will sometimes write $M_\tau(\cdot)$ for $\int_0^\infty M_\alpha(\cdot)\mathrm{d}P_\alpha$.

3.4. THEOREM. *Let $X = (X_\alpha)$ be an L^2-bounded martingale closed by X_∞ with $M_\alpha(X_\infty) = X_\alpha$. Let τ be a stopping time. Then $X_\tau = \lim_{\theta \in \mathscr{P}[0, \infty]} X_{\tau(\theta)}$ exists and is equal to $M_\tau(X_\infty)$.*

Proof. As we noted at the start of this section, $X_{\tau(\theta)} = M_{\tau(\theta)}(X_\infty)$ for $\theta \in \mathscr{P}[0, \infty]$. And we know that the right-hand side converges.

3.5. THEOREM (Optional stopping). *Let σ, τ be times $\sigma \leqslant \tau$. If $X = (X_\alpha)$ is an L^2-bounded martingale closed by X_∞ with $X_\alpha = M_\alpha(X_\infty)$ then $M_\sigma(X_\tau) = X_\sigma$.*

Proof. Since $\sigma \leqslant \tau$ then by 3.2(iii) $M_{\sigma(\theta)} \leqslant M_{\tau(\theta)}$ for $\theta \in \mathscr{P}[0, \infty]$ it follows that $M_\sigma \leqslant M_\tau$. By 3.4 then $M_\sigma(X_\tau) = M_\sigma \cdot M_\tau(X_\infty) = M_\sigma(X_\infty) = X_\sigma$.

3.6. COROLLARY. (i) *Let $X = (X_\alpha)$ be an L^2-martingale and τ a* **bounded** *stopping time. Then $X_\tau = \lim_{\theta \in \mathscr{P}[0, \infty]} X_{\tau(\theta)}$ exists and is equal to $\int_0^s M_\alpha(X_\alpha)\mathrm{d}P_\alpha + X_s(I - P_s)$ where $s = \inf\{\alpha : P_\alpha = I\}$.*

(ii) *If $X = (X_\alpha)$ is an L^2-martingale, $\tau = (P_\alpha)$ is bounded, and $\sigma = (\varphi_\alpha)$, $\sigma \leqslant \tau$, then $M_\sigma(X_\tau) = X_\sigma$.*

Proof. (i) One can consider $\theta \in \mathscr{P}[0, \infty]$ with $s \in \theta$, $s = \alpha_r$, say. For such θ

$$X_{\tau(\theta)} = \sum_{i \leqslant r} X_{\alpha_i} \Delta P_{\alpha_i} + \sum_{r < i} X_{\alpha_i} \Delta P_{\alpha_i} = \sum_{i \leqslant r} X_{\alpha_i} \Delta P_{\alpha_i} + X_{\alpha_{r+1}}(I - P_{\alpha_r}).$$

So $X_{\tau(\theta)} = M_{\tau(\theta_1)}(X_s) + X_{\alpha_{r+1}}(I - P_{\alpha_r})$ where $\theta_1 = \{\alpha_i - \theta : i \leqslant r\}$. Clearly θ_1 is a partition of $[0, s]$ and in the limit as we refine θ we get

$$X_\tau = \int_0^s M_\alpha(X_s) \mathrm{d}P_\alpha + \lim_{\alpha_{r+1} \to s} X_{\alpha_{r+1}}(I - P_s) =$$

$$= \int_0^s M_\alpha(X_s) \mathrm{d}P_\alpha + X_s(I - P_s)$$

because L^p-martingales are L^p-right-continuous [2]. This shows also that for $Y \in L^2(\mathscr{A}_\infty)$, $M_\tau(Y) = \int_0^s M_\alpha(Y) \mathrm{d}P_\alpha + Y_s(I - P_s)$ where $Y_s = M_s(Y)$.

(ii) We use the first part. Since $\sigma \leqslant \tau$ then $M_\sigma \leqslant M_\tau$. So $X_\tau = \int_0^s M_\alpha(X_s) \mathrm{d}P_\alpha +$ $+ X_s(I - P_s) = M_\tau(X_s)$ and hence $M_\sigma(X_\tau) = M_\sigma \cdot M_\tau(X_s) = M_\sigma(X_s)$. But note that $\inf\{\alpha : Q_\alpha = I\} = t \leqslant s$, so $\sigma \leqslant t$ and hence $M_\sigma = M_\sigma \cdot M_t$ and so $M_\sigma(X_\tau) =$ $= M_\sigma(X_s) = M_\sigma \cdot M_t(X_s) = M_\sigma(X_t) = X_\sigma$.

Next we prove a few results about the "time projections" M_τ where τ is a stopping time.

3.7. THEOREM. *Let $\sigma = (Q_\alpha)$, $\tau = (P_\alpha)$ be times with $P_\alpha Q_\alpha = Q_\alpha P_\alpha$, $\forall \alpha$. Then M_σ and M_τ commute and $M_\sigma \wedge M_\tau = M_{\sigma \wedge \tau}$ and $M_\sigma \vee M_\tau = M_{\sigma \vee \tau}$.*

Proof. Let $\theta \in \mathscr{P}[0, \infty]$ and note that

$$P_\alpha + Q_\alpha = P_\alpha \vee Q_\alpha + P_\alpha \wedge Q_\alpha.$$

Then

$$M_{\tau \wedge \sigma(\theta)} = \sum_\theta M_{\alpha_i}(\cdot) \Delta(P_{\alpha_i} \vee Q_{\alpha_i}) = \sum_\theta M_{\alpha_i}(\cdot) \Delta(P_{\alpha_i} + Q_{\alpha_i} - P_{\alpha_i} \wedge Q_{\alpha_i}) =$$

$$= \sum_\theta M_{\alpha_i}(\cdot) \Delta P_{\alpha_i} + \sum_\theta M_{\alpha_i}(\cdot) \Delta Q_{\alpha_i} - \sum_\theta M_{\alpha_i}(\cdot) \Delta(P_{\alpha_i} \wedge Q_{\alpha_i}) =$$

$$= M_{\tau(\theta)} + M_{\sigma(\theta)} - M_{\tau \vee \sigma(\theta)}.$$

Refining θ gives

$$M_{\tau\wedge\sigma} + M_{\tau\vee\sigma} = M_\tau + M_\sigma.$$

Use 3.5 after composing by M_σ or M_τ on the left to get

$$M_{\tau\wedge\sigma} = M_\sigma \cdot M_\tau = M_\tau \cdot M_\sigma = M_\tau \wedge M_\sigma,$$

and from this the other relation we require follows easily.

3.8. Corollary. *Let $X = (X_\alpha)$ be an L^2-martingale and τ a time. Then $(X_{\tau\wedge\alpha})$ is an L^2-martingale.*

Proof. For any $\alpha \in \mathbf{R}^+$ the *time* α is bounded and since its proejctions are just 0 and I they certainly commute with those of τ. Using the relation established in the proof of 3.6(ii)

$$X_{\tau\wedge\alpha} = M_{\tau\wedge\alpha}(X_\alpha) = M_\tau \cdot M_\alpha(X_\alpha).$$

And, if $\beta \leqslant \alpha$,

$$M_\beta(X_{\tau\wedge\alpha}) = M_\beta \cdot M_\tau \cdot M_\alpha(X_\alpha) =$$

$$= M_\tau \cdot M_\beta \cdot M_\alpha(X_\alpha) = M_\tau \cdot M_\beta(X_\beta) = X_{\tau\wedge\beta}.$$

Of course one could just apply 2.5.

3.9. Example. The time projections do not always commute as the following example will show. Let $0 < \alpha_0 < \infty$ and let R, S be non-commuting projections in $\mathscr{A}_{\alpha_0}$. Let

$$R_\alpha = \begin{cases} 0 & \text{if } \alpha \leqslant \alpha_0 \\ R & \text{if } \alpha > \alpha_0 \\ I & \text{if } \alpha = \infty \end{cases},$$

let $\rho = (R_\alpha)$ and $\sigma = (S_\alpha)$ where S_α is defined exactly as R_α except that we replace R with S. It is not dificult to show that for $Y \in L^2(\mathscr{A}_\infty)$,

$$M_\rho(Y) = M_{\alpha_0}(Y)R + Y(I - R)$$

with a similar equation for $M_\sigma(Y)$. It follows that

$$M_\sigma \cdot M_\rho(Y) = M_{\alpha_0}(Y)(R + S - RS) + Y(I - R)(I - S)$$

and

$$(M_\sigma \cdot M_\rho - M_\rho \cdot M_\sigma)(Y) = (M_{\alpha_0}(Y) - Y)(SR - RS).$$

Now the left side of the last equation cannot always be zero for this will imply that $RS = SR$; for example take $(\mathscr{A}_\alpha)$ to be the Clifford filtration and $Y = \psi(u)$ for $u \in L^2(\mathbf{R}^+)$, u real valued and non zero.

QUESTION. Is it still true that $M_\sigma \wedge M_\tau = M_{\tau \wedge \sigma}$ and $M_\sigma \vee M_\tau = M_{\sigma \vee \tau}$ even when $\sigma(\alpha)$ and $\tau(\alpha)$ do not commute? Put another way, is the map $\tau \mapsto M_\tau$ a lattice morphism?

4.

In this section we restrict our attention to the Clifford filtration and, using results from [5], we characterize the *process* $(X_{\tau\wedge\alpha})$. Hereafter $\mathscr{A}_\infty$ denotes the weakly closed Clifford algebra generated by the Fermion fields $\psi(u)$, $u \in L^2(\mathbf{R}^+)$ acting in the Fermion Fock space $\wedge(L^2(\mathbf{R}^+))$ [5]. For $\alpha \in \mathbf{R}^+$, $\mathscr{A}_\alpha$ will denote the (von Neumann) subalgebra of $\mathscr{A}_\infty$ generated by $\psi(u)$ for $u \in L^2([0, \alpha])$. The family $(\mathscr{A}_\alpha)$, $\alpha \in [0, \infty]$, is called the Clifford filtration.

To characterize $(X_{\tau\wedge\alpha})$, for an L^2-martingale X we shall need to consider stopping for some $L^1(\mathscr{A}_\infty)$ processes.

Some notation: let P_1 denote the set of $L^1(\mathscr{A}_\infty)$ valued processes, f, which are Lebesgue measurable and for which $\|f\|_1$ is in $L^1(\mathbf{R}^+)$. We will consider stopping processes $A = (A_\alpha)$, where

$$A_\alpha = \int_0^\alpha f(s)\mathrm{d}s \qquad f \in P_1.$$

Fix $\gamma \in \mathbf{R}^+$ and let $\varepsilon > 0$. If $x \in \mathscr{P}[0, \gamma]$ and $\tau = (P_\alpha)$ is a time, then using summation by parts

$$A_{\tau(x)} = \sum_x A_{\alpha_i}\Delta P_{\alpha_i} = A_\gamma P_\gamma - \sum_x \Delta A_{\alpha_i} P_{\alpha_{i-1}} =$$

$$= \int_0^\gamma f(s)P_\gamma \mathrm{d}s - \int_0^\gamma f(s)g_x(s)\mathrm{d}s.$$

Here $g_x(s) = \sum_{i=1}^n P_{\alpha_{i-1}}\chi_{[\alpha_{i-1},\alpha_i)}(s)$ and we have used the fact that multiplication by an element of $\mathscr{A}_\infty$ is a continuous linear operation on $L^1(\mathscr{A}_\infty)$ ([6], Chapter III, 2.19). So $A_{\tau(x)} = \int_0^\gamma f(s)(P_\gamma - g_x(s))\mathrm{d}s.$

Our next aim is to show that $\lim_x A_{\tau(x)}$ exists and is equal to $\int_0^{\gamma} f(s)(P_{\gamma} - P_s)\mathrm{d}s$.

Given a time τ, Lemma 2.3 of [3] shows how to construct a sequence of (discrete) times which converge pointwise to τ, in fact *as times* they *decrease* to τ. It is not difficult to see that this construction provides us with an *increasing* sequence (θ_n) of partitions of $[0, \infty]$ such that $g_n(s) \to \tau(s)\ \forall s \in [0, \gamma]$ where $g_n(s) = g_{\theta_n}(s)$. We can use this to show that $f(P_{\gamma} - \tau)\chi_{[0,\gamma]} \in P_1$. Suppose first that f is elementary, i.e. $f = h\chi_{[s,t)}$, $h \in L^1(A_s)$. Writing $h = h_1 + h_2$, $h_1 \in A_{\infty}$ and $\|h_2\|_1 < \varepsilon$, we have $\|f(s)(g_n(s) - \tau(s))\|_1 \leqslant \|h_1\|_{\infty}\|g_n(s) - \tau(s)\|_1 + \varepsilon$. So $f(P_{\gamma} - g_n)\chi_{[0,\gamma]} \to f(P_{\gamma} - \tau)\chi_{[0,\gamma]}$ pointwise. Since f and g_n are simple, it is clear that $f(P_{\gamma} - g_n)\chi_{[0,\gamma]} \in P_1$ and that $\|f(P_{\gamma} - g_n)\chi_{[0,\gamma]}\|_1 \leqslant \|f\|_1$. It follows from the dominated convergence theorem ([6], Chapter III, § 6) that $f(P_{\gamma} - \tau)\chi_{[0,\gamma]} \in P_1$. A similar argument works when f is not elementary. So we can integrate with respect to Lebesgue measure, let

$$z_{\gamma} = \int_0^{\gamma} f(s)(P_{\gamma} - \tau(s))\mathrm{d}s.$$

Let $\theta \in \mathscr{P}[0, \gamma]$ and suppose $\theta \supseteq \theta_n$ for some $n \in \mathbf{N}$. Since $\theta \supseteq \theta_n$, $g_{\theta}(s) \geqslant g_n(s)$ and so

$$\|z_{\gamma} - A_{\tau(\theta)}\|_1 = \left\| \int_0^{\gamma} f(s)(g_{\theta}(s) - \tau(s))\mathrm{d}s \right\|_1 =$$

$$= \left\| \int_0^{\gamma} f(s)(\tau(s) - g_{\theta}(s))(\tau(s) - g_n(s))\,\mathrm{d}s \right\|_1 \leqslant$$

$$\leqslant \int_0^{\gamma} \|(f(s)(\tau(s) - g_{\theta}(s)))(\tau(s) - g_n(s))\|_1\,\mathrm{d}s.$$

A recapitulation of one of our previous arguments shows that the last integral tends to zero as θ refines. In other words $\lim_{x \in \mathscr{P}[0,\gamma]} A_{\tau(x)}$ exists and is equal to $\int_0^{\gamma} f(s)(P_{\gamma} - P_s)\,\mathrm{d}s$. Now consider $A_{\tau \wedge \gamma}$. Using the discussion above, if $\alpha_i = \gamma \in x \in$

$\in \mathscr{P}[0,\infty]$ then $A_{\tau\wedge\gamma(x)} = \sum_{\alpha_j\leq\gamma} A_{\alpha_j}\Delta P_{\alpha_j} + A_{\alpha_{i+1}}(I - P_\gamma)$ which in the limit gives $A_{\tau\wedge\gamma} = \int_0^\gamma f(s)(I - P_s)\mathrm{d}s$. When P is a projection we will write $P^\perp$ for $I - P$.

Let $X = (X_\alpha)$ be a centred L^2-martingale and $(\tilde{x}_\alpha)$ the unique $L^2_{loc}(\mathbf{R}^+, \lambda, L^2(\mathscr{A}_\infty))$ process such that for $\alpha \in \mathbf{R}^+$, $X_\alpha = \int_0^\alpha \tilde{x}_s \mathrm{d}\psi_s$. Here $\psi_s = \psi(\chi_{[0,s]})$; see [4] for details. Using the isometry property ([4], 3.5 (c)) it is not difficult to show that for $\alpha \in \mathbf{R}^+$

$$X_{\tau\wedge\alpha} = X_\alpha - \int_0^\alpha \mathrm{d}X_s P_s = \int_0^\alpha \mathrm{d}X_s(I - P_s) = \int_0^\alpha \tilde{x}_s \beta(P_s^\perp)\mathrm{d}\psi_s$$

where β is the parity operator [5], and the "left" stochastic integral on the right-hand side is defined as in [3].

In [4] the sesquilinear process valued form, $\langle X, Y\rangle$, X, $Y \in L^2(\mathscr{A}_\infty)$-martingales, was defined and a characterization of the left stochastic integral of a (suitable) process given as follows.

4.1. Theorem ([5], 7.4). *Let (X_α) be an $L^2(\mathscr{A}_\infty)$ martingale and suppose $f \in \mathscr{P}[0, t]$ for each $t \geq 0$. Then $\left(\int_0^\alpha \mathrm{d}X f\right)$ is the unique centred $L^2(\mathscr{A}_\infty)$-martingale, (N_α), say, such that*

$$\int_0^\alpha \mathrm{d}\langle X, Y\rangle f = \langle N_\alpha, Y_\alpha\rangle$$

for any $L^2(\mathscr{A}_\infty)$ martingale (Y_α).

Here $\mathscr{P}[0, t]$ ([5], 6.4) denotes processes which are the $\|\ \|_\infty$-λ a.e. limit of uniformly bounded simple processes. In fact, the slight extension of the Itô-Clifford integral given in [3] and the discussion of stopping for $L^1(\mathscr{A}_\infty)$-processes given above show that the theorem above holds with a time $\tau = (P_\alpha)$ replacing f. Bearing in mind the definition of $\langle X, Y\rangle$ we have:

4.2. Theorem. *Let $X = (X_\alpha)$ be an L^2-martingale, $\tau = (P_\alpha)$ a time. Then $(X_{\tau\wedge\alpha})$ is the unique centred L^2-martingale, (N_α), say, such that.*

$$\langle X, Y\rangle_{\tau\wedge\alpha} = \langle N_\alpha, Y_\alpha\rangle \quad \textit{for any } L^2\textit{-martingale } (Y_\alpha).$$

Proof. Let $(Y_\alpha) = Y$ be an L^2-martingale. Then

$$\langle X_{\tau\wedge\alpha}, Y_\alpha\rangle = \left\langle \int_0^\alpha dX_s(I - P_s), Y_\alpha \right\rangle =$$

$$= \int_0^\alpha d\langle X, Y\rangle_s (I - P_s) = \langle X, Y\rangle_{\tau\wedge\alpha}$$

by the first part of this section.

4.3. COROLLARY. *Let* $Z_\alpha = \langle X, X\rangle_{\tau\wedge\alpha}$ *and* $Z = (Z_\alpha)$. *Then* $\langle X_{\tau\wedge\alpha}, X_{\tau\wedge\alpha}\rangle = (Z^*)_{\tau\wedge\alpha}$.

REFERENCES

1. APPELBAUM, D. B., The strong Markov property for Fermion Brownian motion, *J. Funct. Anal.*, 65(1986), 273–291.
2. BARNETT, C., Supermartingales on von Neumann algebras, *J. London Math. Soc.(2)*, **24**(1981), 175–181.
3. BARNETT, C.; LYONS, T. J., Stopping non-commutative processes, *Math. Proc. Cambridge Philos. Soc.*, 99(1986), 151–161.
4. BARNETT, C.; STREATER, R. F.; WILDE, I. F., The Itô-Clifford integral, *J. Funct. Anal.*, **48**(1982), 172–212.
5. BARNETT, C.; STREATER, R. F.; WILDE, I. F., The Itô-Clifford integral. IV, *J. Operator Theory*, **11**(1984), 255–271.
6. DUNFORD, N.; SCHWARTZ, J., *Linear operators, Part I*, Interscience Publishers, New York–London, 1958.
7. HUDSON, R. L., The strong Markov property for canonical Wiener process, *J. Funct. Anal.*, **34**(1979), 266–281.
8. KUSSMAUL, A., *Stochastic integration and generalised martingales*, Pitman, 1977.
9. YEADON, F. J., Non-commutative L^p spaces, *Math. Proc. Cambridge Philos. Soc.*, **77**(1975), 91–102.

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