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C^*-ALGEBRAS GENERATED BY PROJECTIVE REPRESENTATIONS OF THE DISCRETE HEISENBERG GROUP

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INTRODUCTION

In recent years, much progress has been made in the classification of C^*-algebras constructed from a countable number of generators with given commutation relations, by the use of K-theoretic methods [9], [17], [20]. The first step of this study was made in the classification up to $*$-isomorphism and strong Morita equivalence of the irrational (and then rational) rotation algebras A_α mainly through the use of K-theory. As was noted in [9] and [20], A_α is the C^*-algebra $C^*(G, \sigma)$ of [26], with $G = \mathbf{Z} \oplus \mathbf{Z}$ and $\sigma : G \times G \to S^1$ the two-cocycle given by $\sigma((m_1, n_1), (m_2, n_2)) = e^{2\pi i \alpha n_1 m_2}$. Rieffel has gone on to discover many interesting properties of these algebras (e.g. cancellation) which allowed him to construct up to equivalence all the projective modules over the A_α along with the corresponding endomorphism rings. A study of the classification of $C^*(G, \sigma)$ for G countable torsion-free abelian and $\sigma : G \times G \to \mathbf{T}$ has been continued in [8] and [9], and Rieffel has recently indicated a method of constructing projective modules which would apply to non-abelian G as well [23]. (Here, as throughout this paper, we consider only those projective modules which are finitely generated.)

One of the simplest non-abelian cases to study is the three-dimensional discrete Heisenberg group H, generated by elements U, V, and W which satisfy $UV = VU$, $VW = WV$, and $UW = VWU$. In [14], the C^*-algebras with generators satisfying $UV = e^{2\pi i\alpha}VU$, $VW = WV$ and $UW = VWU$ were studied in the case where α is irrational, and for them α (mod 1) is a complete invariant. These C^*-algebras are in fact isomorphic to twisted group C^*-algebras for H corresponding to certain multipliers; in this paper we shall study the C^*-algebras of the form $M_n(C^*(H, \sigma))$ where σ is an arbitrary multiplier for H and $n \in \mathbf{N}$. Our main aim is to describe the $*$-isomorphism classes of such algebras; we shall describe the strong Morita equivalence classes of such algebras as well as compute the positive cones of their K_0-groups in a subsequent paper [16]. It is hoped that some of the techniques used here could carry over to twisted group C^*-algebras for more general nilpotent discrete groups. There are three distinct classes of Heisenberg C^*-algebras, depending on the number of generators in $\mathbf{R}$ for the range of a faithful normalized trace on the K_0-group for the C^*-algebra in question. The K-theory of these algebras involves a study of both rational and irrational rotation algebras, and as in [17], [20], and [22], K-theoretic methods will allow us to determine isomorphism types.

Any $C^*(H, \sigma)$ is, as we shall show, $*$-isomorphic to the C^*-algebra generated (universally) by unitary elements U, V, and W satisfying

$$UV = e^{2\pi i\alpha}VU,$$

$$WV = e^{2\pi i\beta}VW,$$

$$UV = WVU,$$

for some $\alpha, \beta \in \mathbf{R}$; we denote such an algebra by $H(\alpha, \beta)$. If τ is any normalized faithful trace for $H(\alpha, \beta)$, $H(\alpha, \beta)$ will be said to be of class 1, 2, or 3 if $\tau^*(K_0(H(\alpha, \beta))) = \mathbf{Z} + \alpha\mathbf{Z} + \beta\mathbf{Z}$ is generated by 1, 2 but not 1, or 3 but not 2 elements of $\mathbf{R}$, respectively. It is not difficult to show that $H(\alpha, \beta)$ is $*$-isomorphic to $H(a\alpha + b\beta, c\alpha + d\beta)$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z})$. The main isomorphism theorem of this paper shows that this condition is also necessary for Heisenberg C^*-algebras to be $*$-isomorphic:

THEOREM 2.9. *Two Heisenberg C^*-algebras $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ are $*$-isomorphic if and only if there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z})$ with $e^{2\pi i\alpha_2} = e^{2\pi i(a\alpha_1 + b\beta_1)}$ and $e^{2\pi i\beta_2} = e^{2\pi i(c\alpha_1 + d\beta_1)}$.*

This theorem contains as corollaries the facts that Heisenberg C^*-algebras of class 1 can be parametrized by $H(1/d, 0)$, $d \in \mathbf{N}$, and Heisenberg C^*-algebras of class 2 can be parametrized by $H(\alpha/q, p/q)$ where $\alpha \in [0, 1/2]$ is irrational, and $p/q \in [0, 1/2]$ is rational in lowest terms. A generalization of Theorem 2.9 can be made which classifies all matrix algebras over Heisenberg C^*-algebras.

All of the Heisenberg C^*-algebras of class 1 are easily seen to be strongly Morita equivalent to $C^*(H)$, the so-called rotation algebra of [1]. The primitive ideal space of this algebra is T_1, and its structure was discussed by Howe in [12]. Each Heisenberg C^*-algebra of class 2 will be shown to be strongly Morita equivalent to a C^*-algebra generated by a certain Anzai skew-product action on the torus, which was studied in [14].

The structure of our work is as follows: In the first section we use a result of Mackey [13] which classifies the similarity classes of 2-cocycles for H with values in $\mathbf{T}$, and allows us to separate the algebras $C^*(H, \sigma)$ for various 2-cocycles σ into the 3 distinct classes mentioned above. We show that the C^*-algebras of classes 2 and 3 are simple and have a unique normalized trace. In section two we briefly discuss a method for studying projective modules over the $H(\alpha, \beta)$ constructed by means of crossed products, which will be an important tool in both this and our subsequent paper. We emphasize the use of strong Morita equivalence bimodules A-X-B for C^*-algebras with unit and the corresponding isomorphisms $M_B^A(X) : \mathrm{K}_0(B) \to \mathrm{K}_0(A)$ and $M_A^B(X) : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ canonically determined by X, and introduce the concept of "coupling constant" when A and B are unital. Examining some of these maps and using the uniqueness of the range of the trace on the K_0-groups for the Heisenberg C^*-algebras we are able to prove the isomorphism theorem. Finally we indicate an extension of Theorem 2.9 to matrix algebras over Heisenberg C^*-algebras.

We would like to thank Professor Marc Rieffel for helpful remarks and for showing us preprints of his work which relate to this subject. A preliminary version of part of this paper was distributed in [15]. Some of the results discussed here were obtained while the author was a member of the M.S.R.I. in Berkeley, California on a N.S.F. postdoctoral fellowship.

1. TWO-COCYCLES FOR H AND THE CLASS OF $C^*(H, \sigma)$

Throughout this paper H represents the discrete Heisenberg group, which we express as $\{(m, n, p) \mid m, n, p \in \mathbf{Z}\}$ where the group structure is given by $(m_1, n_1, p_1) \cdot (m_2, n_2, p_2) = (m_1 + m_2 + p_1 n_2, n_1 + n_2, p_1 + p_2)$; alternatively this group can be described as the subgroup of SL(3, $\mathbf{Z}$) consisting of upper triangular matrices with ones along the diagonal, or as a semidirect product $(\mathbf{Z} \oplus \mathbf{Z}) \rtimes \mathbf{Z}$, where $(\vec{v}_1, a)(\vec{v}_2, b) = (\vec{v}_1 + \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \vec{v}_2, a + b)$. Recall that a 2-cocycle for H with values in $\mathbf{T} = S^1$ is a map $\sigma : H \times H \to \mathbf{T}$ satisfying

$$\sigma(x, yz)\sigma(y, z) = \sigma(xy, z)\sigma(x, y).$$

A cocycle σ is called a coboundary if

$$\sigma(x, y) = b(x)b(y)b(xy)^{-1} \quad \text{for some map } b : H \to \mathbf{T}.$$

Two 2-cocycles σ_1 and σ_2 are called cohomologous if their quotient is a coboundary. Cocycles with values in $\mathbf{T}$ are also called *multipliers*, after [13].

We define the C^*-algebra $C^*(H, \sigma)$ as in [26], 2.24; recall that the isomorphism class $C^*(H, \sigma)$ depends only on the cohomology class of σ in $H^2(G; \mathbf{T})$. In our case, $C^*(H, \sigma) \cong C^*_{\mathrm{red}}(H, \sigma)$ [26], Section 5; this allows us to avoid a number of technicalities. If A is an automorphism of H, then $C^*(H, \sigma) \cong C^*(H, \sigma A)$ where $\sigma A(g_1, g_2) = \sigma(A(g_1), A(g_2))$. This is seen by mapping U_g into $W_{A^{-1}(g)}$, where $U_g \in C^*(H, \sigma)$, $W_g \in C^*(H, \sigma A)$. This is true as well for an arbitrary group G, but is especially helpful here since the quotient group Aut(H)/Inn(H) of automorphisms of H modulo the subgroup of inner automorphisms (conjugacies) is known to be GL(2, $\mathbf{Z}$). As will be seen, this considerably facilitates the classification of the $C^*(H, \sigma)$. First let us attack the problem of finding the cohomology classes of two-cocycles for H with values in $\mathbf{T}$. Though the following results can be worked out by using exact sequences in cohomology and the universal coefficient theorem, I prefer to use the following more explicit method to compute cohomology classes of two-cocycles for H.

PROPOSITION 1.1. *Let $\nu : H \times H \to \mathbf{T}$ be a two-cocycle for H with values in $\mathbf{T}$. Then there exist $\lambda, \mu \in \mathbf{T}$ such that the two-cocycle $\sigma = \sigma(\lambda, \mu)$ defined by*

$$\sigma((m_1, n_1, p_1),(m_2, n_2, p_2)) = \lambda^{m_2 p_1 + \frac{p_1(p_1-1)}{2} n_2} \mu^{n_1(m_2 + p_1 n_2) + \frac{p_1 n_2(n_2-1)}{2}}$$

is cohomologous to ν.

Proof. We noted in the beginning of this section that H can be expressed as the semidirect product $(\mathbf{Z} \oplus \mathbf{Z}) \times \mathbf{Z}$ where $(\vec{v}_1, a)(\vec{v}_2, b) = (\vec{v}_1 + \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \vec{v}_2, a + b)$. Hence by Theorem 9.4 of [13], any multiplier ν for H is cohomologous to a multiplier $\nu' = \nu'(\gamma, \omega, g)$ of the form

$$\nu'((m_1, n_1, p_1), (m_2, n_2, p_2)) =$$

$$= \gamma((m_1, n_1), \begin{pmatrix} 1 & p_1 \\ 0 & 1 \end{pmatrix} (m_2, n_2))\, \omega(p_1, p_2) g((m_2, n_2, p_1))$$

where γ is a multiplier for $\mathbf{Z} \oplus \mathbf{Z}$, ω is a multiplier for $\mathbf{Z}$, and g is a function from H to $\mathbf{T}$ satisfying

(a) $$\gamma\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \vec{v}_1, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \vec{v}_2\right) = \frac{\gamma(\vec{v}_1, \vec{v}_2) g(\vec{v}_1 + \vec{v}_2, t)}{g(\vec{v}_1, t) g(\vec{v}_2, t)}$$

(b) $$g(\vec{v}, t_1 + t_2) = g(\vec{v}, t_1) g\left(\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \vec{v}_1, t_2\right).$$

Clearly, if ω_1 is cohomologous to ω_2 then $\nu'(\gamma, \omega_1, g)$ is cohomologous to $\nu'(\gamma, \omega_2, g)$. It is an easy calculation that if γ_1 is cohomologous to γ_2, i.e. if

$\gamma_1(\vec{v}_1, \vec{v}_2) = \gamma_2(\vec{v}_1, \vec{v}_2)\dfrac{b(\vec{v}_1 + \vec{v}_2)}{b(\vec{v}_1)b(\vec{v}_2)}$, then, with $g_2(\vec{v}, t) = \dfrac{b\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\vec{v}\right)}{b(\vec{v})} g_1(\vec{v}, t)$, $\nu'(\gamma_1, \omega, g_1)$ is cohomologous to $\nu'(\gamma_2, \omega, g_2)$. Thus to derive the cohomology class of multipliers for H, it suffices to find parametrizations $[\omega]$ for $H^2(\mathbf{Z}; \mathbf{T})$ and $[\gamma]$ for $H^2(\mathbf{Z} \oplus \mathbf{Z}; \mathbf{T})$ and to find all functions g satisfying (a) and (b) for a given γ. The structures of $H^2(\mathbf{Z}; \mathbf{T})$ and $H^2(\mathbf{Z} \oplus \mathbf{Z}; \mathbf{T})$ are well known; all cocycles for $\mathbf{Z}$ with values in $\mathbf{T}$ are coboundaries, and any cocycle for $\mathbf{Z} \oplus \mathbf{Z}$ with values in $\mathbf{T}$ is cohomologous to one of the form $\gamma'((m_1, n_1), (m_2, n_2)) = \mu^{n_1 m_2}$ for some $\mu \in \mathbf{T}$. Fixing γ in such a form, we now find appropriate g. By the cocycle identity (b), the value of g on $(\mathbf{Z} \oplus \mathbf{Z}) \times \mathbf{Z}$ is determined by its values on the set $(\mathbf{Z} \oplus \mathbf{Z}, 1)$, since $g(\vec{v}, n)$ can be expressed as a finite product $g(\vec{v}_1, 1)g(\vec{v}_2, 1) \ldots$ $\ldots g(\vec{v}_n, 1)$ for appropriate choice of the $\vec{v}_i$. Hence our parametrization of multipliers of H up to cohomology will be completed upon calculating an appropriate function $\delta : \mathbf{Z} \to \mathbf{T}$ with $\delta(n) = g((0, n), 1)$. Computations using equations (a) and (b) show that $g(m, n, 1) = \lambda^m \mu^{\frac{n(n-1)}{2}} \delta^n$, where $\lambda = g((1, 0), 1)$ and $\delta = g((0, 1), 1)$. It follows that

$$g(m, n, p) = \lambda^{pm} \lambda^{\frac{p(p-1)n}{2}} \mu^{\frac{pn(n-1)}{2}} \delta^{pn}.$$

Hence

$$\nu'((m_1, n_1, p_1), (m_2, n_2, p_2)) = \gamma((m_1, n_1), (m_2 + p_1 n_2, n_2))g(m_2, n_2, p_1) =$$

$$= \mu^{n_1(m_2 + p_1 n_2) + \frac{p_1 n_2(n_2-1)}{2}} \lambda^{m_2 p_1 + \frac{p_1(p_1-1)n_2}{2}} \delta^{p_1 n_2}.$$

But ν' is now easily seen to be cohomologous to $\sigma = \sigma(\lambda, \mu)$; just define $b : H \to \mathbf{T}$ by $b(m, n, p) = \delta^p$ and compute $b(h_1)b(h_2)b(h_1, h_2)^{-1}$ for $h_1, h_2 \in H$. Thus δ, which appears in the relation $UW = \delta VWU$, has no bearing on the cohomology class of ν'.

The proof of the proposition shows that it is fairly easy to determine λ and μ when the 2-cocycle ν has certain symmetric properties:

COROLLARY 1.2. *If ν is a 2-cocycle for H with*

$$\nu((1, 0, 0), (0, 1, 0)) = 1, \quad \nu((0, 1, 0), (1, 0, 0)) = \mu,$$

$$\nu((1, 0, 0), (0, 0, 1)) = 1, \quad \textit{and} \quad \nu((0, 0, 1), (1, 0, 0)) = \lambda,$$

then ν is cohomologous to $\sigma(\lambda, \mu)$, where $\sigma(\lambda, \mu)$ is the cocycle of Proposition 1.1.

The proof follows from repeated use of the Proposition and Theorem 9.4 of [13]; we leave it as an exercise to the reader.

As a result of Proposition 1.1 and Corollary 1.2, any $C^*(H, \sigma)$ is $*$-isomorphic to the algebra generated by three unitary elements with the following relations, for some $\lambda, \mu \in \mathbf{T}$:

$$UV = \lambda VU,$$

$$WV = \mu VW,$$

$$UW = VWU.$$

We shall denote $C^*(H, \sigma(\lambda, \mu))$ by $H(\alpha, \beta)$, where $\lambda = e^{2\pi i\alpha}$, $\mu = e^{2\pi i\beta}$. For future reference let us now prove the following simple proposition, which allows us to reduce further the number of cases which must be studied (the proposition is related to work of Brenken [3] on $*$-automorphisms of two-dimensional non-commutative tori).

PROPOSITION 1.3. *Let* $\lambda = e^{2\pi i\alpha}$ *and* $\mu = e^{2\pi i\beta}$ *for* $\alpha, \beta \in \mathbf{R}$. *Then for any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$, $C^*(H, \sigma(\lambda, \mu)) = H(\alpha, \beta)$ *is* $*$*-isomorphic to* $C^*(H, \sigma(\lambda^d\mu^b, \lambda^c\mu^a)) = H(b\beta + d\alpha, a\beta + c\alpha)$.

Proof. Let A_M denote the automorphism of H defined on the generators by $A_M((1,0,0)) = (\det M, 0, 0)$, $A_M((0,1,0)) = (0, a, c)$, $A_M((0,0,1)) = (0, b, d)$. Then if we denote by σA_M the two-cocycle for H defined by $\sigma A_M(g_1, g_2) = \sigma(A_M(g_1), A_M(g_2))$, we know that $C^*(H, \sigma)$ is $*$-isomorphic to $C^*(H, \sigma A_M)$, and $C^*(H, \sigma A_M)$ is defined by

$$W' = W^a U^c,$$

$$U' = W^b U^d,$$

$$V' = V^{\det M}.$$

We see that

$$U'V' = W^b U^d V^{\det M} = W^b V^{\det M} \lambda^{d \det M} U^d =$$

$$= \lambda^{\det M} \mu^{b \det M} V^{\det M} W^b U^d = (\lambda^d \mu^b)^{\det M} V^{\det M} W^b U^d = (\lambda^d \mu^b)^{\det M} V' U',$$

$$W'V' = W^a U^c V^{\det M} = W^a \lambda^{c \det M} V^{\det M} U^c =$$

$$= \lambda^{c \det M} \mu^{a \det M} V^{\det M} W^a U^c = (\lambda^c \mu^a)^{\det M} V' W',$$

$$U'W' = W^b U^d W^a U^c = W^b \varkappa_1 V^{ad} W^a U^d U^c = \qquad (\text{some } \varkappa_1 \in \mathbf{T})$$

$$= \varkappa_1 \mu^{abd} V^{ad} W^a W^b U^c U^d = \varkappa_1 \mu^{adb} V^{ad} \varkappa_2 V^{-bc} W^a U^c W^b U^d = \qquad (\varkappa_2 \in \mathbf{T})$$

$$= \gamma V^{ad-bc} W^a U^c W^b U^d = \gamma V' W' U'. \qquad (\gamma \in \mathbf{T})$$

Hence σA_M is cohomologous to $\sigma((\lambda^d\mu^b)^{\det M}, (\lambda^c\mu^a)^{\det M})$, so that by Corollary 1.2 $C^*(H, \sigma(\lambda, \mu))$ is $*$-isomorphic to $C^*(H, \sigma((\lambda^d\mu^b)^{\det M}, (\lambda^c\mu^a)^{\det M}))$. If $M \in \mathrm{SL}(2, \mathbf{Z})$, we are done since $\det M = 1$. If $\det M = -1$, then, as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$, and the proposition is proved for matrices in $\mathrm{SL}(2, \mathbf{Z})$, $C^*(H, \sigma((\lambda^d\mu^b)^{-1}, (\lambda^c\mu^a)^{-1})$ is $*$-isomorphic to $C^*(H, \sigma(\lambda^d\mu^b, \lambda^c\mu^a))$. This proves the desired result. ▨

We now wish to introduce a preliminary classification of the $C^*(H, \sigma)$. In order to do this, let us show that the range of any faithful normalized trace τ on $\mathrm{K}_0(C^*(H, \sigma(\lambda, \mu)))$ is equal to $\Pi^{-1}(\langle\lambda, \mu\rangle) = \mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\beta$, where $\langle\lambda, \mu\rangle$ is the subgroup of $\mathbf{T}$ generated by μ and λ and $\Pi : \mathbf{R} \to \mathbf{R}/\mathbf{Z} = \mathbf{T}$ is the natural projection. Let $\langle U, V\rangle$ and $\langle V, W\rangle$ be the C^*-subalgebras of $C^*(H, \sigma(\lambda, \mu))$ generated by U, V and V, W, respectively. We can write $C^*(H, \sigma(\lambda, \mu))$ as the crossed product $\langle U, V\rangle \times \mathbf{Z}$ or as the crossed product $\langle V, W\rangle \times \mathbf{Z}$. In either event, by the Pimsner-Voiculescu exact sequence, the group $\mathrm{K}_0(C^*(H, \sigma(\lambda, \mu)))$ is generated by the insertions of $\mathrm{K}_0(\langle U, V\rangle)$ and $\mathrm{K}_0(\langle V, W\rangle)$ into it. The desired result follows from this, together with known facts about the images of normalized traces on $\mathrm{K}_0(\langle U, V\rangle)$, $\mathrm{K}_0(\langle V, W\rangle)$ [9], [20]. This fact gives meaning to the classification which follows:

DEFINITION 1.4. Let $C^*(H, \sigma)$ be brought into the form $C^*(H, \sigma(\lambda, \mu))$ for some $\lambda, \mu \in \mathbf{T}$. We say that $C^*(H, \sigma(\lambda, \mu))$ is of class 1 if $\Pi^{-1}(\langle\lambda, \mu\rangle)$ can be generated by one element of $\mathbf{R}$, class 2 if $\Pi^{-1}(\langle\lambda, \mu\rangle)$ can be generated by two elements of $\mathbf{R}$ but not by one, and is of class 3 if $\Pi^{-1}(\langle\lambda, \mu\rangle)$ can be generated by three but not by two elements of $\mathbf{R}$.

The paragraph preceding Definition 1.4 guarantees that the class is an isomorphism invariant for the algebras $C^*(H, \sigma)$. The following proposition follows from Proposition 1.3 and gives a standard form for Heisenberg C^*-algebras of classes 1 and 2.

PROPOSITION 1.5. *If $C^*(H, \sigma)$ is of class 1, then $C^*(H, \sigma)$ is $*$-isomorphic to $H(1/n, 0)$ for some $n \in \mathbf{N}$. If $C^*(H, \sigma)$ is of class 2, then it is $*$-isomorphic to $H(\alpha, p/q)$ for some irrational α and some $p, q \in \mathbf{Z}$.*

Proof. If $C^*(H, \sigma)$ is of class 1, it is of the form $H(p/q, r/s)$ for $p, q, r, s \in \mathbf{Z}$ with $(p, q) = (r, s) = 1$. Then, with n the least common multiple of q and s, it is easily shown using Proposition 1.3 and elementary number-theoretic arguments that there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p/q \\ r/s \end{pmatrix} = \begin{pmatrix} 1/n \\ 0 \end{pmatrix} \bmod 1$, so that $H(p/q, r/s) \cong H(1/n, 0)$.

If $C^*(H, \sigma)$ is of class 2, then it is $*$-isomorphic to $H(\beta_1, \beta_2)$ where at least one of the β_i is irrational, say β_1. Since $\mathbf{Z} + \beta_1\mathbf{Z} + \beta_2\mathbf{Z}$ has two generators,

$\beta_2 \in \mathbf{Q} + \mathbf{Q}\beta_1$. In other words there exist integers h, l, m and m' with

$$\beta_2 = \frac{n}{m}\beta_1 + \frac{l}{m'}.$$

We may assume that $(n, m) = 1$. Let a and b be integers such that $am + bn = -1$. Then $\begin{pmatrix} -m & b \\ n & a \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$. By Proposition 1.4, the C^*-algebra $C^*(H, \sigma(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}))$ is $*$-isomorphic to the C^*-algebra

$$C^*(H, \sigma(e^{(2\pi i\beta_1)a}\, e^{(2\pi i\beta_2)b},\ e^{(2\pi i\beta_1)n}\, e^{(2\pi i\beta_2)-m})) =$$

$$= C^*(H, \sigma(e^{2\pi i a\beta_1} e^{2\pi i b\beta_2},\ e^{2\pi i n\beta_1} e^{2\pi i(-m)\left(\frac{n}{m}\beta_1 + \frac{l}{m'}\right)})) =$$

$$= C^*(H, \sigma(e^{2\pi i\alpha},\ e^{2\pi i p/q})) = H(\alpha, p/q),$$

where

$$\alpha = a\beta_1 + b\beta_2 = \left(a + \frac{bn}{m}\right)\beta_1 + \frac{l}{m'},$$

which is irrational, and

$$p/q = \frac{-m}{m'} \cdot l. \qquad ▨$$

We shall see in the next section that class 1 Heisenberg C^*-algebras are strongly Morita equivalent to the group C^*-algebra for the Heisenberg group $C^*(H)$, the rotation algebra of [1]. Those of class 2 will be shown to be strongly Morita equivalent to C^*-algebras generated by minimal Anzai transformations on the two-torus, and are therefore simple. We now introduce another approach towards showing that class 2 Heisenberg C^*-algebras are simple, and this method can be applied to class 3 Heisenberg C^*-algebras as well.

Let $\sigma(\lambda, \mu)$ be the cocycle discussed at the beginning of this section, and let $\psi : H \times H \to \mathbf{Z} \oplus \mathbf{Z}$ be given by

$$\psi((m_1, n_1, p_1), (m_2, n_2, p_2)) =$$

$$= \left(p_1\left(m_1 + \frac{(p_1 - 1)}{2} n_2\right), p_1 n_2 \frac{(n_2 - 1)}{2} + (m_2 + n_2 p_1) n_2\right).$$

It is easy to verify that ψ is a cocycle, and denoting by Π the map from $\mathbf{Z} \oplus \mathbf{Z} \to \mathbf{T}$ given by $\Pi(m, n) = \lambda^m \mu^n$, we see that the diagram

$$\begin{array}{ccc} H \times H & \xrightarrow{\psi} & \mathbf{Z} \oplus \mathbf{Z} \\ & \searrow^{\sigma} & \downarrow \Pi \\ & & \mathbf{T} \end{array}$$

commutes.

Now form the central group extension $H \times_\psi (\mathbf{Z} \oplus \mathbf{Z})$. Since H is a nilpotent torsion-free discrete group, $H \times_\psi (\mathbf{Z} \oplus \mathbf{Z}) = \Gamma$ is a nilpotent torsion-free discrete group. Hence, by the work of Howe [12] the C^*-algebra $C^*(\Gamma)$ has a T_1 primitive ideal space, i.e. any primitive ideal of $C^*(\Gamma)$ is maximal. We can define a unitary representation of Γ on $L^2(H)$ by setting

$$U_{(g_1, m_1)} f(g) = \Pi(m_1)\sigma(g_1, g_1^{-1} g) f(g_1^{-1} g)$$

(here $\Gamma = H \times_\psi (\mathbf{Z} \oplus \mathbf{Z})$ so that $g_1 \in H, m_1 \in \mathbf{Z} \oplus \mathbf{Z}$).

Hence there exists a representation φ of $C^*(\Gamma)$ on $L^2(H)$. We claim that $\varphi(C^*(\Gamma)) = C^*(H, \sigma)$. It is clear that $\varphi(C^*(\Gamma))$ is generated by $U_{(1,0,0,0,0)}, U_{(0,1,0,0,0)}$, and $U_{(0,0,1,0,0)}$. By inspection we see that $U_{(1,0,0,0,0)}$, $U_{(0,1,0,0,0)}$, and $U_{(0,0,1,0,0)}$ correspond to V, W, and U respectively, in the left regular representation of $C^*(H, \sigma)$. Thus $\varphi(C^*(\Gamma)) = C^*_r(H, \sigma)$. But since H is amenable, $C^*_r(H, \sigma) \cong C^*(H, \sigma) \cong$ $\cong H(\alpha, \beta)$. Assume that $H(\alpha, \beta)$ is of class 2 or 3 and that α is irrational. It is not hard to see that the representation φ of Γ described above is a factorial representation in this case. In fact let $\mathcal{M}$ be the von Neumann algebra acting on $L^2(H)$ generated by U, V, and W, and let $\mathcal{N}$ be the von Neumann subalgebra of $\mathcal{M}$ generated by U and V. It is easily shown that $\mathcal{M}$ is $*$-isomorphic to $\mathcal{N} \times \mathbf{Z}$ where the action of $\mathbf{Z}$ on $\mathcal{N}$ is generated by $\mathrm{Ad}\, W$. Since $\mathcal{N}$ is a factor (by our choice of α) and since $\mathrm{Ad}\, W^n$ is an outer action on $\mathcal{N}$ for all $n \in \mathrm{N}$, $\mathcal{M} = \mathcal{N} \times \mathbf{Z}$ is a factor. It follows that the kernel of φ, $\mathcal{J}$, is a primitive ideal. Since Γ is nilpotent and torsion-free, by the work of Howe $\mathcal{J}$ is maximal. In other words, $C^*(\Gamma)/\mathcal{J} \cong H(\alpha, \beta)$ is simple. Also, by the same work of Howe [12, p. 297, Proposition 3], this simple quotient has a unique trace. The above ideas complete the proof of

THEOREM 1.6. *Let $C^*(H, \sigma(\lambda, \mu)) = H(\alpha, \beta)$ be a Heisenberg C^*-algebra of class 2 or 3. Then $H(\alpha, \beta)$ is simple and has a unique normalized trace.*

REMARK 1.7. The referee has noted that Theorem 1.6 can also be proved using the arguments of Slawny [24] for non-commutative tori. We add that by applying the methods of Howe as in the proof above, one can show that if N is a finitely generated torsion-free nilpotent group, $C^*(N, \sigma)$ is simple and has a unique normalized trace whenever the following two conditions hold:

1) σ can be lifted to a cocycle $\hat{\sigma}$ taking values in $\mathbf{R}$;
2) the left regular representation of $C^*(N, \sigma)$ is factorial.

2. STRONG MORITA EQUIVALENCE AND ITS APPLICATIONS TO THE CLASSIFICATION OF HEISENBERG C^*-ALGEBRAS

In this section we discuss the isomorphism problem for Heisenberg C^*-algebras and their matrix algebras. As in the case of irrational rotation algebras the range of a normalized trace on the K_0-group provides an important isomorphism invariant. It is a complete invariant for Heisenberg C^*-algebras of classes 1 and 3. Distinguishing between the various class 2 C^*-algebras and their matrix algebras necessitates the use of further invariants, however, and so we first review the relationship between strong Morita equivalence and projective modules over unital C^*-algebras and their subalgebras.

We recall the idea of strong Morita equivalence, due to Rieffel [19]. Let A and B be algebras. A complete left A- and right B-bimodule X is said to be a *strong Morita equivalence* bimodule if it has A and B valued inner products satisfying

(1) $\langle x, y\rangle_A z = x\langle y, z\rangle_B$.

(2) The representation of A (respectively B) on X is a continuous representation by operators which are bounded for $\langle\ ,\ \rangle_B$ (respectively $\langle\ ,\ \rangle_A$).

(3) The linear span of $\langle X, X\rangle_A$ is dense in A and similarly for $\langle X, X\rangle_B$ in B.

If such an X exists, A and B are said to be *strongly Morita equivalent*. Following [2] an A-B strong Morita equivalence bimodule X_1 is said to be equivalent to an A-B strong Morita equivalence bimodule X_2 if they are isomorphic as A-B equivalence bimodules. If A-X-B and B-Y-C are equivalence bimodules, then there is a natural composition of bimodules providing the equivalence bimodule A-$X\otimes_B Y$-C. It is shown in [2], [19] that the category whose objects are equivalence bimodules of C^*-algebras and whose morphisms are equivalence classes of equivalence bimodules is a category with inverses.

When unital C^*-algebras A and B are strongly Morita equivalent, the work of [20, Section 2] shows that projections in A can be represented as projections in some $M_n(B)$ and conversely. The correspondence between finitely generated projective A- and B-modules described by the tensoring-by-X operation above is exactly the map of projections described in [20]. In this way we obtain a linear isomorphism between $K_0(A)$ and $K_0(B)$, which we denote by $M_A^B(X)$, and a linear isomorphism $M_B^A(X)$ between $K_0(B)$ and $K_0(A)$. Given an equivalence bimodule A-X-B we define an equivalence bimodule B-$\tilde{X}$-A by setting $\tilde{X} = X, b\cdot x = xb^*$, $x\cdot a = = a^*x$, $\langle x, y\rangle_A^{\text{new}} = \langle x, y\rangle_B^{\text{old}}$, $\langle x, y\rangle_B^{\text{new}} = \langle x, y\rangle_A^{\text{old}}$. It is clear that $M_A^B(\tilde{X}) = M_A^B(X)$ and $M_B^A(\tilde{X}) = M_B^A(X)$. By Lemma 6.22 of [19], A-$X \otimes_B \tilde{X}$-A is equivalent to the natural equivalence bimodule A-A-A where right and left inner products are given by $\langle x, y\rangle = x^*y$, $\langle x, y\rangle = xy^*$, respectively. Similarly B-$\tilde{X} \otimes_A X$-B is equivalent to B-B-B. Hence

$$M_{A_{\text{left}}}^{A^{\text{right}}}(X \otimes_B \tilde{X}) = \text{Id}(K_0(A)) \quad \text{and} \quad M_B^B(\tilde{X} \otimes_A X) = \text{Id}(K_0(B)),$$

but on the other hand

$$M_{B_{\text{left}}}^{A^{\text{right}}}(X \otimes \tilde{X}) = M_B^A(X)M_A^B(X).$$

Similarly $M_A^B(X)\,M_B^A(X) = \text{Id}(K_0(B))$ thus the isomorphisms $M_A^B(X)$ and $M_B^A(X)$ are inverse to each other.

DEFINITION 2.1. Let A-X-B be a strong Morita equivalence bimodule where A and B are C^*-algebras with unit. Let τ be a faithful trace on A, and let $\text{Ind}_X(\tau)$ be the trace on B induced by X, as defined in [20, Proposition 2.2]. We define the *coupling constant from A to B for τ determined by X*, denoted by $C_A^B(X)(\tau)$, to be the positive number $\text{Ind}_X(\tau)(\text{Id}_B)\tau(\text{Id}_A)^{-1}$. Similarly, if τ' is a faithful trace on B, we define *the coupling constant from B to A for τ' determined by X* denoted by $C_B^A(X)(\tau')$, to be the positive number $\text{Ind}_X(\tau')(\text{Id}_A)\tau'(\text{Id}_B)^{-1}$. (This concept is related to the work of [5, Section 3], and most importantly to [20, Proposition 2.5 and 2.6].)

We note in passing that the coupling constant remains unchanged under normalization, i.e. if A-X-B and $\tau \in \text{Trace}\, A$ are as above, and if $K \in \mathbf{R}^+$, then

$$C_A^B(X)(K\tau) = \text{Ind}_X(K\tau)(\text{Id}_B)(K\tau(\text{Id}_A))^{-1} =$$

$$= K\,\text{Ind}_X(\tau)(\text{Id}_B)K^{-1}\tau(\text{Id}_A)^{-1} = C_A^B(X)(\tau).$$

We also note that if we denote the normalization of the trace $\text{Ind}_X\tau$ by $n(\text{Ind}_X\tau)$, then

(1) $$n(\text{Ind}_X(\tau))^*(K_0(B)) = (C_A^B(X)(\tau))^{-1}\tau^*(K_0(A)),$$

(2) $$n(\text{Ind}_X(\tau))^*(M_A^B(X)[p]) = (C_A^B(X)(\tau))^{-1}\tau^*([p]), \quad \text{for every } [p] \in K_0(A).$$

This is because $\text{Ind}_X(\tau)^*(K_0(B)) = \tau^*(K_0(A))$, and $n(\text{Ind}_X(\tau)) = (C_A^B(X)(\tau))^{-1}\text{Ind}_X(\tau)$.

EXAMPLE 2.2. Let α be an irrational number, and suppose that $\begin{pmatrix} a & b \\ q & p \end{pmatrix} \in$ $\in \mathrm{GL}(2, \mathbf{Z})$, and let $V_\alpha(p, q)$ be the left A_β-right A_α-module described in [22, Section 1], where $\beta = \dfrac{a\alpha + b}{q\alpha + p}$ and A_α, A_β are irrational rotation algebras with normalized traces τ_α and τ_β. The groups $K_0(A_\alpha)$ and $K_0(A_\beta)$ can be identified with $\mathbf{Z} + \alpha\mathbf{Z}$ and $\mathbf{Z} + \beta\mathbf{Z}$ respectively. The coupling constants are given by

$$C_{A_\beta}^{A_\alpha}(V_\alpha(p, q))(\tau_{A_\beta}) = |q\alpha + p|^{-1}$$

and

$$C_{A_\alpha}^{A_\beta}(V_\alpha(p, q))(\tau_{A_\alpha}) = |q\alpha + p|,$$

as can be seen by the proof of [20, Theorem 4].

EXAMPLE 2.3. Let a and q be relatively prime and choose p, b such that $ap + bq = 1$. Let $A = C(\mathbf{T}^2)$ and let $X(q, a)$ be the left $C(\mathbf{T}^2)$-module consisting of all continuous maps h from $\mathbf{R} \times \mathbf{T}$ to $\mathbf{C}$ satisfying $h(t - q, s) = e^{2\pi i a s} h(t, s)$. From the results of [22] we know that $\mathrm{End}_{C(\mathbf{T}^2)} X(q, a) = A_{p/q}$, the rational rotation algebra, so that $X(q, a)$ is a left $C(\mathbf{T}^2)$-right $A_{p/q}$-strong Morita equivalence bimodule. Furthermore if we let $K_0(C(\mathbf{T}^2))$ be parametrized by $\{(m, n) \mid m, n \in \mathbf{Z}\}$ where m represents dimension and n represents twist, following the notation of [22, Section 3], then

$$M_{C(\mathbf{T}^2)}^{A_{p/q}}(X(q, a))((q, -a)) = [\mathrm{Id}]_{K_0(A_{p/q})}.$$

This is just a restatement of the fact that $X(q, a)$ is a projective module with dimension q and twist $-a$, which is proved in [22]. It follows that if τ is any faithful normalized trace on $C(\mathbf{T}^2)$, $C_{C(\mathbf{T}^2)}^{A_{p/q}}(\tau) = q$. We now state the following easy

PROPOSITION 2.4. *Let A-X-B and B-Y-C be two strong Morita equivalence bimodules, where A, B and C are all C^*-algebras with unit. Let A-$X \otimes_B Y$-C be the strong Morita equivalence between A and C formed from X and Y via the tensor product construction. Let τ be a faithful trace on A. Then*

$$C_A^C(X \otimes_B Y)(\tau) = C_B^C(Y)(\mathrm{Ind}_X(\tau)) C_A^B(X)(\tau).$$

Proof. The conclusion follows from noting that

$$\mathrm{Ind}_Y(\mathrm{Ind}_X(\tau)) = \mathrm{Ind}_{X \otimes_B Y}(\tau),$$

since then

$$C_A^C(X \otimes_B Y)(\tau) = \mathrm{Ind}_{X\otimes_B Y}(\tau)(\mathrm{Id}_C)\tau(\mathrm{Id}_A)^{-1} = \mathrm{Ind}_Y(\mathrm{Ind}_X(\tau))(\mathrm{Id}_C)\tau(\mathrm{Id}_A)^{-1} =$$

$$= \mathrm{Ind}_Y(\mathrm{Ind}_X(\tau))(\mathrm{Id}_C)\mathrm{Ind}_X(\tau)(\mathrm{Id}_B)^{-1}\mathrm{Ind}_X(\tau)(\mathrm{Id}_B)\tau(\mathrm{Id}_A)^{-1} =$$

$$= C_B^C(Y)(\mathrm{Ind}_X(\tau))C_A^B(X)(\tau).$$

To prove the equality of the traces, one simply uses formulas for $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_C$ given in [18, p.186] to verify that $\mathrm{Ind}_{X\otimes_B Y}(\tau)(\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_C) = \mathrm{Ind}_Y(\mathrm{Ind}_X(\tau))(\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_C)$. ▨

We move on to discuss the construction of crossed product equivalence bimodules for crossed products of unital C^*-algebras, from equivalence bimodules for the original C^*-algebras. This has been discussed extensively in [4], [7] and [15]. We include here an outline of a treatment very close to that of Combes [4], which is slightly different from the work of Curto, Muhly, and Williams [7].

DEFINITION 2.5. Let X be a left module over the C^*-algebra A which is a left A-rigged space in the sense of [19, Definition 2.8], and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of the locally compact group G on A. We say that (A, α, G) is a *unitarily covariant system* with respect to X if there exists a strongly continuous action of G on X, i.e. homomorphism $U : G \to \mathrm{Aut}(X)$, which satisfies

$$\text{(1)} \qquad \langle U_g x, U_g y\rangle_A = \alpha(g)(\langle x, y\rangle_A) \quad \forall g \in G,\ \forall x, y \in X,$$

$$\text{(2)} \qquad U_g a U_{g^{-1}} = \alpha(g)(a) \text{ as an endomorphism of } X, \quad \forall g \in G, \forall a \in A.$$

If X is a finitely generated projective module over A, then the condition above implies that the map $\alpha(g)^* : \mathrm{K}_0(A) \to \mathrm{K}_0(A)$ determined by the Morita equivalence between A and itself defined by the automorphism $\alpha(g)$ (see [2, Section 3]) acts as the identity on $[P_X]_{\mathrm{K}_0(A)}$ where P_X is the projection in some $M_n(A)$ determined by X.

We now state the following result, which is a slight variation of the main results of [4] and [7] for unital C^*-algebras.

THEOREM 2.6 ([4], [7]). *Let A-X-B be a strong Morita equivalence bimodule where A and B are unital C^*-algebras. Let α be a continuous homomorphism of the unimodular locally compact group G into* Aut(A), *and suppose (A, α, G) is a unitarily covariant system with respect to X. Then there exists a continuous action β of G on B such that (B, β, G) is unitarily covariant with respect to X, and the crossed product C^*-algebras $A \times_\alpha G$ and $B \times_\beta G$ are strongly Morita equivalent.*

Sketch of proof. The existence of the action of G on B is actually contained in ([4], p. 292, Proposition and Remark 2), but we include a few details for our case. Given $g \in G$, $b \in B$, define $\beta(g)(b)$ as an endomorphism of X acting on the right by setting $x \cdot \beta(g)(b) = U_g((U_{g^{-1}}x)b)$, $\forall x \in X$. Since A and B have units, and X is complete, we can identify B with $\mathrm{End}_A(X)$, and we claim $\beta(g)(b) \in B$. Since

$$(ax)\beta(g)(b) = U_g((U_{g^{-1}})(ax))b) = U_g((\alpha(g^{-1})(a)[U_{g^{-1}}x])b) =$$

$$= U_g(\alpha(g^{-1})(a)[(U_{g^{-1}}x)b]) = a(U_g[(U_{g^{-1}}x)b]) = a(x\beta(g)(b)),$$

$\beta(g)(b)$ is an element of $\mathrm{End}_A(X)$ and hence defines an element of B. It is easy to verify that (B, β, G) is a unitarily covariant system for X. The conclusion of Theorem 2.6 now follows from application of [7, Theorem 1] or the theorem of [4, p. 299]. For further use we note that the equivalence bimodule between $A \times_\alpha G$, and $B \times_\beta G$ can be constructed as follows: Let $\mathscr{A} = C_c(G, A)$ and let $\mathscr{B} = C_c(G, B)$ denote the $*$-algebras of continuous functions with compact support from G to A and B respectively. Let $\mathscr{X} = C_c(G, X)$ where $C_c(G, X)$ is the set of continuous functions with compact support from G to X. We shall give $\mathscr{X}$ the structure of an $\mathscr{A}$-$\mathscr{B}$ equivalence bimodule.

The left and right module actions of $\mathscr{A}$ and $\mathscr{B}$ are defined on $\mathscr{X}$ by settting

$$f \cdot x(g) = \int f(g_1)U_{g_1}(x(g_1^{-1}g))\,dg_1,$$

$$x \cdot h(g) = \int x(g_1)\beta(g_1)(h(g_1^{-1}g))\,dg_1,$$

for $f(g), x(g), h(g) \in \mathscr{A}, \mathscr{X}, \mathscr{B}$ respectively. Then for every $f \in \mathscr{A}$, $x \in \mathscr{X}$, $h \in \mathscr{B}$,

$$(f \cdot x) \cdot h = f \cdot (x \cdot h).$$

The $\mathscr{A}$- and $\mathscr{B}$- valued inner products on $\mathscr{X}$ are defined by

$$\langle f_1, f_2 \rangle_{\mathscr{A}}(g) = \int \langle f_1(g_1),\ U^g(f_2(g_1^{-1}g)) \rangle_A\,dg_1,$$

$$\langle f_1, f_2 \rangle_{\mathscr{B}}(g) = \int \beta(g_1)\langle f_1(g_1), f_2(g_1^{-1}g) \rangle_B\,dg_1.$$

Upon suitably structuring and completing $\mathscr{X}$ we obtain the desired equivalence bimodule, which hereafter we denote by $X \times G$.

REMARK 2.7. If G is discrete, it is clear from the definition of coupling constant given in Definition 2.1 that if $\tau_{A\times G}$ is a trace on $A\times G$ which reduces to the trace τ_A on A, then $C^{B\times G}_{A\times G}(X\times G)(\tau_{A\times G}) = C^B_A(X)(\tau_A)$.

EXAMPLE 2.8. This example will be useful in proving the isomorphism theorems. Let $A = C(\mathbf{T}^2)$ and let $X(q, a)$ be the left $C(\mathbf{T}^2)$-module consisting of all continuous maps from $\mathbf{R}\times\mathbf{T}\to\mathbf{C}$ satisfying $h(t-q, s) = e^{2\pi i a s}h(t, s)$, where $(a, q) = 1$.

From the results of [22] we know that $\mathrm{End}_{C(\mathbf{T}^2)}X(q, a) = A_{p/q}$, a rational rotation algebra, where $p\in\mathbf{Z}$ is such that $ap + bq = 1$ for some $b\in\mathbf{Z}$. Let V and W be generators of $A_{p/q}$ such that $WV = e^{2\pi i p/q}VW$. Recall that the action of $A_{p/q}$ on $X(q, a)$ is defined by

$$h\cdot V(t, s) = h(t, s)e^{2\pi i t/q},$$

$$h\cdot W(t, s) = h(t-p, s)e^{2\pi i b s}.$$

(These formulas come from examination of the formulas found in [22], 3.4, 3.5, 3.6.)

Let α be the action of $\mathbf{Z}$ on $C(\mathbf{T}^2)$ given by Anzai's skew product on the torus: $\alpha(1)f(t, s) = f(t+\alpha, s+t)$, $f\in C(\mathbf{T}^2)$. In order to apply Theorem 2.6 to $C(\mathbf{T}^2)\times_\alpha\mathbf{Z} = H_\alpha$ we must find an $\alpha(1)$-equivariant automorphism of $X(q, a)$. On defining $Qh(t, s) = e^{2\pi i g(t)}h(t+\alpha, s+t)$, where $g(t) = -\dfrac{a}{2q}t^2 + \dfrac{a}{2}t$ is a real-valued function, a calculation shows that

$$Q f(t, s)\, Q^{-1}\cdot h(t, s) = f(t+\alpha, t+s)h(t, s) = \alpha(1)(f)\cdot h(t, s),$$

for $f\in C^*(\mathbf{T}^2)$ and $h\in X(q, a)$.

Furthermore $\langle Qh, Qh\rangle_{C(\mathbf{T}^2)} = \alpha(1)(\langle h, h\rangle_{C(\mathbf{T}^2)})$. Hence the conditions of Theorem 2.6 are satisfied, and we calculate

$$Q((Q^{-1}h)V) = h(s, t)e^{2\pi i\alpha/q}V,$$

$$Q((Q^{-1}h)W) = h(s, t)\varkappa V\cdot W \quad \text{for some } \varkappa\in\mathbf{T};$$

in other words, the action of $\mathbf{Z}$ on $A_{p/q}$ guaranteed by Theorem 2.6 is given by

$$\beta(V) = e^{2\pi i\alpha/q}V,$$

$$\beta(W) = \varkappa VW.$$

Thus by Corollary 1.2, $A_{p/q}\times_\beta\mathbf{Z}$ is $*$-isomorphic to $H(\alpha/q, p/q)$, as defined in Section 1. Hence the application of Theorem 2.6 to our situation shows that H_α

is strongly Morita equivalent to $H(\alpha/q, p/q)$. We denote the equivalence bimodule constructed there by $X(q, a) \times_{\alpha/q} \mathbf{Z}$. An application of Remark 2.7 shows us that the projection in some $M_n(H_\alpha)$ corresponding to the projective module $X(q, a) \times_{\alpha/q} \mathbf{Z}$ is just the projection in some $M_n(C(\mathbf{T}^2))$ corresponding to the projective module $X(q, a)$. Following Rieffel, we say this projection has trace (dimension) q and twist $-a$. Remark 2.7 also shows that $C_{H_\alpha}^{H(\alpha/q,p/q)}(X(q, a)\times_{\alpha/q} \mathbf{Z})(\tau) = q$, where τ is the unique normalized trace on H_α, because $C_{C(\mathbf{T}^2)}^{A_{p/q}}(X(q, a))(\tilde{\tau}) = q$, where $\tilde{\tau}$ is any faithful normalized trace on $C(\mathbf{T}^2)$. ▨

Two direct applications of Example 2.8 and Proposition 1.5 show that any Heisenberg C^*-algebra of class 1 (which is $*$-isomorphic to $H(1/n, 0)$ for some $n \in \mathbf{N}$) is strongly Morita equivalent to the rotation algebra, $C^*(H)$, and any Heisenberg C^*-algebra of class 2 (which is $*$-isomorphic to $H(\beta, p/q)$ for some irrational β and $p, q \in \mathbf{Z}$) is strongly Morita equivalent to a C^*-algebra generated by an Anzai skew product action on the torus.

We are now in a position to prove the first of our main isomorphism theorems:

THEOREM 2.9. *Let $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ be two Heisenberg C^*-algebras. Then $H(\alpha_1, \beta_1)$ is $*$-isomorphic to $H(\alpha_2, \beta_2)$ if and only if there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ $\in \mathrm{GL}(2, \mathbf{Z})$ with*

$$e^{2\pi i\alpha_2} = e^{2\pi i(a\alpha_1 + c\beta_1)} \quad \text{and} \quad e^{2\pi i\beta_2} = e^{2\pi i(b\alpha_1 + d\beta_1)}.$$

The proof of the theorem will be divided into several lemmas. Given any Heisenberg C^*-algebra, we determine whether it is of class 1, 2 or 3 by examining the range of a trace on the K_0-group. The following result for class 1 Heisenberg C^*-algebras comes directly from the work of Section 1:

LEMMA 2.10. *Let H_1 and H_2 be Heisenberg C^*-algebras of class 1, so that $H_1 = H(p_1/q_1, r_1/s_1)$ and $H_2 = H(p_2/q_2, r_2/s_2)$, where $(p_i/q_i) = 1$ and $(r_i, s_i) = 1$, $i = 1, 2$. Then $H_1 \cong H_2$ if and only if $\mathrm{l.c.m.}(q_1, s_1) = \mathrm{l.c.m.}(q_2, s_2)$, and $H_i \cong H\left(\frac{1}{\mathrm{l.c.m.}(q_i, s_i)}, 0\right)$.*

Proof. The fact that $H_i \cong H\left(\frac{1}{\mathrm{l.c.m.}(q_i, s_i)}, 0\right)$ was proved in Proposition 1.5 of the previous section. Thus if $\mathrm{l.c.m.}(q_1, s_1) = \mathrm{l.c.m.}(q_2, s_2) = n$ we have $H_1 \cong H_2 \cong H(1/n, 0)$. Now assuming $H_1 \cong H_2$, let τ be any normalized faithful trace on $H_1 \cong H_2$. By the results of Elliott [8] combined with the proof of Proposition 1.7, we see that $\tau^*(K_0(H_1)) = \frac{1}{\text{l.c.m.}(q_1, s_1)} \mathbf{Z}$, $\tau^*(K_0(H_2)) = \frac{1}{\text{l.c.m.}(q_2, s_2)} \mathbf{Z}$. Since $\tau^*(K_0(H_1)) = \tau^*(K_0(H_2))$, this implies that $\text{l.c.m.}(q_1, s_1) = \text{l.c.m.}(q_2, s_2)$. ▨

Before moving on to study the isomorphism problem for Heisenberg C^*-algebras of class 2, we review those crossed product C^*-algebras generated by certain Anzai skew-products on the torus, those given by $T(z, w) = (\lambda z, zw)$ where $(z, w) \in \mathbf{T}^2$ and $\lambda = e^{2\pi i\alpha}$ for irrational α. These C^*-algebras, which we denote here by H_α, were classified in [14, Section 3]. The Pimsner-Voiculescu exact sequence tells us that $K_0(H_\alpha) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ and is generated by three elements: the identity projection, the so-called Bott generator in $M_2(\langle V, W\rangle)$, which we denote by e_1, of trace 1 and twist -1, and the projection in $\langle U, V\rangle = A_\alpha$ of trace α, which we denote by P_α. We assume that $\alpha \in (0, 1/2)$ and write the generators of $K_0(H_\alpha)$ in vector form as $(1, 0, 0) = [P_\alpha]$, $(0, 1, 0) = [\text{Id}]$, and $(0, 0, 1) = [\text{Id}] - [e_1]$. If τ is the unique normalized faithful trace on H_α, then $\tau((1, 0, 0)) = \alpha$, $\tau((0, 1, 0)) = 1$, and $\tau((0, 0, 1)) = 0$.

By Proposition 1.5 a more general Heisenberg C^*-algebra of class 2 will be $*$-isomorphic to one of the form $H(\beta, p/q)$, for some irrational β and $p, q \in \mathbf{Z}$ with $(p, q) = 1$. The range of the unique normalized faithful trace on $K_0(H(\beta, p/q)) = \mathbf{Z} + \beta\mathbf{Z} + p/q\mathbf{Z} = 1/q\mathbf{Z} + \beta\mathbf{Z}$. This range is not a complete isomorphism invariant for Heisenberg C^*-algebras of class 2, but it does limit the possibilities, and along with the strong Morita equivalence bimodules constructed in Example 2.8, can be used to prove the following lemma, which is a preliminary version of Theorem 2.9.

LEMMA 2.11. *Let $H(\beta_1, p_1/q_1)$ and $H(\beta_2, p_1/q_1)$ be two Heisenberg C^*-algebras of class 2, where β_1 and β_2 are irrational and $(p_1, q_1) = 1$, $(p_2, q_2) = 1$ for $p_i, q_i \in \mathbf{Z}$, $i = 1, 2$. Then $H(\beta_1, p_1/q_1)$ is $*$-isomorphic to $H(\beta_2, p_2/q_2)$ if and only if $q_1 = q_2 = q$, $p_2 = \pm p_1 \bmod q$, and $q\beta_1 = \pm q\beta_2 \bmod 1$.*

Proof. Let us first show sufficiency. If $p_1 = p_2 = p$, $q_1 = q_2 = q$, $q\beta_1 = q\beta_2 \bmod 1$, then there exists an integer $i < q$ with $\beta_i + i/q = \beta_2 \bmod 1$. We then note that there is a matrix $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbf{Z})$ which gives the isomorphism between $H(\beta_1, p/q)$ and $H(\beta_2, p/q)$ by Proposition 1.3. The proof of sufficiency in the remaining cases also uses Proposition 1.3 and we omit details.

Now we prove the necessity. Let $H_1 = H(\beta_1, p_1/q_1)$ be $*$-isomorphic to $H_2 = H(\beta_2, p_2/q_2)$ where β_1 and β_2 are irrational and $(p_1, q_1) = 1, (p_2, q_2) = 1$. Then H_1 is strongly Morita equivalent to $H_{q_1\beta_1}$ and H_2 is strongly Morita equivalent to $H_{q_2\beta_2}$, so both are simple and have unique normalized traces τ_1 and τ_2.

The image of these traces on the K_0 groups is an isomorphism invariant. Hence if $H_1 \cong H_2$, then

$$\tau_1^*(K(H_1)) = \mathbf{Z} + p_1/q_1\mathbf{Z} + \beta_1\mathbf{Z} = \tau_2^*(K_0(H_2)) = \mathbf{Z} + p_2/q_2\mathbf{Z} + \beta_2\mathbf{Z}$$

as subgroups of $\mathbf{R}$. This implies that $q_1 = q_2 = q$, and that $q\beta_1 = \pm q\beta_2 \bmod 1$. Without loss of generality, we may assume $q\beta_1 = q\beta_2$ (since $H(\beta_2, p_2/q_2) \cong = H(-\beta_2, p_2/q_2)$ in any case). Hence we have reduced our problem to showing that if $H(\beta_1, p_1/q) \cong H(\beta_2, p_2/q)$ where $q\beta_1 = q\beta_2 \bmod 1$, then $p_1 = \pm p_2 \bmod q$. We now examine the strong Morita equivalences

$$H_\alpha = H_{q\beta_1}\text{-}X(q, a_1) \times_{\beta_1} \mathbf{Z}\text{-}H(\beta_1, p_1/q_1)$$

and

$$H_\alpha = H_{q\beta_2}\text{-}X(q, a_2) \times_{\beta_2} \mathbf{Z}\text{-}H(\beta_2, p_2/q_2),$$

for $a_1, a_2, b_1, b_2 \in \mathbf{Z}$ with $a_1p_1 + b_1q = 1$, $a_2p_2 + b_2q = 1$, where the equivalence bimodules are as constructed in Example 2.8.

Forming the equivalence bimodule

$$H(\beta_2, p_2/q)\text{-}X(q, a_2) \times_{\beta_2} \mathbf{Z}\text{-}H_{q\beta_2} = H_\alpha,$$

as in the remarks preceding Definition 2.1, we note that the isomorphism of $H(\beta_2, p_2/q)$ with $H(\beta_1, p_1/q)$ allows us to form the strong Morita equivalence bimodule

$$H_\alpha = H_{q\beta_1}\text{-}X(q, a_1) \times_{\beta_1} \mathbf{Z} \underset{H_1 \cong H_2}{\otimes} \widetilde{X(q, a_2)} \times_{\beta_2} \mathbf{Z}\text{-}H_{q\beta_2} = H_\alpha,$$

which we shall denote by $\mathscr{V}$.

With respect to our notation given earlier, we see that $M_{H_\alpha\text{left}}^{H_\alpha\text{right}}(\mathscr{V}) = = M_{H(\beta_2, p_2/q)}^{H_\alpha}(\widetilde{X(q, a_2)} \times_{\beta_2} \mathbf{Z}) M_{H_\alpha}^{H(\beta_1, p_1/q)}(X(q, a_1) \times_{\beta_1} \mathbf{Z})$. To ease notation, we write $M_A^B(\mathscr{V}) = M_A^B$, for C^*-algebras A and B, with the bimodule $\mathscr{V}$ fixed as above. Then by Example 2.3 and Remark 2.7 we know that, in terms of the standard generators for $K_0(H_\alpha)$ discussed previously,

$$M_{H(\beta_2, p_2/q)}^{H_\alpha}([\mathrm{Id}]_{K_0(H(\beta_2, p_2/q))}) = (0, q, -a_2), \tag{1}$$

$$M_{H(\beta_1, p_1/q)}^{H_\alpha}([\mathrm{Id}]_{K_0(H(\beta_1, p_1/q))}) = (0, q, -a_1). \tag{2}$$

These two mappings along with Proposition 2.4 show us that the coupling constants are $C_{H_\alpha}^{H(\beta_1, p_1/q)}(\rho) = C_{H(\beta_2, p_2/q)}^{H_\alpha}(\tau) = 1/q$ (where ρ, τ are the unique normalized traces on $H(\beta_2, p_2/q) \cong H(\beta_1, p_1/q)$ and H_α respectively), so that by Proposition 2.4

$C_{H_\alpha \text{left}}^{H_\alpha \text{right}}(\rho) = q \cdot 1/q = 1$, i.e. the normalized trace on H_α (right) is equal to the trace induced on H_α (right) from the strong Morita equivalence bimodule $\mathscr{V}$ (since this latter trace is normal hence equals the unique normalized trace τ). Hence, by our equation (2) after Definition 2.1, the map τ^* is $M_{H_\alpha \text{left}}^{H_\alpha \text{right}}(\mathscr{V})$-invariant where $\tau^*\colon \mathrm{K}_0(H_\alpha) \to \mathbf{R}$, i.e.

$$\tau^*(M_{H_\alpha \text{left}}^{H_\alpha \text{right}}([p]_{\mathrm{K}_0(H_\alpha)})) = \tau^*([p]_{\mathrm{K}_0(H_\alpha)}).$$

Hence, if we wish to write $M_{H_\alpha \text{left}}^{H_\alpha \text{right}}(\mathscr{V})$ as an element of $\mathrm{GL}(3, \mathbf{Z})$ with respect to our standard generators for $\mathrm{K}_0(H_\alpha)$ we must have

$$M_{H_\alpha \text{left}}^{H_\alpha \text{right}}(\mathscr{V}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix}$$

(since if the first two rows were not as above, τ^* would not be $M_{H_\alpha \text{left}}^{H_\alpha \text{right}}$-invariant). Hence we write $M_{H_\alpha \text{left}}^{H_\alpha \text{right}}$ as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}$, where $c = \pm 1$, with respect to the standard generators for $\mathrm{K}_0(H_\alpha)$, so that

$$M_{H_\alpha \text{left}}^{H_\alpha \text{right}} \begin{pmatrix} 0 \\ q \\ -a_1 \end{pmatrix} = M_{H(\beta_2, p_2/q)}^{H_\alpha} M_{H_\alpha}^{H(\beta_1, p_1/q)} \begin{pmatrix} 0 \\ q \\ -a_1 \end{pmatrix} = \qquad \text{(by equation 2)}$$

$$(3) \qquad = M_{H(\beta_2, p_2/q)}^{H_\alpha}([\mathrm{Id}]_{\mathrm{K}_0(H(\beta_1, p_1/q))}) = \qquad \text{(since } H(\beta_2, p_2/q) \cong H(\beta_1, p_1/q))$$

$$= M_{H(\beta_2, p_2/q)}^{H_\alpha}([\mathrm{Id}]_{\mathrm{K}_0(H(\beta_2, p_2/q))}) = \begin{pmatrix} 0 \\ q \\ -a_2 \end{pmatrix}. \qquad \text{(by equation 2)}$$

Hence

$$(4) \qquad M_{H_\alpha \text{left}}^{H_\alpha \text{right}} \begin{pmatrix} 0 \\ q \\ -a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ -a_2 \end{pmatrix}.$$

Hence either

$$(5)\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \begin{pmatrix} 0 \\ q \\ -a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ -a_2 \end{pmatrix}$$

or

$$(6)\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{pmatrix} \begin{pmatrix} 0 \\ q \\ -a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ -a_2 \end{pmatrix}.$$

Hence either

$$(7)\qquad a_2 = -qb + a_1$$

or

$$(8)\qquad a_2 = -qb - a_1.$$

If $a_2 = (-b)q + a_1$, we note that $P = p_1$, $B = (b_1 + bp_1)$ satisfy

$$a_2 P + Bq = a_2 p_1 + (b_1 + bp_1)q = a_2 p_1 + b_1 q + bq \cdot p_1 =$$
$$= a_2 p_1 + b_1 q + p_1(a_1 - a_2) = b_1 q + p_1 a_1 = 1.$$

Hence $a_2 P + Bq = 1$, which implies that $P = p_2 + kq$, $B = b_2 - kp_2$ for some integer k. Hence $p_1 = p_2 + kq$ so that $p_1 = p_2 \bmod q$. On the other hand, if (8) holds one easily shows that $p_2 = -p_1 \bmod q$, as desired. ▨

REMARK 2.12. Lemma 2.11 uses the following idea, which is a simplification of the proof in [22] that rational rotation algebras A_{p_1/q_1} and A_{p_2/q_2} are isomorphic if and only if $p_1/q_1 = \pm p_2/q_2 \bmod 1$. (This fact was initially proved in [11].) Form the bimodules

$$C^*(\mathbf{T}^2)\text{-}X(q_1, a_1)\text{-}A_{p_1/q_1}$$

and

$$C^*(\mathbf{T}^2)\text{-}X(q_2, a_2)\text{-}A_{p_2/q_2}$$

of [26]. Then

$$C^*(\mathbf{T}^2)\text{-}X(q_1, a_1) \underset{A_{p_1/q_1} \cong A_{p_2/q_2}}{\otimes} \tilde{X}(q_2, a_2)\text{-}C^*(\mathbf{T}^2)$$

is a strong Morita equivalence bimodule for $C^*(\mathbf{T}^2)$ which we denote by $\mathscr{X}$:

$$C^*(\mathbf{T}^2)\text{-}\mathscr{X}\text{-}C^*(\mathbf{T}^2).$$

We see that $C^{C^*(\mathbf{T}^2)}_{C^*(\mathbf{T}^2)}(\mathscr{X})(\tau) = 1$ where τ is the trace on $\mathbf{T}^2$ determined by Haar measure, and $q_1 = q_2$. This implies that the matrix

$$M^{C^*(\mathbf{T}^2)\,\text{right}}_{C^*(\mathbf{T}^2)\,\text{left}}(\mathscr{X}) \colon \mathrm{K}_0(C^*(\mathbf{T}^2)) \to \mathrm{K}_0(C^*(\mathbf{T}^2))$$

must be of the form $\begin{pmatrix} 1 & 0 \\ a & \pm 1 \end{pmatrix}$, where a is an integer, with respect to our standardized generators for $\mathrm{K}_0(C^*(\mathbf{T}^2))$ as discussed in Example 2.3. Then as in Lemma 2.11 one computes that

$$M^{C^*(\mathbf{T}^2)}_{C^*(\mathbf{T}^2)}(\mathscr{X}) \begin{pmatrix} q \\ -a_1 \end{pmatrix} = \begin{pmatrix} q \\ -a_2 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 0 \\ a & +1 \end{pmatrix} \begin{pmatrix} q \\ -a_1 \end{pmatrix} = \begin{pmatrix} q \\ -a_2 \end{pmatrix}$$

where a_1, a_2, b_1, b_2 are integers such that

$$a_1 p_1 + b_1 q = 1, \quad a_2 p_2 + b_2 q = 1.$$

But the arguments of Lemma 2.11 show that p_1 must be equal $\pm p_2 \bmod q$ in such a situation. Hence

$$A_{p_1/q_1} \cong A_{p_2/q_2} \Rightarrow p_1/q_1 = \pm p_2/q_2 \bmod 1,$$

as desired. ▨

We turn now to class 3 Heisenberg C^*-algebras. It is not surprising to find that here the trace completely determines the isomorphism class.

Let τ be the unique normalized trace on the class 3 C^*-algebra $H(\alpha, \beta)$. We saw in Section 1 that $\tau_*(\mathrm{K}_0(H(\alpha, \beta))) = \mathbf{Z} + \alpha\mathbf{Z} + \beta\mathbf{Z}$. This fact allows us to prove the following

LEMMA 2.13. *Let $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ be two class 3 Heisenberg C^*-algebras. Then the following three statements are equivalent:*

(i) $H(\alpha_1, \beta_1) = H(\alpha_2, \beta_2)$.

(ii) $\mathbf{Z} + \mathbf{Z}\alpha_1 + \mathbf{Z}\beta_1 = \mathbf{Z} + \mathbf{Z}\alpha_2 + \mathbf{Z}\beta_2 \subset \mathbf{R}$.

(iii) *There exists* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$ *with*

$$b\alpha_1 + d\beta_1 = \alpha_2 \bmod 1,$$

$$a\alpha_1 + c\beta_1 = \beta_2 \bmod 1.$$

Proof. The direction (i) $\Rightarrow$ (ii) is clear from the tracial invariant mentioned above. The direction (iii) $\Rightarrow$ (i) was shown in Proposition 1.4. Thus we need only show the implication (ii) $\Rightarrow$ (iii). If $\mathbf{Z} + \alpha_1\mathbf{Z} + \beta_1\mathbf{Z} = \mathbf{Z} + \alpha_2\mathbf{Z} + \beta_2\mathbf{Z}$ where (α_i, β_i) are pairs of linearly independent irrational numbers, then there exist $M, A, B, N, C, D \in \mathbf{Z}$ with

$$M + A\alpha_1 + B\beta_1 = \alpha_2,$$

$$N + C\alpha_1 + D\beta_1 = \beta_2.$$

Without loss of generality we may assume M and N are zero, so that $A\alpha_1 + B\beta_1 = \alpha_1$, $C\alpha_1 + D\beta_1 = \beta_2$. We now claim that in fact $AD - BC = \pm 1$. Since $\mathbf{Z} + \alpha_1\mathbf{Z} + \beta_1\mathbf{Z} = \mathbf{Z} + \alpha_2\mathbf{Z} + \beta_2\mathbf{Z}$, there exist $l, r, s, x, t, w \in \mathbf{Z}$ with

$$l + r\alpha_2 + s\beta_2 = \alpha_1,$$

$$x + t\alpha_2 + w\beta_2 = \beta_1.$$

Hence, in succession, we have the following statements:

$$l + r(A\alpha_1 + B\beta_1) + s(C\alpha_1 + D\beta_1) = \alpha_1,$$

$$x + t(A\alpha_1 + B\beta_1) + w(C\alpha_1 + D\beta_1) = \beta_1;$$

because of the independence of $1, \alpha_1$, and β_1,

$$l = x = 0,$$

$$(rA + sC) = (tB + wD) = 1,$$

$$(rB + sD) = (tA + wC) = 0;$$

$$\begin{pmatrix} r & s \\ t & w \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \pm 1 \quad \text{and} \quad \begin{pmatrix} r & s \\ t & w \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}.$$

Taking $A = b$, $B = d$, $C = a$, $D = c$, we obtain the desired result. ∎

We have completed the proofs of our preliminary lemmas and are now in a position to prove the main theorem:

Proof of Theorem 2.9. The fact that $H(\alpha_1, \beta_1) \cong H(\alpha_2, \beta_2)$ if there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$ with $e^{2\pi i\alpha_2} = e^{2\pi i(a\alpha_1 + b\beta_1)}$ and $e^{2\pi i\beta_2} = e^{2\pi i(c\alpha_1 + d\beta_1)}$ is merely

Proposition 1.3 restated. For the proof of $\Rightarrow$, we note that if $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ are isomorphic, they are of the same class. The theorem then follows for Heisenberg C^*-algebras of class 1 and 3 from Lemmas 2.10 and 2.13 respectively. As for class 2 Heisenberg C^*-algebras, the result follows using Lemma 2.11 in conjunction with Proposition 1.5. ▨

Note added in revision: After reading the original version of this paper [15], Hong-sheng Yin remarked in the preprint "Classification of C^*-crossed products associated with characters on free groups" that he could provide a shorter proof of the above theorem, also using K-theoretic invariants. For our purposes, we prefer the above approach, since the isomorphism theorem for all matrix algebras of Heisenberg C^*-algebras follows directly from this method, and since we use these constructions in our subsequent paper [16], where we determine a one-to-one correspondence between strong Morita equivalence classes of Heisenberg C^*-algebras and the real projective plane, and prove that the cancellation property of Rieffel holds for Heisenberg C^*-algebras of class 2 and 3.

To conclude, we state the generalization of Theorem 2.9 to matrix algebras over $H(\alpha, \beta)$:

THEOREM 2.14. *Let* $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{R}, n_1, n_2 \in \mathbf{N}$. *Then* $M_{n_1}(H(\alpha_1, \beta_1)) \cong M_{n_2}(H(\alpha_2, \beta_2))$ *if and only if* $n_1 = n_2$ *and there exists* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$ *with*

$$e^{2\pi i\alpha_2} = e^{2\pi i(a\alpha_1 + b\beta_1)},$$

and

$$e^{2\pi i\beta_2} = e^{2\pi i(c\alpha_1 + d\beta_1)}.$$

Proof. The proof involves many of the techniques of Theorem 2.9, including dividing up the situation into the three cases class 1, class 2 and class 3, as wel as using the matrix method determined by strong Morita bimodules; as before a key role is played by the range of a normalized trace on the K_0-group of the C^*-algebras. We give here the proof for the class 2 case only and leave the other cases to the reader.

Suppose that $\mathcal{M}_1 = M_{n_1}(H(\alpha_1, \beta_1)) \cong M_{n_2}(H(\alpha_2, \beta_2)) = \mathcal{M}_2$ with class $\mathcal{M}_1 =$ class $\mathcal{M}_2 = 2$. Then there exist matrices $A_j = \begin{pmatrix} m_j & n_j \\ r_j & s_j \end{pmatrix}$, irrational numbers ρ_j, and relatively prime non-negative integers p_j, q_j with $0 \leqslant p_j < q_j$, $\begin{pmatrix} m_j & n_j \\ r_j & s_j \end{pmatrix} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = \begin{pmatrix} \rho_j \\ p_j/q_j \end{pmatrix} \bmod 1$, such that $\mathcal{M}_j \cong M_{n_j}(H(\rho_j, p_j/q_j))$, $j = 1, 2$.

We will show that $n_1 = n_2 = n$, $q_1 = q_2 = q$, $p_1 = \pm p_2 \bmod q$ and $q\rho_1 = \pm q\rho_2 \bmod 1$. This will complete the proof for class 2 Heisenberg algebras, since in this case $(e^{2\pi i\alpha_1}, e^{2\pi i\beta_1})$ and $(e^{2\pi i\alpha_2}, e^{2\pi i\beta_2})$ would be conjugate under a matrix of the form $A_1^{-1}\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1\end{pmatrix} A_2 \in GL(2, \mathbf{Z})$.

If $\mathcal{A}_1 \cong \mathcal{A}_2$ then the ranges of normalized traces on the K_0-groups must be equal, i.e.

$$\frac{1}{n_1}\left(\frac{1}{q_1}\mathbf{Z} + \rho_1 \mathbf{Z}\right) = \frac{1}{n_2}\left(\frac{1}{q_2}\mathbf{Z} + \rho_2 \mathbf{Z}\right).$$

Hence $n_1 q_1 = n_2 q_2$ and $q_1\rho_1 = \pm q_2\rho_2 \bmod 1$. Let $\delta_1 = q_1\rho_1$, $\delta_2 = q_2\rho_2$. We then have the following chain of strong Morita equivalences, following the notation of Example 2.8:

$$H_{\delta_1} - X(q_1, a_1) \times_{\rho_1} \mathbf{Z} - H(\rho_1, p_1/q_1),$$

$$H(\rho_1, p_1/q_1) - \bigotimes_{i=1}^{n_1} H(\rho_1, p_1/q_1) - M_{n_1}H((\rho_1, p_1/q_1) \cong$$

$$\cong M_{n_2}(H(\rho_2, p_2/q_2) - \bigoplus_{i=1}^{n_2} H(\rho_2, p_2/q_2) - H(\rho_2, p_2/q_2),$$

$$H(\rho_1, p_2/q_2) - X(q_2, a_1) \times_{\rho_2} \mathbf{Z} - H_{\delta_2}.$$

Hence there exists a strong Morita equivalence bimodule

$$H_{\delta_1} - \mathcal{Y} - H_{\delta_2},$$

and we calculate (using the method of Lemma 2.11)

$$M_{H_{\delta_1}}^{H_{\delta_2}}(\mathcal{Y})\begin{pmatrix} 0 \\ n_1 q_1 \\ -n_1 a_1\end{pmatrix} = \begin{pmatrix} 0 \\ n_2 q_2 \\ -n_2 a_2\end{pmatrix},$$

in terms of the standard generators for $K_0(H_{\delta_1})$. Hence n_1 divides $n_2 q_2$, n_2 divides $n_1 q_1$. Since $(a_1, q_1) = (a_2, q_2) = 1$, this implies that n_1 divides n_2 and n_2 divides n_1, so that $n_1 = n_2 = n$. Since $n_1 q_1 = n_2 q_2$ we obtain $q_1 = q_2 = q$, which implies $q\rho_1 = \pm q\rho_2 \bmod 1$, so that we need only show $p_1 = \pm p_2 \bmod q$. Since $n_1 = n_2 = n$,

$q_1 = q_2 = q$, we see that

$$M_{H_{\delta_1}}^{H_{\delta_2}}(\mathscr{Y})\begin{pmatrix} 0 \\ q \\ -a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ -a_2 \end{pmatrix}.$$

Since $C_{H_{\delta_1}}^{H_{\delta_2}}(\mathscr{Y})(\tau) = 1$, $M_{H_{\delta_1}}^{H_{\delta_2}}(\mathscr{Y})$ is of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & \pm 1 \end{pmatrix}$. As in the proof of Lemma 2.11, we see that $p_1 = p_2 \bmod q$ or $p_1 = -p_2 \bmod q$, which is what we desired to show. ▨

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