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THE SPECTRUM OF INTEGRAL OPERATORS ON LEBESGUE SPACES

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1. INTRODUCTION

Let Ω be a measure space. Assume T is a linear transformation mapping the integrable simple functions on Ω into measurable functions on Ω. If T has an extension to a bounded linear operator on the Lebesgue space $L^p(\Omega)$, then denote this extension by T_p. In many examples T_p exists for p in an interval of real numbers. In this situation it is of interest to know how the sets $\sigma(T_p)$, the spectrum of the operator T_p, are related.

As an example, let Ω be a locally compact (LC) unimodular group equipped with Haar measure. A function $f \in L^1(\Omega)$ determines a bounded convolution operator on each of the spaces $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, by the rule

$$(T_f)_p(g)(x) = \int_{\Omega} f(xy^{-1})g(y)\,\mathrm{d}y \qquad (g \in L^p(\Omega)).$$

Then the question is: how are the sets $\sigma((T_f)_p)$ related for $1 \leqslant p \leqslant +\infty$? In the case where Ω is abelian it is known that for all p, $1 \leqslant p \leqslant +\infty$, $\sigma((T_f)_p)$ is the same as the spectrum of f in the Banach algebra $L^1(\Omega)$; see [8], Theorem 13.3 for the case $\Omega = \mathbf{R}^m$.

In this paper we consider the general question concerning the spectrum of the integral operators on Lebesgue spaces determined by certain kernels. We study both the spectrum and the Fredholm spectrum of these operators. Banach algebra methods prove useful here. The applications in Sections 3 and 4 are based on a strong result concerning the spectrum of the operators in the image of a representation of certain Banach $*$-algebras. This result is proved in Section 2.

In Section 3, for Ω a unimodular LC group with polynomial growth we consider the spectrum of the convolution operators $(T_f)_p$. The results of this section extend the work of T. Pytlik [12] and A, Hulanicki [7] who studied the spectrum of the operator $(T_f)_2$.

In Section 4 we consider the spectrum of integral operators acting on L^p-spaces where the kernel involved satisfies the properties

$$\operatorname{ess\,sup}_x \int_\Omega |K(x, y)| \, dy < +\infty$$

$$\operatorname{ess\,sup}_y \int_\Omega |K(x, y)| \, dx < +\infty.$$

Kernels with these properties form a Banach algebra A_1. The results of Section 4 are proved for certain subalgebras of A_1.

2. A RESULT ON BANACH $*$-ALGEBRAS

When A is a Banach algebra, let $r_A(b)$ denote the spectral radius of $b \in A$, and let $\sigma_A(b)$ denote the spectrum of b in A.

In [7] A. Hulanicki proves a useful result concerning Banach $*$-algebras.

HULANICKI'S THEOREM. *Let A be a Banach $*$-algebra and S a $*$-subalgebra of A (not necessarily closed). Assume that $T : A \to B(H)$ is a faithful $*$-representation of A on a Hilbert space H. If*

$$r_A(f) = \|T_f\| \quad \textit{for all } f = f^* \in S,$$

then

$$\sigma_A(f) = \sigma(T_f) \quad \textit{for all } f = f^* \in S.$$

It is easy to see that the conclusion of Hulanicki's Theorem holds for all $f \in S$. If in addition S is a Banach algebra, then Hulanicki's Theorem implies that S is symmetric. In fact, Pytlik uses this result in [12] to prove that certain convolution Banach $*$-subalgebras of $L^1(G)$ are symmetric.

In this section we prove two results concerning Banach $*$-algebras that are related to Hulanicki's Theorem. The hypotheses of these results are restrictive, but the conclusions are extremely strong. We apply these theorems in Sections 3 and 4.

We need some notation and terminology. We assume throughout that A and B are unital Banach algebras. When $\varphi : A \to B$ is an algebra homomorphism from A into B, then it is assumed that φ maps the unit of A onto the unit of B. The set of invertible elements of A is denoted by $\mathrm{Inv}(A)$.

DEFINITION. The algebra A is *quotient inverse closed* if whenever $\varphi : A \to B$ is a continuous algebra homomorphism, then $f \in A$ and $\varphi(f) \in \mathrm{Inv}(B)$ imply that $f + \ker(\varphi) \in \mathrm{Inv}(A/\ker(\varphi))$. A $*$-algebra A is *$*$-quotient inverse closed* if whenever φ is as above, then $f = f^* \in A$ and $\varphi(f) \in \mathrm{Inv}(B)$ imply that $f + \ker(\varphi) \in \mathrm{Inv}(A/\ker(\varphi))$.

Conditions are given in [3] that imply that a commutative Banach algebra is quotient inverse closed. In particular, this is true for regular commutative semisimple Banach algebras [3], Theorem 1. B^*-algebras are $*$-quotient inverse closed by [13], Theorem (4.8.3) and Theorem (4.9.2).

NOTE. Assume that A is quotient inverse closed, $\varphi : A \to B$ is a continuous algebra homomorphism, and K is a closed ideal of B.

$$\text{If } \varphi(f) + K \in \mathrm{Inv}(B/K), \text{ then } f + \varphi^{-1}(K) \in \mathrm{Inv}(A/\varphi^{-1}(K)).$$

To verify this, define $\psi : A \to B/K$ by $\psi(g) = \varphi(g) + K$. Then ψ is a continuous algebra homomorphism of A into B/K with $\ker(\psi) = \varphi^{-1}(K)$. If $\varphi(f) + K = \psi(f) \in \mathrm{Inv}(B/K)$, then $f + \varphi^{-1}(K) = f + \ker(\psi) \in \mathrm{Inv}(A/\ker(\psi)) = \mathrm{Inv}(A/\varphi^{-1}(K))$.

A similar result holds when A is $*$-quotient inverse closed and $f = f^* \in A$.

When A is a Banach $*$-algebra with unit and f is a normal element of A, let $A\langle f\rangle$ be the closed $*$-subalgebra of A generated by f and 1.

LEMMA. *Assume that A is a symmetric Banach $*$-algebra. Let φ be a continuous algebra homomorphism of A into a Banach algebra B. Let $J = \ker(\varphi)$.*

Assume $f \in A$, $f = f^*$, and $\exists \{f_n\} \subset A$ such that each $f_n = f_n^*$, $A_n = A\langle f_n\rangle$ is quotient inverse closed, $n \geq 1$, and $\|f_n - f\| \to 0$. If $\varphi(f) \in \mathrm{Inv}(B)$, then $f + J \in \mathrm{Inv}(A/J)$.

Proof. For $\varepsilon > 0$, set $D_\varepsilon = \{\lambda \in \mathbf{C} : |\lambda| \leq \varepsilon\}$. Since $\varphi(f) \in \mathrm{Inv}(B)$, $\exists \varepsilon > 0$ such that D_ε is disjoint from $\sigma_B(\varphi(f))$. By [13], Theorem (1.6.16) we may assume that $\sigma_B(\varphi(f_n)) \subset \{\lambda : |\lambda| > \varepsilon\}$ for all n. Set $J_n = A_n \cap J$. In this situation it always holds that for $g \in A_n$

$$\sigma_B(\varphi(g)) \subset \sigma_{A/J}(g + J) \subset \sigma_{A_n/J_n}(g + J_n).$$

Then since $A_n = A\langle f_n\rangle$ is quotient inverse closed,

$$\sigma_{A_n/J_n}(g + J_n) = \sigma_B(\varphi(g)).$$

Thus for $g \in A_n$

$$\sigma_B(\varphi(g)) = \sigma_{A/J}(g + J) = \sigma_{A_n/J_n}(g + J_n).$$

Applying this equality to f_n, we have $\sigma_{A/J}(f_n + J) \subset \{\lambda : |\lambda| > \varepsilon\}$. If $\lambda \in \sigma_{A/J}((f_n + J)^{-1})$, then $\lambda^{-1} \in \sigma_{A/J}(f_n + J)$, so that $|\lambda^{-1}| \geq \varepsilon$ and $|\lambda| \leq \varepsilon^{-1}$. Therefore

$$r_{A/J}((f_n + J)^{-1}) \leq \varepsilon^{-1} \quad (n \geq 1).$$

Now J need not be a $*$-ideal of A. But since primitive ideals in a symmetric Banach $*$-algebra are $*$-ideals (this follows from [13], Theorem (4.7.14)), the intersection of all primitive ideals of A containing J is a closed $*$-ideal of A. Call this ideal I. Also note that

$$r_{A/I}(g + I) = r_{A/J}(g + J) \quad (g \in A);$$

see [1], BA.2.3 and BA.2.4. Then A/I is a symmetric Banach $*$-algebra, and

$$r_{A/I}((f_n + I)^{-1}) \leqslant \varepsilon^{-1} \quad (n \geqslant 1).$$

Since A/I is symmetric, by [11], (5.3), p. 268, $r_{A/I}$ is submultiplicative on the set of self-adjoint elements of A/I. Therefore

$$\begin{aligned} &r_{A/I}((1 + I) - (f + I)(f_n + I)^{-1}) \leqslant r_{A/I}(f_n - f + I)r_{A/I}((f_n + I)^{-1}) \leqslant \\ (1) \quad &\leqslant \|f_n - f\|\varepsilon^{-1} \quad \text{for all } n \geqslant 1. \end{aligned}$$

Then from standard Banach algebra theory, [13], Lemma (1.4.18), $f + I$ is invertible in A/I. But then $f + J \in \mathrm{Inv}(A/J)$.

The Lemma immediately implies the following result.

THEOREM 2.1. *Let A be a symmetric Banach $*$-algebra. If for each $f = f^* \in A$ there exists a sequence $\{f_n\}$ of self-adjoint elements of A such that $A\langle f_n\rangle$ is quotient inverse closed for all n and $\|f_n - f\| \to 0$, then A is $*$-quotient inverse closed.*

When A is both symmetric and $*$-quotient inverse closed, then strong conclusions can be drawn concerning the spectrum of continuous images. This is the content of the next result.

THEOREM 2.2. *Assume that A is a symmetric Banach $*$-algebra and A is $*$-quotient inverse closed. Let $\varphi : A \to B$ be a continuous homomorphism. Set $J = \ker(\varphi)$.*

(1) *If f is a normal element of A and $\varphi(f) \in \mathrm{Inv}(B)$, then $f + J \in \mathrm{Inv}(A/J)$;*

(2) *If f is a normal element of A, then $\sigma_{A/J}(f + J) = \sigma_B(\varphi(f))$;*

(3) *For any $g \in A$,*

$$\sigma_{A/J}(g + J) = \sigma_B(\varphi(g)) \cup \overline{\sigma_B(\varphi(g^*))}.$$

Proof. Assume that $f \in A$ is normal. Then $A\langle f\rangle$ is a commutative unital symmetric Banach $*$-algebra. Let D be a maximal commutative subalgebra of B containing $\varphi(A\langle f\rangle)$. Suppose $\varphi(f) \in \mathrm{Inv}(B)$. Then $\varphi(f) \in \mathrm{Inv}(D)$. Thus, for every nonzero multiplicative linear functional (m.l.f.) ψ on D, $\psi(\varphi(f)) \neq 0$. Now $\psi\circ\varphi$ is a nonzero m.l.f. on $A\langle f\rangle$. Therefore by the symmetry of this algebra, $\psi\circ\varphi(f^*) = \overline{\psi\circ\varphi(f)}$. This implies that $\psi(\varphi(f^*)) \neq 0$ for all nonzero m.l.f.'s on D. Therefore $\varphi(f^*) \in \mathrm{Inv}(D)$. It follows that $\varphi(f^*f) \in \mathrm{Inv}(B)$. Since A is $*$-quotient inverse closed, we have $f^*f+J \in \mathrm{Inv}(A/J)$. It follows that $f+J \in \mathrm{Inv}(A/J)$. This proves (1).

(2) is an immediate consequence of (1).

To prove (3), first note that since A is symmetric, we have (as in the proof of the Lemma) that the ideal I which is the intersection of all primitive ideals containing J is a closed $*$-ideal of A. Then $g + I \in \mathrm{Inv}(A/I)$ if and only if $g^* + I \in \mathrm{Inv}(A/I)$. Then by [1], BA.2.4, $g + J \in \mathrm{Inv}(A/J)$ if and only if $g^* + J \in \mathrm{Inv}(A/J)$. Therefore,

$$g + J \in \mathrm{Inv}(A/J) \Leftrightarrow g^*g + J \text{ and } gg^* + J \in \mathrm{Inv}(A/J) \Leftrightarrow$$

$$\Leftrightarrow \varphi(g^*g) \text{ and } \varphi(gg^*) \in \mathrm{Inv}(B) \Leftrightarrow \varphi(g) \text{ and } \varphi(g^*) \in \mathrm{Inv}(B).$$

These equivalences prove that

$$\sigma_{A/J}(g + J) = \sigma_B(\varphi(g)) \cup \overline{\sigma_B(\varphi(g^*))}.$$

3. ALGEBRAS OF CONVOLUTION OPERATORS

Assume that Ω is a locally compact unimodular topological group equipped with Haar measure. For $f \in L^1(G)$, $g \in L^p(G)$, let

$$(T_f)_p(g) = f * g \in L^p,$$

[6], (20.19). Then $f \to (T_f)_p$ is continuous algebra isomorphism of $L^1(\Omega)$ into $B(L^p(\Omega))$. Also, let $\mathrm{BC}(\Omega)$ denote the Banach space of bounded continuous functions on Ω (uniform norm). For $f \in L^1(\Omega)$, $g \in \mathrm{BC}(\Omega)$, we have again that $f*g \in \mathrm{BC}(\Omega)$, [6], (20.19). Let

$$(T_f)_0(g) = f*g \quad (g \in \mathrm{BC}(\Omega)).$$

We prove a general theorem concerning representations of $L^1(\Omega)$ such as the representation $f \to (T_f)_p$, $1 \leqslant p \leqslant +\infty$, $p = 0$, described above.

Certain restrictions must be made on Ω. We make the assumptions:

(1) Ω has polynomial growth; see [7] or [10].

(2) Either,

(i) Ω is symmetric, (meaning $L^1(\Omega)$ is symmetric), or

(ii) Ω is compactly generated and ω is a polynomial weight on Ω, [12], p. 900.

Now when (1) and (2)(i) hold, let A be the algebra $L^1(\Omega)$ with unit adjoined if Ω is not discrete (in which case $L^1(\Omega)$ has a unit); and when (1) and (2)(ii) hold, let A be the algebra $L^1(\Omega, \omega)$, [12], p. 900, the L^1-algebra on Ω with respect to the weight ω, again with unit adjoined if Ω is not discrete. Theorem 2.1 and Theorem 2.2 apply to either of these algebras. First, A is symmetric in either case. When (2)(i) holds, this follows by definition; and when (2)(ii) holds, it follows from a theorem of Pytlik, [12], Corollary 7. Secondly, if $f \in L^1(\Omega) \cap L^2(\Omega)$, $f = f^*$ (here $f^*(x) = f(x^{-1})$), and f vanishes outside of some compact set, then by [2], (4.2) and Remarks p. 307 and [3], Theorem 1, $A\langle f\rangle$ is quotient inverse closed. Thus, whenever $g = g^* \in A$, then $\exists \{g_n\} \subset A$ with $g_n = g_n^*$, $n \geqslant 1$, and with $A\langle g_n\rangle$ quotient inverse closed such that $\|g_n - g\| \to 0$. This verifies that the hypotheses of Theorem 2.1 hold. As a result, we have the theorem below. Here if $T \in B(X)$ and $K(X)$ denotes the closed ideal of compact operators on X, then

$$\omega(T) = \{\lambda \in \mathbf{C} : \lambda - T \text{ is not invertible in } B(X) \text{ modulo } K(X)\}.$$

THEOREM 3.1. *Let A be either of the two types of Banach $*$-algebras described above. Then A is $*$-quotient inverse closed. Assume $\pi : A \to B(X)$ is a continuous representation of A on a Banach space X with $J = \ker(\pi)$.*

(1) *For $f \in A$, f normal,*

$$\sigma_{A/J}(f + J) = \sigma(\pi(f)).$$

(2) *For general $g \in A$,*

$$\sigma_{A/J}(g + J) = \sigma(\pi(g)) \cup \overline{\sigma(\pi(g^*))}.$$

Now assume π is faithful ($J = \{0\}$) and that Ω is not compact.

(3) *For $f \in A$, f normal,*

$$\sigma_A(f) = \sigma(\pi(f)) = \omega(\pi(f)).$$

(4) *For general $g \in A$,*

$$\sigma_A(g) = \sigma(\pi(g)) \cup \overline{\sigma(\pi(g^*))} = \omega(\pi(g)) \cup \overline{\omega(\pi(g^*))}.$$

Proof. (1) and (2) follow directly from Theorem 2.1 and Theorem 2.2. Assume $\lambda \in \mathbf{C}$ and $f \in L^1(\Omega)$ (or $L^1(\Omega, \omega)$) with $0 \neq \pi(\lambda 1 + f) \in K(X)$. Then $\exists h = h^* \in L^1(\Omega)$ (or $L^1(\Omega, \omega)$) such that $h \neq 0$ and $\pi(h) \in K(X)$. Now

$$\sigma_A(\pi(h)) = \sigma_A(h) = \sigma((T_h)_2).$$

Thus the compact operator $\pi(h)$ has an isolated non-zero real eigenvalue λ_0. If e is the spectral projection in A corresponding to the eigenvalue λ_0, $e = \frac{1}{2\pi i}\int_\gamma (\lambda - h)^{-1} d\lambda$ for a suitable closed path γ about λ_0, then clearly $(T_e)_2$ is the spectral idempotent in $B(L^2)$ relative to the isolated point λ_0 of $\sigma((T_h)_2)$. By [9], p. 443 $(T_e)_2$ is the projection on the eigenspace of $(T_h)_2$ corresponding to λ_0. Thus $h*e = \lambda_0 e$ and $e*L^2$ is finite dimensional. Then L^2*e is finite dimensional. This implies that $L^1(\Omega)$ has a non-zero finite dimensional square integrable representation. A result of A. Weil shows that this can happen only when Ω is compact, [14], p. 70. The contradiction proves that $J = \pi^{-1}(K(X)) = \{0\}$. Therefore (3) and (4) follow from Theorem 2.2 and the note preceding the lemma in § 2 with $K = K(X)$.

Parts (3) and (4) of Theorem 3.1 apply to the faithful representations $f \to (T_f)_p$, $1 \leqslant p \leqslant +\infty$, $p = 0$. The spectral relationships concerning these representations have some interesting properties that hold with only minimal hypotheses on Ω. We note these next.

PROPOSITION 3.2. *Let Ω be a unimodular σ-finite locally compact group. Let A be $L^1(\Omega)$ with unit adjoined if Ω is not discrete. For all $f \in A$:*

(1) $\sigma_A(f) = \sigma((T_f)_1)$;

(2) *If* $p^{-1} + q^{-1} = 1$ *with* $1 \leqslant p < +\infty$, *then* $\sigma((T_{f^*})_q) = \overline{\sigma((T_f)_p)}$;

(3) $\sigma_A(f) = \sigma((T_f)_\infty)$;

(4) $\sigma((T_f)_p) \subset \sigma_A(f)$, $1 \leqslant p \leqslant +\infty$.

Proof. (1) follows from [13], Theorem (1.4.15).

(2). Let $\tilde{f}(x) = f(x^{-1})$, $x \in \Omega$. If $h \in L^p$ and $g \in L^q$, then $\langle T_f h, g\rangle = \langle h, T_{\tilde{f}} g\rangle$ where $\langle u, v\rangle = \int u(x)v(x)\,dx$. Thus, $T_{\tilde{f}}$ is the transpose operator of T_f. As is well known, [9], Theorem 2.4, this implies

$$\sigma((T_f)_p) = \sigma((T_{\tilde{f}})_q).$$

Now $f^* = (\tilde{f})^-$. It is easy to verify that

$$\sigma((T_{f^*})_q) = \overline{\sigma((T_{\tilde{f}})_q)}.$$

This proves (2).

(3). Apply (2) when $p = 1$ and $q = \infty$. Thus, by (1),

$$\sigma_A(f^*) = \sigma((T_{f^*})_1) = \overline{\sigma((T_f)_\infty)}.$$

It is easy to see that $\sigma_A(f^*) = \overline{\sigma_A(f)}$, [13], Lemma (4.1.1). Thus,

$$\sigma_A(f) = \sigma((T_f)_\infty).$$

(4) is elementary.

4. ALGEBRAS OF HILLE-TAMARKIN OPERATORS

Throughout this section Ω is a metric space with metric d, and μ is a positive regular σ-finite Borel measure on Ω. For convenience we denote the integral over Ω of a function f with respect to μ as $\int f(x)\,dx$. When Γ is a Borel subset of Ω, let $\chi(\Gamma)$ be the characteristic function of Γ.

For $m \geqslant 0$, let

$$\Gamma[m] = \{(x,y) \in \Omega \times \Omega : d(x,y) \leqslant m\}.$$

Also, for $x \in \Omega$, let

$$\Gamma[m]_x = \{y \in \Omega : (x,y) \in \Gamma[m]\} = \{y \in \Omega : (y,x) \in \Gamma[m]\}.$$

We make the following basic assumption:

(4.1) $\exists\, C > 0$ and $\exists\, \beta > 0$ such that $\mu(\Gamma[m]_x) \leqslant Cm^\beta$ for all $x \in \Omega$.

All kernels are assumed to be measurable functions on $\Omega \times \Omega$, and two kernels are considered equal if they are equal a.e. Let A_1 be the linear space of all kernels $K(x,y)$ such that

$$\|K\|_1 = \max\left(\operatorname*{ess\,sup}_x \int |K(x,y)|\,dy,\ \operatorname*{ess\,sup}_y \int |K(x,y)|\,dx\right) < +\infty.$$

Let A_2 be the linear space of all kernels $K(x,y)$ such that

$$\|K\|_2 = \max\left(\operatorname*{ess\,sup}_x \left(\int |K(x,y)|^2\,dy\right)^{1/2},\ \operatorname*{ess\,sup}_y \left(\int |K(x,y)|^2\,dx\right)^{1/2}\right) < +\infty.$$

By [8], Theorem 11.5, the spaces $(A_1, \|\cdot\|_1)$ and $(A_2, \|\cdot\|_2)$ are Banach spaces. For a kernel $K(x,y)$, define

$$K^*(x,y) = \overline{K(y,x)} \quad (x,y \in \Omega).$$

Then A_1 is a Banach $*$-algebra with involution $K \to K^*$ and multiplication

$$K*J(x,y) = \int K(x,z)J(z,y)\,dz \quad (K, J \in A_1).$$

It is straightforward to verify that $K \in A_1$ determines bounded linear operators K_1 and K_∞ on $L^1(\Omega)$ and $L^\infty(\Omega)$ respectively by setting

$$K_p(f)(x) = \int K(x,y)f(y)\,dy \quad (f \in L^p),$$

for $p = 1$ and $p = \infty$. Then it follows from the Riesz Convexity Theorem that the same formula defines a bounded operator for $1 \leqslant p \leqslant \infty$; see [5], p. 525 and Exercise 3, p. 527.

Let D be the set of all kernels $K(x, y)$ in A_1 such that $K(x, y) = 0$ for a.a. $(x, y) \notin \Gamma[m]$ for some m.

4.2. NOTE. Assume $K, J \in D$, $K(x, y) = 0$ for a.a. $(x, y) \notin \Gamma[q]$, and $J(x,y) = 0$ for a.a. $(x, y) \notin \Gamma[p]$. Then $K{*}J(x, y) = 0$ for a.a. $(x, y) \notin \Gamma[p + q]$. [Proof: We may assume $K(x, y) = 0$ for all $(x, y) \notin \Gamma[p]$ and $J(x, y) = 0$ for all $(x, y) \notin \Gamma[q]$. Fix $(x, y) \notin \Gamma[p + q]$. If for some $z \in \Omega$, $K(x, z) \neq 0$ and $J(z, y) \neq 0$, then $d(x, z) \leqslant p$ and $d(z, y) \leqslant q$. Then $d(x, y) \leqslant p + q$, a contradiction. Thus, $K(x, z)J(z, y) = 0$ for all $z \in \Omega$, so $K{*}J(x, y) = 0$.]

Fix $\delta, 0 < \delta \leqslant 1$, and define a weight function on $\Omega \times \Omega$ by

$$w(x, y) = (1 + d(x, y))^{\delta}.$$

Let A_w be the set of all kernels K such that $Kw \in A_1$, and define $\|K\|_w = \|Kw\|_1$. Then $(A_w, \|\cdot\|_w)$ is a Banach space.

4.3. NOTE. For all $x, y, z \in \Omega$,

$$1 \leqslant w(x, y) \leqslant w(x, z)w(z, y).$$

Thus, for $K, J \in A_w$,

$$\|K{*}J\|_w \leqslant \|K\|_w \|J\|_w.$$

[Proof: Since $d(x, y) \leqslant d(x, z) + d(z, y)$, we have $w(x, y) = (1 + d(x, y))^{\delta} \leqslant (1 + d(x, z))^{\delta}(1 + d(z, y))^{\delta} = w(x, z)w(z, y)$. The second assertion follows from this inequality.]

Let A_1^0 be the closed subalgebra of all $K \in A_1$ such that

$$\lim_{m \to \infty} \|\chi(\Gamma[m]^c)K\|_1 = 0.$$

Define A_w^0 in a similar fashion (use $\|\cdot\|_w$ in place of $\|\cdot\|_1$). In what follows we shall be interested in the Banach $*$-algebras $A_{1,2}^0 = A_1^0 \cap A_2$ and $A_{w,2}^0 = A_w^0 \cap A_2$ where the norms are $\|K\|_{1,2} = \max(\|K\|_1, \|K\|_2)$ and $\|K\|_{w,2} = \max(\|K\|_w, \|K\|_2)$, respectively. That these norms are algebra norms follows from the next lemma.

LEMMA 4.4. *For* $J, K \in A_{1,2} = A_1 \cap A_2$,

$$\|J{*}K\|_2 \leqslant \max(\|J\|_1 \|K\|_2, \|J\|_2 \|K\|_1).$$

For $J, K \in A_{w,2} = A_w \cap A_2$,

$$\|J*K\|_2 \leqslant \max(\|J\|_w\|K\|_2, \|J\|_2\|K\|_w).$$

Proof. We give the proof of the first assertion only; the proof of the second is the same.

Assume $h \in L^2(\Omega)$. Then

$$\int |J*K(x, y)|\,|h(y)|\,\mathrm{d}y \leqslant \int\left(\int |J(x, z)|\,|K(z, y)|\,\mathrm{d}z\right)|h(y)|\,\mathrm{d}y =$$

$$= \int |J(x, z)|\left(\int |K(z, y)|\,|h(y)|\,\mathrm{d}y\right)\mathrm{d}z \leqslant \left(\int |J(x, z)|\left(\int |K(z, y)|^2\,\mathrm{d}y\right)^{1/2}\|h\|_2\right)\mathrm{d}z \leqslant$$

$$\leqslant \|J\|_1\|K\|_2\|h\|_2.$$

Taking the sup over $h \in L^2(\Omega)$, $\|h\|_2 \leqslant 1$, we have

$$\left(\int |J*K(x, y)|^2\,\mathrm{d}y\right)^{1/2} \leqslant \|J\|_1\|K\|_2.$$

Then

$$\operatorname*{ess\,sup}_x\left(\int |J*K(x, y)|^2\,\mathrm{d}y\right)^{1/2} \leqslant \|J\|_1\|K\|_2.$$

A similar argument interchanging x and y yields

$$\operatorname*{ess\,sup}_y\left(\int |J*K(x, y)|^2\,\mathrm{d}x\right)^{1/2} \leqslant \|J\|_2\|K\|_1.$$

This proves the lemma.

THEOREM 4.5. *If* $K = K^* \in D \cap A_{1,2}$, *then* $A_{1,2}\langle K\rangle$ *is quotient inverse closed. If* $K = K^* \in D \cap A_{w,2}$, *then* $A_{w,2}\langle K\rangle$ *is quotient inverse closed.*

Proof. We prove the second statement only; the proof of the first is similar. Let $u(z)$ and $v(z)$ be the entire functions $u(z) = e^{iz} - 1$, $v(z) = u(z)/z$, $z \neq 0$, $v(0) = i$. Let $U_n = u(nK)$ and $V_n = v(nK)$. Then $U_n = n(V_n*K)$. To show that $A_{w,2}\langle K\rangle$ is quotient inverse closed, it suffices by [3, Theorem 1] to prove that it is semisimple and regular. The semisimplicity follows from [13], Theorem (4.1.19). We will show that $\exists\, M > 0$ and $\exists\, \alpha > 0$ such that

$$\|U_n\|_{w,2} \leqslant Mn^\alpha \quad (n \geqslant 1). \tag{1}$$

The argument in [2], (4.2) shows that this condition implies that $A_{w,2}\langle K\rangle$ is regular. (This argument is based on an idea of J. Dixmier in [4].)

Now we verify that (1) holds. Note that the operator $(V_n)_2$ is normal on $L^2(\Omega)$ and $\|(V_n)_2\| \leqslant 2$. Let

$$I_n(y) = \left(\int |V_n * K(x,y)|^2 \mathrm{d}x\right)^{1/2} \leqslant \left(\int \left|\int V_n(x,z)K(z,y)\,\mathrm{d}z\right|^2 \mathrm{d}x\right)^{1/2} =$$

$$= \|(V_n)_2(K(\cdot,y))\|_2 \leqslant 2\|K(\cdot,y)\|_2.$$

Therefore

$$\operatorname*{ess\,sup}_y \left(\int |U_n(x,y)|^2\,\mathrm{d}x\right)^{1/2} = \operatorname*{ess\,sup}_y nI_n(y) \leqslant 2n\|K\|_2.$$

This inequality plus a similar argument interchanging the roles of x and y yield

$$\|U_n\|_2 \leqslant 2n\|K\|_2. \tag{2}$$

Now assume $K(x,y) = 0$ for a.a. $(x,y) \notin \Gamma[m]$.

Let

$$\varepsilon_n = \|\chi(\Gamma[m(n^2-1)]^c)U_n\|_w, \quad n \geqslant 1.$$

By hypothesis and Note 4.2 $K^{(p)}(x,y) = 0$ if $(x,y) \notin \Gamma[m(n^2-1)]$ for $1 \leqslant p \leqslant n^2 - 1$. We may assume $\|K\|_w \leqslant 1$. Then using the series expansion for $u(z)$ we have

$$\varepsilon_n \leqslant \left\| \sum_{p=n^2}^{\infty} (1/p!)(\mathrm{i}n)^p K^{(p)} \right\|_w \leqslant \sum_{p=n^2}^{\infty} (1/p!) n^p.$$

As is shown in the proof of [4], Lemma 6, this implies $\varepsilon_n \to 0$ as $n \to \infty$.

Let

$$J_n(y) = \int |U_n(x,y)| \chi(\Gamma[m(n^2-1)])(x,y) w(x,y)\,\mathrm{d}x =$$

$$= n\int \left|\int V_n(x,z)K(z,y)\,\mathrm{d}z\right| \chi(\Gamma[m(n^2-1)])(x,y)w(x,y)\,\mathrm{d}x \leqslant$$

$$\leqslant n\|(V_n)_2(K(\cdot,y))\|_2 \|\chi(\Gamma[m(n^2-1)])(\cdot,y)w(\cdot,y)\|_2 \leqslant$$

$$\leqslant 2n\|K(\cdot,y)\|_2 (C(m(n^2-1))^{\beta})^{1/2}(1 + m(n^2-1))^{\delta}.$$

(This last inequality uses (4.1).)

Thus, $\exists\,\gamma > 0$ and $M_1 > 0$ such that

$$\operatorname*{ess\,sup}_y J_n(y) \leqslant M_1 n^\gamma \quad (n \geqslant 1).$$

A similar argument interchanging x and y completes the proof that

$$\|\chi(\Gamma[m(n^2 - 1)])U_n\|_w \leqslant M_1 n^\gamma, \quad n \geqslant 1.$$

Therefore

$$(3) \qquad \|U_n\|_w \leqslant M_1 n^\gamma + \varepsilon_n, \quad n \geqslant 1.$$

Combining (2) and (3), we have $\exists\, M > 0$ and $\exists\,\alpha > 0$ such that (1) holds.

In order for Theorem 2.1 and Theorem 2.2 to hold it is necessary to work with a symmetric Banach $*$-algebra. We prove that the algebra $A_{w,2}$ is symmetric. We begin with a lemma.

Denote the spectral radius of a kernel K in A_1, A_w, or $A_{w,2}$, by $r_1(K)$, $r_w(K)$, or $r_{w,2}(K)$, respectively. We use the same type of notation to denote the spectrum of a kernel in one of these algebras.

LEMMA 4.6. *If* $K \in A_w$, *then* $r_w(K) = r_1(K)$. *If* $K \in A_{w,2}$, *then* $r_{w,2}(K) = r_1(K)$.

Proof. Suppose we know that for all K in either of the two algebras above that

$$(1) \qquad r_w(K) \leqslant \|K\|_1.$$

Then $r_w(K)^n = r_w(K^{(n)}) \leqslant \|K^{(n)}\|_1$, so $r_w(K) \leqslant \|K^{(n)}\|_1^{1/n} \to r_1(K)$. Thus, $r_w(K) \leqslant r_1(K)$. Since the opposite inequality always holds (A_w and $A_{w,2}$ are subalgebras of A_1), we have

$$(2) \qquad \text{If (1) holds, then } r_w(K) = r_1(K).$$

Suppose (1) holds for all $K \in A_{w,2}$. By Lemma 4.4 for $J, K \in A_{w,2}$

$$\|J * K\|_2 \leqslant \max(\|J\|_w \|K\|_2, \|J\|_2 \|K\|_w).$$

Thus, for $K \in A_{w,2}$

$$\|K^{(2n)}\|_{w,2} = \max(\|K^{(2n)}\|_w, \|K^{2(n)}\|_2) \leqslant \max(\|K^{(2n)}\|_w, \|K^{(n)}\|_w \|K^{(n)}\|_2).$$

It follows that

$$r_{w,2}(K) \leqslant \max(r_w(K), [r_w(K) r_{w,2}(K)]^{1/2}).$$

Therefore by this inequality and (2)

$$r_{w,2}(K) \leqslant r_w(K) = r_1(K).$$

It remains to verify (1). For each $\varepsilon, 0 < \varepsilon \leqslant 1$, define

$$w_\varepsilon(x, y) = (1 + \varepsilon d(x, y))^\delta \quad (x, y \in \Omega).$$

The proof of Note 4.3 works for w_ε in place of w, so

$$1 \leqslant w_\varepsilon(x, y) \leqslant w_\varepsilon(x, z) w_\varepsilon(z, y) \quad (x, y, z \in \Omega),$$

and A_{w_ε} is a Banach algebra with norm $\|K\|_{w_\varepsilon} = \|Kw_\varepsilon\|_1$. Also, $w_\varepsilon \leqslant w \leqslant \varepsilon^{-\delta} w_\varepsilon$ on $\Omega \times \Omega$. Then $\|J\|_w \leqslant \|J\|_{w_\varepsilon} \varepsilon^{-\delta}$ for all J, and for $n \geqslant 1$,

$$\|K^{(n)}\|_w^{1/n} \leqslant \|K^{(n)}\|_{w_\varepsilon}^{1/n} \varepsilon^{-\delta/n}.$$

This proves

(3) $$r_w(K) \leqslant r_{w_\varepsilon}(K) \leqslant \|K\|_{w_\varepsilon}.$$

Assume $K \in A_w$. Now since $0 < \delta \leqslant 1$,

$$t^\delta \leqslant (1 + t)^\delta \leqslant 1 + t^\delta \quad (t \geqslant 0).$$

Also, $1 \leqslant w_\varepsilon(x, y)$ for all x, y. Therefore

$$\operatorname*{ess\,sup}_x \int |K(x, y)| \, dy \leqslant \operatorname*{ess\,sup}_x \int |K(x, y)| w_\varepsilon(x, y) \, dy \leqslant$$

$$\leqslant \operatorname*{ess\,sup}_x \int |K(x, y)| (1 + \varepsilon^\delta d(x, y)^\delta) \, dy \leqslant$$

$$\leqslant \operatorname*{ess\,sup}_x \int |K(x, y)| \, dy + \varepsilon^\delta \left(\operatorname*{ess\,sup}_x \int |K(x, y)| d(x, y)^\delta \, dy \right).$$

Similar inequalities hold with x and y reversed. This proves

$$\lim_{\varepsilon \to 0} \|K\|_{w_\varepsilon} = \|K\|_1.$$

Combining this with (3) yields (1).

THEOREM 4.7. *If* $K = K^* \in A_{w,2}$, *then*

$$\sigma_{w,2}(K) = \sigma(K_2).$$

Thus, $A_{w,2}$ *is a symmetric Banach* $*$*-algebra.*

Proof. Assume $K = K^* \in A_{w,2}$. First we prove that $r_1(K) = \|K_2\|$. Now

$$\|K^{(n+1)}\|_1 \leqslant \|\chi(\Gamma[2^n])K^{(n+1)}\|_1 + \|\chi(\Gamma[2^n]^c)K^{(n+1)}\|_1.$$

Since $1 \leqslant w(x, y)(2^n)^{-\delta}$ whenever $(x, y) \notin \Gamma[2^n]$, we have

$$\|\chi(\Gamma[2^n]^c)K^{(n+1)}\|_1 \leqslant \|K^{(n+1)}\|_w(2^{-\delta})^n. \tag{1}$$

Now

$$\int |K^{(n+1)}(x, y)|\chi(\Gamma[2^n])(x, y)\, dx = \int \left| \int K^{(n)}(x, z)K(z, y)\, dz \right| \chi(\Gamma[2^n])(x, y)\, dx \leqslant$$

$$\leqslant \|K_2^n(K(\cdot, y))\|_2 \|\chi(\Gamma[2^n])(\cdot, y)\|_2 \leqslant \|K_2^n\| \, \|K\|_2(C(2^n)^\beta)^{1/2}$$

(This last inequality uses (4.1).)

A similar argument interchanging the roles of x and y gives

$$\|\chi(\Gamma[2^n])K^{(n+1)}\|_1 \leqslant \|K_2^n\| \, \|K\|_2(C2^{n\beta})^{1/2}. \tag{2}$$

Combining (1) and (2) we have that $\exists\, M > 0$ such that

$$\|K^{(n+1)}\|_1^{1/n+1} \leqslant \|K_2\|^{n/n+1}M^{1/n+1}(2^{\beta/2})^{n/n+1} + \|K^{(n+1)}\|_w^{1/n+1}(2^{-\delta})^{n/n+1}.$$

Therefore

$$r_1(K) \leqslant 2^{\beta/2}\|K_2\| + 2^{-\delta}r_w(K).$$

By Lemma 4.6 $r_w(K) = r_1(K)$ so this implies

$$r_1(K) \leqslant (1 - 2^{-\delta})^{-1}2^{\beta/2}\|K_2\|.$$

Then

$$r_1(K) = r_1(K^{(n)})^{1/n} \leqslant ((1 - 2^{-\delta}))^{-1}2^{\beta/2})^{1/n}\|K_2\|,$$

and letting $n \to +\infty$, we have $r_1(K) \leqslant \|K_2\|$. The reverse inequality holds by [13], Lemma (4.4.6). Applying Lemma 4.6 gives

$$r_{w,2}(K) = \|K_2\|.$$

It follows from Hulanicki's Theorem that $\sigma_{w,2}(K) = \sigma(K_2)$ for all $K = K^*$ in the algebra $A_{w,2}$. This implies the symmetry of $A_{w,2}$.

Now we apply Theorem 2.1 and Theorem 2.2.

THEOREM 4.8. *The algebra* $A^0_{w,2}$ *is* $*$*-quotient inverse closed. Assume* $K \in A^0_{w,2}$ *and* $1 \leqslant p \leqslant +\infty$.

(1) $\sigma_{w,2}(K) = \sigma(K_p)$ *when* K *is normal.*

(2) $\sigma_{w,2}(K) = \sigma(K_p) \cup \overline{\sigma((K^*)_p)}$ *in general.*

(3) $\omega(K_2) = \omega(K_p)$ *when* $K = K^*$.

(4) $\omega(K_2) = \omega(K_p) \cup \overline{\omega((K^*)_p)}$ *in general.*

Proof. By Theorem 4.7, $A_{w,2}$ is a symmetric Banach $*$-algebra. For $K = K^* \in A^0_{w,2}$ setting $K_m = \chi(\Gamma[m])K$, we have $\|K_m - K\|_{w,2} \to 0$ as $m \to \infty$. By Theorem 4.5, $A_{w,2}\langle K_m \rangle$ is quotient inverse closed for all m. Therefore $A^0_{w,2}$ is $*$-quotient inverse closed by Theorem 2.1. Then (2) follows from Theorem 2.2.

Again, assume $K = K^* \in A^0_{w,2}$. By (1)

$$\sigma_{w,2}(K) = \sigma(K_p) = \sigma(K_2) \subseteq \mathbf{R}, \quad 1 \leqslant p \leqslant +\infty.$$

In this situation if $\lambda_0 \in \sigma(K_p)$ and $\lambda_0 \notin \omega(K_p)$ for some p, then λ_0 is an isolated point of $\sigma(K_p)$ and the spectral projection corresponding to the set $\{\lambda_0\}$ has finite dimensional range; see [1], Theorem R.2.4 for a very general statement of this fact. Thus, λ_0 is an isolated point in $\sigma_{w,2}(K)$ and $\sigma(K_q)$, $1 \leqslant q \leqslant +\infty$. If E is the spectral idempotent in $A_{w,2}$ corresponding to the set $\{\lambda_0\}$ in $\sigma_{w,2}(K)$, then E_q is the spectral idempotent in $B(L^q(\Omega))$ corresponding to the set $\{\lambda_0\}$ in $\sigma(K_q)$. Since E_p has finite dimensional range, $EA_{w,2}E$ is finite dimensional, and this implies that E_q has finite dimensional range for all q, $1 \leqslant q \leqslant +\infty$. Then $\lambda_0 \notin \omega(K_q)$, $1 \leqslant q \leqslant +\infty$. Therefore $\omega(K_q) = \omega(K_2)$ for all q, which proves (3).

The proof that (4) follows from (3) is straightforward.

NOTE. Let $C_0(\Omega)$ denote the Banach space of continuous complex-valued functions on Ω that vanish at ∞ on Ω. Denote by $A_{1,c}$ the set of all kernels $K \in A_1$ such that for every $g \in C_0(\Omega)$, both

$$\int K(x, y) g(y)\, dy \quad \text{and} \quad \int K^*(x, y) g(y)\, dy$$

are in $C_0(\Omega)$. It is easy to check that $A_{1,c}$ is a closed $*$-subalgebra of A_1. Thus, $A_{w,2,c} = A_{w,2} \cap A_{1,c}$ is a symmetric Banach $*$-algebra. Furthermore, setting

$$K_c(f)(x) = \int K(x, y) f(y)\, dy \quad (f \in C_0(\Omega)),$$

we have $K \to K_c$ is a faithful continuous representation of $A_{w,2,c}$ into $B(C_0(\Omega))$. It is not hard to see that if $K(x, y) \in C_0(\Omega \times \Omega)$ and $K \in A_1$, then $K \in A_{1,c}$.

Now for any $m \geq 1$ define

$$\tau_m(x, y) = \begin{cases} 1 & (x, y) \in \Gamma[m-1] \\ m - d(x, y) & (x, y) \in \Gamma[m] \setminus \Gamma[m-1] \\ 0 & (x, y) \notin \Gamma[m]. \end{cases}$$

If $K \in C_0(\Omega \times \Omega)$, then $\tau_m K \in C_0(\Omega \times \Omega), m \geq 1$. If $K = K^* \in C_0(\Omega \times \Omega)$ and $K \in A^0_{w,2}$, then $\tau_m K \in D$ and $\|\tau_m K - K\|_{w,2} \to 0$ as $m \to \infty$. Thus, $A_{w,2,c}$ is $*$-quotient inverse closed by Theorem 2.1. Applying Theorem 2.2 we have that $\sigma_{w,2,c}(K) = \sigma(K_c)$.

5. TWO EXAMPLES

In this section we look briefly at two very different examples of algebras of Hille-Tamarkin operators. In each case we give a simple sufficient condition that an operator be in $A^0_{w,2}$.

5.1. EXAMPLE. Let Ω be the set of all positive integers equipped with counting measure μ and metric $d(n, m) = |n - m|$. Then for $m \geq 0$

$$\Gamma[m] = \{(j, j + k) : 1 \leq j, \max(-m, 1 - j) \leq k \leq m\}.$$

Thus, $\Gamma[m]$ is a set of $2m + 1$ diagonals in $\Omega \times \Omega$. Since for any sequence $\{a_k\}$, $\sum_{k=1}^{\infty} |a_k|^2 \leq \left(\sum_{k=1}^{\infty} |a_k|\right)^2$, it follows that in this case $A_1 \subset A_2$ and $\|K\|_1 \geq \|K\|_2$ for $K \in A_1$. The weight function has the form

$$w(j, k) = (1 + |j - k|)^\delta \quad (j, k \in \Omega).$$

We have the following simple criteria for a kernel to be in $A^0_w = A^0_{w,2}$.

5.2. Assume $K(i, j)$ is a kernel. For each integer k let $M_k = \sup\{|K(j, j + k)| : \max(1, 1 - k) \leq j < \infty\}$ (the sup over the k^{th} diagonal of K). If $\sum_{k=-\infty}^{+\infty} M_k(1 + |k|)^\delta = M < +\infty$, then $K \in A^0_w = A^0_{w,2}$.

Proof. For any j

$$\sum_{k=1}^{+\infty} |K(j, k)|(1 + |j - k|)^\delta = \sum_{k=-j+1}^{+\infty} |K(j, j + k)|(1 + |k|)^\delta \leq$$

$$\leq \sum_{k=-\infty}^{+\infty} M_k(1 + |k|)^\delta = M.$$

Thus,

$$\sup_{j \geq 1} \sum_{k=1}^{+\infty} |K(j, k)| w(j, k) \leqslant M.$$

This inequality plus a similar inequality with j and k reversed imply $\|K\|_w \leqslant M$. Therefore $K \in A_w$, and essentially the same computation verifies $K \in A_w^0 = A_{w,2}^0$.

5.3. EXAMPLE. Let $\Omega = [0, +\infty)$ equipped with Lebesgue measure μ. Let d be the usual metric on $[0, +\infty)$. The weight function $w(x, y)$ is of the form

$$w(x, y) = (1 + |x - y|)^{\delta} \quad (x, y \in \Omega).$$

5.4. Let $K(x, y)$ be a kernel on $\Omega \times \Omega$. If $\exists\, \alpha > 1$ and $\exists\, M > 0$ such that

$$|K(x, y)| \leqslant M(1 + |x - y|)^{-\alpha} \quad (x, y \in \Omega),$$

then $K \in A_{w,2}^0$ for an appropriate choice of δ, $0 < \delta \leqslant 1$.

Proof. Choose δ such that $0 < \delta < \min(1, \alpha - 1)$. Let w be the corresponding weight function (defined as above). By hypothesis

$$|K(x, y)| w(x, y) \leqslant M(1 + |x - y|)^{-(\alpha - \delta)} \quad (x, y \in \Omega).$$

Note that $\alpha - \delta > 1$. This inequality shows that it suffices to prove that when $\beta > 1$, then

$$(1 + |x - y|)^{-\beta} \in A_{1,2}^0.$$

The verification of this is a straightforward computation.

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