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ANALYTIC TOEPLITZ OPERATORS WITH SELF-COMMUTATORS HAVING TRACE CLASS SQUARE ROOTS

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1.

For $\varphi \in H^{\infty}$, $\varphi \not\equiv$ const, let $S = T_{\varphi}$ be the analytic Toeplitz operator on H^2 with symbol φ. (Concerning these and subnormal operators in general, see [2].) Let D denote the self-commutator of S, so that

$$S^*S - SS^* = D \geqslant 0. \tag{1.1}$$

It is known that S is purely subnormal. Hence, since S is also hyponormal, $\mathrm{Re}(S) = (1/2)(S + S^*)$ is absolutely continuous; see [6], p. 42. If N on $L^2(0,2\pi)$ denotes the minimal normal extension of S, then, in general, $\mathrm{Re}(N)$ is not absolutely continuous. In fact, $\mathrm{Re}(N)$ need not have an absolutely continuous part even if D is of trace class; see [7]. However, there will be proved the following

THEOREM 1. *Let $S = T_{\varphi}$ $(\varphi \not\equiv \mathrm{const})$ be an analytic Toeplitz operator with self-commutator D of* (1.1) *satisfying*

$$D^{\frac{1}{2}} \text{ is of trace class.} \tag{1.2}$$

If N is the minimal normal extension of S, then the absolutely continuous part $(\mathrm{Re}(N))_{\mathrm{a}}$ of $\mathrm{Re}(N)$ is unitarily equivalent to the direct sum of $\mathrm{Re}(S)$ with itself, that is,

$$(\mathrm{Re}(N))_{\mathrm{a}} \cong \mathrm{Re}(S) \oplus \mathrm{Re}(S). \tag{1.3}$$

More generally, if a and b are real and $a^2 + b^2 > 0$ then

$$(a\mathrm{Re}(N) + b\mathrm{Im}(N))_{\mathrm{a}} \cong (a\mathrm{Re}(S) + b\mathrm{Im}(S)) \oplus (a\mathrm{Re}(S) + b\mathrm{Im}(S)). \tag{1.4}$$

Further, whenever N^*N (or, equivalently, S^*S) has an absolutely continuous part then

$$(1.5) \qquad (N^*N)_a \cong (S^*S)_a \oplus (S^*S)_a.$$

Since (1.4) follows immediately from (1.3) if S is replaced by $e^{it}S$ (t real) then only (1.3) and (1.5) need be established. Before beginning the proof of Theorem 1 it will be convenient to prove the following

LEMMA. *Let G denote a bounded Hermitian matrix*

$$(1.6) \qquad G = (g_{ij}), \quad g_{ij} = \bar{g}_{ji} \quad (i, j = 1, 2, \ldots)$$

on the unilateral sequence space ℓ^2 and let $\bar{G} = (\bar{g}_{ij})$. Then G and $\bar{G}$ are unitarily equivalent, that is

$$(1.7) \qquad \bar{G} \cong G.$$

Proof of Lemma. The Lemma may be well-known. In case the underlying Hilbert space is finite-dimensional, this is surely the case. In fact, the assertion is then seen to be a consequence of the fact that the eigenvalues of G are real and that the characteristic equations of G and of $\bar{G}$ have the same roots with corresponding multiplicities. Further, even in the infinite dimensional case, it is seen that λ is an eigenvalue of G with unit eigenvector $x = (x_1, x_2, \ldots)$ if and only if λ is an eigenvalue of $\bar{G}$ with unit eigenvector $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$. The Lemma then follows readily if G has a pure point spectrum. It appears then that only the existence of a continuous spectrum for G offers any possible difficulties. The following argument treats the general case.

First, one has $G \cong D_0 \oplus J_1 \oplus J_2 \oplus \ldots$, where D_0 is a real diagonal matrix and $J_1, J_2, \ldots$ are Jacobi matrices

$$J_n = \begin{bmatrix} a_{1n} & b_{1n} & 0 & \ldots \\ b_{1n} & a_{2n} & b_{2n} & \ldots \\ 0 & \bar{b}_{2n} & a_{3n} & \ldots \\ . & . & . & \ldots \end{bmatrix},$$

where the a_{in} are real and $b_{in} \neq 0$; see [8], pp. 282—298, also [9].

Of course, D_0 or some of the J_n may be absent. In addition, and this is a crucial point, it may be supposed that the elements $b_{1n}, b_{2n}, \ldots$ also are real; see [8], p. 547. Clearly the matrices $D_0, J_1, J_2, \ldots$ may be combined into a single matrix $G' = (g'_{ij})$, $i, j = 1, 2, \ldots$, of intermingled blocks. Thus, if A and B are distinct matrices of the set $D_0, J_1, J_2, \ldots$, and if a_i is a nonzero element of A'

(where A' is regarded as a "block" in G' corresponding to A) occurring in either the i-th row or the i-th column of G' then both the i-th row and i-th column of G' contain no elements of B' (B' corresponding to B as A' does to A). Thus there exists a unitary matrix $U:\ell^2 \to \ell^2$ for which $U^*GU = G'$, where G' is real. Hence $\bar{U}^*\bar{G}\,\bar{U} = \bar{G}' = G'$, so that G and $\bar{G}$ are both unitarily equivalent to G' and hence to one another. This completes the proof of the Lemma.

2.

Proof of Theorem 1. For the moment, let S be an arbitrary purely subnormal operator (not necessarily an analytic Toeplitz operator) on a separable infinite-dimensional Hilbert space H and let N be its minimal normal extension on $K \supset H$. Then N has the representation

$$N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix} \quad \text{on } K = H \oplus H^\perp; \tag{2.1}$$

see [1] and [2], pp. 129 ff. The operator T, the dual of S, is purely subnormal on $H^\perp$ with the minimal normal extension N^* on K. Also, $\sigma(T) = \{\bar{z} : z \in \sigma(S)\}$ and S is the dual of T. Simple calculations show that

$$S^*S - SS^* = XX^* \quad \text{and} \quad T^*T - TT^* = X^*X \tag{2.2}$$

and that

$$\mathrm{Re}(N) = \begin{bmatrix} \mathrm{Re}(S) & 0 \\ 0 & \mathrm{Re}(T) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \quad \text{on } K = H \oplus H^\perp. \tag{2.3}$$

In case $(XX^*)^{1/2}$ is of trace class, that is, if X is of trace class, it then follows from the Rosenblum-Kato theory (see [7]) that

$$(\mathrm{Re}(N))_a \cong \begin{bmatrix} \mathrm{Re}(S) & 0 \\ 0 & \mathrm{Re}(T) \end{bmatrix}. \tag{2.4}$$

For the case at hand, let $S = T_\varphi$ with $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \not\equiv a_0$. Then there exists a unitary operator U_1 for which

$$U_1^* T_\varphi U_1 = A = \begin{bmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_0 & 0 & \dots \\ a_2 & a_1 & a_0 & \dots \\ \cdot & \cdot & \cdot & \dots \end{bmatrix},$$

where the lower triangular matrix A is bounded on ℓ^2, and hence

$$\text{(2.5)} \qquad \mathrm{Re}(T_\varphi) \cong \mathrm{Re}(A).$$

It was shown by Conway ([1], p. 198) that the dual of T_φ is unitarily equivalent to T_{φ^*}, where $\varphi^*(z) = \sum_{n=0}^{\infty} \bar{a}_n z^n$. Consequently,

$$\text{(2.6)} \qquad \mathrm{Re}(\text{Dual of } T_\varphi) \cong \mathrm{Re}(T_{\varphi^*}).$$

Also, analogous to (2.5), one has

$$\text{(2.7)} \qquad \mathrm{Re}(T_{\varphi^*}) \cong \mathrm{Re}(\bar{A}).$$

In view of the Lemma, $\mathrm{Re}(\bar{A})$ $(= \overline{\mathrm{Re}(A)}) \cong \mathrm{Re}(A)$ and so by relations (2.5)—(2.7), $\mathrm{Re}(\text{Dual of } T_\varphi) \cong \mathrm{Re}(T_\varphi)$.

Thus, for $S = T_\varphi$ and $T =$ Dual of T_φ, one has $\mathrm{Re}(T) \cong \mathrm{Re}(S)$ and hence $\mathrm{Re}(S) \oplus \mathrm{Re}(T) \cong \mathrm{Re}(S) \oplus \mathrm{Re}(S)$. Thus, by (2.3), there exists a unitary operator W for which

$$\text{(2.8)} \qquad \mathrm{Re}(N) = W^* \begin{bmatrix} \mathrm{Re}(S) & 0 \\ 0 & \mathrm{Re}(S) \end{bmatrix} W + \frac{1}{2} \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

Actually, (2.8) is valid when S is any analytic Toeplitz operator T_φ, not necessarily subject to the restriction (1.2). In case $S^*S - SS^* = D = XX^*$ satisfies (1.2), however (that is, if X is of trace class), one has also (2.4) and hence (1.3) of Theorem 1.

The proof of (1.5) of Theorem 1 is similar. In fact, one now has $T^*T \cong$ $\cong (T_{\varphi^*})^* T_{\varphi^*} \cong \bar{A}^*\bar{A} = \overline{A^*A}$ and $S^*S = (T_\varphi)^* T_\varphi \cong A^*A$. Again, by the Lemma, $\overline{A^*A} \cong A^*A$ and so $T^*T \cong S^*S$. As before, a simple calculation shows that

$$N^*N = NN^* = \begin{bmatrix} S^*S & 0 \\ 0 & T^*T \end{bmatrix} + \begin{bmatrix} 0 & S^*X \\ X^*S & 0 \end{bmatrix},$$

and (1.5) follows by an argument analogous to that used in obtaining (1.3). This completes the proof of Theorem 1.

Clearly, if $S = T_\varphi$ of Theorem 1 is an isometry then the absolutely continuous parts of S^*S and of N^*N (N now being unitary) are absent. It will be shown in Theorem 2 below that, at least under a certain extra restriction, the absolutely continuous parts of these operators are always present whenever T_φ is not a multiple of an isometry.

3.

THEOREM 2. *Let* $\varphi(z)=\sum_{n=0}^{\infty}a_nz^n\not\equiv a_0$ *be bounded* in $|z|<1$ *and let* $S=T_\varphi$ *be the associated analytic Toeplitz operator satisfying* (1.2) *as in Theorem* 1. *In addition, suppose that* $\sum_{n=0}^{\infty}a_n e^{int}$ *is the Fourier series of a function* $\psi(t)$ *of bounded variation on* $[0,2\pi]$ *and that*

$$|\psi(t)|\not\equiv \text{const},\quad 0\leqslant t\leqslant 2\pi. \tag{3.1}$$

Then $(N^*N)_a$ *and* $(S^*S)_a$ *exist and satisfy* (1.5).

Proof of Theorem 2. It is known that $\psi(t)=\lim_{r\to 1-0}\varphi(re^{it})$ is continuous, and even absolutely continuous, and that also $\sum_{n=0}^{\infty}|a_n|<\infty$; see [10], vol. I, pp. 285—286. Clearly, $\psi(t)=\sum_{n=0}^{\infty}a_ne^{int}$, $0\leqslant t\leqslant 2\pi$, is the image of the unit circle under the mapping $\varphi(z)$ with $z=e^{it}$, and the original hypothesis on ψ amounts to supposing that this image is a rectifiable curve. (Portions of this curve may, of course, be multiply covered.)

Since $\psi(t)$ is absolutely continuous, so also is $|\psi(t)|^2$. Since N is the operator of multiplication by $\psi(t)$ on $L^2(0,2\pi)$, then N^*N is that of multiplication by the absolutely continuous function $|\psi(t)|^2$ on $L^2(0,2\pi)$. Consequently, (3.1) now becomes precisely the condition that N^*N has an absolutely continuous part, and the proof of Theorem 2 is complete.

4.

ADDENDUM (12. 6. 1986). The author is indebted to the referee for pointing out the reference to V. V. Peller [4] and to the fact noted there that if $S=T_\varphi$ is the Toeplitz operator belonging to $\varphi(e^{it})=\sum_{n=0}^{\infty}a_ne^{int}$ in H^∞, then the associated Hankel operator is nuclear (i.e., that the above condition (1.2) is satisfied) if and only if φ belongs to the Besov space B_1^1 consisting of functions in $L(-\pi,\pi)$ for which

$$\int_{-\pi}^{\pi}t^{-2}\int_{-\pi}^{\pi}|f(e^{ix+it})+f(e^{ix-it})-2f(e^{ix})|\,dx\,dt<\infty.$$

In the preprint [3], Peller gives this and other equivalent conditions for the nuclearity (i.e., trace class) property of the associated Hankel matrix $B = (a_{i+j-1})$, where $i, j = 1, 2, \ldots$. A useful survey of Hankel operators together with numerous references can be found in [5].

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