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SOME REMARKS ON THE GROUPOID APPROACH TO WIENER-HOPF OPERATORS

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INTRODUCTION

Let P be a closed convex cone in $\mathbf{R}^n$ such that $\operatorname{sp} P = \mathbf{R}^n$. The Wiener-Hopf operator with symbol $f \in C_c(\mathbf{R}^n)$ on P is the operator in $\mathscr{L}(L^2(P))$ defined by the formula:

$$[W_P(f)\xi](t) = \int_P f(t-s)\xi(s)\mathrm{d}s;$$

the C^*-subalgebra of $\mathscr{L}(L^2(P))$ generated by $\{W_P(f) \mid f \in C_c(\mathbf{R}^n)\}$ is denoted by $\mathscr{W}(P)$ (the C^*-algebra of the Wiener-Hopf operators on P). A fruitful method of studying $\mathscr{W}(P)$ was discovered by P. Muhly and J. Renault in [7]. They found a locally compact groupoid (with a canonical Haar system) $\mathfrak{G}$ having the property that $C^*_{\rm red}(\mathfrak{G})$ (the reduced C^*-algebra associated to $\mathfrak{G}$) is isomorphic to $\mathscr{W}(P)$ and obtained information about $\mathscr{W}(P)$ from the structure of $\mathfrak{G}$. In fact, $C^*_{\rm red}(\mathfrak{G}) = C^*(\mathfrak{G})$ because the groupoid $\mathfrak{G}$ is amenable in the sense of Renault. Details may be found in [8].

In the present paper we pursue this approach to the study of $\mathscr{W}(P)$. We shall describe a new presentation for $\mathfrak{G}$, which has the advantage that it can be studied with the use of quite sophisticated tools from convex analysis. More precisely, we shall construct and study another groupoid — let us call it $\mathfrak{g}$ — which also has the property that $C^*_{\rm red}(\mathfrak{g})$ (or, equivalently $C^*(\mathfrak{g})$ is isomorphic to $\mathscr{W}(P)$; a posteriori it can be seen that $\mathfrak{G}$ and $\mathfrak{g}$ are iso- morphic.

The crucial point is the following: the closed invariant subsets of $\mathfrak{G}^0$ (or $\mathfrak{g}^0$) give information about the ideal structure of $\mathscr{W}(P)$ ($\mathfrak{G}^0$ and $\mathfrak{g}^0$ are the unit spaces of $\mathfrak{G}$ and $\mathfrak{g}$, respectively); that is why a good description of $\mathfrak{G}^0$ (or $\mathfrak{g}^0$) is necessary. In [7], Muhly and Renault emphasized a class of elements of $\mathfrak{G}^0$ which is "indexed" by $\mathscr{F}(P)$ (the set of faces of P) and found cases — among them, the polyhedral and the self-dual cones when this class exhausts $\mathfrak{G}^0$. Making a parallel study for $\mathfrak{g}$, we find a class of elements of $\mathfrak{g}^0$ indexed by $\mathscr{F}(\hat{P})$, with $\hat{P}$ the "dual" of P, and we find a sufficient condition on P which assures that this larger class of elements exhausts $\mathfrak{g}^0$ (the condition is: "$\hat{P}$ is tame" — Proposition 6.1). It is likely that "most cones" satisfy this condition, but all we can say at this moment is that polyhedral cones, all cones in $\mathbf{R}^3$ and the forward light cone in any dimension satisfy it. The description of $\mathfrak{g}^0$ seems to be difficult when P is not tame (see Observation 6.2.2°).

The subjects of the various sections of the paper are as follows: in Section 1 we make the simple, but important observation that additional conditions concerning the continuity axiom are needed when we want to reduce the Haar system of a locally compact groupoid to a locally closed set. In Section 2 we consider in an axiomatic manner the construction made in [7] and show that the axiom (M4) (which initially appears to be just a technical condition) leads us in a natural way to the groupoid $\mathfrak{g}$. In order for this machinery to work, a certain condition (M) on P turns out to be necessary and sufficient. In Section 3 we show that (M) is always fulfilled when P is a closed convex cone in $\mathbf{R}^n$ with $\operatorname{sp} P = \mathbf{R}^n$ (the setting of Section 2 is a little bit more general). Section 4 is devoted to the construction of the previously mentioned class of elements of $\mathfrak{g}^0$, which is "indexed" by $\mathscr{F}(\hat{P})$. In Section 5 we introduce the (new) notion of "tame convex set"; this is a preliminary for Section 6 where we show that if $\hat{P}$ is tame then the class exhibited in Section 4 exhausts $\mathfrak{g}^0$. Finally, in Section 7 we se that when $\hat{P}$ is tame, the closed invariant subsets of $\mathfrak{g}^0$ are in bijective correspondence with the subsets of $\mathscr{F}(\hat{P})$ which are hereditary and closed relative to a certain natural Hausdorff distance.

It is very probable that using the results of Section 7 and some facts about fields of C^*-algebras, one can show that $\mathscr{W}(P)$ is postliminary when $\hat{P}$ is tame. However, this is not done in the present paper. Another subject for further research is given by the fact that generating systems — the axiomatic presentation of the construction made in [7] — can be used to study C^*-subalgebras of $\mathscr{L}(L^2(P))$ which are greater than $\mathscr{W}(P)$. This is, in fact, already done in [3] in the discrete case (the facts about generating systems are stated for "a closed cone P in a locally compact second countable unimodular group G"; in [3] P is $\mathbf{N}^2$ and G is $\mathbf{Z}^2$).

When writing this paper, the author was not aware of the work of A. Dynin, [4], where the connection between $\mathscr{W}(P)^\wedge$ and $\mathscr{F}(\hat{P})$ is established in the case o completely tangible cones. The methods used by Dynin are completely different from the groupoid approach to the problem.

The examples of cones with tame dual mentioned above (polyhedral cones, 3-dimensional cones, light cones) are completely tangible, but the exact relation between the cones with tame dual and the completely tangible ones is not clear at this moment. In [2] (which is an appendix to [4]) it is shown that the intersections of finitely many smooth cones and the closures of homogeneous cones of finite type are completely tangible.

Finally, the author wishes to thank those who advised him during the preparation of his graduation paper, on which this paper is based.

1. THE CONTINUITY AXIOM FOR REDUCED GROUPOIDS

1.1. Let $\mathfrak{g}$ be a locally compact groupoid. If V is a non-void, locally closed ubset of $\mathfrak{g}^0$, then the reduction of $\mathfrak{g}$ to V, denoted by $\mathfrak{g}\,|\,V$ is defined to be $\{x \in \mathfrak{g} \mid d(x) \in V, r(x) \in V\}$. $\mathfrak{g}\,|\,V$ is also a locally compact groupoid ([7], 2.2.5, p. 14).

Let us suppose that on $\mathfrak{g}$ we have a left Haar system $\lambda = (\lambda^u)_{u \in \mathfrak{g}^0}$; by defining:

$$\sigma^v = \lambda^v \,|\, \{x \in g^v \mid d(x) \in V\}, \quad \forall\, v \in V, \tag{1}$$

we obtain a natural candidate for a left Haar system on $\mathfrak{g}\,|\,V$. It is easy to verify that $(\sigma^v)_{v \in V}$ is left translation-invariant; as it is pointed out in [7], 2.5, p. 16, an additional condition is needed in order that $(\sigma^v)_{v \in V}$ satisfy the support axiom (required by any Haar system—see [7], 2.4, p. 15 or [8], Chapter I, Definition 2.2). In this section we make the simple but important observation that additional conditions are necessary in connection with the continuity axiom, too.

1.2. NOTATION. We shall consider the particular case of the reduction of a transformation group to a closed set. Thus let G be a locally compact group U a locally compact space and $G \times U \to U$ a continuous action on the left. (The groupoid we obtain is $G \times U$, the domain and codomain of (t, u) are u and tu, respectively.) On G we take a left Haar measure μ which induces a left Haar system $\lambda = (\lambda^u)_{u \in U}$ on $G \times U$ by the formula:

$$\int_{(G \times U)^u} f \mathrm{d}\lambda^u = \int_G f(t^{-1}, tu) \mathrm{d}\mu(t), \quad \forall\, u \in U, \forall\, f \in C_c((G \times U)^u)$$

(we have $(G \times U)^u = \{x \in G \times U \mid r(x) = u\} = \{(t^{-1}, tu) \mid t \in G\}$). The conjugated right Haar system $(\lambda_u)_{u \in U}$ is given by:

$$\int_{(G \times U)_u} f \mathrm{d}\lambda_u = \int_G f(t, u) \mathrm{d}\mu^{-1}(t), \quad \forall u \in U, \forall\, f \in C_c((G \times U)_u)$$

(we have $(G \times U)_u = \{x \in G \times U \mid d(x) = u\} = G \times \{u\}$; here μ^{-1} is the right Haar measure conjugate to μ).

Now let V be a non-void closed subset of U. For any $v \in V$ we define:

$$G_v = \{t \in G \mid tv \in V\}; \tag{2}$$

it is clear that G_v is closed and contains e, the unit of G. We denote the reduction of $G \times U$ to V by $\mathfrak{g}$. Obviously for any $v \in V$, $\mathfrak{g}_v = G_v \times \{v\}$ and hence $\sigma_v = \lambda_v \mid \mathfrak{g}_v$ integrates according to the formula:

$$\int_{\mathfrak{g}_v} f \mathrm{d}\sigma_v = \int_{G_v} f(t, v) \mathrm{d}\mu^{-1}(t), \quad \forall f \in C_c(\mathfrak{g}_v). \tag{3}$$

The next proposition gives equivalent conditions for $(\sigma_v)_{v \in V}$ to be a right Haar system (which is clearly equivalent to the fact that $(\sigma^v)_{v \in V}$ is a left Haar system) in terms of the sets $(G_v)_{v \in V}$ of (2).

1.3. PROPOSITION *Using the above notation we have that:*
1° $(\sigma_v)_{v \in V}$ *satisfies the support axiom if and only if* $\operatorname{supp} \mu^{-1} \mid G_v = G_v$, $\forall v \in V$.
2° $(\sigma_v)_{v \in V}$ *satisfies the continuity axiom if and only if* $V \ni v \mapsto \int_{G_v} f \mathrm{d}\mu^{-1}$ *is continuous for every* $f \in C_c(G)$ *if and only if* $V \ni v \mapsto \chi_{G_v} \in L^\infty(\mu^{-1})$ *is continuous when* $L^\infty(\mu^{-1})$ *is regarded as the dual of* $L^1(\mu^{-1})$ *with the* w* *topology.*

Proof. 1° comes out from (3). The equivalence between the last two statements of 2° is clear ($C_c(G)$ is dense in $L^1(\mu^{-1})$). It remains to prove the equivalence between the first two ones.

"$\Rightarrow$" We take $f \in (C_c(G)$ and $v_0 \in V$. Let φ be in $C_c(U)$ such that $\varphi = 1$ on a neighbourhood of v_0. We define a θ in $C_c(\mathfrak{g})$ by $\theta(t, v) = f(t)\varphi(v)$. By the hypothesis, $V \ni v \mapsto \int_{\mathfrak{g}_v} \theta \mathrm{d}\sigma_v = \varphi(v) \int_{G_v} f \mathrm{d}\mu^{-1}$ is continuous and this function coincides with $v \mapsto \int_{G_v} f \mathrm{d}\mu^{-1}$ on a neighbourhood of v_0. Hence for fixed f, $V \ni v \mapsto \int_{G_v} f \mathrm{d}\mu^{-1}$ is continuous at every point of V.

"$\Leftarrow$" We have to show that $V \ni v \mapsto \int_{\mathfrak{g}_v} \theta \mathrm{d}\sigma_v$ is continuous for every θ in $C_c(\mathfrak{g})$. When θ is of the form $\theta(t, v) = f(t)\varphi(v)$ with $f \in C_c(G)$, $\varphi \in C_c(U)$, this is a direct consequence of the hypothesis and the equality (3). It is clear that the set of those θ for which the desired continuity holds is a linear subspace of $C_c(\mathfrak{g})$ and so we obtain that this set is sequentially dense in $C_c(\mathfrak{g})$ with the inductive limit topology.

Now let us fix θ in $C_c(\mathfrak{g})$ and v_0 in V. Consider a sequence $(\theta_n)_{n=1}^{\infty}$ in $C_c(\mathfrak{g})$ converging to θ in the inductive limit topology and having the desired property. Consider a compact subset $\mathscr{K}$ of $\mathfrak{g}$ such that $\operatorname{supp}\theta_n \subseteq \mathscr{K}, \forall n$, a positive $f \in C_c(G)$ which is identically one on $\mathrm{pr}_G(\mathscr{K})$ and a compact neighbourhood V_0 of v_0. For any n and $v \in V_0$:

$$\left|\int_{\mathfrak{g}_v}\theta\mathrm{d}\sigma_v - \int_{\mathfrak{g}_{v_0}}\theta\mathrm{d}\sigma_{v_0}\right| \leqslant \left|\int_{\mathfrak{g}_v}\theta\mathrm{d}\sigma_v - \int_{\mathfrak{g}_v}\theta_n\mathrm{d}\sigma_v\right| +$$

$$+\left|\int_{\mathfrak{g}_v}\theta_n\mathrm{d}\sigma_v - \int_{\mathfrak{g}_{v_0}}\theta_n\mathrm{d}\sigma_{v_0}\right| + \left|\int_{\mathfrak{g}_{v_0}}\theta_n\mathrm{d}\sigma_{v_0} - \int_{\mathfrak{g}_{v_0}}\theta\mathrm{d}\sigma_{v_0}\right|.$$

The first term of the sum can be written $\left|\int_{G_v}\theta(t, v) - \theta_n(t, v)\mathrm{d}\mu^{-1}(t)\right|$ and is less than $\|\theta - \theta_n\|_\infty\int_{G_v} f\mathrm{d}\mu^{-1}$; so is the third term and we get:

$$\left|\int_{\mathfrak{g}_v}\theta\mathrm{d}\sigma_v - \int_{\mathfrak{g}_{v_0}}\theta\mathrm{d}\sigma_{v_0}\right| \leqslant 2\|\theta - \theta_n\|_\infty \sup_{w\in V_0}\int_{G_w} f\mathrm{d}\mu^{-1} + \tag{4}$$

$$+\left|\int_{\mathfrak{g}_v}\theta_n\mathrm{d}\sigma_v - \int_{\mathfrak{g}_{v_0}}\theta_n\mathrm{d}\sigma_{v_0}\right|.$$

Having (4), the proof is carried out with the usual argument involving three times ε (note that $\sup_{w\in V_0}\int_{G_w} f\mathrm{d}\mu^{-1}$ is finite being the supremum of a continuous function on a compact set).

1.4. Examples when $(\sigma_v)_{v\in V}$ is not a Haar system. 1° Let $\mathbf{R}$ act on itself by translations. If we make the reduction to $\mathbf{Z}$, then $G_v = \mathbf{Z}$, $\forall v \in \mathbf{Z}$ and the support axiom is not satisfied.

2° Let $\mathbf{Z}$ act on $\mathbf{R}$ by translations. If we make the reduction to $[0, 1]$, then $G_0 = \{0, 1\}$, $G_1 = \{-1, 0\}$ and $G_v = \{0\}$, $\forall v \in (0, 1)$. The continuity axiom is not satisfied (take for instance $f = \chi_{\{1\}} \in C_c(\mathbf{Z})$).

1.5. Definition. Let $(\mathfrak{g}, \lambda)$ be a locally compact groupoid with left Haar system and V a locally closed non-void subset of $\mathfrak{g}^0$. If $(\sigma^v)_{v\in V}$ from (1) is a left Haar system for $\mathfrak{g}\,|\,V$, we say that *λ can be reduced to V.*

2. $\mathscr{W}(P)$ WRITTEN AS THE REDUCED C^*-ALGEBRA OF A GROUPOID

2.1. The setting in which we are working is slightly different from the one of Section 3 of [7] and is described as follows: we consider a locally compact second countable unimodular group G and a left Haar measure μ on G. We say that a closed subset F of G is *solid* if $F \neq \emptyset$ and $\operatorname{supp}\mu|F = F$ (this clearly happens when $\emptyset \neq F = \operatorname{clos} \mathring{F}$). We consider (and fix) a solid closed subset P of G which is a cone; that is, $PP \subseteq P$ and $P \ni e$, the unit of G. We shall constantly write Q for P^{-1}. For any f in $C_c(G)$ we define the Wiener-Hopf operator with symbol f on P to be:

$$W_P(f) \in \mathscr{L}(L^2(\mu\,|\,P)).$$

$$[W_P(f)\xi](t) = \int_P f(ts^{-1})\xi(s)\mathrm{d}\mu(s).$$

That is, W_P is the compression to $L^2(\mu\,|\,P)$ of the regular left representation $\Lambda : C_c(G) \to \mathscr{L}(L^2(\mu))$. The C^*-subalgebra of $\mathscr{L}(L^2(\mu\,|\,P))$ generated by $\operatorname{Ran} W_P$ is denoted by $\mathscr{W}(P)$.

2.2. THE DEFINITION OF A GENERATING SYSTEM.

2.2.1. The axioms which appear in the next definition represent a transformation of the statements of Lemma 3.3, p. 33, [7]; the novelty consists in the axiom (M4) which surprisingly turns out to be more than a technical condition (as we shall see in 2.3.1).

DEFINITION. *A generating system over* P is a system $(U, G\times U \to U, \tau; V)$ where: U is a locally compact space, $G\times U \to U$ is a continuous action on the left, $\tau : G \to U$ is a function and V is the closure of $\tau(P)$ in U, having the following properties:

(M1) $s\tau(t) = \tau(st)$, $\forall s, t \in G$.

(M2) V is compact.

(M3) $t \in G$ and $\tau(t) \in V$ imply $t \in P$.

(M4) The canonical Haar system on the transformation group $G\times U$ can be reduced to V.

2.2.2. OBSERVATIONS. 1° The function $\tau : G \to U$ in the above definition is automatically continuous because $\tau(t) \equiv t\tau(e)$ (and the action of G on U is continuous).

2° Taking Proposition 1.3 into account, we may express the fulfilment of (M4) in terms of the sets $(G_v)_{v\in V}$ given by (2). More precisely, (M4) is satisfied if and only if every G_v is solid (recall that G_v is closed and contains e) and the function $V \ni v \mapsto \chi_{G_v} \in L^\infty(\mu)$ is continuous when $L^\infty(\mu)$ is regarded as the dual of $L^1(\mu)$ with the w* topology (here we have $\mu = \mu^{-1}$).

Let us also notice that $G_{\tau(t)} = Pt^{-1}$, $\forall t \in P$; indeed, we have $s \in G_{\tau(t)}$ if and only if

$$s\tau(t) \in V \overset{(M1)}{\Leftrightarrow} \tau(st) \in V \overset{(M3)}{\Leftrightarrow} st \in P \Leftrightarrow s \in Pt^{-1}.$$

In [7], Muhly and Renault have emphasized (in a particular case) the connection between generating systems and Wiener-Hopf operators. We rewrite, in Lemma 2.2.3 and Proposition 2.2.4, one of their results (see 3.6 and 3.7, p. 38—40, [7]) using the above terminology. Proposition 2.2.4 explains the term "generating system".

2.2.3. LEMMA. *Let* $(U, G\times U \to U, \tau;\ V)$ *be a generating system over* P. *Let* $\mathfrak{g}$ *be the reduction of* $G\times U$ *to* V *(a locally compact groupoid); due to* (M4), $\mathfrak{g}$ *has a canonical left Haar system. For every* $f \in C_c(G)$ *we denote by* $\tilde{f}$ *the function* $\mathfrak{g} \ni (t, v) \mapsto f(t)$, *which is in* $C_c(\mathfrak{g})$ *because* V *is compact.*

Let us consider a v *in* V; *the Dirac measure* δ_v *on* V *induces a measure* ν^{-1} *on* $\mathfrak{g}$ *(see* 2.11, p. 21, [7]*). There exists a unitary operator* $T: L^2(\nu^{-1}) \to L^2(\mu|G_v)$, T *and* T^* *being described as follows:*

$$\begin{cases} \text{for any } \eta \in C_c(g),\ T\eta = \zeta, \text{ where } \zeta(t) \equiv \eta(t, v); \\ \text{for any } \xi \in C_c(G_v),\ T^*\xi = \eta, \text{ where } \eta(t, w) = \begin{cases} \xi(t) & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases} \end{cases}$$

Moreover we have:

$$T[(\operatorname{Ind}\delta_v)\tilde{f}]T^* = W_{G_v}(f), \quad \forall f \in C_c(G),$$

where $\operatorname{Ind}\delta_v : C_c(G) \to (L^2(\nu^{-1}))$ *is the induced representation* (2.12, p. 22, [7]) *and* $W_{G_v}(f)$ *is the Wiener-Hopf operator with symbol* f *on* G_v.

2.2.4. PROPOSITION. *We keep the same notation as in the preceding lemma. There exists a canonical isometric *-representation* $\pi : C^*_{\mathrm{red}}(\mathfrak{g}) \to \mathscr{L}(L^2(\mu\,|\,P))$ *such that* $\pi(\tilde{f}) = W_P(f)$, $\forall f \in C_c(G)$. $\operatorname{Ran}\pi$ *is called the* C^**-subalgebra of* $\mathscr{L}(L^2(\mu\,|\,P))$ *canonically generated by* $(U, G\times U \to U, \tau;\ V)$.

2.2.5. REMARK. The representation of 2.2.4 is obtained by putting $v = e$ in Lemma 2.2.3 and proving that $\operatorname{Ind}\delta_v$ is isometric on $C^*_{\mathrm{red}}(\mathfrak{g})$ (we note, of course, that $G_{\tau(e)} = P$). It is clear that $\operatorname{Ran}\pi \supseteq \mathscr{W}(P)$. If we obtain for some particular generating system that $\operatorname{Ran}\pi = \mathscr{W}(P)$, then we have a "presentation" for $\mathscr{W}(P)$ as the reduced C^*-algebra of a groupoid (this is what we need).

2.3. The existence of generating systems is equivalent to the Condition (M)

2.3.1. Let us take the particular case when $G = \mathbf{R}^2$ and $P = [0, \infty)^2$. When considering generating systems, we should keep in mind the following example of an action $G \times U \to U$ together with a function $\tau : G \to U$; $G \times G \to G$ is defined to be $(t, u) \mapsto t + u$ and $\tau(t) \equiv t$. Here $\mathrm{clos}(P) = P$ is not compact, i.e. (M2) is not fulfilled; we overcome this difficulty replacing G with its one-point compactification $G \cup \{\infty\}$. The action becomes

$$(t, u) \mapsto \begin{cases} t + u & \text{if } u \in G \\ \infty & \text{if } u = \infty \end{cases};$$

τ remains the same. We have $V = P \cup \{\infty\}$ and it is easily seen that conditions (M1), (M2) and (M3) are satisfied.

For (M4) we have to look at the sets $(G_v)_{v \in V}$ (Observation 2.2.2); they are $G_t = P - t$, $\forall t \in P$ and $G_\infty = G$. Obviously, each G_v is solid. Using the uniform continuity of the functions in $C_c(G)$ we see that $V \ni v \mapsto \chi_{G_v} \in L^\infty(\mu)$ is w*-continuous on $\tau(P) = P$. What remains to be verified is whether this last function is continuous at ∞, or equivalently if:

$$\lim_{\substack{t \to \infty \\ t \in P}} \int_{P-t} f(s)\mathrm{d}s = \int_{\mathbf{R}^2} f(s)\mathrm{d}s, \quad \forall f \in C_c(\mathbf{R}^2). \tag{5}$$

When $t_k \to \infty$ as in Figure 1.1, then (5) takes place because for every f in $C_c(\mathbf{R}^2)$ we have $\mathrm{supp} f \subseteq P - t_k$ for sufficiently large k (in the picture we represent $t_k + Q = -(P - t_k)$ and $-\mathrm{supp} f$). But when $t_k \to \infty$ on a vertical line, as in Figure 1.2, then it may happen for some f that $\int_{P-t_k} f(s)\mathrm{d}s = 0, \forall k$ and still $\int_{\mathbf{R}^2} f(s)\mathrm{d}s \neq 0$.

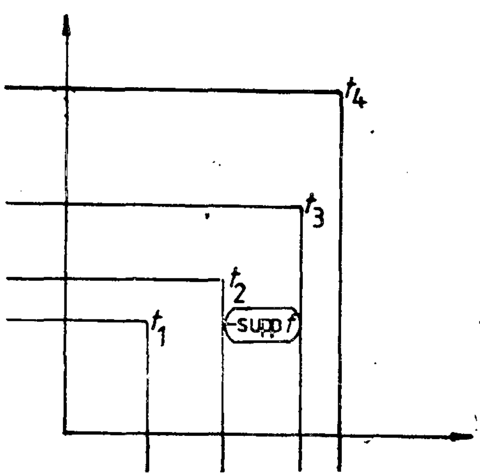

Figure 1.1

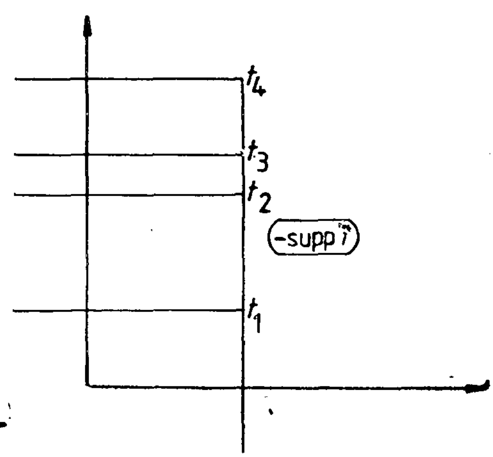

Figure 1.2

So axiom (M4) is the one which causes trouble when we are looking for generating systems. The one-point compactification does not satisfy it because "it has not enough points at infinity". In the general case, writing an arbitrary $v \in V$ as $\lim_\alpha \tau(t_\alpha)$, with $(t_\alpha)_\alpha$ a net in P, we see that it is necessary that $\chi_{P-t_\alpha} \xrightarrow[\alpha]{w^*} \chi_{G_v}$. Hence P must be compactified by adding one point for each "type of convergence to ∞" where the type of convergence to ∞ determined by a net $(t_\alpha)_\alpha$ in P must be connected with w*-lim χ_{P-t_α}. In this way we are led to the construction made in Proposition 2.3.4.

2.3.2. Definition. We say that P *satisfies condition* (M) if every element of w*-clos$\{\chi_{tQ} \mid t \in P\} \subseteq L^\infty(\mu)$ is of the form χ_A with A a closed solid subset of G (recall that $Q = P^{-1}$).

2.3.3. Observation. The set A which appears in the preceding definition is uniquely determined. Indeed, let A and B be closed solid subsets of G such that $\chi_A = \chi_B$ μ-a.e. Then $\mu(A \setminus B) = \mu(B \setminus A) = 0$. But $A \setminus B$ is open in A and $\operatorname{supp} \mu \mid A = A$, hence the equation $\mu(A \setminus B) = 0$ implies $A \setminus B = \emptyset$. Hence $A \subseteq B$ and by symmetry we obtain $A = B$.

2.3.4. Proposition and Definition. *There exist generating systems over P if and only if P satisfies condition* (M). *More precisely (for the sufficiency) if* (M) *is fulfilled we can construct the following generating system, called* **minimal**: *U is the unit ball of $L^\infty(\mu)$ with the w* topology; $G \times U \to U$ is given by $(t, h) \mapsto h \circ L_{t^{-1}}$ where $L_{t^{-1}} : G \to G$ is $L_{t^{-1}}(s) \equiv t^{-1}s$; $\tau : G \to U$ is defined to be $\tau(t) \equiv \chi_{tQ}$.*

Proof. "$\Rightarrow$" Let $(U, G \times U \to U, \tau; V)$ be a generating system over P. An arbitrary element of w*-clos$\{\chi_{tQ} \mid t \in P\}$ can be written as $\lim_\alpha \chi_{t_\alpha Q}$, where $(t_\alpha)_\alpha$ is a net in P for which $v = \lim_\alpha \tau(t_\alpha) \in V$ exists. This happens because V is compact, hence every net in V has a convergent subnet. (M4) implies that $\chi_{Pt_\alpha^{-1}} \xrightarrow[\alpha]{w^*} \chi_{G_v}$ (see also Observation 2.2.2) and using the unimodularity of G we obtain $\chi_{t_\alpha Q} \xrightarrow[\alpha]{w^*} \chi_{G_v^{-1}}$. We may take $A = G_v^{-1}$ (A is solid due to (M4)).

"$\Leftarrow$" We assume (M) satisfied and we prove that the minimal generating system described in the proposition makes sense. It is easily seen that $(t, h) \mapsto h \circ L_{t^{-1}}$ is a continuous action on the left. The verification of (M1) is trivial and (M2) is satisfied because U itself is compact (by the Alaoglu theorem; we also note that U is metrizable because G is supposed to be second countable).

We prove that (M3) is fulfilled. Let t be in G such that $\tau(t) \in V$. Thus there exists a sequence $(t_n)_{n=1}^\infty$ in P such that $\chi_{t_n Q} \xrightarrow[n\to\infty]{w^*} \chi_{tQ}$. It is clear that $t_n Q \supseteq Q$ for every n and this implies $tQ \supseteq Q$ (otherwise $\emptyset \neq Q\setminus tQ$ is open in Q. We consider an open relatively compact subset D of G intersecting $Q\setminus tQ$ and we see that $0 < \mu((Q\setminus tQ) \cap D) < \infty$ hence $\int_{t_nQ} \chi_{(Q\setminus tQ)\cap D}\, d\mu \xrightarrow[n\to\infty]{} \int_{tQ} \chi_{(Q\setminus tQ)\cap D}\, d\mu$, a contradiction). But from $tQ \supseteq Q$ ($\ni e$) we obtain $t^{-1} \in Q$, i.e. $t \in P$.

Now we take (M4). We shall use a

LEMMA. *Let v be an arbitrary element of V and let A be the unique closed solid subset of G with $\chi_A = v$ (see Observation 2.3.3). Then $G_v = A^{-1}$.*

Proof of the lemma. "$\subseteq$" Let t be in G_v. We have $tv = t\chi_A = \chi_{tA} \in V$. Let $(t_n)_{n=1}^{\infty}$ be a sequence in P such that $\chi_{t_nQ} \xrightarrow[n\to\infty]{w^*} \chi_{tA}$. Using the same argument as in the proof of (M3) we see that $tA \supseteq Q$. Hence $tA \ni e$, i.e. $t \in A^{-1}$.

"$\supseteq$" Let t be in A^{-1}. Consider a sequence $(t_n)_{n=1}^{\infty}$ in P such that $\chi_{t_nQ} \xrightarrow[n\to\infty]{w^*} \chi_A$. Then $t\chi_{t_nQ} \xrightarrow[n\to\infty]{w^*} t\chi_A$ (the action of G on U is continuous), that is $\chi_{tt_nQ} \xrightarrow[n\to\infty]{w^*} \chi_{tA}$. We need to show that $\chi_{tA} \in V$ ($\Rightarrow t \in G_v$); to that end we shall prove that the sequence $(tt_n)_{n=1}^{\infty}$ is "asymptotically" in P.

Let $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ be a base at e, every V_n being open and relatively compact. Using an argument of the same type with the one which proved (M3), we see that for every n there exists a k_n such that $m \geqslant k_n$ implies $tt_mQ \cap \cap V_n \neq \emptyset$; of course, we can arrange that $k_1 < k_2 < k_3 < \dots$. So we obtain the subsequence $(t_{k_n})_{n=1}^{\infty}$ of $(t_n)_{n=1}^{\infty}$ with the property that $tt_{k_n}Q \cap V_n \neq \emptyset$, $\forall n$. For every n we take $q_n \in Q$ such that $tt_{k_n}q_n \in V_n$ and define $p_n = q_n^{-1} \in P$. We have $tt_{k_n}q_n \to e$, therefore $P_n(tt_{k_n})^{-1} \xrightarrow[n\to\infty]{} e$; using again the continuity of the action, from $\chi_{tt_{k_n}Q} \xrightarrow[n\to\infty]{w^*} \chi_{tA}$ we get $p_n(tt_{k_n})^{-1}\chi_{tt_{k_n}Q} \xrightarrow[n\to\infty]{w^*} \chi_{tA}$, i.e. $\chi_{p_nQ} \xrightarrow[n\to\infty]{w^*} \chi_{tA}$. Hence χ_{tA} is in V.

From the lemma it is clear that every G_v is solid. Moreover, consider a convergent sequence $v_n \xrightarrow[n\to\infty]{} v$ in V and an $f \in C_c(G)$. Let A and $(A_n)_{n=1}^{\infty}$ be closed solid subsets of G such that $v = \chi_A$ and $v_n = \chi_{A_n}$, $\forall n$. To end the proof we have to show that $\int_{G_{v_n}} f\, d\mu \xrightarrow[n\to\infty]{} \int_{G_v} f\, d\mu$; this is equivalent to $\int_{A_n} (f \circ \mathrm{Inv})\, d\mu \xrightarrow[n\to\infty]{} \int_{A} (f \circ \mathrm{Inv})\, d\mu$ (where $\mathrm{Inv}\colon\ G \to G$ is $\mathrm{Inv}(t) \equiv t^{-1}$). But our hypothesis is $\chi_{A_n} \xrightarrow[n\to\infty]{w^*} \chi_A$; so the last limit does clearly exist.

2.4. THE MINIMAL GENERATING SYSTEM. The next proposition shows that the minimal generating system can be used to study $\mathscr{W}(P)$.

2.4.1. PROPOSITION. *We use the notation of Proposition* 2.2.4 *for the minimal generating system. Then* $\{\tilde{f} \mid f \in C_c(G)\} \subseteq C_c(\mathfrak{g}) \subseteq C^*_{\mathrm{red}}(\mathfrak{g})$ *generates* $C^*_{\mathrm{red}}(\mathfrak{g})$ *as a* C^**-algebra. Consequently* $\operatorname{Ran}\pi$ *(the* C^**-subalgebra of* $\mathscr{L}(L^2(\mu \mid P))$ *canonically associated to the minimal generating system) is* $\mathscr{W}(P)$.

Proposition 2.4.1 is very close to Proposition 3.5, p. 35, [7]; in fact, its proof is just a transcription of the proof of the quoted proposition and that is why we omit it.

In order to justify the term "minimal generating system" we mention (without going into details) that one can define an injective homomorphism from the groupoid associated to the minimal generating system into the groupoid associated to an arbitrary generating system, and that this homomorphism induces a surjective C^*-algebra homomorphism in the opposite direction. We also mention without proof the following uniqueness property:

2.4.2. PROPOSITION. *Let* $(U, G\times U \to U, \tau; V)$ *be a generating system over* P. *We denote by* $\mathfrak{g}'$ *the reduction of* $G\times U$ *to* V *(*$\mathfrak{g}'$ *is a locally compact groupoid with a canonical Haar system). The following are equivalent:*

1° *The* C^**-subalgebra of* $\mathscr{L}(L^2(\mu|P))$ *canonically associated to* $(U, G\times U \to U, \tau; V)$ *is* $\mathscr{W}(P)$.

2° *If* $v_1, v_2 \in V$ *and* $G_{v_1} = G_{v_2}$, *then* $v_1 = v_2$.

3° $\mathfrak{g}'$ *is topologically isomorphic (in the sense of* 2.9, p. 20, [7]*) with the groupoid associated to the minimal generating system.*

Proposition 2.4.2 explains (a posteriori) the numerous similarities between the generating system presented in [7] and the minimal one.

3. ANY CLOSED CONVEX SOLID CONE IN $\mathbf{R}^n$ SATISFIES (M)

3.1. We maintain the notations used in Section 2 with the following specializations: $G = \mathbf{R}^n$ for some $n \in \mathbf{N}^*$ and P has the property that $\lambda P \subseteq P$, $\forall \lambda \in [0, \infty)$, i.e. P is what one usually calls a convex cone. Of course, we shall write the law of composition on G additively. It is not difficult to observe that in this case the condition on P to be solid is tantamount to $\operatorname{sp} P = \mathbf{R}^n$ or equivalently to $\mathring{P} \neq \emptyset$ (see also [1], Lemma 2.1, p. 8). In what follows we shall work with the *dual* of P, which is the closed convex cone:

$$\hat{P} = \{\xi \in \mathbf{R}^n \mid \langle x, \xi\rangle \geq 0, \ \forall x \in P\}. \tag{6}$$

Our purpose is to prove the following statement, at which we arrive in Corollary 3.4.5:

(7) $$\text{“}\{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n,\ A \textit{ closed and convex}\} \textit{ is } \text{w}^*\textit{-compact”}.$$

It is obvious that $t + Q \supseteq Q, \forall t \in P$; so (7) implies $\text{w}^*\text{-clos}\{\chi_{t+Q} \mid t \in P\} \subseteq$ $\subseteq \{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n$, A closed and convex$\}$ which is more than (M). In order to obtain (7) we shall develop another description for $\{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n$, A closed and convex$\}$ where the topology is given by a Hausdorff distance between closed convex cones.

3.2. Hausdorff distances. The left, right and symmetrized Hausdorff distance between two arbitrary compact convex non-void subsets A and B of $\mathbf{R}^n$ are defined respectively by the formulas:

$$\begin{cases} d_{\mathrm{H,l}}(A, B) = \inf\{r > 0 \mid A + r\mathbf{B} \supseteq B\} \\ d_{\mathrm{H,r}}(A, B) = \inf\{r > 0 \mid B + r\mathbf{B} \supseteq A\}\ (= d_{\mathrm{H,l}}(B, A)) \\ d_{\mathrm{H}}(A, B) = \max(d_{\mathrm{H,l}}(A, B),\ d_{\mathrm{H,r}}(A, B)). \end{cases}$$

Here $\mathbf{B}$ stands for the closed unit ball of $\mathbf{R}^n$ with the Euclidean norm. It is known that d_{H} is a metric on the set $\mathscr{K}$ of compact convex non-void subsets of $\mathbf{R}^n$ and that $(\mathscr{K}, d_{\mathrm{H}})$ is complete. Besides, a theorem of Blaschke which we shall repeatedly use says that for any $K \in \mathscr{K}$, the set $\{A \in \mathscr{K} \mid A \subseteq K\}$ is d_{H}-compact. A related (but simpler) result we also mention is that for any closed convex non-void subset F of $\mathbf{R}^n$, $\{A \in \mathscr{K} \mid A \subseteq F\}$ is d_{H}-closed. For the details, the reader may consult [5], Section 7, p. 19—20.

Identifying the closed convex cones in $\mathbf{R}^n$ with their intersections with $\mathbf{B}$, we obtain a metric on the set of closed convex cones; more precisely we put (C_1 and C_2 are arbitrary convex cones):

(8) $$\begin{cases} d_{\mathrm{H,C,l}}(C_1, C_2) = d_{\mathrm{H,l}}(C_1 \cap \mathbf{B}, C_2 \cap \mathbf{B}) \\ d_{\mathrm{H,C,r}}(C_1, C_2) = d_{\mathrm{H,r}}(C_1 \cap \mathbf{B}, C_2 \cap \mathbf{B}) \\ d_{\mathrm{H,C}}(C_1, C_2) = d_{\mathrm{H}}(C_1 \cap \mathbf{B}, C_2 \cap \mathbf{B}). \end{cases}$$

As a corollary to the Blaschke theorem we obtain without difficulty that the set of closed convex cones in $\mathbf{R}^n$ with $d_{\mathrm{H,C}}$ is compact. A useful property of $d_{\mathrm{H,C,l}}$ (and, by symmetry, of $d_{\mathrm{H,C,r}}$) is the following: for every $t \in C_2$ there exists $s \in C_1$ such that $|s - t| \leqslant d_{\mathrm{H,C,l}}(C_1, C_2)|t|$ (C_1 and C_2 are as in (8); this property is obtained considering $\frac{1}{|t|} t \in C_2 \cap \mathbf{B}$ and the definition of $d_{\mathrm{H,l}}(C_1 \cap \mathbf{B},\ C_2 \cap \mathbf{B}) =$ $= d_{\mathrm{H,C,l}}(C_1, C_2)$).

3.3. A first set which can be put into bijection with $\{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n, A$ closed and convex$\}$ is of course $\{A \mid Q \subseteq A \subseteq \mathbf{R}^n, A$ closed and convex$\}$ (this comes out from Observation 2.3.3). Another one is given by the next

PROPOSITION AND DEFINITION. *For any closed convex subset A of $\mathbf{R}^n$ which contains Q we define its* ***support function*** σ_A *to be:*

$$\sigma_A : \hat{P} \to [0, \infty], \quad \sigma_A(\xi) = \sup_{x \in A} \langle x, \xi \rangle, \ \ \forall \xi \in \hat{P}.$$

σ_A *is positively homogeneous (with the convention* $0 \cdot \infty = 0$*), subadditive and lower semicontinuous.*

The mapping $A \mapsto \sigma_A$ *is a bijection between* $\{A \mid Q \subseteq A \subseteq \mathbf{R}^n, A$ *closed and convex*$\}$ *and the set of positively homogeneous, subadditive and lower semicontinuous functions from* $\hat{P}$ *into* $[0, \infty]$. *Its inverse associates to the function* φ *the set:*

$$A = \{x \in \mathbf{R}^n \mid \langle x, \xi \rangle \leqslant \varphi(\xi), \ \ \forall \xi \in \hat{P}\}.$$

The proof of the proposition and the definition of σ_A can be found, with some slight changes, in [9]. We only mention that the main tool used in the proof is the separation (Hahn-Banach) theorem in $\mathbf{R}^n$ and $\mathbf{R}^{n+1}$.

3.4. CONES "OF EPIGRAPH TYPE".

3.4.1. DEFINITIONS. 1° For any $\varphi : \hat{P} \to [0, \infty]$, positively homogeneous, subadditive and lower semicontinuous we define its *epigraph:*

$$\tilde{E}(\varphi) = \{(\xi, \nu) \in \mathbf{R}^{n+1} \mid \nu \geqslant \varphi(\xi)\}.$$

(It is easy to see that $\tilde{E}(\varphi)$ is a closed convex cone in $\mathbf{R}^{n+1}$.)

2° A closed convex cone $\tilde{C} \subseteq \mathbf{R}^{n+1}$ is said to be "*of epigraph type*" if it has the property:

$$\text{"}(\xi, \mu) \in \tilde{C} \text{ and } \nu \geqslant \mu \text{ (a real number)"} \Rightarrow \text{"}(\xi, \nu) \in \tilde{C}\text{"}.$$

3° We denote by $\tilde{\mathscr{E}}$ the set of closed convex cones in $\mathbf{R}^{n+1}$ which are of epigraph type and are contained in $\hat{P} \times [0, \infty)$.

3.4.2. LEMMA. *The mapping* $\varphi \mapsto \tilde{E}(\varphi)$ *is a bijection between the set of positively homogeneous, subadditive and lower semicontinuous functions from* $\hat{P}$ *into* $[0, \infty]$ *and* $\tilde{\mathscr{E}}$. *Its inverse associates to* $\tilde{C} \in \tilde{\mathscr{E}}$ *the function:*

$$\varphi(\xi) = \begin{cases} \inf\{\mu \geqslant 0 \mid (\xi, \mu) \in \tilde{C}\} & \text{if } \exists\, \mu \geqslant 0 \text{ such that } (\xi, \mu) \in \tilde{C} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Clear.

3.4.3. LEMMA. *Let us denote by $\widetilde{d_{\mathrm{H,C}}}$ the (symmetrized) Hausdorff distance on the set of closed convex cones in* $\mathbf{R}^{n+1}$. *Then $\tilde{\mathscr{E}}$ is $\widetilde{d_{\mathrm{H,C}}}$-closed (and hence $\widetilde{d_{\mathrm{H,C}}}$-compact).*

Proof. Let $(\tilde{C}_k)_{k=1}^{\infty}$ and $\tilde{C}$ be closed convex cones in $\mathbf{R}^{n+1}$ such that $\tilde{C}_k \xrightarrow{\widetilde{d_{\mathrm{H,C}}}} \tilde{C}$ and each $\tilde{C}_k$ is in $\tilde{\mathscr{E}}$. Then $\tilde{C}_k \subseteq \hat{P}\times[0,\infty)$, $\forall k \geqslant 1$, implies $\tilde{C} \subseteq \hat{P}\times[0,\infty)$ (see 3.2, the result stated after the Blaschke theorem). Further let (ξ, μ) be in $\tilde{C}$ and consider an arbitrary $\nu > \mu$; we shall prove that $(\xi, \nu) \in \tilde{C}$. We write $|\xi| + \nu = a$ (a will be used as a majorant). Consider an arbitrary $\varepsilon \in (0, 1)$ such that $\varepsilon < \frac{\nu - \mu}{a}$ and a k for which $\widetilde{d_{\mathrm{H,C}}}(\tilde{C}_k, \tilde{C}) < \varepsilon$. There exists $(\xi_k, \mu_k) \in \tilde{C}_k$ such that $|(\xi_k, \mu_k) - (\xi, \mu)| \leqslant \varepsilon|(\xi, \mu)| \leqslant \varepsilon a$. Then $|\mu_k - \mu| \leqslant \varepsilon a$ implies $\mu_k \leqslant \mu + \varepsilon a \leqslant \nu$ and hence $(\xi_k, \nu) \in \tilde{C}_k$ ($\tilde{C}_k$ is of epigraph type). Now there exists $(\eta, \lambda) \in \tilde{C}$ such that $|(\eta, \lambda) - (\xi_k, \nu)| \leqslant \varepsilon\, |(\xi_k, \nu)|$. But $|(\xi_k, \nu)| \leqslant |\xi_k| + \nu \leqslant |\xi| + \nu + |\xi - \xi_k| \leqslant 2a$ and we get $|\lambda - \nu| \leqslant 2\varepsilon a$, $|\eta - \xi| \leqslant |\eta - \xi_k| + |\xi_k - \xi| \leqslant 2\varepsilon a + \varepsilon a = 3\varepsilon a$, and hence $|(\lambda, \eta) - (\xi, \nu)| \leqslant 5\varepsilon a$. Making ε tend to 0 we obtain that $(\xi, \nu) \in \operatorname{clos}\tilde{C} = \tilde{C}$.

3.4.4. PROPOSITION. *The mapping $\tilde{E}(\sigma_A) \mapsto \chi_A$ is continuous from $(\tilde{\mathscr{E}}, \widetilde{d_{\mathrm{H,C}}})$ into $(L^{\infty}(\mathbf{R}^n), \mathrm{w}^*)$.*

Proof. Taking into account Proposition 3.3 and Lemma 3.4.2, we have to prove that if $(\varphi_k)_{k=1}^{\infty}$ and φ are positively homogeneous, subadditive and lower semicontinuous functions from $\hat{P}$ into $[0, \infty]$ such that $\tilde{E}(\varphi_k) \xrightarrow[k\to\infty]{\widetilde{d_{\mathrm{H,C}}}} \tilde{E}(\varphi)$, then, putting $A_k = \{x \in \mathbf{R}^n \mid \langle x, \xi\rangle \leqslant \varphi_k(\xi),\ \forall \xi \in \hat{P}\}$ $\forall k \geqslant 1$ and $A = \{x \in \mathbf{R}^n \mid \langle x, \xi\rangle \leqslant \varphi(\xi),\ \forall \xi \in \hat{P}\}$, we have $\chi_{A_k} \xrightarrow[k\to\infty]{\mathrm{w}^*} \chi_A$. In fact with $(\varphi_k)_{k=1}^{\infty}$, φ, $(A_k)_{k=1}^{\infty}$ and A as above, we shall prove:

(a) $\forall x \in \mathbf{R}^n \setminus A$, $\exists k_0$ such that $x \in \mathbf{R}^n \setminus A_k$, $\forall k \geqslant k_0$; and

(b) $\forall x \in A$, $\exists k_0$ such that $x \in A_k$, $\forall k \geqslant k_0$.

Indeed, (a), (b) and the fact that $\lambda_n(\partial A) = 0$ (where λ_n is the Lebesgue measure on $\mathbf{R}^n$) imply via the dominated convergence theorem that $\chi_{A_k} \xrightarrow[k\to\infty]{\mathrm{w}^*} \chi_A$.

(a) Let x be in $\mathbf{R}^n \setminus A$; there exists $\xi \in \hat{P}$ such that $\langle x, \xi\rangle > \varphi(\xi)$. Consider an $\varepsilon > 0$ such that $\langle x, \xi\rangle - \varphi(\xi) > \varepsilon(1 + |x|)|(\xi, \varphi(\xi))|$ and a k_0 such that $k \geqslant k_0$ implies $\widetilde{d_{\mathrm{H,C}}}(\tilde{E}(\varphi_k), \tilde{E}(\varphi)) < \varepsilon$. We fix an arbitrary $k \geqslant k_0$; there exists $(\xi_k, \nu_k) \in \tilde{E}(\varphi_k)$ for which $|(\xi_k, \nu_k) - (\xi, \varphi(\xi))| \leqslant \varepsilon|(\xi, \varphi(\xi))|$. We have:

$$\varphi_k(\xi_k) \leqslant \nu_k \leqslant \varphi(\xi) + \varepsilon|(\xi, \varphi(\xi))| < \langle x, \xi\rangle - \varepsilon|x|\,|(\xi, \varphi(\xi))| =$$

$$= \langle x, \xi_k\rangle + \langle x, \xi - \xi_k\rangle - \varepsilon|x|\,|(\xi, \varphi(\xi))| \leqslant \langle x, \xi_k\rangle$$

(the last inequality holds because $\langle x, \xi - \xi_k \rangle \leqslant |x|\,|\xi - \xi_k| \leqslant |x|\varepsilon|(\xi, \varphi(\xi))|$). Finally $\varphi_k(\xi_k) < \langle x, \xi_k \rangle$ implies $x \in \mathbf{R}^n \setminus A_k$.

(b) We begin by noticing that for any $x \in \mathring{A}$ we have $\langle x, \xi \rangle < \varphi(\xi)\ \forall \xi \in \in \hat{P} \setminus \{0\}$; indeed for every $\xi \neq 0$ in $\hat{P}$ we have $x + \varepsilon\xi \in A$ for a sufficiently small $\varepsilon > 0$, hence $\langle x, \xi \rangle < \langle x + \varepsilon\xi, \xi \rangle \leqslant \varphi(\xi)$. This implies in turn that for any $x \in \mathring{A}$ there exists a $\delta > 0$ such that $\varphi(\xi) \geqslant \delta|\xi| + \langle x, \xi \rangle, \forall \xi \in \hat{P}$; indeed, we may take $\delta = \inf\{\varphi(\xi) - \langle x, \xi \rangle \mid \xi \in \hat{P}, |\xi| = 1\}$ which is strictly positive being the infimum of a lower semicontinuous strictly positive function on a compact set.

We fix from now on a point $x \in \mathring{A}$ for which we shall prove (b). Obviously, we can find $\theta > 0$ such that $y = (1 + \theta)x \in \mathring{A}$. We apply to y the remark made in the preceding paragraph and we get a $\delta > 0$ such that $\varphi(\xi) \geqslant \delta|\xi| + \langle y, \xi \rangle$, $\forall \xi \in \hat{P}$. We take $0 < \varepsilon < (\min(\delta, \theta))/(1 + \delta + |y|)$ and k_0 such that $\widetilde{d_{\mathrm{H,C}}}(\tilde{E}(\varphi_k), \tilde{E}(\varphi)) < \varepsilon$ for any $k \geqslant k_0$; we consider arbitrary $k \geqslant k_0$ and $\xi \in \hat{P}$ and we shall prove that $\langle x, \xi \rangle \leqslant \varphi_k(\xi)$ (this clearly implies $x \in A_k, \forall k \geqslant k_0$).

If $\varphi_k(\xi) = \infty$ we have nothing to prove, so we shall assume $\varphi(\xi)$ finite. Then $(\xi, \varphi_k(\xi)) \in \tilde{E}(\varphi_k)$ and we can find $(\eta, \nu) \in \tilde{E}(\varphi)$ such that $|(\eta, \nu) - (\xi, \varphi_k(\xi))| \leqslant \leqslant \varepsilon|(\xi, \varphi_k(\xi))|$. We obtain:

$$\varphi_k(\xi) \geqslant \nu - \varepsilon|(\xi, \varphi_k(\xi))| \geqslant \varphi(\eta) - \varepsilon|(\xi, \varphi_k(\xi))|. \tag{9}$$

But we can also write:

$$\varphi(\eta) \geqslant \delta|\eta| + \langle y, \eta \rangle \geqslant \delta|\xi| + \langle y, \xi \rangle - \delta|\xi - \eta| - \langle y, \xi - \eta \rangle \geqslant$$
$$\geqslant \delta|\xi| + \langle y, \xi \rangle - (\delta + |y|)|\xi - \eta| \geqslant$$
$$\geqslant \delta|\xi| + \langle y, \xi \rangle - (\delta + |y|)\varepsilon|(\xi, \varphi_k(\xi))|.$$

Introducing these estimations in (9) we get:

$$\varphi_k(\xi) \geqslant \delta|\xi| + \langle y, \xi \rangle - (\delta + |y|)\varepsilon|(\xi, \varphi_k(\xi))| - \varepsilon|(\xi, \varphi_k(\xi))|$$

and consequently:

$$(1 + \theta)\varphi_k(\xi) = \theta\varphi_k(\xi) + \varphi_k(\xi) \geqslant$$
$$\geqslant \theta\varphi_k(\xi) + \delta|\xi| + \langle y, \xi \rangle - (1 + \delta + |y|)\varepsilon|(\xi, \varphi_k(\xi))|,$$

where the last inequality takes place because $\theta\varphi_k(\xi) + \delta|\xi| \geqslant (\min(\delta, \theta))|(\xi, \varphi_k(\xi))|$ and $\min(\delta, \theta) \geqslant \varepsilon(1 + \delta + |y|)$. Finally we see that $\varphi_k(\xi) \geqslant (1/(1 + \theta))\langle y, \xi \rangle = = \langle x, \xi \rangle$.

Proposition 3.4.4 and Lemma 3.4.3 have the following immediate consequences:

3.4.5. COROLLARY. 1° $\{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n$, *A closed and convex*$\}$ *is* w*-*compact.*

2° $\chi_A \mapsto \tilde{E}(\sigma_A)$ *is a homeomorphism from* $\{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n$, *A closed and convex*$\}$ *with* w* *onto* $(\mathscr{E}, \widetilde{d_{\mathrm{H,C}}})$.

4. WHAT IS w*-CLOS$\{\chi_{t+Q} \mid t \in P\}$?

4.1. We continue to work with the same closed convex solid cone $P \subseteq \mathbf{R}^n$ from Section 3. Taking into account the results of Sections 2 and 3, we have at our disposal a locally compact groupoid with Haar system $(\mathfrak{g}, \lambda)$ such that its reduced C^*-algebra is isomorphic to $\mathscr{W}(P)$. (Because $\mathbf{R}^n$ is amenable, $C^*(\mathfrak{g}, \lambda) = C^*_{\mathrm{red}}(\mathfrak{g}, \lambda)$.) The next step is to obtain information about $\mathscr{W}(P)$ from the structure of $(\mathfrak{g}, \lambda)$. In connection with this, an important fact is that the closed invariant subsets of $\mathfrak{g}^0$ give rise to ideals in the associated C^*-algebra (see 2.16, p. 26, [7] or Proposition 4.5, p. 101, [8]). For the moment we only try to describe explicitely $\mathfrak{g}^0$, which is equal to w*-clos$\{\chi_{t+Q} \mid t \in P\}$ by Proposition 2.3.4. In Section 6 we shall succeed in doing this for P in a large class of cones; in this section we shall only emphasize a class of elements of $\mathfrak{g}^0$ which is connected to the facial structure of $\hat{P}$ (Proposition 4.6.2). In order to make things look natural we shall first present (in Subsection 4.4) a more visible connection between $\mathfrak{g}^0$ and the faces of P, which was put into evidence by Muhly and Renault in [7], and after that we shall extend it. But first of all we need some convexity preliminaries.

4.2. FACES OF A CONVEX SET. We shall use for "face" the following definition ([4], Section 2, p. 31):

4.2.1. DEFINITION. Let C be a convex non-void subset of $\mathbf{R}^n$. A subset F of C is called a *face* of C if it is non-void, convex and extremal (in the sense that $t_1, t_2 \in C$, $\lambda \in (0,1)$ and $(1-\lambda)t_1 + \lambda t_2 \in F$ imply $t_1, t_2 \in F$). We denote the set of faces of C by $\mathscr{F}(C)$.

Clearly $\mathscr{F}(C) \neq \emptyset$ because $\mathscr{F}(C) \ni C$. The intersection of an arbitrary family of faces of C is still a face of C, if it is non-void, and this is why we can define for every non-void subset A of C the face of C generated by A, which is $\bigcap_{\substack{F \in \mathscr{F}(C) \\ F \supseteq A}} F$. We shall use the following well-known result:

4.2.2. PROPOSITION. *Let C be a non-void convex subset of $\mathbf{R}^n$ and let m be the dimension of C ($m = \dim \mathrm{fl}\, C$, where $\mathrm{fl}\, C$ is the affine variety generated by C). Then every $F \in \mathscr{F}(C)$ but C has dimension strictly less than m and is contained in $C \setminus \mathrm{int}_{\mathrm{fl}\, C} C$. In fact,* $\bigcup_{\substack{F \in \mathscr{F}(C) \\ F \neq C}} F$ *equals* $C \setminus \mathrm{int}_{\mathrm{fl}\, C} C$.

4.2.3. COROLLARY. *If C is a non-void closed convex subset of $\mathbf{R}^n$, then all the faces of C are closed. If, moreover, C is a closed convex cone, then all the faces of C are closed convex cones.*

Proof. Assume C closed and fix an $F \in \mathscr{F}(C)$. It is clear that F is a face of $(\mathrm{fl}\, F) \cap C$ and that $\dim F \leqslant \dim((\mathrm{fl}\, F) \cap C) \leqslant \dim(\mathrm{fl}\, F) = \dim F$. The proposition applied to $(\mathrm{fl}\, F) \cap C$ yields $F = (\mathrm{fl}\, F) \cap C$ (a closed set). The rest is trivial.

4.3. DUALITY FOR CLOSED CONVEX CONES. We have already introduced the notion of "the dual of a closed convex cone" (relation (6) in 3.1). We introduce now a somewhat more complicated concept, which will be needed in the proof of 4.6.2.

4.3.1. DEFINITION. Consider a couple $R \subseteq V \subseteq \mathbf{R}^n$ where R is a closed convex cone and V is a linear subspace. *The dual of R in V* is by definition the following closed convex cone contained in V:

$$\mathscr{D}_V(R) = \{\xi \in V \mid \langle x, \xi \rangle \geqslant 0, \ \forall x \in R\}.$$

4.3.2. REMARK. Let R be a closed convex cone in $\mathbf{R}^n$. Among the various "duals" of R, the most important are the extreme ones: $\mathscr{D}_{\mathbf{R}^n}(R)$ $(= \hat{R})$ and $\mathscr{D}_{\mathrm{sp}\, R}(R)$. Every other $\mathscr{D}_V(R)$ can be recaptured from them according to the relations:

$$\mathscr{D}_V(R) = \hat{R} \cap V = \mathscr{D}_{\mathrm{sp}\, R}(R) + (V \ominus \mathrm{sp}\, R).$$

4.3.3. PROPOSITION. *The following relations concerning dualization hold (R, R_1, R_2 are closed cones, V is a linear subspace of $\mathbf{R}^n$):*

1° $R \subseteq V \subseteq \mathbf{R}^n \Rightarrow \mathscr{D}_V(\mathscr{D}_V(R)) = R.$

2° $R \subseteq V \subseteq \mathbf{R}^n \Rightarrow \mathscr{D}_V(-R) = -\mathscr{D}_V(R).$

3° $R_1, R_2 \subseteq V \subseteq \mathbf{R}^n \Rightarrow \mathscr{D}_V(\mathrm{clos}(R_1 + R_2)) = \mathscr{D}_V(R_1) \cap \mathscr{D}_V(R_2)$ *and* $\mathscr{D}_V(R_1 \cap R_2) = \mathrm{clos}(\mathscr{D}_V(R_1) + \mathscr{D}_V(R_2)).$

4° $R_1 \subseteq R_2 \subseteq V \subseteq \mathbf{R}^n \Rightarrow \mathscr{D}_V(R_1) \supseteq \mathscr{D}_V(R_2).$

Proof. See [1], Theorems 2.1 and 2.2, p. 5–6.

We now introduce an operator between $\mathscr{F}(R)$ and $\mathscr{F}(\hat{R})$ which turns out to be very important for our purposes (to motivate its appearance look for instance at Lemma 4.4.2). As in the case of the dualization operation, we shall work in a slightly more general context.

4.3.4. PROPOSITION AND DEFINITION. *Consider a couple $R \subseteq V \subseteq \mathbf{R}^n$ as in Definition 4.3.1. For $F \in \mathscr{F}(R)$, set $\Theta_R^V(F) = \mathscr{D}_V(R) \cap F^{\perp}$. Then $\Theta_R^V(F)$ lies in $\mathscr{F}(\mathscr{D}_V(R))$, and Θ_R^V is a monotone map (relative to containment) from $\mathscr{F}(R)$ into $\mathscr{F}(\mathscr{D}_V(R))$.*

Proof. Immediate.

4.3.5. OBSERVATION. Let R and V be as in 4.3.1. Then $\Theta_R^V(F)$ can also be written:

$$\Theta_R^V(F) = \mathscr{D}_V(R) \cap \mathscr{D}_V(-F), \quad \forall F \in \mathscr{F}(R). \tag{10}$$

Applying $\mathscr{D}_V$ and taking into account point 3° of 4.3.3, we get:

$$\mathscr{D}_V(\Theta_R^V(F)) = \mathrm{clos}(R - F), \quad \forall F \in \mathscr{F}(R). \tag{11}$$

(10) and (11) together yield:

$$\Theta_{\mathscr{D}_V(R)}^V(\Theta_R^V(F)) = R \cap \mathrm{clos}(F - R), \quad \forall F \in \mathscr{F}(R). \tag{12}$$

Indeed we have:

$$\Theta_{\mathscr{D}_V(R)}^V(\Theta_R^V(F)) \overset{(10)}{=} \mathscr{D}_V(\mathscr{D}_V(R)) \cap \mathscr{D}_V(-\Theta_R^V(F)) =$$

$$= R \cap [-\mathscr{D}_V(\Theta_R^V(F))] \overset{(11)}{=} R \cap \mathrm{clos}(F - R).$$

Relation (12) is an explicitation for the operator $\Theta_{\mathscr{D}_V(R)}^V \circ \Theta_R^V$ on $\mathscr{F}(R)$. It is clear from (12) that this operator carries each face of R into a greater one. However, $\Theta_{\mathscr{D}_V(R)}^V \circ \Theta_R^V$ is not always the identity of $\mathscr{F}(R)$ (in 4.5 an example is given when Θ_R^V is not surjective). The next proposition presents two statements weaker than "$\Theta_{\mathscr{D}_V(R)}^V \circ \Theta_R^V = \mathrm{id}\,\mathscr{F}(R)$" which do not fail to be true and are needed in the sequel.

4.3.6. PROPOSITION. *Let R and V be as in* 4.3.1. *Then:*

(13) 1° $\Theta_R^V \circ \Theta_{\mathscr{D}_V(R)}^V \circ \Theta_R^V = \Theta_R^V$.

2° *For any $F \in \mathscr{F}(R)$ we have:* $\Theta_{\mathscr{D}_V(R)}^V(\Theta_R^V(\mathscr{F})) = R \Leftrightarrow F = R$.

Proof. 1° We fix an $F \in \mathscr{F}(R)$ and denote $\Theta_R^V(F)$ by Φ. We have $\Theta_R^V(\Theta_{\mathscr{D}_V(R)}^V(\Phi)) \supseteq \Phi$ (see 4.3.5) and this is "$\supseteq$" in equation (13) applied to F. Applying Θ_R^V to the similar inclusion $\Theta_{\mathscr{D}_V(R)}^V(\Theta_R^V(F)) \supseteq F$ we obtain "$\subseteq$", too.

2° For any $F \in \mathscr{F}(R)$ we have:

$$\Theta_{\mathscr{D}_V(R)}^V(\Theta_R^V(F)) = F \overset{(12)}{\Leftrightarrow} R \cap \mathrm{clos}(F - R) = R \Leftrightarrow \mathrm{clos}(F - R) \supseteq R \overset{(a)}{\Leftrightarrow}$$

$$\overset{(a)}{\Leftrightarrow} \mathrm{clos}(F - R) \supseteq \mathrm{sp}\, R \overset{(b)}{\Leftrightarrow} F - R \supseteq \mathrm{sp}\, R \overset{(c)}{\Leftrightarrow} F = R,$$

where "$\Leftarrow$" of (a), (b), (c) are trivial and "$\Rightarrow$" are proved as follows:

(a) $\operatorname{clos}(F - R)$ is a cone which contains R (by the hypothesis) and $-R$ (obviously); hence it contains $R - R = \operatorname{sp} R$.

(b) It is known that $\operatorname{int}_{\operatorname{sp} R}(F - R) = \operatorname{int}_{\operatorname{sp} R}[\operatorname{clos}(F - R)]$; using the hypothesis we obtain $F - R \supseteq \operatorname{int}_{\operatorname{sp} R}(F - R) \supseteq \operatorname{sp} R$.

(c) Any $t \in R$ can be written as $f - r$ with $f \in F$ and $r \in R$; hence $F \ni (1/2)f = (1/2)(t + r)$ and the extremality of F implies $t \in F$.

4.4. W*-CLOS $\{\chi_{t+Q} \mid t \in P\}$ IS CONNECTED WITH $\mathscr{F}(P)$. In [7], Muhly and Renault emphasized a class of elements in the unit space of the groupoid they used which is "indexed" by $\mathscr{F}(P)$. Keeping in mind that their groupoid is isomorphic to the one associated to the minimal generating system, it is not hard to find the following result:

4.4.1. PROPOSITION. *For any $F \in \mathscr{F}(P)$ and $a \in P$ we have that*

$$\chi_{\operatorname{clos}(a+F+Q)} \text{ is in } \text{w*-clos}\{\chi_{t+Q} \mid t \in P\}.$$

The proof is just a transcription of the one of Proposition 3.11, p. 45, [7], and we omit it. (In fact, Proposition 4.4.1 and 4.4.3 are not used in the sequel, but are introduced to make things look natural.)

Now, the minimal generating system is advantageous because we can find out accurately "by what must we divide" the sets in 4.4.1:

4.4.2. LEMMA. *Let F, G be in $\mathscr{F}(P)$ and a, b be in $\mathbf{R}^n$. Then $\chi_{\operatorname{clos}(a+F+Q)}$ equals $\chi_{\operatorname{clos}(b+G+Q)}$ in $L^\infty(\mathbf{R}^n)$ if and only if $\Theta_P^{\mathbf{R}^n}(F) = \Theta_P^{\mathbf{R}^n}(G) \perp (a - b)$.*

Proof. We shall use the fact that $\chi_{\operatorname{clos}(a+F+Q)} = \chi_{\operatorname{clos}(b+G+Q)}$ in $L^\infty(\mathbf{R}^n)$ if and only if $\operatorname{clos}(a + F + Q) = \operatorname{clos}(b + G + Q)$ (Observation 2.3.3).

"$\Rightarrow$" Denote $\operatorname{clos}(a + F + Q) = \operatorname{clos}(b + G + Q)$ by A. For any $\xi \in \hat{P}$ we have:

$$\sup_{t \in A} \langle t, \xi \rangle = \sup_{f \in F,\, q \in Q} \langle a + f + q, \xi \rangle = \langle a, \xi \rangle + \sup_{f \in F} \langle f, \xi \rangle =$$

$$= \begin{cases} \langle a, \xi \rangle & \text{if } \xi \in \Theta_P^{\mathbf{R}^n}(F) \\ \infty & \text{otherwise.} \end{cases}$$

Analogously we see that:

$$\sup_{t \in A} \langle t, \xi \rangle = \begin{cases} \langle b, \xi \rangle & \text{if } \xi \in \Theta_P^{\mathbf{R}^n}(G) \\ \infty & \text{otherwise.} \end{cases}$$

Therefore $\Theta_P^{\mathbf{R}^n}(F) = \Theta_P^{\mathbf{R}^n}(G) = \{\xi \in \hat{P} \mid \sup_{t \in A} \langle t, \xi \rangle < \infty\}$. In addition, we obtain $\langle a, \xi \rangle = \langle b, \xi \rangle$, $\forall \xi \in \Theta_P^{\mathbf{R}^n}(F) = \Theta_P^{\mathbf{R}^n}(G)$, that is $(a - b) \perp \Theta_P^{\mathbf{R}^n}(F) = \Theta_P^{\mathbf{R}^n}(G)$.

"$\Leftarrow$" It suffices to prove:

(a) $\Theta_P^{\mathbf{R}^n}(F) = \Theta_P^{\mathbf{R}^n}(G) \Rightarrow \operatorname{clos}(F + Q) = \operatorname{clos}(G + Q)$; and

(b) $c \perp \Theta_P^{\mathbf{R}^n}(F) \Rightarrow \operatorname{clos}(c + F + Q) = \operatorname{clos}(F + Q)$.

Indeed, if $\Theta_P^{\mathbf{R}^n}(F) = \Theta_P^{\mathbf{R}^n}(G) \perp (a - b)$, then we get $\operatorname{clos}((a - b) + F + Q) = \operatorname{clos}(F + Q) = \operatorname{clos}(G + Q)$ and it is clear that $\operatorname{clos}(a + F + Q) = \operatorname{clos}(b + G + Q)$.

(a) It clearly suffices to prove "$\subseteq$". Note that $F \subseteq P \cap \operatorname{clos}(F + Q) = \Theta_{\hat{P}}^{\mathbf{R}^n}(\Theta_P^{\mathbf{R}^n}(F)) = \Theta_{\hat{P}}^{\mathbf{R}^n}(\Theta_P^{\mathbf{R}^n}(G)) = P \cap \operatorname{clos}(G + Q) \subseteq \operatorname{clos}(G + Q)$. So $\operatorname{clos}(G + Q)$ is a closed convex cone which contains F and Q, and (a) is done.

(b) It is sufficient to prove "$\subseteq$" (c may be replaced with $-c$). In fact it is sufficient to show that c is in $\operatorname{clos}(F + Q)$ (because $\operatorname{clos}(F + Q)$ is a cone); so what we need is $\Theta_P^{\mathbf{R}^n}(F)^{\perp} \subseteq \operatorname{clos}(F + Q)$. But this comes out from

$$\Theta_P^{\mathbf{R}^n}(F)^{\perp} \subseteq \widehat{\Theta_P^{\mathbf{R}^n}(F)} \overset{(11)}{=} \operatorname{clos}(P - F).$$

Using Lemma 4.4.2 it is easy to establish the following reformulation of Proposition 4.4.1:

4.4.3. Proposition. *We have an injective mapping:*

$$\bigcup_{\Phi \in \operatorname{Ran} \Theta_P^{\mathbf{R}^n} \subseteq \mathscr{F}(\hat{P})} \{\Phi\} \times \operatorname{clos} \operatorname{proj}_{\operatorname{sp} \Phi}(P) \to \text{w*-clos}\{\chi_{t+Q} \mid t \in P\} \tag{14}$$

defined in the following manner: for any $\Phi \in \operatorname{Ran} \Theta_P^{\mathbf{R}^n}$ *and* $a \in \operatorname{clos} \operatorname{proj}_{\operatorname{sp} \Phi}(P)$ *one takes an arbitrary* $F \in \mathscr{F}(P)$ *such that* $\Theta_P^{\mathbf{R}^n}(F) = \Phi$ *and sends* (Φ, a) *into* $\chi_{\operatorname{clos}(a+P+Q)}$.

4.5. Example. Let P be the following cone in $\mathbf{R}^3$:

$$P = \{t = (t_1, t_2, t_3) \in \mathbf{R}^3 \mid t_3 \geqslant \sqrt{t_1^2 + t_2^2},\ t_2 \geqslant 0\}.$$

Its dual is easily computed to be:

$$\hat{P} = \{\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{R}^3 \mid \tau_3 \geqslant \sqrt{\tau_1^2 + [\min(\tau_2, 0)]^2}\};$$

Figure 2 represents the intersection of P (respectively $\hat{P}$) with the plane with equation $t_3 = 1$ (respectively $\tau_3 = 1$).

It is not hard to verify geometrically that besides $\operatorname{Ran}\Theta_P^{\mathbf{R}^n}$, $\mathscr{F}(\hat{P})$ contains the sets $\Phi_+ = \{(\lambda, 0, \lambda) \mid \lambda \in [0, \infty)\}$ and $\Phi_- = \{(-\lambda, 0, \lambda) \mid \lambda \in [0, \infty)\}$. On the other hand, in this particular case one can compute w*-clos$\{\chi_{t+Q} \mid t \in P\}$ by ad-hoc

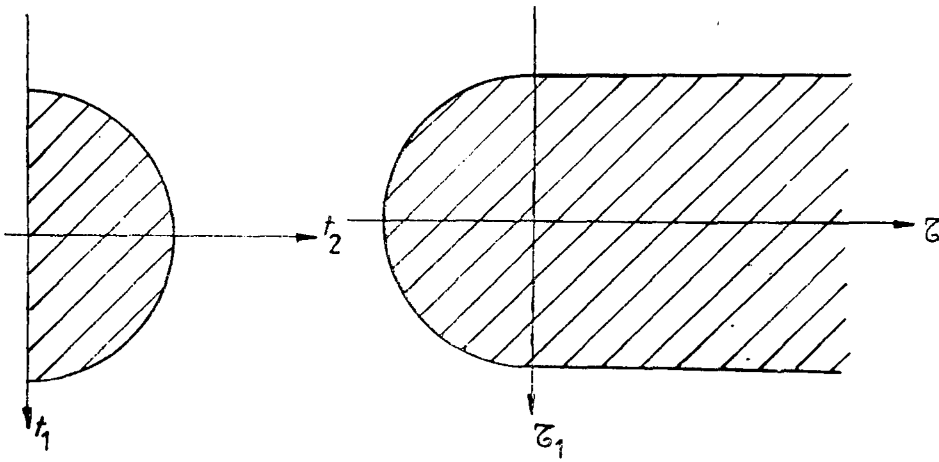

Figure 2

methods and besides the range of the function (14), the following two sets of limits are obtained: $\{\{t \mid t_3 \leqslant c + t_1\} \mid c \geqslant 0\}$ and $\{\{t \mid t_3 \leqslant c - t_1\} \mid c \geqslant 0\}$ (so in this case the sets of type clos$(a + F + Q)$ do not exhaust the unit space of the groupoid). We note that the extra limits are translates of $\hat{\Phi}_-$ and $\hat{\Phi}_+$, respectively. This fact leads to the supposition that the function (14) can be extended to a set where the union is made after $\Phi \in \mathscr{F}(\hat{P})$, (and not only after $\Phi \in \operatorname{Ran}\Theta_P^{\mathbf{R}^n} \subset \mathscr{F}(\hat{P})$).

4.6. w*-CLOS$\{\chi_{t+Q} \mid t \in P\}$ IS CONNECTED WITH $\mathscr{F}(\hat{P})$.

4.6.1. We mention without proof the following remarks:

LEMMA. *If Φ is in $\operatorname{Ran}\Theta_P^{\mathbf{R}^n}$, then:*

1° $\operatorname{clos}\operatorname{proj}_{\mathrm{sp}\,\Phi}(P) = \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$.

2° *For any $F \in \mathscr{F}(P)$ such that $\Theta_P^{\mathbf{R}^n}(F) = \Phi$ and any $a \in \operatorname{clos}\operatorname{proj}_{\mathrm{sp}\,\Phi}(P)$ we have* $\operatorname{clos}(a + F + Q) = a - \hat{\Phi}$.

This lemma enables us to write the function (14) as:

$$\begin{cases} \displaystyle\bigcup_{\Phi \in \operatorname{Ran}\Theta_P^{\mathbf{R}^n}} \{\Phi\} \times \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi) \to \text{w*-clos}\{\chi_{t+Q} \mid t \in P\} \\ (\Phi, a) \mapsto \chi_{a - \hat{\Phi}}\,. \end{cases}$$

So we are led to the next proposition.

4.6.2. PROPOSITION. *We have an injective mapping:*

$$(15)\qquad \begin{cases} \bigcup_{\Phi\in\mathscr{F}(\hat{P})} \{\Phi\}\times\mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)\to \mathrm{w}^*\text{-clos}\{\chi_{t+Q}\mid t\in P\} \\ (\Phi, a)\to\chi_{a-\hat{\Phi}} \end{cases}$$

(which extends the one described in Proposition 4.4.3*).*

Proof. The fact that $\chi_{a-\hat{\Phi}}$ belongs to $\mathrm{w}^*\text{-clos}\{\chi_{t+Q}\mid t\in P\}$ when Φ and a are as in (15) is the difficult part of the proof and is based on three lemmas.

LEMMA 1. *Let C be a closed convex cone in $\mathbf{R}^n$ which is not $\{0\}$. Then "$s<_C t \overset{\mathrm{def}}{\Leftrightarrow} t-s\in C$" defines a preorder relation on* sp*C. In* (sp$C$, $<_C$) *one can find sequences $(t_k)_{k=1}^{\infty}$ which are increasing and cofinal (*i.e., $t_1<_C t_2<_C\ldots$ *and for any $t\in \mathrm{sp}C$ there exists a k such that $t<_C t_k$).*

Proof of Lemma 1. We prove only the last statement. Let t_1 be in $\mathrm{int}_{\mathrm{sp}\,C}C$ and put $t_k=kt_1$, $\forall k$; then clearly $t_1<_C t_2<_C\ldots$. In addition, for any $t\in\mathrm{sp}C$ we have $\lim t_1-\dfrac{1}{k}t=t_1\in\mathrm{int}_{\mathrm{sp}\,C}C$, hence $t_1-\dfrac{1}{k}t\in C$ for sufficiently large k. But $t_1-\dfrac{1}{k}t\in C$ implies $t<_C t_k$.

LEMMA 2. *Let Φ and Φ_1 be in $\mathscr{F}(\hat{P})$ such that $\Phi\subseteq\Phi_1$ and:*

$$(16)\qquad \Theta^{\mathrm{sp}\,\Phi_1}_{\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)}(\Theta^{\mathrm{sp}\,\Phi_1}_{\Phi_1}(\Phi))=\Phi.$$

Then we have:

$$\mathrm{w}^*\text{-clos}\{\chi_{a-\hat{\Phi}}\mid a\in\mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)\}\subseteq\mathrm{w}^*\text{-clos}\{\chi_{a_1-\hat{\Phi}_1}\mid a_1\in\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)\}.$$

Proof of Lemma 2. We shall assume $\Phi\neq\Phi_1$ ($\Phi=\Phi_1$ is trivial). We put $G=\Theta^{\mathrm{sp}\,\Phi_1}_{\Phi_1}(\Phi)\in\mathscr{F}(\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1))$; G is not $\{0\}$ because $G=\{0\}$ and (16) imply $\Phi=\Phi_1$. We apply Lemma 1 to G and we get a sequence $(g_k)_{k=1}^{\infty}$ which is increasing and cofinal in (sp$G$, $<_G$). It is easy to see that for any $g', g''\in G$ we have: $g'<_G g''\Rightarrow$ $\Rightarrow g'-\hat{\Phi}_1\subseteq g''-\hat{\Phi}_1$. This implies $\bigcup_{k=1}^{\infty} g_k-\hat{\Phi}_1=G-\hat{\Phi}_1$ (and of course $g_1-\hat{\Phi}_1\subseteq g_2-\hat{\Phi}_1\subseteq\ldots$). Applying the dominated convergence theorem we get $\chi_{g_k-\hat{\Phi}_1}\xrightarrow[k\to\infty]{\mathrm{w}^*}\chi_{\mathrm{clos}(G-\hat{\Phi}_1)}$. But $\Phi=\Theta^{\mathrm{sp}\,\Phi_1}_{\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)}(G)=\Phi_1\cap-\hat{G}$, therefore $\hat{\Phi}=$ $=\mathrm{clos}(\hat{\Phi}_1-G)$ by point 3° of 4.3.3 and what we have obtained is $\chi_{g_k-\hat{\Phi}_1}\xrightarrow[k\to\infty]{\mathrm{w}^*}\chi_{-\hat{\Phi}}$.

Now we consider an arbitrary $a_1 \in \mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)$. We denote $\mathrm{proj}_{\mathrm{sp}\,\Phi}(a_1)$ by a; it is clear that $a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$. Let us also remark that $a_1 - \hat{\Phi} = a - \hat{\Phi}$ (because $a - a_1 \in \Phi^{\perp} = \hat{\Phi} \cap (-\hat{\Phi})$). $\chi_{g_k - \hat{\Phi}_1} \xrightarrow[k\to\infty]{w^*} \chi_{-\hat{\Phi}}$ implies $\chi_{(g_k + a_1) - \hat{\Phi}} \xrightarrow[k\to\infty]{w^*} \chi_{a_1 - \hat{\Phi}} = \chi_{a - \hat{\Phi}}$ and so we see that:

$$(17) \qquad \chi_{a - \hat{\Phi}} \in \text{w*-clos}\{\chi_{a_1 - \hat{\Phi}_1} \mid a_1 \in \mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)\}, \qquad \forall a \in \mathrm{proj}_{\mathrm{sp}\,\Phi}(\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)).$$

A simple procedure of passing to the limit yields that (17) extends to the case when $a \in \mathrm{clos}\,\mathrm{proj}_{\mathrm{sp}\,\Phi}(\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1))$. But using again the relation $\hat{\Phi} = \mathrm{clos}(\hat{\Phi}_1 - G)$ we can show that this last set is the whole $\mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$. Indeed, for any $t \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi) = \hat{\Phi} \cap \mathrm{sp}\,\Phi = \mathrm{clos}(\hat{\Phi}_1 - G) \cap \mathrm{sp}\,\Phi$ we may consider two sequences, $(s_k)_{k=1}^{\infty}$ in $\hat{\Phi}_1$ and $(h_k)_{k=1}^{\infty}$ in G such that $s_k - h_k \xrightarrow[k\to\infty]{} t$ and we may project this limit onto $\mathrm{sp}\,\Phi$. We obtain $\mathrm{proj}_{\mathrm{sp}\,\Phi}\, s_k \xrightarrow[k\to\infty]{} t$ (because $G \perp \mathrm{sp}\,\Phi$), hence $\mathrm{proj}_{\mathrm{sp}\,\Phi}(\mathrm{proj}_{\mathrm{sp}\,\Phi_1} s_k) \xrightarrow[k\to\infty]{} t$; but $\mathrm{proj}_{\mathrm{sp}\,\Phi_1} s_k$ is in $\mathscr{D}_{\mathrm{sp}\,\Phi_1}(\Phi_1)$ for every k.

LEMMA 3. *Let Φ be a face of $\hat{P}$ which is not $\hat{P}$. We can find the faces Φ_0, $\Phi_1, \ldots, \Phi_m$ of $\hat{P}$ such that $\hat{P} = \Phi_0 \underset{\neq}{\supset} \Phi_1 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} \Phi_m = \Phi$ and such that:*

$$(18) \qquad \Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\mathscr{D}_{\mathrm{sp}\,\Phi_{k-1}}(\Phi_{k-1})}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\Phi_{k-1}}(\Phi_k)) = \Phi_k, \qquad \forall\, 1 \leqslant k \leqslant m.$$

Proof of Lemma 3. We shall build $(\Phi_k)_k$ recursively. First of all we put $\Phi_0 = \hat{P}$. Let us suppose further that for a certain $k \geqslant 1$ we have built $\Phi_{k-1} \in \mathscr{F}(\hat{P})$ such that $\Phi_{k-1} \supseteq \Phi$. If $\Phi_{k-1} = \Phi$, then we stop; if not, we define

$$(19) \qquad \Phi_k = \Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\mathscr{D}_{\mathrm{sp}\,\Phi_{k-1}}(\Phi_{k-1})}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\Phi_{k-1}}(\Phi)) \in \mathscr{F}(\Phi_{k-1}).$$

It is clear that $\Phi_k \in \mathscr{F}(P)$ (because $\mathscr{F}(\Phi_{k-1}) \subseteq \mathscr{F}(P)$) and that $\Phi_k \supseteq \Phi$. In this way we construct a (finite or infinite) sequence $(\Phi_k)_k$ of faces of $\hat{P}$, all of them containing Φ. The manner in which we started assures us that at least Φ_1 is constructed.

We now observe that for any k for which Φ_{k-1} and Φ_k are constructed, we have $\Phi_{k-1} \supseteq \Phi_k$ because $\Phi_k \in \mathscr{F}(\Phi_{k-1})$. If Φ_k were equal to Φ_{k-1}, then (19) and point 2° of 4.3.6 would imply $\Phi_{k-1} = \Phi$, a contradiction. This observation implies that the sequence we have built is finite (because for any k for which

Φ_{k-1} and Φ_k exist, we have $\dim \Phi_k < \dim \Phi_{k-1}$, according to Proposition 4.2.2), hence it is of the form $\hat{P} = \Phi_0 \underset{\neq}{\supset} \Phi_1 \underset{\neq}{\supset} \dots \underset{\neq}{\supset} \Phi_m = \Phi$.

Finally we see that for any $1 \leqslant k \leqslant m$:

$$\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\mathscr{D}_{\mathrm{sp}\,\Phi_{k-1}}(\Phi_{k-1})}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\Phi_{k-1}}(\Phi_k)) \overset{(19)}{=}$$

$$\overset{(19)}{=} \Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\mathscr{D}_{\mathrm{sp}\,\Phi_{k-1}}(\Phi_{k-1})}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\Phi_{k-1}}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\mathscr{D}_{\mathrm{sp}\,\Phi_{k-1}}(\Phi_{k-1})}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\Phi_{k-1}}(\Phi)))) \overset{(13)}{=}$$

$$\overset{(13)}{=} \Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\mathscr{D}_{\mathrm{sp}\,\Phi_{k-1}}(\Phi_{k-1})}(\Theta^{\mathrm{sp}\,\Phi_{k-1}}_{\Phi_{k-1}}(\Phi)) \overset{(19)}{=} \Phi_k .$$

Now it is easy to show that $\chi_{a-\hat{\Phi}} \in \text{w}^*\text{-clos}\{\chi_{t+Q} \mid t \in P\}$ for every $\Phi \in \mathscr{F}(P)$ and $a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$. Indeed, ($\Phi$ being fixed) we consider a sequence $\hat{P} = \Phi_0 \underset{\neq}{\supset} \Phi_1 \underset{\neq}{\supset} \dots \dots \underset{\neq}{\supset} \Phi_m = \Phi$ as in Lemma 3 and we apply Lemma 2 m times.

It remains to prove that if Φ, Ψ are in $\mathscr{F}(\hat{P})$ and $a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$, $b \in \mathscr{D}_{\mathrm{sp}\,\Psi}(\Psi)$ are such that $a - \hat{\Phi} = b - \hat{\Psi}$, then $\Phi = \Psi$ and $a = b$. Here we use

LEMMA 4. *Let C be a closed convex cone in $\mathbf{R}^n$. If $s \in \mathbf{R}^n$ is such that $s + C$ is still a cone, then $s + C = C$ and $s \in C \cap (-C)$.*

Proof of Lemma 4. $0 \in s + C \Rightarrow s \in -C$; on the other hand $0 \in C \Rightarrow \Rightarrow s \in s + C \Rightarrow 2s \in s + C \Rightarrow s \in C$, hence $s \in C \cap (-C)$. But then $s + C \subseteq C$, $-s + C \subseteq C$ and therefore $s + C = C$.

From $a - \hat{\Phi} = b - \hat{\Psi}$, which can be also written $\hat{\Phi} = (a - b) + \hat{\Psi}$, and Lemma 4, we deduce that $\hat{\Phi} = \hat{\Psi}$ and $a - b \in \hat{\Phi} \cap (-\hat{\Phi}) - \Phi^{\perp}$. But $\hat{\Phi} = \hat{\Psi}$ implies $\Phi = \Psi$ (point 1° of 4.3.3). Finally it is clear that $a - b \in \mathrm{sp}\,\Phi$ and $a - b \in \Phi^{\perp}$ hence $a = b$.

5. TAME POINTED CONES IN $\mathbf{R}^n$

5.1. POINTED CONES AND CUTS THROUGH THEM. A closed convex cone $C \subseteq \mathbf{R}^n$ is said to be *pointed* if $C \cap (-C) = \{0\}$ (see [1], Section 2, p. 7). It is easy to see that "pointed" is the dual notion for "solid". Indeed, let P be a closed convex cone in $\mathbf{R}^n$; then $\hat{P} \cap (-\hat{P}) = P^{\perp}$ and this is why "$\hat{P}$ is pointed" is equivalent to "$\mathrm{sp}\,P = \mathbf{R}^n$", i.e. to "$P$ is solid".

5.1.1. PROPOSITION AND DEFINITION. *Let C be a closed convex pointed cone in $\mathbf{R}^n$.*

1° *For any* $x \in \hat{C}$ *we have:* $x \in \mathring{C} \Leftrightarrow \langle x, \xi \rangle > 0,\ \forall \xi \in C \setminus \{0\}$.

2° *We assume* $C \neq \{0\}$. *For any* $x \in \mathring{C}$, *the set:*

$$\Sigma_x = \{\xi \in C \mid \langle x, \xi \rangle = 1\}$$

is called ***the cut through C determined by*** x. Σ_x *is convex and compact, with* $\dim \Sigma_x = (\dim C) - 1$ *and has the property that* $\{\lambda\xi \mid \lambda \in [0, \infty), \xi \in \Sigma_x\} = C$.

Proof. 1° "$\Rightarrow$" For any $\xi \in C \setminus \{0\}$ we have $x - \varepsilon\xi \in \hat{C}$ for a sufficiently small $\varepsilon > 0$. Then $\langle x - \varepsilon\xi, \xi \rangle \geqslant 0$, i.e. $\langle x, \xi \rangle \geqslant \varepsilon|\xi|^2 > 0$.

"$\Leftarrow$" If $C = \{0\}$, it is clear; if not, then $\inf\{\langle x, \xi \rangle \mid \xi \in C, |\xi| = 1\} \equiv \delta$ is strictly positive (the infimum of a continuous function on a compact set is attained). Then $\langle x, \xi \rangle \geqslant \delta|\xi|, \forall \xi \in C$ and consequently $\{y \in \mathbf{R}^n \mid |y - x| \leqslant \delta\} \subseteq C$.

2° We saw in 1°, part "$\Leftarrow$", that there exists $\delta > 0$ such that $\langle x, \xi \rangle \geqslant \delta|\xi|, \forall \xi \in C$. Then Σ_x is contained in $\{\xi \in \mathbf{R}^n \mid |\xi| \leqslant 1/\delta\}$, i.e. it is bounded. The other stated properties of Σ_x are immediate.

5.1.2. OBSERVATION. The study of a pointed cone can be reduced to that of a cut through it. Let us consider for instance the facial structure. If $C \subseteq \mathbf{R}^n$ is a closed convex pointed cone then it is easy to see that $0 \in \mathbf{R}^n$ is an extreme point of C (and in fact the only extreme point). Now let us consider an $x \in \mathring{C}$ and the cut Σ_x through C. It is not hard to prove that there exists a bijection between $\mathscr{F}(C)\setminus\{\{0\}\}$ and $\mathscr{F}(\Sigma_x)$, described as follows:

$$\begin{cases} \mathscr{F}(C)\setminus\{\{0\}\} \ni \Phi \mapsto \Phi \cap \Sigma_x \in \mathscr{F}(\Sigma_x) \\ \mathscr{F}(\Sigma_x) \ni \Gamma \mapsto \{\lambda\xi \mid \lambda \in [0, \infty), \xi \in \Gamma\} \in \mathscr{F}(C)\setminus\{\{0\}\}. \end{cases}$$

This bijection is a homeomorphism when $\mathscr{F}(C)\setminus\{\{0\}\}$ and $\mathscr{F}(\Sigma_x)$ are considered with the Hausdorff metrics $d_{\mathrm{H},C}$ and d_{H} respectively. In fact it can be shown that for any $\Phi, \Psi \in \mathscr{F}(C)\setminus\{\{0\}\}$ we have:

$$d_{\mathrm{H},C}(\Phi, \Psi) \leqslant 2|x| d_{\mathrm{H}}(\Phi \cap \Sigma_x, \Psi \cap \Sigma_x),$$

$$d_{\mathrm{H}}(\Phi \cap \Sigma_x, \Psi \cap \Sigma_x) \leqslant (2/\delta)(1 + |x|/\delta) d_{\mathrm{H},C}(\Phi, \Psi),$$

where in the last relation δ is $\inf\{\langle x, \xi \rangle \mid \xi \in C, |\xi| = 1\}$ which appears in the proof of 5.1.1.

5.2. TAME CLOSED CONVEX SETS. The term "tame" is ad-hoc. It is justified by the fact that cones for which the dual is not tame seem to be "wild" indeed as far as Wiener-Hopf operators are concerned.

5.2.1. DEFINITION. A closed convex set $C \subset \mathbf{R}^n$ is said to be *tame* if it is non-void and the mapping:

$$(20) \qquad [0,1] \times C \times C \to C, \quad (\lambda, \xi, \eta) \mapsto (1-\lambda)\xi + \lambda\eta$$

is open (C is taken with its natural topology and $[0,1]\times C\times C$ with the product topology).

5.2.2. OBSERVATIONS. 1° Generally, in order to prove that C is tame we show that the mapping (20) is open at every $(\lambda, \xi, \eta) \in [0,1] \times C \times C$. The fact that (20) is open at a (fixed) (λ, ξ, η) has a useful expression in terms of sequences. More precisely, the map is open at (λ, ξ, η) if and only if for any sequence $(\zeta_k)_{k=1}^{\infty}$ in C, convergent to $\zeta = (1-\lambda)\xi + \lambda\eta$, there exist a k_0 and sequences $(\xi_k)_{k=k_0}^{\infty}$, $(\eta_k)_{k=k_0}^{\infty}$ (in C) and $(\lambda_k)_{k=k_0}^{\infty}$ (in $[0,1]$) such that $\xi_k \xrightarrow[k\to\infty]{} \xi$, $\eta_k \xrightarrow[k\to\infty]{} \eta$, $\lambda_k \xrightarrow[k\to\infty]{} \lambda$ and $(1-\lambda_k)\xi_k + \lambda_k\eta_k = \zeta_k$, $\forall k \geq k_0$.

2° It is obvious from the definition that the property of being tame is invariant under affine isomorphisms.

Routine verifications involving the criterion with sequences (point 1° of 5.2.2) yield the following useful results:

5.2.3. LEMMA. *Let C be a non-void closed convex subset of $\mathbf{R}^n$.*

1° *For any $\zeta \in C$,*

$$\{(\lambda, \xi, \eta) \mid (1-\lambda)\xi + \lambda\eta = \zeta \text{ and (20) is open at } (\lambda, \xi, \eta)\}$$

is a closed subset of $[0,1] \times C \times C$.

2° *The mapping (20) is open at $(\lambda, \xi, \eta) \in [0,1] \times C \times C$ if and only if for any $\varepsilon > 0$ the set:*

$$\{(1-\lambda')\xi' + \lambda'\eta' \mid \lambda' \in [0,1], \xi', \eta' \in C, |\xi' - \xi| \leq \varepsilon, |\eta' - \eta| \leq \varepsilon\} =$$

$$= \mathrm{co}(\{\xi' \in C \mid |\xi' - \xi| \leq \varepsilon\} \cup \{\eta' \in C \mid |\eta' - \eta| \leq \varepsilon\})$$

is a neighbourhood of $(1-\lambda)\xi + \lambda\eta$ in C.

Let us also notice that the mapping (20) is in fact open at "almost all" triplets (λ, ξ, η):

5.2.4. PROPOSITION. *The mapping (20) is open at every (λ, ξ, η) for which $\lambda \in \{0,1\}$, or $\xi = \eta$, or $(1-\lambda)\xi + \lambda\eta \in \mathring{C}$.* (N.B.: *The last alternative takes place whenever $\lambda \in (0,1)$ and either ξ or η is in $\mathring{C}$; taking into account Observation 2°, 5.3.2, we may always assume that $\mathring{C} \neq \emptyset$ — and hence that $C = \mathrm{clos}\,\mathring{C}$*).

Proof. The proof of the first two alternatives is immediate. When proving the third one, we shall assume that $\lambda \in (0, 1)$ and $\xi \neq \eta$. Using the fact that the open segment determined by a point of C and one of $\mathring{C}$ is in $\mathring{C}$, we deduce from $(1 - \lambda)\xi + \lambda\eta \in \mathring{C}$ that $(1 - \mu)\xi + \mu\eta \in \mathring{C}$, $\forall \mu \in (0, 1)$. It is clear that we can find $\xi_k \xrightarrow[k\to\infty]{} \xi$ and $\eta_k \xrightarrow[k\to\infty]{} \eta$ on the open segment determined by ξ and η such that $(1 - \lambda)\xi_k + \lambda\eta_k = (1 - \lambda)\xi + \lambda\eta$, $\forall k$; thus point 1° of 5.2.4 implies that we may assume from the beginning that $\xi \in \mathring{C}$.

Consider an $r > 0$ such that $\{\xi' \in \mathbf{R}^n \mid |\xi' - \xi| < r\} \subseteq C$. Then for any $0 < \varepsilon < r$ we have that

$$\{(1 - \lambda')\xi' + \lambda'\eta' \mid \lambda' \in [0, 1],\ \xi', \eta' \in C,\ |\xi' - \xi| \leqslant \varepsilon,\ |\eta' - \eta| \leqslant \varepsilon\} \supseteq$$
$$\supseteq \{(1 - \lambda)\xi' + \lambda\eta \mid \xi' \in C,\ |\xi' - \xi| \leqslant \varepsilon\} =$$
$$= \{\zeta' \in \mathbf{R}^n \mid |\zeta' - (1 - \lambda)\xi - \lambda\eta| \leqslant (1 - \lambda)\varepsilon\}$$

is a neighbourhood of $(1 - \lambda)\xi + \lambda\eta$ even in $\mathbf{R}^n$. Finally, point 2° of 5.2.3 applies.

5.3. Closed convex tame pointed cones.

5.3.1. Lemma. *Let C be a closed convex pointed cone in $\mathbf{R}^n$ which is not $\{0\}$. Let x be in $\mathring{C}$ and denote by Σ_x the cut through C determined by x. Then C is tame if and only if so is Σ_x.*

Proof. It is a routine check, using the criterion with sequences.

5.3.2. Lemma. *Let C be a non-void compact convex subset of $\mathbf{R}^n$.*
1° *If every point of $C \setminus \mathrm{int}_{\mathrm{ri}\,C} C$ is an extreme point of C then C is tame.*
2° *If $\dim C \leqslant 2$, then C is tame.*

Proof. 1° Taking into account Observation 2°, 5.2.2, we may assume that $\dim C = n$, and the hypothesis becomes that every point of ∂C is an extreme point of C. In this case it is obvious that every $(\lambda, \xi, \eta) \in [0, 1] \times C \times C$ satisfies the hypotheses of Proposition 5.2.4.

2° If $\dim C \leqslant 1$, then C satisfies the hypothesis of 1°, so we shall assume $\dim C = 2$. Using again 5.2.2 we may also assume that $n = 2$.

We fix an arbitrary $(\lambda, \xi, \eta) \in [0, 1] \times C \times C$ at which we shall prove that (20) is open. Due to Proposition 5.2.4 we may assume that $\lambda \in (0, 1)$, $\xi \neq \eta$ and $(1 - \lambda)\xi + \lambda\eta \equiv \zeta \in \partial C$. Let Φ be the face of C generated by ζ (see 4.2). We have $\dim \Phi \leqslant 1$ because $\zeta \in \partial C$ (Proposition 4.2) and $\dim \Phi \geqslant 1$ because ζ is not an extreme point (recall that $\lambda \in (0, 1)$, $\xi \neq \eta$); hence $\dim \Phi = 1$ and Φ is the closed segment defined by some ξ_0 and η_0 in C. Using point 1° of 5.2.3 we may assume that ξ and η lie in the open segment defined by ξ_0 and η_0.

It is easy to see that C cannot meet both open half-planes determined by the line through ξ_0 and η_0 (Figure 3.1). So C is contained in one of the closed

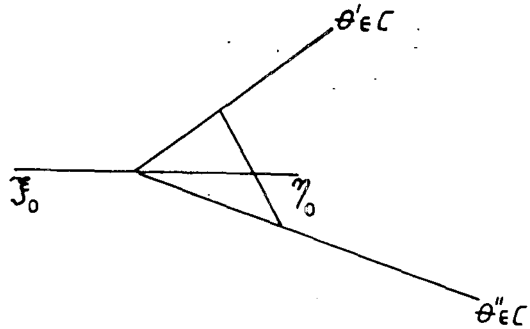

Figure 3.1

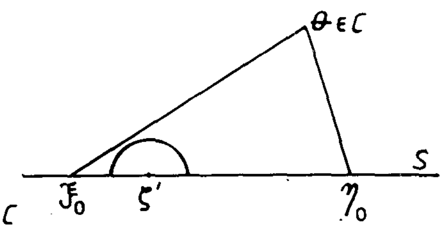

Figure 3.2

half-planes determined by this line; we denote this half-plane by S. Another fact we observe easily is that for any ζ' in the open segment defined by ξ_0 and η_0 there exists $\varepsilon > 0$ such that $C \cap \{\zeta'' \in \mathbf{R}^2 \mid |\zeta'' - \zeta| \leqslant \varepsilon\} = S \cap \{\zeta'' \in \mathbf{R}^2 \mid |\zeta'' - \zeta'| \leqslant \varepsilon\}$ (Figure 3.2). This implies that for sufficiently small $\varepsilon > 0$ we have $\mathrm{co}(\{\xi' \in C \mid |\xi' - \xi| \leqslant \varepsilon\} \cup \{\eta' \in C \mid |\eta' - \eta'| \leqslant \varepsilon\}) \supseteq \{\zeta'' \in C \mid |\zeta'' - \zeta'| \leqslant \varepsilon\}$ and we only have to apply point 2° of 5.2.3.

5.3.3. Proposition. 1° *Any pointed polyhedral cone is tame.*

2° *Any closed convex pointed cone with dimension* $\leqslant 3$ *is tame.*

3° *The forward light cone in any dimension is tame.*

Remark. A (closed) convex cone C is polyhedral if it is finitely generated, i.e. $C = \left\{\sum_{j=1}^{r} \lambda_j \xi_j \mid \lambda_1, \ldots, \lambda_r \geqslant 0\right\}$ for some $\xi_1, \ldots, \xi_r$ in $\mathbf{R}^n$. We shall use the fact that the dual of a polyhedral cone is still polyhedral; that is, a polyhedral cone can also be written $C = \{\xi \in \mathbf{R}^n \mid \langle x_j, \xi\rangle \geqslant 0,\ \forall\, 1 \leqslant j \leqslant s\}$ for some $x_1, \ldots, x_s \in \mathbf{R}^n$ (generators of $\hat{C}$). For a reference see [6], Theorem 2.12, p. 83.

Proof. 2° and 3° result from Lemmas 5.3.1 and 5.3.2. For 1°, let us consider a pointed polyhedral cone $C \subseteq \mathbf{R}^n$. We shall assume $\dim C \geqslant 2$ (we may assume even $\dim C \geqslant 4$, because of 2°). Due to Proposition 5.2.4, it suffices to show that the mapping (20) is open at a fixed $(\lambda, \xi, \eta) \in [0, 1] \times C \times C$ with $\lambda \in (0, 1)$ and $\xi \neq \eta$. Making a rotation (which is an affine isomorphism) we may assume that ξ and η give the direction of the n-th axis of the coordinate system, i.e. that $\xi^0 = \eta^0$, where for instance $\xi^0 \in \mathbf{R}^{n-1}$ is obtained from ξ by deletion of the n-th component. Using point 1° of 5.2.3 we reduce ourselves to proving that (20) is open at a (λ, ξ', η') with ξ' and η' in the open segment defined by ξ and η, "very close to" ξ and η, respectively, and such that $(1 - \lambda)\xi' + \lambda\eta' = (1 - \lambda)\xi + \lambda\eta \equiv \zeta$ (ξ' and η' are fixed from now on).

Let $x_1, \ldots, x_s$ in $\mathbf{R}^n$ be such that $C = \{\xi \in \mathbf{R}^n \mid \langle x_j, \xi \rangle \geqslant 0, \forall 1 \leqslant j \leqslant s\}$. We may arrange that $|x_1| = \ldots = |x_s| = 1$. For any $1 \leqslant j \leqslant s$ we have $\langle x_j, \xi' \rangle = 0$ if and only if $\langle x_j, \eta' \rangle = 0$; for instance it is not possible that $\langle x_j, \xi' \rangle = 0$ and $\langle x_j, \eta' \rangle > 0$ because $\zeta' \mapsto \langle x_j, \zeta' \rangle$ is affine on the closed segment defined by ξ and η and we would obtain $\langle x_j, \xi \rangle < 0$ (absurd). So we can write $\{1, \ldots, s\} = S_1 \cup S_2$ such that $\langle x_j, \xi' \rangle = \langle x_j, \eta' \rangle = 0$, $\forall j \in S_1$ and $\langle x_j, \xi' \rangle > 0$, $\langle x_j, \eta' \rangle > 0$, $\forall j \in S_2$.

For any ζ' in the closed segment defined by ξ' and η' we consider the closed onvex subset of $\mathbf{R}^{n-1}$:

$$M_{\zeta'} = \{\tau \in \mathbf{R}^{n-1} \mid (\tau, 0) + \zeta' \in C\};$$

it is clear that $M_{\zeta'} \ni 0$ and that $\tau \leftrightarrow (\tau, 0) + \zeta'$ is an affine isomorphism between $M_{\zeta'}$ and $C \cap \{\theta \in \mathbf{R}^n \mid \theta_n = \zeta'_n\}$. The key of the proof is the fact that if $\varepsilon > 0$ is such that:

$$\varepsilon < \min\{\langle x_j, \xi' \rangle, \langle x_j, \eta' \rangle\}, \quad \forall j \in S_2,$$

then $M_{\zeta'} \cap \{\tau \in \mathbf{R}^{n-1} \mid |\tau| \leqslant \varepsilon\}$ does not depend on ζ'. Indeed we have:

$$\tau \in M_{\zeta'} \Leftrightarrow \langle x_j, (\tau, 0) + \zeta' \rangle \geqslant 0, \quad \forall 1 \leqslant j \leqslant s \Leftrightarrow$$

$$\Leftrightarrow \langle x_j^0, \tau \rangle \geqslant - \langle x_j, \zeta' \rangle \quad \forall 1 \leqslant j \leqslant s.$$

If $|\tau| \leqslant \varepsilon$ and $j \in S_2$, then $\langle x_j, \zeta' \rangle \geqslant \min\{\langle x_j, \xi' \rangle, \langle x_j, \eta' \rangle\} > \varepsilon$, hence we automatically get $\langle x_j^0, \tau \rangle \geqslant -|x_j^0| \, |\tau| \geqslant -\varepsilon \geqslant -\langle x_j, \zeta' \rangle$. So we obtain that:

$$\tau \in M_{\zeta'} \text{ and } |\tau| \leqslant \varepsilon \Leftrightarrow \langle x_j^0, \tau \rangle \geqslant 0, \quad \forall j \in S_1 \text{ and } |\tau| \leqslant \varepsilon.$$

At this moment it is geometrically clear that for sufficiently small $\varepsilon > 0$ we have $\mathrm{co}(\{\xi'' \in C \mid |\xi'' - \xi'| \leqslant \varepsilon\} \cup \{\eta'' \in C \mid |\eta'' - \eta'| \leqslant \varepsilon\} \supseteq \{\zeta'' \in C \mid |\zeta'' - \zeta'| \leqslant \varepsilon\}$ and the use of 5.2.3 (point 2°) ends the proof.

Proposition 5.3.3 tends to support the idea that "most pointed closed convex cones are tame". However, there exist pointed "wild" cones, as we shall presently see.

5.3.4. PROPOSITION. *If K is a non-void compact convex subset of $\mathbf{R}^n$ which is tame, then $\mathscr{F}(K)$ is d_{H}-closed (hence d_{H}-compact; d_{H} is the Hausdorff distance defined in 3.2). Consequently, if C is a closed convex pointed tame cone, then $\mathscr{F}(C)$ is $d_{\mathrm{H},C}$-closed (and compact).*

Proof. Let $(F_k)_{k=1}^{\infty}$ be a sequence in $\mathscr{F}(K)$ which converges in the d_{H} metric to a non-void compact convex set A. We know that $A \subseteq K$ (this is part of the Blaschke theorem). We need to show that A is extreme in K.

Consider a $\zeta \in A$ written as $(1-\lambda)\xi + \lambda\eta$ with $\lambda \in (0, 1)$ and $\xi, \eta \in K$. For any k there exists $\zeta_k \in F_k$ such that $|\zeta_k - \zeta| \leqslant d_{\mathrm{H}}(A, F_k)$; hence we have $\zeta_k \xrightarrow[k\to\infty]{} \zeta$ and the criterion with sequences (Observation 1°, 5.2.2) gives us the sequences $(\xi_k)_{k=k_0}^{\infty}$, $(\eta_k)_{k=k_0}^{\infty}$, $(\lambda_k)_{k=k_0}^{\infty}$ converging to ξ, η, λ respectively and such that $(1-\lambda_k)\xi_k + \lambda_k\eta_k = \zeta_k$, $\forall k \geqslant k_0$. We may of course assume $\lambda_k \in (0, 1)$, $\forall k \geqslant k_0$. Then the extremality of the F_k's implies that ξ_k and η_k are in F_k, $\forall k \geqslant k_0$. Finally, for every $k \geqslant k_0$ we may find ξ_k' and η_k' in A such that $|\xi_k - \xi_k'| \leqslant d_{\mathrm{H}}(A, F_k)$, $|\eta_k - \eta_k'| \leqslant d_{\mathrm{H}}(A, F_k)$; it is obvious that $\xi_k' \xrightarrow[k\to\infty]{} \xi$, $\eta_k' \xrightarrow[k\to\infty]{} \eta$, hence ξ and η are in $\operatorname{clos} A = A$.

The statement about cones derives from the one about compact sets, Lemma 5.3.1 and Observation 5.1.2.

5.3.5. Example. Let B be the closed convex pointed cone in $\mathbf{R}^4$ which intersects $\{\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbf{R}^4 \mid \xi_4 = 1\}$ after the set:

$$\Sigma = \operatorname{co}(\{(0, 0, -1, 1), (0, 0, 1, 1)\} \cup \{(\xi_1, \xi_2, 0, 1) \mid \xi_1^2 + (\xi_2 - 1)^2 = 1\}).$$

It is clear that $0 \in \mathbf{R}^4$ is an accumulation point of the set of extreme points of Σ, without being itself extreme. Thus Proposition 5.3.4 and Observation 5.1.2 imply hat neither Σ nor B can be tame.

One can compute the analytic expression of B, too; it is:

$$B = \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \mid |\xi_3| \leqslant \xi_4, \xi_1^2 + \xi_2^2 \leqslant 2\xi_2(\xi_4 - |\xi_3|)\}.$$

6. WHAT IS w*-CLOS$\{\chi_{t+Q} \mid t \in P\}$? (CONTINUED)

The reason we have studied tame cones so thoroughly is given by the next proposition:

6.1. Proposition. *Let P be a closed convex solid cone in $\mathbf{R}^n$ such that $\hat{P}$ is tame. Then*

$$\text{(21)} \qquad \text{w*-clos}\{\chi_{t+Q} \mid t \in P\} = \{\chi_{a-\hat{\Phi}} \mid \Phi \in \mathscr{F}(\hat{P}), a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)\}.$$

Proof. "$\supseteq$" is the main result of Section 4 (see Proposition 4.6.2). In order to prove "$\subseteq$" we recall that w*-clos$\{\chi_{t+Q} \mid t \in P\} \subseteq \{\chi_A \mid Q \subseteq A \subseteq \mathbf{R}^n$, A closed and convex$\}$ which is w*-compact and homeomorphic by $\chi_A \mapsto \tilde{E}(\sigma_A)$ to the set $\tilde{\mathscr{E}}$ defined in 3.4.1 (see Corollary 3.4.5).

We are going to show that the image of the set on the right side of (21) through $\chi_A \mapsto \tilde{E}(\sigma_A)$ is $\widetilde{d_{\mathrm{H,C}}}$-compact. This will imply that the set on the right side of (21) is w*-compact. But this set contains $\{\chi_{a-\hat{\hat{P}}} \mid a \in \hat{\hat{P}} \cap \operatorname{sp} \hat{P}\}$ (corresponding to $P \in \mathscr{F}(\hat{P})$) which is exactly $\{\chi_{t+Q} \mid t \in P\}$ and so "$\subseteq$" will become clear.

LEMMA 1. *Let φ be a positively homogeneous, subadditive, lower semicontinuous function from $\hat{P}$ into $[0, \infty]$. Then φ is of the form $\sigma_{a-\hat{\Phi}}$ for some $\Phi \in \mathscr{F}(\hat{P})$ and $a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi)$ if and only if it is additive.*

Proof of Lemma 1. "$\Rightarrow$" For any $\xi \in \hat{P}$ we have:

$$\varphi(\xi) = \sup_{t \in \hat{\Phi}} \langle a - t, \xi \rangle = \langle a, \xi \rangle - \inf_{t \in \hat{\Phi}} \langle t, \xi \rangle = \begin{cases} \langle a, \xi \rangle & \text{if } \xi \in \Phi \\ \infty & \text{if } \xi \in \hat{P} \setminus \Phi. \end{cases}$$

This expression and the fact that Φ is a face yield immediately the additivity of φ.

"$\Leftarrow$" Let us put $\Phi = \{\xi \in \hat{P} \mid \varphi(\xi) < \infty\}$. If a $\zeta \in \Phi$ is written as $(1 - \lambda)\xi + \lambda\eta$ with $\lambda \in (0, 1)$ and $\xi, \eta \in \hat{P}$, then $(1 - \lambda)\varphi(\xi) + \lambda\varphi(\eta) = \varphi(\zeta) < \infty$, hence ξ and η are in Φ. It is obvious that Φ is convex and non-void ($\Phi \ni 0$), thus Φ is a face of $\hat{P}$.

We can extend φ to $\operatorname{sp}\Phi = \Phi - \Phi$ by the formula: $\tilde{\varphi}(\xi - \eta) = \varphi(\xi) - \varphi(\eta)$ $\forall \xi, \eta \in \Phi$. It is easy to see that the definition of $\tilde{\varphi}$ makes sense that $\tilde{\varphi}$ is linear and that indeed $\tilde{\varphi}$ is an extension of φ. The linearity of $\tilde{\varphi}$ implies the existence of an $a \in \operatorname{sp}\Phi$ such that $\varphi(\xi) = \langle a, \xi \rangle$, $\forall\, \xi \in \Phi$, and the positivity of $\tilde{\varphi}$ implies that a is in $\hat{\Phi}$. Finally, φ is of the form:

$$\varphi(\xi) = \begin{cases} \langle a, \xi \rangle, & \text{if } \xi \in \Phi \\ \infty\ , & \text{if } \xi \in \hat{P} \setminus \Phi \end{cases}.$$

Looking again at the proof of "$\Rightarrow$", we see that $\varphi = \sigma_{a-\hat{\Phi}}$.

In order to finish the proof of the proposition, it suffices to take $(\varphi_k)_{k=1}^{\infty}$ and φ positively homogeneous, subadditive and lower semicontinuous functions from $\hat{P}$ into $[0, \infty]$ such that $\tilde{E}(\varphi_k) \xrightarrow[k \to \infty]{\widetilde{d_{H,C}}} \tilde{E}(\varphi)$ and each φ_k is additive and to show that φ is additive, too. We first prove a lemma about $(\varphi_k)_{k=1}^{\infty}$ and φ (which are fixed from now on).

LEMMA 2. 1° *If $\xi_k \xrightarrow[k \to \infty]{} \xi$ in P, then* $\liminf_{k \to \infty} \varphi_k(\xi_k) \geqslant \varphi(\xi)$.

2° *For any $\xi \in \hat{P}$ with $\varphi(\xi) < \infty$, there exists a sequence $(\xi_k)_{k=1}^{\infty}$ in $\hat{P}$ such that $\xi_k \xrightarrow[k \to \infty]{} \xi$ and $\varphi_k(\xi_k) \xrightarrow[k \to \infty]{} \varphi(\xi)$.*

Proof of Lemma 2. 1° The case $\varphi(\xi) = 0$ is clear, so we shall assume $\varphi(\xi) > 0$. We take an arbitrary α with $0 < \alpha < \varphi(\xi)$ and we prove that $\varphi_k(\xi_k) > \alpha$ for sufficiently large k. Let us suppose that there exist $1 \in \sigma(1) < \sigma(2) < \dots$ such that $\varphi_{\sigma(k)}(\xi_{\sigma(k)}) \leqslant \alpha$, $\forall k$. For any k we have $(\xi_{\sigma(k)}, \alpha) \in \tilde{E}(\varphi_{\sigma(k)})$. Therefore we can find $(\eta_k, \beta_k) \in \tilde{E}(\varphi)$ such that $|(\eta_k, \beta_k) - (\xi_{\sigma(k)}, \alpha)| \leqslant \widetilde{d_{\mathrm{H,C}}}(\tilde{E}(\varphi_{\sigma(k)}), \tilde{E}(\varphi))\, |(\xi_{\sigma(k)}, \alpha)|$. We then have $|(\xi_{\sigma(k)}, \alpha) - (\eta_k, \beta_k)| \xrightarrow[k\to\infty]{} 0$ because $(\xi_{\sigma(k)})_{k=1}^{\infty}$ is bounded and $\widetilde{d_{\mathrm{H,C}}}(\tilde{E}(\varphi_{\sigma(k)}), \tilde{E}(\varphi)) \xrightarrow[k\to\infty]{} 0$. This clearly implies that $(\xi, \alpha) = \lim_{k\to\infty} (\eta_k, \beta_k) \in \tilde{E}(\varphi)$, thus $\alpha \geqslant \varphi(\xi)$, contradiction.

2°. For any k there exists $(\xi_k, \nu_k) \in \tilde{E}(\varphi_k)$ such that $|(\xi_k, \nu_k) - (\xi, \varphi(\xi))| \leqslant \leqslant \widetilde{d_{\mathrm{H,C}}}(\tilde{E}(\varphi_k), \tilde{E}(\varphi))\, |(\xi, \varphi(\xi))|$. Thus $\xi_k \xrightarrow[k\to\infty]{} \xi$ and $\nu_k \xrightarrow[k\to\infty]{} \varphi(\xi)$; but $\nu_k \geqslant \varphi_k(\xi_k)$, $\forall k$, and so we find that $\limsup_{k\to\infty} \varphi_k(\xi_k) \leqslant \lim_{k\to\infty} \nu_k = \varphi(\xi)$. On the other hand point 1° says that $\liminf_{k\to\infty} \varphi_k(\xi_k) \geqslant \varphi(\xi)$ and hence $\varphi_k(\xi_k) \xrightarrow[k\to\infty]{} \varphi(\xi)$.

Now we are able to show that φ is superadditive (hence additive). Let ξ and η be arbitrary in $\hat{P}$. If $\varphi(\xi + \eta) = \infty$ then clearly $\varphi(\xi + \eta) \geqslant \varphi(\xi) + \varphi(\eta)$, so we shall assume $\varphi(\xi + \eta) < \infty$. Let $(\zeta_k)_{k=1}^{\infty}$ be a sequence in $\hat{P}$ convergent to $(1/2)(\xi + \eta)$, such that $\varphi_k(\zeta_k) \xrightarrow[k\to\infty]{} \varphi((1/2)(\xi + \eta))\,(=(1/2)\varphi(\xi + \eta) < \infty$; we have used Lemma 2.2°). $\hat{P}$ is tame by the hypothesis, hence there exist sequences $(\xi_k)_{k=k_0}^{\infty}$, $(\eta_k)_{k=k_0}^{\infty}$, $(\lambda_k)_{k=k_0}^{\infty}$ converging to ξ, η and $1/2$ respectively such that $(1 - \lambda_k)\xi_k + \lambda_k\eta_k = \zeta_k$, $\forall k \geqslant k_0$. Applying φ_k in this equality we get $\varphi_k(\zeta_k) = = \varphi_k((1 - \lambda_k)\xi_k + \lambda_k\eta_k) = (1 - \lambda_k)\varphi_k(\xi_k) + \lambda_k\varphi_k(\eta_k)$. Finally we pass to $\liminf$ and taking Lemma 2.1° into account we obtain $(1/2)\varphi(\xi + \eta) \geqslant (1/2)\varphi(\xi) + + (1/2)\varphi(\eta)$, exactly what we needed.

6.2. OBSERVATIONS. 1° Among the cones considered in Proposition 6.1 are to be found: a) every polyhedral solid cone; b) every closed convex solid cone in $\mathbf{R}^3$, c) the forward light cone in any dimension. (This comes out from Proposition 5.3.3.)

2° Equality (21) does not hold for $P = \hat{B}$ where B is the cone considered in 5.3.5. In fact it is easily seen in the general case that $\mathrm{w}^*\text{-clos}\{\chi_{t+Q} \mid t \in P\}$ contains $\chi_{a-\hat{\Phi}}$ for every $\Phi \in d_{\mathrm{H,C}}\text{-clos}\, \mathscr{F}(\hat{P})$ and $a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$; when $P = \hat{B}$, $\mathscr{F}(\hat{P}) = \mathscr{F}(B)$ is not $d_{\mathrm{H,C}}$-closed because there exists a cut Σ through B for which $\mathscr{F}(\Sigma)$ is not d_{H}-closed (see also Observation 5.1.2).

Moreover, $\mathrm{w}^*\text{-clos}\{\chi_{t-\hat{B}} \mid t \in \hat{B}\}$ can be computed by ad-hoc methods and the surprising result is that besides $\{\chi_{a-\hat{\Phi}} \mid \Phi \in d_{\mathrm{H,C}}\text{-clos}\, \mathscr{F}(B),\ a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)\}$ it contains some sets which are not translated cones. This fact suggests that making a precise description of $\mathrm{w}^*\text{-clos}\{\chi_{t+Q} \mid t \in P\}$ when $\hat{P}$ is not tame is a quite difficult problem.

3° It can be shown that "$\Theta^{\mathbf{R}^n}_{\hat{P}} \circ \Theta^{\mathbf{R}^n}_{P} = \mathrm{id}\,\mathscr{F}(P)$" always holds when P is polyhedral; hence for a polyhedral P the functions (14) and (15) are the same and we have (taking also Propositions 5.3.3 and 6.1 into account):

$$\text{w}^*\text{-clos}\{\chi_{t+Q} \mid t \in P\} =$$

$$= \{\chi_A \mid \exists\ F \in \mathscr{F}(P) \text{ and } a \in \operatorname{clos} \operatorname{proj}_{\operatorname{sp} \Theta^{\mathbf{R}^n}_P (F)} (P) \text{ such that } A = \operatorname{clos}(a + F + Q)\}.$$

7. THE CLOSED INVARIANT SUBSETS OF $\mathfrak{g}^0$ WHEN $\hat{P}$ IS TAME

In this section we consider a closed convex solid cone P in $\mathbf{R}^n$ such that $\hat{P}$ is tame. Proposition 6.1 says that the groupoid $\mathfrak{g}$ parametrizing $\mathscr{W}(P)$, which was found in Sections 2 and 3, has unit space equal to $\{\chi_{a-\hat{\Phi}} \mid \Phi \in \mathscr{F}(\hat{P}), a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi)\}$. Thus taking into account the lemma used in the proof of 2.3.4 we see that:

$$\mathfrak{g} = \{(t, \chi_{a-\hat{\Phi}}) \mid \Phi \in \mathscr{F}(\hat{P}),\ a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi),\ t \in \hat{\Phi} - a\}.$$

The source and cosource of $(t, \chi_{a-\hat{\Phi}}) \in \mathfrak{g}$ are $\chi_{a-\hat{\Phi}}$ and $\chi_{t+(a-\hat{\Phi})} = \chi_{\operatorname{proj}_{\operatorname{sp}\Phi}(t+a)-\Phi}$, respectively. Now by Proposition 4.6.2, $\chi_{a-\hat{\Phi}} = \chi_{b-\hat{\Psi}}$ if and only if $\Phi = \Psi$ and $a = b$ (where $\Phi, \Psi \in \mathscr{F}(\hat{P})$, $a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi)$, $b \in \mathscr{D}_{\operatorname{sp}\Psi}(\Psi)$) and this implies that each $\{\chi_{a-\hat{\Phi}} \mid a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi)\}$ is a union of orbits of $\mathfrak{g}^0$. If we also notice that for an arbitrary $\Phi \in \mathscr{F}(\hat{P})$ and an $a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi)$, $x = (a, \chi_{-\hat{\Phi}}) \in \mathfrak{g}$ has $d(x) = \chi_{-\hat{\Phi}}$, $r(x) = = \chi_{a-\hat{\Phi}}$, the following result becomes clear:

7.1. LEMMA. *There exists a bijection between the subsets of $\mathscr{F}(\hat{P})$ and the invariant subsets of $\mathfrak{g}^0$ which associates to $\mathscr{F}_0 \subseteq \mathscr{F}(\hat{P})$ the set $\{\chi_{a-\hat{\Phi}} \mid \Phi \in \mathscr{F}_0, a \in \mathscr{D}_{\operatorname{sp}\Phi}(\Phi)\}$.*

In order to distinguish among the subsets of $\mathscr{F}(\hat{P})$ those which give rise to closed invariant subsets of $\mathfrak{g}_0$, we introduces a topology on $\mathscr{F}(\hat{P})$ in the following manner:

7.2. PROPOSITION AND DEFINITION. *For every $\mathscr{F}_0 \subseteq \mathscr{F}(\hat{P})$ we define:*

$$\overline{\mathscr{F}_0} = \{\Phi \in \mathscr{F}(\hat{P}) \mid \inf_{\Psi \in \mathscr{F}_0} d_{\mathrm{H,c,r}}(\Phi, \Psi) = 0\}$$

($d_{\mathrm{H,c,r}}$ is defined in (8), *Section* 3.2*). Then $\mathscr{F}_0 \mapsto \overline{\mathscr{F}_0}$ is a closure operator on*

$\mathcal{F}(\hat{P})$, and hence there exists a unique topology $\mathcal{U}$ on $\mathcal{F}(\hat{P})$ such that $\mathcal{F}_0 \subseteq \mathcal{F}(\hat{P})$ is $\mathcal{U}$-closed if and only if $\mathcal{F}_0 = \bar{\mathcal{F}}_0$. This closure operator can be described as follows:

$$\bar{\mathcal{F}}_0 = \{\Phi \in \mathcal{F}(\hat{P}) \mid \Phi \subseteq \operatorname{clos} \bigcup_{\Psi \in \mathcal{F}_0} \Psi\} =$$

$$= \{\Phi \in \mathcal{F}(\hat{P}) \mid \operatorname{int}_{\mathrm{sp}\,\Phi}(\Phi) \cap \operatorname{clos} \bigcup_{\Psi \in \mathcal{F}_0} \Psi \neq \emptyset\}.$$

Proof. $\mathcal{F}_0 \mapsto \bar{\mathcal{F}}_0$ trivially satisfies the closure axioms. (Let us check for instance that $\bar{\bar{\mathcal{F}}}_0 \subseteq \bar{\mathcal{F}}_0$. We take an element $\Theta \in \bar{\bar{\mathcal{F}}}_0$ and an $\varepsilon > 0$. There exists a $\Psi \in \bar{\mathcal{F}}_0$ such that $d_{\mathrm{H,C,r}}(\Theta, \Psi) < \varepsilon/2$ and a $\Phi \in \mathcal{F}_0$ such that $d_{\mathrm{H,C,r}}(\Psi, \Phi) < \varepsilon/2$; so $d_{\mathrm{H,C,r}}(\Theta, \Phi) \leqslant d_{\mathrm{H,C,r}}(\Theta, \Psi) + d_{\mathrm{H,C,r}}(\Psi, \Phi) < \varepsilon$).

We further fix a subset $\mathcal{F}_0$ of $\mathcal{F}(\hat{P})$ and show about an arbitrary $\Phi \in \mathcal{F}(\hat{P})$ that:

(a) $$\inf_{\Psi \in \mathcal{F}_0} d_{\mathrm{H,C,r}}(\Phi, \Psi) = 0 \Rightarrow \Phi \subseteq \operatorname{clos} \bigcup_{\Psi \in \mathcal{F}_0} \Psi.$$

Indeed, for any $\xi \in \Phi$ and $\varepsilon > 0$ we can find a $\Psi \in \mathcal{F}_0$ such that $d_{\mathrm{H,C,r}}(\Phi, \Psi) < \varepsilon/(1 + |\xi|)$ and then an $\eta \in \Psi$ such that $|\xi - \eta| \leqslant d_{\mathrm{H,C,r}}(\Phi, \Psi)|\xi| < \varepsilon$.

(b) $$(\operatorname{int}_{\mathrm{sp}\,\Phi}\Phi) \cap (\operatorname{clos} \bigcup_{\Psi \in \mathcal{F}_0} \Psi) \neq \Phi \Rightarrow \inf_{\Psi \in \mathcal{F}_0} d_{\mathrm{H,C,r}}(\Phi, \Psi) = 0.$$

Indeed, let us take a point ξ in the non-void intersection of (b). For any $k \geqslant 1$ there exist $\Psi_k \in \mathcal{F}_0$ and $\xi_k \in \Psi_k$ such that $|\xi - \xi_k| \leqslant 1/k$. But $\mathcal{F}(\hat{P})$ is $d_{\mathrm{H,C}}$-compact (Proposition 5.3.4) and that is why we can extract a $d_{\mathrm{H,C}}$-convergent subsequence $(\Psi_{\sigma(k)})_{k=1}^{\infty}$ of $(\Psi_k)_{k=1}^{\infty}$. We denote its limit by Ψ. For any k there exists $\eta_{\sigma(k)} \in \Psi$ such that $|\eta_{\sigma(k)} - \xi_{\sigma(k)}| \leqslant d_{\mathrm{H,C}}(\Psi_{\sigma(k)}, \Psi)\, |\xi_{\sigma(k)}|$ and we obtain $\xi = \lim_{k \to \infty} \eta_{\sigma(k)} \in \Psi$.

Now $\Psi \ni \xi$ and $\xi \in \operatorname{int}_{\mathrm{sp}\,\Phi}\Phi$, imply together that $\Psi \supseteq \Phi$. Indeed, for any $\eta \in \Phi$ we have $\lim_{k \to \infty} \xi - (1/k)\eta = \xi \in \operatorname{int}_{\mathrm{sp}\,\Phi}\Phi$ and the sequence is in $\mathrm{sp}\,\Phi$. Hence for a sufficiently large (fixed) k we have $\xi - (1/k)\eta \equiv \zeta \in \Phi \subseteq \hat{P}$. But then $k\xi = \eta + k\zeta$ and the extremality of Ψ yields $\eta \in \Psi$.

Finally we remark that $d_{\mathrm{H,C,r}}(\Phi, \Psi_{\sigma(k)}) \leqslant d_{\mathrm{H,C,r}}(\Psi, \Psi_{\sigma(k)}) \leqslant d_{\mathrm{H,C}}(\Psi, \Psi_{\sigma(k)})$, $\forall k \geqslant 1$, hence $d_{\mathrm{H,C,r}}(\Phi, \Psi_{\sigma(k)}) \xrightarrow[k \to \infty]{} 0$. So (b) is proved and this clearly ends the proof of the proposition.

7.3. PROPOSITION. *Let* $\mathscr{F}_0$ *be a subset of* $\mathscr{F}(\hat{P})$. *The following are equivalent:*

1° *The invariant subset of* $\mathfrak{g}^0$ *associated to* $\mathscr{F}_0$ *in* 7.1 *is closed.*

2° $\mathscr{F}_0$ *is* $d_{H,C}$*-closed in* $\mathscr{F}(\hat{P})$ *and hereditary (that is, if* $\Phi, \Psi \in \mathscr{F}(\hat{P})$ *are such that* $\Phi \subseteq \Psi$ *and* $\Psi \in \mathscr{F}_0$, *then* $\Phi \in \mathscr{F}_0$, *too).*

3° $\mathscr{F}_0$ *is* $\mathscr{U}$*-closed.*

Proof. 1° $\Rightarrow$ 2° In order to show that $\mathscr{F}_0$ is $d_{H,C}$-closed, it suffices to observe that $\Phi_k \xrightarrow[k\to\infty]{d_{H,C}} \Phi \Rightarrow \Phi_k \times [0, \infty) \xrightarrow[k\to\infty]{\widetilde{d_{H,C}}} \Phi \times [0,\infty) \Leftrightarrow \tilde{E}(\sigma_{-\hat{\Phi}_k}) \xrightarrow[k\to\infty]{\widetilde{d_{H,C}}} \tilde{E}(\sigma_{-\hat{\Phi}}) \Leftrightarrow \chi_{-\hat{\Phi}_k} \xrightarrow[k\to\infty]{w^*} \chi_{-\hat{\Phi}}$ (here we used Corollary 3.4.5 and the description of $\sigma_{a-\hat{\Phi}}$ given in the proof of Lemma 1, Proposition 6.1). For the heredity, let us consider $\Phi, \Psi \in \mathscr{F}(\hat{P})$ such that $\Phi \subseteq \Psi \in \mathscr{F}_0$. $\hat{\Psi}$ is a closed convex solid cone in $\mathbf{R}_n$ and $\Phi \in \mathscr{F}(\hat{\hat{\Psi}})$. Applying Proposition 4.6.2 to this situation we obtain

$$\{\chi_{a-\hat{\Phi}} \mid a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)\} \subseteq \mathrm{w}^*\text{-clos}\{\chi_{b-\hat{\Psi}} \mid b \in \hat{\Psi}\} =$$

$$= \mathrm{w}^*\text{-clos}\{\chi_{b-\hat{\Psi}} \mid b \in \mathscr{D}_{\mathrm{sp}\,\Psi}(\Psi)\}$$

(which implies $\Phi \in \mathscr{F}_0$).

2° $\Rightarrow$ 3° Let Φ be in $\mathscr{F}_0$ and consider a sequence $(\Psi_k)_{k=1}^{\infty}$ in $\mathscr{F}_0$ such that $d_{H,C,r}(\Phi, \Psi_k) \xrightarrow[k\to\infty]{} 0$. Because $\mathscr{F}(\hat{P})$ is $d_{H,C}$-compact we may assume (replacing $(\Psi_k)_{k=1}^{\infty}$ by a subsequence) that there exists $\Psi = d_{H,C}\text{-}\lim_{k\to\infty} \Psi_k \in d_{H,C}\text{-clos}\, \mathscr{F}_0 = \mathscr{F}_0$. On the other hand, $d_{H,C,r}(\Phi, \Psi) = \lim_{k\to\infty} d_{H,C,r}(\Phi, \Psi_k) = 0$ hence $\Phi \subseteq \Psi$ and finally $\Phi \in \mathscr{F}_0$ because $\mathscr{F}_0$ is hereditary.

3° $\Rightarrow$ 1° Let us suppose that $\chi_{Q_k - \hat{\Phi}_k} \xrightarrow[k\to\infty]{w^*} \chi_{a-\hat{\Phi}}$ where the Φ_k's are in $\mathscr{F}_0$, Φ is in $\mathscr{F}(\hat{P})$, $a_k \in \mathscr{D}_{\mathrm{sp}\,\Phi_k}(\Phi_k)$, $\forall k \geq 1$ and $a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)$. Then $\hat{E}(\sigma_{a_k - \hat{\Phi}_k}) \xrightarrow[k\to\infty]{\widetilde{d_{H,C}}} \tilde{E}(\sigma_{a-\hat{\Phi}})$ by Corollary 3.4.5. Looking again at the proof of Lemma 1, Proposition 6.1, we see that

$$\sigma_{a-\hat{\Phi}}(\xi) = \begin{cases} \langle a, \xi \rangle & \text{if } \xi \in \Phi \\ \infty & \text{if } \xi \in \hat{P} \setminus \Phi \end{cases}$$

and that we have a similar expression for any $\sigma_{a_k - \hat{\Phi}_k}$.

Consider an arbitrary $\xi \in \Phi$. We know that $(\xi, \langle a, \xi \rangle) \in \tilde{E}(\sigma_{a-\hat{\Phi}})$, so for any k we can find $(\xi_k, v_k) \in \tilde{E}(\sigma_{a_k - \hat{\Phi}_k})$ such that $|(\xi, \langle a, \xi \rangle) - (\xi_k, v_k)| \leq$

$\leqslant \widetilde{d_{H,C}}(\tilde{E}(\sigma_{a-\hat{\Phi}}), \tilde{E}(\sigma_{a_k-\hat{\Phi}_k}))|(\xi, \langle a, \xi\rangle)|$. It is clear that $\xi_k \xrightarrow[k\to\infty]{} \xi$ and that $\xi_k \in \Phi_k$, $\forall\, k \geqslant 1$, so ξ is in clos $\bigcup_{k=1}^{\infty} \Phi_k$. Finally we have $\Phi \subseteq$ clos $\bigcup_{\Psi\in\mathscr{F}_0} \Psi$ and Proposition 7.2 implies that $\Phi \in \overline{\mathscr{F}}_0 = \mathscr{F}_0$. Hence $\{\chi_{a-\hat{\Phi}} \mid \Phi \in \mathscr{F}_0,\ a \in \mathscr{D}_{\mathrm{sp}\,\Phi}(\Phi)\}$ is (w*-) closed.

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