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ON LATTICES OF INVARIANT SUBSPACES OF OPERATOR ALGEBRAS

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In this paper we continue to develop the approach to the theory of operator algebras introduced in [5] and [7]. Two classes of reflexive algebras play a central role in this theory: W^*-algebras and CSL-algebras (including the particular case of nest algebras). Usually, when making a study of CSL-algebras or non self-adjoint algebras in general, one considers the pairs $(\mathscr{A}, \mathrm{Lat}\,\mathscr{A})$ in the same way as one considers the pairs $(\mathscr{A}, \mathscr{A}')$ in the context of W^*-algebras. Much work has also been done to investigate the Lie algebras $\mathscr{D}(\mathscr{A})$ of all continuous derivations of $\mathscr{A}$ The structure of $\mathscr{D}(\mathscr{A})$ is usually very complicated but in many important cases it is in fact very simple. Thus it is well-known that all derivations of W^*-algebras are inner. Christensen [2] and Wagner [8] have proved that the same is true of nest and quasitriangular algebras. But already in the case of CSL-algebras Gilfeather, Hopenwasser and Larson [3] have shown that these may have non-inner derivations although none of these derivations are implemented by bounded operators.

Although a knowledge of the structure of $\mathscr{D}(\mathscr{A})$ gives much useful information about the structure of $\mathscr{A}$ and of $\mathrm{Lat}\,\mathscr{A}$, there is a Lie subalgebra $\mathrm{Ad}\,\mathscr{A}$ of $\mathscr{D}(\mathscr{A})$ which is more closely linked to $\mathscr{A}$ and to $\mathrm{Lat}\,\mathscr{A}$ and whose structure can be established with greater ease. This Lie subalgebra consists of all bounded operators which generate derivations of $\mathscr{A}$. As well as the obvious connection between $\mathscr{A}$ and $\mathrm{Ad}\,\mathscr{A}$, there is also a close link between $\mathrm{Lat}\,\mathscr{A}$ and $\mathrm{Ad}\,\mathscr{A}$.

(i) All operators A in $\mathrm{Ad}\,\mathscr{A}$ generate one-parameter groups of one-to-one mappings of $\mathrm{Lat}\,\mathscr{A}$ on itself ($M \mapsto \exp(tA)M$).

(ii) For every subspace L in $\mathrm{Lat}\,\mathscr{A}$ the set $\mathrm{Ad}\,\mathscr{A}(L) = \{A \in \mathrm{Ad}\,\mathscr{A} : AL \subseteq L\}$ is a Lie subalgebra of $\mathrm{Ad}\,\mathscr{A}$ and $\mathscr{A} = \bigcap_{L \in \mathrm{Lat}\,\mathscr{A}} \mathrm{Ad}\,\mathscr{A}(L)$ if $\mathscr{A}$ is reflexive.

An understanding of the structure of $\mathrm{Ad}\,\mathscr{A}$ makes it possible to obtain a clearer description of $\mathrm{Lat}\,\mathscr{A}$. This can be done by establishing the structure of orbits in $\mathrm{Lat}\,\mathscr{A}$ with respect to $\mathrm{Ad}\,\mathscr{A}$. This grasp of the structure of $\mathrm{Lat}\,\mathscr{A}$ in turn leads to deeper understanding of the structure of $\mathscr{A}$. It has therefore been suggested

[5] that in the general case of operator algebra $\mathscr{A}$ it would be more useful to consider the triplets ($\mathscr{A}$, Lat $\mathscr{A}$, Ad $\mathscr{A}$).

In many cases, however, these triplets degenerate into pairs. For example, if $\mathscr{A}$ is a W^*-algebra, then Lat $\mathscr{A}$ is the set of all projections in $\mathscr{A}'$, and Ad $\mathscr{A} = \mathscr{A} + \mathscr{A}'$; as a result of this, the triplet turns into the pair ($\mathscr{A}$, $\mathscr{A}'$). If $\mathscr{A}$ is a CSL-algebra, then Ad $\mathscr{A} = \mathscr{A}$ and the triplet becomes the pair ($\mathscr{A}$, Lat $\mathscr{A}$). But, in the case of an arbitrary operator algebra, Ad $\mathscr{A}$ is not usually equal to $\mathscr{A} + \mathscr{A}'$ and does not contain Lat $\mathscr{A}$; in this case, therefore, the triplet does not degenerate into a pair.

One of the simplest classes $\mathscr{R}_1$ of this type of algebra consists of all the reflexive algebras $\mathscr{A}$ such that the quotient Lie algebra Ad $\mathscr{A}/\mathscr{A}$ is not trivial and that for every L in Lat $\mathscr{A}$ the codimension of Ad $\mathscr{A}(L)$ in Ad $\mathscr{A}$ is less than or equal to 1.

No CSL- or W^*-algebras (except for the factors $B(H) \otimes I_2$) belong to $\mathscr{R}_1$. As a general rule, Lie algebras Ad $\mathscr{A}$, for $\mathscr{A} \in \mathscr{R}_1$, are much larger than $\mathscr{A} + \mathscr{A}'$; as a result, these algebras $\mathscr{A}$ have many non-inner derivations implemented by bounded operators.

The triplets ($\mathscr{A}$, Lat $\mathscr{A}$, Ad $\mathscr{A}$) were investigated in [5] for the case when $\mathscr{A} \in \mathscr{R}_1$ and when the quotient Lie algebras Ad $\mathscr{A}/\mathscr{A}$ are finite-dimensional. A new method of constructing reflexive operator algebras was introduced in [5] and [7] which provided many examples of algebras from $\mathscr{R}_1$. The structure of Ad $\mathscr{A}/\mathscr{A}$ when $\mathscr{A} \in \mathscr{R}_1$ in the general case was investigated in [6].

In this article we consider arbitrary operator algebras $\mathscr{A}$ and show that Lat $\mathscr{A}$ is the disjoint union of sets $\{\mathscr{L}_n\}_{n=0}^{\infty}$, where $\mathscr{L}_n = \{L \in \text{Lat}\,\mathscr{A} : \text{codim}(\text{Ad}\,\mathscr{A}(L)) = n\}$ and that all $\mathscr{L}_n$ are invariant under Ad $\mathscr{A}$. For example, Lat $\mathscr{A} = \mathscr{L}_0$ if $\mathscr{A}$ is a CSL-algebra, and Lat $\mathscr{A} = \mathscr{L}_0 \cup \mathscr{L}_1$ if $\mathscr{A}$ belongs to $\mathscr{R}_1$. The analysis of the structure of Lie subalgebras of codimension 1 given in [6] makes it possible to describe the structure of orbits in $\mathscr{L}_1$. This, in turn, allows us to establish some interesting properties of $\mathscr{L}_0 \cup \mathscr{L}_1$ for arbitrary operator algebras, especially for the case when $\mathscr{L}_2 = \emptyset$. The link between the orbits in Lat $\mathscr{A}$ and the closed Lie ideals in Ad $\mathscr{A}$ is investigated.

We shall also investigate the effect which the existence of simple orbits in Lat $\mathscr{A}$ has on the structure of $\mathscr{A}$. Theorem 3.1 shows that if ω is a simple orbit, then there exist closed operators F and G and a projection P such that $P\mathscr{A}P$ is a subalgebra of $\mathscr{A}(F, G)$, so that Lat $\mathscr{A}(F, G) \subseteq$ Lat $P\mathscr{A}P \subseteq$ Lat $\mathscr{A}$. (The algebras $\mathscr{A}(F, G)$ were considered in [7].) The case when $P\mathscr{A}P$ contains a finite rank operator is particularly interesting. Here one can prove that ω lies in $\mathscr{L}_1$ and that Lat $P\mathscr{A}P$ = Lat $\mathscr{A}(F, G)$. One can also describe the structure of Lat $P\mathscr{A}P$ in detail and in some cases one can even show that $P\mathscr{A}P = \mathscr{A}(F, G)$.

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1. PRELIMINARIES AND NOTATION

Let $B(H)$ be the algebra of all bounded operators on a Hilbert space H and let $\mathscr{A}$ be a subalgebra of $B(H)$. $\operatorname{Lat}\mathscr{A}$ is the set of all closed subspaces invariant under operators from $\mathscr{A}$ and $\operatorname{Alg}\operatorname{Lat}\mathscr{A}$ is the subalgebra of all operators in $B(H)$ which leave every member of $\operatorname{Lat}\mathscr{A}$ invariant. $\mathscr{A}$ is called *reflexive* if $\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$.

Let M be a linear subspace in H and let B be a linear operator on H. We shall denote

(i) the domain of B by $D(B)$;

(ii) the set $\{Bx : x \in M\}$ by BM, if $M \subseteq D(B)$;

(iii) the closure of M by $\operatorname{sp}(M)$;

(iv) the closure of B by $\operatorname{cl}(B)$, if B is a closable operator;

(v) by $\{B\}$ the one-dimensional subspace in $B(H)$ generated by B, if $B \in B(H)$.

If M is closed and if B is bounded and has a bounded inverse, then $\operatorname{sp}(BM) = BM$.

If M and L are two linear subspaces in H, then by $M + L$ we shall denote the linear subspace generated by all sums $x + y$ where $x \in M$ and $y \in L$. If $M \cap L = \{0\}$, then their direct sum will be denoted by $M \dotplus L$. Let L be a linear subspace of M and suppose that there exists a linear subspace N in M such that $\dim N = n$ and that $M = L \dotplus N$. Then we say that L has codimension n in M and denote it by $\operatorname{codim} L = n$.

For any operators A and B from $B(H)$ let

$$[A,B] = AB - BA.$$

Then $[A,B]$ is a Lie multiplication on $B(H)$ and

$$\|[A,B]\| \leqslant 2\|A\|\|B\|.$$

A subspace $\mathscr{B}$ in $B(H)$ closed in the norm topology is called *a normed Lie subalgebra of* $B(H)$ if $[A,B] \in \mathscr{B}$ for all A and B from $\mathscr{B}$. If $\mathscr{B}$ is weakly closed, then it is called *a weakly closed Lie subalgebra of* $B(H)$. If $\mathscr{B}$ and $\mathscr{D}$ are normed (weakly closed) Lie subalgebras of $B(H)$ and if $\mathscr{D} \subset \mathscr{B}$, then $\mathscr{D}$ is called *a closed (weakly closed) Lie subalgebra of* $\mathscr{B}$. If in addition $[B,C] \in \mathscr{D}$ for every $B \in \mathscr{B}$ and for every $C \in \mathscr{D}$, then $\mathscr{D}$ is called *a closed (weakly closed) Lie ideal of* $\mathscr{B}$.

DEFINITION. Let $\mathscr{A}$ be a subalgebra of $B(H)$ and let $M \in \operatorname{Lat}\mathscr{A}$.

$$\operatorname{Ad}\mathscr{A} = \{B \in B(H) : [B, A] \in \mathscr{A} \text{ for all } A \in \mathscr{A}\};$$

$$\operatorname{Ad}\mathscr{A}(M) = \{B \in \operatorname{Ad}\mathscr{A} : BM \subseteq M\}.$$

Obviously every operator from $\mathrm{Ad}\,\mathscr{A}$ generates a bounded derivation on $\mathscr{A}$.

LEMMA 1.1. *Let $\mathscr{A}$ be a subalgebra of $B(H)$ closed in the norm topology (the weak topology) and let $M \in \mathrm{Lat}\,\mathscr{A}$.*

(i) *$\mathrm{Ad}\,\mathscr{A}$ is a normed (weakly closed) Lie subalgebra of $B(H)$. $\mathscr{A}$ is a closed (weakly closed) Lie ideal of $\mathrm{Ad}\,\mathscr{A}$ and the commutant $\mathscr{A}'$ is a weakly closed Lie ideal of $\mathrm{Ad}\,\mathscr{A}$;*

(ii) *$\mathrm{Ad}\,\mathscr{A}(M)$ is a closed (weakly closed) Lie subalgebra of $\mathrm{Ad}\,\mathscr{A}$ and $\mathscr{A} \subseteq \mathrm{Ad}\,\mathscr{A}(M)$;*

(iii) *There exists a maximal closed (weakly closed) Lie ideal $I(M)$ of $\mathrm{Ad}\,\mathscr{A}$ in $\mathrm{Ad}\,\mathscr{A}(M)$ such that*

1) $\mathscr{A} \subseteq I(M) \subseteq \mathrm{Ad}\,\mathscr{A}(M)$;

2) *$I \subseteq I(M)$ for any Lie ideal I of $\mathrm{Ad}\,\mathscr{A}$ in $\mathrm{Ad}\,\mathscr{A}(M)$;*

(iv) *If $\mathscr{A}$ is reflexive, then $\bigcap_{M \in \mathrm{Lat}\,\mathscr{A}} \mathrm{Ad}\,\mathscr{A}(M) = \mathscr{A}$.*

Proof. It is easy to check that for all operators A, B and C

$$[[B, C], A] = [B, [C, A]] - [C, [B, A]]. \tag{1.1}$$

If B and C belong to $\mathrm{Ad}\,\mathscr{A}$ and if A belongs to $\mathscr{A}$, then $[[B, C], A] \in \mathscr{A}$, so that $[B, C] \in \mathrm{Ad}\,\mathscr{A}$. If $B_n \in \mathrm{Ad}\,\mathscr{A}$ and if $\{B_n\}$ converges to B in the norm topology, then

$$\|[B, A] - [B_n, A]\| = \|[B - B_n, A]\| \leqslant 2\|B - B_n\|\|A\| \to 0.$$

Since $[B_n, A] \in \mathscr{A}$ and since $\mathscr{A}$ is closed, $[B, A] \in \mathscr{A}$. Hence $B \in \mathrm{Ad}\,\mathscr{A}$ and $\mathrm{Ad}\,\mathscr{A}$ is a normed Lie subalgebra of $B(H)$. If $\mathscr{A}$ is weakly closed, then it is also easy to show that $\mathrm{Ad}\,\mathscr{A}$ is weakly closed. If $C \in \mathscr{A}'$, then, by (1.1), $[B, C] \in \mathscr{A}'$ for every $B \in \mathrm{Ad}\,\mathscr{A}$. Since $\mathscr{A}'$ is always weakly closed, (i) is proved.

(ii) is obvious. Ordering all closed Lie ideals of $\mathrm{Ad}\,\mathscr{A}$ in $\mathrm{Ad}\,\mathscr{A}(M)$ by inclusion and using Zorn's lemma, we obtain the proof of (iii) in the usual way.

Finally, by (ii) $\bigcap_{M \in \mathrm{Lat}\,\mathscr{A}} \mathrm{Ad}\,\mathscr{A}(M)$ contains $\mathscr{A}$ and consists of all operators from $\mathrm{Ad}\,\mathscr{A}$ which leave every member of $\mathrm{Lat}\,\mathscr{A}$ invariant. Since $\mathscr{A}$ is reflexive, $\bigcap_{M \in \mathrm{Lat}\,\mathscr{A}} \mathrm{Ad}\,\mathscr{A}(M) = \mathscr{A}$ which completes the proof.

DEFINITION. For all $n \geqslant 0$ let

$$\mathscr{L}_n = \{M \in \mathrm{Lat}\,\mathscr{A} : \mathrm{codim}(\mathrm{Ad}\,\mathscr{A}(M)) = n\}$$

where $\mathrm{Ad}\,\mathscr{A}(M)$ is considered as a subspace of $\mathrm{Ad}\,\mathscr{A}$. Then $\mathscr{L}_0$ consists of all $M \in \mathrm{Lat}\,\mathscr{A}$ such that $\mathrm{Ad}\,\mathscr{A}(M) = \mathrm{Ad}\,\mathscr{A}$. By $\mathscr{R}_1$ we shall denote the class of all reflexive algebras $\mathscr{A}$ such that $\mathscr{L}_1 \neq \emptyset$ and that $\mathrm{Lat}\,\mathscr{A} = \mathscr{L}_0 \cup \mathscr{L}_1$.

By sl(2, **C**) we shall denote *the simple Lie algebra* of all complex matrices $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Put

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Then

$$[e_0, e_+] = e_+, \quad [e_0, e_-] = -e_-, \quad [e_+, e_-] = 2e_0.$$

THEOREM 1.2. ([6]). *If $M \in \mathscr{L}_1$, then*

(i) $1 \leqslant \operatorname{codim} I(M) \leqslant 3$ *and the quotient Lie algebra* $\operatorname{Ad}\mathscr{A}/I(M)$ *is isomorphic to* sl(2, **C**) *or to a Lie subalgebra of* sl(2, **C**);

(ii) *if* $\operatorname{codim} I(M) = 2$, *then there exist operators B_- and B_0 in* $\operatorname{Ad}\mathscr{A}$ *such that*

$$\operatorname{Ad}\mathscr{A} = \{B_-\} \dotplus \operatorname{Ad}\mathscr{A}(M), \quad \operatorname{Ad}\mathscr{A}(M) = \{B_0\} \dotplus I(M),$$

$$[B_0, B_-] \equiv -B_- \mod I(M);$$

(iii) *if* $\operatorname{codim} I(M) = 3$, *then* $\operatorname{Ad}\mathscr{A}/I(M)$ *is isomorphic to* sl(2, **C**) *and there exist operators B_-, B_0 and B_+ in* $\operatorname{Ad}\mathscr{A}$ *such that*

$$\operatorname{Ad}\mathscr{A} = \{B_-\} \dotplus \operatorname{Ad}\mathscr{A}(M), \quad \operatorname{Ad}\mathscr{A}(M) = \{B_0\} \dotplus \{B_+\} \dotplus I(M),$$

$$[B_0, B_+] \equiv B_+ \mod I(M), \quad [B_0, B_-] \equiv -B_- \mod I(M),$$

$$[B_+, B_-] \equiv 2B_0 \mod I(M).$$

By SL(2, **C**) we shall denote *the Lie group* of all matrices $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ such that $\det g = 1$.

Let $h = \begin{pmatrix} r/2 & s \\ p & -r/2 \end{pmatrix} \in \mathrm{sl}(2, \mathbf{C})$. Then for $\lambda \in \mathbf{C}$

$$g(\lambda h) = \exp(\lambda h) \in \mathrm{SL}(2, \mathbf{C}).$$

In order to calculate $g(\lambda h)$ we shall consider different cases. The eigenvalues of h are

$$\mu_{1,2} = \pm\sqrt{ps + r^2/4}.$$

I. Let $ps + r^2/4 \neq 0$ and let $p \neq 0$. Set $\Delta = (2p\sqrt{ps + r^2/4})^{-1}$. Then

$$
\begin{aligned}
g_{11} &= \Delta[e^{\lambda\mu_1}(\mu_1 + r/2) - e^{\lambda\mu_2}(\mu_2 + r/2)],\\
g_{12} &= \Delta(\mu_1 + r/2)(\mu_2 + r/2)(e^{\lambda\mu_2} - e^{\lambda\mu_1}),\\
g_{21} &= \Delta p^2(e^{\lambda\mu_1} - e^{\lambda\mu_2}),\\
g_{22} &= \Delta p[e^{\lambda\mu_2}(\mu_1 + r/2) - e^{\lambda\mu_1}(\mu_2 + r/2)].
\end{aligned}
\tag{1.2}
$$

II. Let $ps + r^2/4 = 0$ but $rp \neq 0$. Then

$$g_{11} = 1 + \lambda r/2, \quad g_{12} = \lambda s, \quad g_{21} = \lambda p, \quad g_{22} = 1 - \lambda r/2. \tag{1.3}$$

III. If $r = s = 0$ and $p \neq 0$, then

$$g_{11} = g_{22} = 1, \quad g_{12} = 0, \quad g_{21} = \lambda p. \tag{1.4}$$

I. If $p = 0$ and $r \neq 0$, then $g_{11} = e^{\lambda r/2}$, $g_{22} = g_{11}^{-1}$, $g_{21} = 0$, $g_{12} = s(g_{11} - g_{22})/r$, and if $p = r = 0$, then $g_{11} = g_{22} = 1$, $g_{21} = 0$, $g_{12} = \lambda s$.

Now let $S^2 = \mathbf{C} \cup \infty$. For every $t \in S^2$ and for every $g \in \mathrm{SL}(2, \mathbf{C})$ set

$$\pi(g)t = (g_{11}t + g_{12})/(g_{21}t + g_{22}). \tag{1.5}$$

It is well-known that π is a representation of $\mathrm{SL}(2, \mathbf{C})$ into the group of homeomorphisms of S^2.

2. ORBITS IN LAT $\mathscr{A}$ WITH RESPECT TO AD $\mathscr{A}$

Throughout this section we assume that an algebra $\mathscr{A}$ is closed in the norm topology. For any bounded operators A and B set

$$\operatorname{ad} B(A) = [B, A] \quad \text{and} \quad \exp(\operatorname{ad} B)(A) = \sum_{n=0}^{\infty} \operatorname{ad}^n B(A)/n!.$$

It is well-known that

$$\exp(B)\, A \exp(-B) = \exp(\operatorname{ad} B)\,(A). \tag{2.1}$$

For any subspace M and for any bounded operator B set

$$M_B = \exp(B)\, M.$$

LEMMA 2.1. *Let* $B \in \operatorname{Ad} \mathscr{A}$.

(i) *The mapping* $A \in \mathscr{A} \mapsto \exp(B)\, A \exp(-B)$ *is an automorphism of* $\mathscr{A}$.

(ii) *The transformation* $M \mapsto M_B$ *maps* $\operatorname{Lat} \mathscr{A}$ *onto itself.*

(iii) *The mapping* $C \in \operatorname{Ad} \mathscr{A} \mapsto \exp(B)\, C \exp(-B)$ *is an automorphism of* $\operatorname{Ad} \mathscr{A}$ *and*

$$\exp(B) \operatorname{Ad} \mathscr{A}(M) \exp(-B) = \operatorname{Ad} \mathscr{A}(M_B).$$

Proof. (i) follows immediately from formula (2.1). For any $A \in \mathscr{A}$ by (2.1),

$$AM_B = A \exp(B)\, M = \exp(B)\, A'M \subseteq M_B$$

where $A' = \exp(-\operatorname{ad} B)(A) \in \mathscr{A}$. Thus M_B is invariant under all operators from $\mathscr{A}$. Since $\exp(B)$ is invertible, M_B is closed and therefore $M_B \in \operatorname{Lat} \mathscr{A}$. For any $M \in \operatorname{Lat} \mathscr{A}$ the subspace M_{-B} belongs to $\operatorname{Lat} \mathscr{A}$ and $\exp(B) M_{-B} = M$. Therefore $\exp(B)$ maps $\operatorname{Lat} \mathscr{A}$ onto itself.

By Lemma 1.1 (i), $\operatorname{Ad} \mathscr{A}$ is closed in the norm topology. Hence, by (2.1), $\exp(B) C \exp(-B) \in \operatorname{Ad} \mathscr{A}$ for any $C \in \operatorname{Ad} \mathscr{A}$. Since $\exp(\operatorname{ad} B)$ is invertible, $C \mapsto \exp(\operatorname{ad} B)(C)$ is an automorphism of $\operatorname{Ad} \mathscr{A}$. If $C \in \operatorname{Ad} \mathscr{A}(M)$, then

$$\exp(B) C \exp(-B) M_B = \exp(B) CM \subseteq M_B$$

which completes the proof of the lemma.

DEFINITION. We say that a subset ω in $\operatorname{Lat} \mathscr{A}$ *is invariant under* $\operatorname{Ad} \mathscr{A}$ if for every $M \in \omega$ and for every $B \in \operatorname{Ad} \mathscr{A}$ the subspace M_B belongs to ω. If ω is invariant and if it does not contain any invariant subset, then it *is called an orbit.*

LEMMA 2.2. (i) $\mathscr{L}_n$ *are invariant under* $\operatorname{Ad} \mathscr{A}$.

(ii) $\omega = \{\exp(B_1) \ldots \exp(B_k) M : B_1, \ldots, B_k$ *belong to* $\operatorname{Ad} \mathscr{A}\}$ *is the orbit which contains* M.

(iii) *If* ω_1 *and* ω_2 *are two orbits, then* $\omega_1 \cap \omega_2 = \emptyset$.

Proof. If $M \in \mathscr{L}_n$, then there exists a subspace N in $\operatorname{Ad} \mathscr{A}$ such that $\dim N = n$ and that $\operatorname{Ad} \mathscr{A} = N \dotplus \operatorname{Ad} \mathscr{A}(M)$. By Lemma 2.1 (iii)

$$\operatorname{Ad} \mathscr{A} = \exp(B) \operatorname{Ad} \mathscr{A} \exp(-B) = \exp(B) N \exp(-B) \dotplus \operatorname{Ad} \mathscr{A}(M_B).$$

Since $\dim(\exp(B) N \exp(-B)) = n$, we have that $M_B \in \mathscr{L}_n$ and (i) is proved. Obviously, ω is the smallest set which contains M and is invariant under $\operatorname{Ad} \mathscr{A}$. If $M_1 = \exp(B_1) \ldots \exp(B_k) M$, then $M = \exp(-B_k) \ldots \exp(-B_1) M_1$, so that ω does not contain any invariant subset. Therefore ω is an orbit.

Since ω_1 and ω_2 are invariant under $\mathrm{Ad}\,\mathscr{A}$, $\omega_1 \cap \omega_2$ is also invariant under $\mathrm{Ad}\,\mathscr{A}$. Since $\omega_1 \cap \omega_2 \subseteq \omega_1$ and since ω_1 is a smallest invariant subset, we have that either $\omega_1 \cap \omega_2 = \emptyset$ or $\omega_1 \cap \omega_2 = \omega_1$. But in the last case $\omega_1 \subset \omega_2$ which contradicts the condition that ω_2 is also a smallest invariant subset. This completes the proof of the lemma.

Let ω be an orbit in $\mathrm{Lat}\,\mathscr{A}$. Set

$$I(\omega) = \bigcap_{L \in \omega} \mathrm{Ad}\,\mathscr{A}(L),$$

$$K(\omega) = \bigcap_{L \in \omega} L,$$

$$M(\omega) = \mathrm{sp}(\sum_{L \in \omega} L),$$

where $\sum_{L \in \omega} L$ is the linear space of all linear combinations of elements from all $L \in \omega$.

LEMMA 2.3. (i) *$I(\omega)$ is a closed Lie ideal of* $\mathrm{Ad}\,\mathscr{A}$ *and $I(\omega) = I(L)$ for any $L \in \omega$. If I is a Lie ideal of* $\mathrm{Ad}\,\mathscr{A}$ *such that $\mathscr{A} \subseteq I \subseteq \mathrm{Ad}\,\mathscr{A}(L)$, then $I \subseteq I(\omega)$.*

(ii) *The subspaces $K(\omega)$ and $M(\omega)$ belong to $\mathscr{L}_0$. If $L \in \omega$, then for any N from $\mathscr{L}_0$ such that $N \subseteq L$ $(L \subseteq N)$, we have that $N \subseteq K(\omega)$ $(M(\omega) \subseteq N)$.*

Proof. By Lemma 1.1 (ii), $I(\omega)$ is a closed Lie subalgebra of $\mathrm{Ad}\,\mathscr{A}$. By Lemma 2.1 (iii), for any $B \in \mathrm{Ad}\,\mathscr{A}$, for any complex t, for any $L \in \omega$ and for any $A \in I(\omega)$

$$\exp(tB)A\exp(-tB) \in \exp(tB)\,\mathrm{Ad}\,\mathscr{A}(L)\exp(-tB) = \mathrm{Ad}\,\mathscr{A}(L_{tB}).$$

Since the map $L \mapsto L_{tB}$ maps ω onto itself, $\exp(tB)A\exp(-tB) \in I(\omega)$. Since $I(\omega)$ is closed, using formula (2.1) and differentiating with respect to t, we obtain that $[B, A] \in I(\omega)$. Therefore $I(\omega)$ is a closed Lie ideal of $\mathrm{Ad}\,\mathscr{A}$.

If I is a closed Lie ideal of $\mathrm{Ad}\,\mathscr{A}$ such that $\mathscr{A} \subseteq I \subseteq \mathrm{Ad}\,\mathscr{A}(L)$ for a subspace $L \in \omega$, then, by (2.1) and by Lemma 2.1(iii),

$$\exp(B)I\exp(-B) = I \subseteq \mathrm{Ad}\,\mathscr{A}(L_B)$$

for any $B \in \mathrm{Ad}\,\mathscr{A}$. From Lemma 2.2(ii) it follows that $I \subseteq \mathrm{Ad}\,\mathscr{A}(M)$ for all $M \in \omega$, so that $I \subseteq I(\omega)$. Therefore $I(L) \subseteq I(\omega)$. On the other hand, by Lemma 1.1, $I(\omega) \subseteq I(L)$. Therefore $I(\omega) = I(L)$ and (i) is proved.

Since all subspaces in $\mathrm{Lat}\,\mathscr{A}$ are closed, $K(\omega)$ is a closed subspace. If $x \in K(\omega)$, then for any $B \in \mathrm{Ad}\,\mathscr{A}$, for any $L \in \omega$ any for any complex t

$$\exp(tB)x \in \exp(tB)L = L_{tB} \in \omega.$$

Therefore $\exp(tB)x \in K(\omega)$. Since $K(\omega)$ is closed, differentiation by t gives us that, $Bx \in K(\omega)$. Hence $K(\omega) \in \mathscr{L}_0$.

If $N \in \mathscr{L}_0$ and if $N \subseteq L$ for a certain $L \in \omega$, then $N = \exp(B)N \subseteq L_B$ for any $B \in \operatorname{Ad}\mathscr{A}$. Therefore $N \subseteq M$ for any $M \in \omega$, so that $N \subseteq K(\omega)$. In the same way one can prove (ii) for $M(\omega)$. The lemma is proved.

Definition. By ψ we shall denote the mapping

$$\psi : \omega \mapsto I(\omega)$$

of the set of all orbits in $\operatorname{Lat}\mathscr{A}$ into the set of all closed Lie ideals of $\operatorname{Ad}\mathscr{A}$.

In an early paper [5] we obtained the structure of orbits in $\mathscr{L}_1$ for the case when the quotient Lie algebra $\operatorname{Ad}\mathscr{A}/\mathscr{A}$ is finite-dimensional. But using Theorem 1.2 and repeating the argument of Theorem 3.2 and of Theorem 3.5 [5], one can obtain that the structure of orbits ω in $\mathscr{L}_1$ is the same even in the case when $\operatorname{Ad}\mathscr{A}/\mathscr{A}$ is infinite dimensional.

Theorem 2.4. *Let $\omega \subset \mathscr{L}_1$ and let $M \in \omega$.*

(i) *Let* $\operatorname{codim} I(M) = 2$ *and let the operators B_- and B_0 be the same as in Theorem* 1.2(ii). *Then there exists a subspace L_∞ in ω such that*

1) $B_0 L_\infty \subseteq L_\infty$ *and* $\omega = \{L_t = \exp(B_-/t)L_\infty : t \in S^2\diagdown\{0\}\}$;

2) $\operatorname{Ad}\mathscr{A}(L_t) = \{B_- + tB_0\} \dotplus I(\omega)$, $\operatorname{Ad}\mathscr{A}(L_\infty) = \{B_0\} \dotplus I(\omega)$;

3) $L_t \neq L_u$ *if* $t \neq u$. *For every* $B = pB_- + rB_0 + A \in \operatorname{Ad}\mathscr{A}$, *where* $A \in I(\omega)$ *and p and r are complex numbers,*

$$\exp(B)L_t = L_u,$$

where $u = \pi(g(h))t$, $g(h) = \exp h$ *and* $h = \begin{pmatrix} r/2 & 0 \\ p & -r/2 \end{pmatrix}$.

(ii) *Let* $\operatorname{codim} I(\omega) = 3$ *and let the operators B_-, B_0 and B_+ be the same as in Theorem* 1.2(iii). *Then there exist subspaces L_0 and L_∞ in ω such that*

1) $\omega = [L_t : t \in S^2\}$ *and* $L_t = \exp(B_-/t)L_\infty = \exp(tB_+)L_0$, *for* $t \neq 0, \infty$.

2) $\operatorname{Ad}\mathscr{A}(L_t) = \{B_- + tB_0\} \dotplus \{B_0 - tB_+\} \dotplus I(\omega)$ *and* $\operatorname{Ad}\mathscr{A}(L_\infty) = \{B_0\} \dotplus \dotplus \{B_+\} \dotplus I(\omega)$;

3) $L_t \neq L_u$ *if* $t \neq u$. *For every* $B = pB_- + rB_0 + sB_+ + A$, *where* $A \in I(\omega)$ *and p, r and s are complex numbers,*

$$\exp(B)L_t = L_u,$$

where $u = \pi(g)t$, $g = \exp h$ *and* $h = \begin{pmatrix} r/2 & s \\ p & -r/2 \end{pmatrix}$.

Remark 2.5. If $\operatorname{codim} I(M) = 1$, i.e., $\operatorname{Ad}\mathscr{A}(M) = I(\omega)$, then there exists an operator B_- in $\operatorname{Ad}\mathscr{A}$ such that $\operatorname{Ad}\mathscr{A} = \{B_-\} \dotplus I(\omega)$. Let $B = tB_- + A$, where

$A \in I(\omega)$. From the Campbell-Hausdorff formula [1] it can be derived thatthere exists an operator $A_1 \in I(\omega)$ such that

$$\exp B = \exp(tB_- + A) = \exp(tB_-)\exp(A_1).$$

Therefore $M_B = M_{tB_-}$. Since $\mathrm{Ad}\,\mathscr{A}(M)$ is a closed Lie ideal in $\mathrm{Ad}\,\mathscr{A}$, it follows from Lemma 2.1(iii), that $\mathrm{Ad}\,\mathscr{A}(M_B) = \mathrm{Ad}\,\mathscr{A}(M)$. From this and from Lemma 2.2(ii) we obtain that $\omega = \{M_{tB_-} : t \in \mathbf{C}\}$, but we do not know whether all subspace sM_{tB_-} are different.

Now let L and M belong to $\mathscr{L}_1$ and let $\mathrm{Ad}\,\mathscr{A}(L) \neq \mathrm{Ad}\,\mathscr{A}(M)$. Then $\mathscr{B} = \mathrm{Ad}\,\mathscr{A}(L) \cap \mathrm{Ad}\,\mathscr{A}(M)$ is contained in $\mathrm{Ad}\,\mathscr{A}(L \cap M)$ and in $\mathrm{Ad}\,\mathscr{A}(\mathrm{sp}(L + M))$. Since $\mathrm{codim}\,\mathscr{B} \leqslant 2$, we have that

$$L \cap M \in \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2 \quad \text{and} \quad \mathrm{sp}(L + M) \in \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2. \tag{2.2}$$

In the following two theorems we shall consider the case when the subspaces $L \cap M$ and $\mathrm{sp}(L + M)$ belong to $\mathscr{L}_1$.

THEOREM 2.6. *Let subspaces L and M belong to $\mathscr{L}_1$, let $\mathrm{Ad}\,\mathscr{A}(L) \neq \mathrm{Ad}\,\mathscr{A}(M)$ and let $N = \mathrm{sp}(L + M) \in \mathscr{L}_1$.*

(i) $\mathrm{Ad}\,\mathscr{A}(N)$ *coincides with one of the Lie subalgebras* $\mathrm{Ad}\,\mathscr{A}(L)$ *or* $\mathrm{Ad}\,\mathscr{A}(M)$.

(ii) *If* $\mathrm{Ad}\,\mathscr{A}(N) = \mathrm{Ad}\,\mathscr{A}(M)$, *then*

a) $M(\omega_L) \subseteq N$, *where ω_L is the orbit which contains L,*

b) $N = \mathrm{sp}(M \dotplus L_1)$ *for any $L_1 \in \omega_L$ such that* $\mathrm{Ad}\,\mathscr{A}(M) \neq \mathrm{Ad}\,\mathscr{A}(L_1)$.

Proof. Since $\mathrm{Ad}\,\mathscr{A}(L) \neq \mathrm{Ad}\,\mathscr{A}(M)$, $\mathrm{codim}\,\mathscr{B} = 2$ and there exist operators $B_1 \in \mathrm{Ad}\,\mathscr{A}(L)$ and $B_2 \in \mathrm{Ad}\,\mathscr{A}(M)$ such that

$$\begin{gathered}\mathrm{Ad}\,\mathscr{A}(L) = \{B_1\} \dotplus \mathscr{B}, \quad \mathrm{Ad}\,\mathscr{A}(M) = \{B_2\} \dotplus \mathscr{B}, \\ \mathrm{Ad}\,\mathscr{A} = \{B_1\} \dotplus \mathrm{Ad}\,\mathscr{A}(M) = \{B_2\} \dotplus \mathrm{Ad}\,\mathscr{A}(L) = \{B_1\} \dotplus \{B_2\} \dotplus \mathscr{B}.\end{gathered} \tag{2.3}$$

Since $\mathrm{codim}\,(\mathrm{Ad}\,\mathscr{A}(N)) = 1$, there exist complex numbers t_1 and t_2 such that

$$\mathrm{Ad}\,\mathscr{A}(N) = \{t_1B_1 + t_2B_2\} \dotplus \mathscr{B}.$$

Suppose that $\mathrm{Ad}\,\mathscr{A}(N) \neq \mathrm{Ad}\,\mathscr{A}(L)$. Then $t_2 \neq 0$. Since $L \subseteq N$,

$$(t_1B_1 + t_2B_2)x \in N$$

for every $x \in L$. But $B_1x \in L$. Therefore $B_2L \subseteq N$. Since $B_2M \subseteq M$, we obtain that $B_2(L \dotplus M) \subseteq N$, so that $B_2N \subseteq N$. Therefore $B_2 \in \mathrm{Ad}\,\mathscr{A}(N)$ and

$$\mathrm{Ad}\,\mathscr{A}(M) = \{B_2\} \dotplus \mathscr{B} = \mathrm{Ad}\,\mathscr{A}(N).$$

(i) is proved.

Let $\operatorname{Ad}\mathscr{A}(N) = \operatorname{Ad}\mathscr{A}(M)$. Since $L \subseteq N$ and since $B_2 \in \operatorname{Ad}\mathscr{A}(N)$,

$$L_{tB_2} = \exp(tB_2)L \subseteq \exp(tB_2)N = N \tag{2.4}$$

for every complex t. In order to prove a) we shall consider three different cases in which $\operatorname{codim}(I(\omega_L))$ is 1, 2 or 3.

1) $I(\omega_L) = \operatorname{Ad}\mathscr{A}(L)$. It follows from Remark 2.5 and from formula (2.3) that $\omega_L = \{L_{tB_2} : t \in \mathbf{C}\}$. By (2.4), all $L_{tB_2} \in N$. Therefore $M(\omega_L) \subseteq N$.

2) $\operatorname{codim} I(\omega_L) = 2$. Let the operators B_- and B_0 be the same as in Theorem 1.2(ii). Then there exist p and r such that

$$B_2 = pB_- + rB_0 + A,$$

where $A \in I(\omega_L)$, and, by (2.3), $p \neq 0$. By (2.4), $L_{xB_2} \subseteq N$ for all complex x. It follows from Theorem 2.4(i) that $\omega_L = \{L_t : t \in S^2\diagdown\{0\}\}$ and that $L = L_{t_0}$ for a certain $t_0 \in S^2\diagdown\{0\}$. Then

$$L_{xB_2} = \exp(xB_2)\, L_{t_0} = L_t$$

where $t = \pi(g)t_0$, $g = \exp(xh)$ and $h = \begin{pmatrix} r/2 & 0 \\ p & -r/2 \end{pmatrix}$. If $r \neq 0$, then, by (1.2) and by (1.5),

$$t = e^{xr}t_0 r/(pt_0(e^{xr} - 1) + r).$$

If $r = 0$, then, by (1.4) and by (1.5),

$$t = t_0/(xpt_0 + 1).$$

If $r = 0$, then, since $p \neq 0$, we have that t can be any element from $S^2\diagdown\{0\}$. Therefore all subspaces L_t from ω_L are contained in N. Hence $M(\omega_L) \subseteq N$.

If $r \neq 0$ and if $r - pt_0 \neq 0$, then t can be any element from $S^2\diagdown\{0\}$ apart from $t_1 = r/p$. Therefore all L_t, $t \neq r/p$, are contained in N. It follows from Theorem 2.4(i) that if t_i converge to r/p, then the projections P_{t_i} on L_{t_i} converge to the projection $P_{r/p}$ on $L_{r/p}$ in the norm topology. Therefore for any $y \in L_{r/p}$ the sequence $\{P_{t_i}y\}$ converges to y and $P_{t_i}y \in L_{t_i} \subseteq N$. Since N is closed, $y \in N$, so that $L_{r/p} \subseteq N$. Thus, $L_t \subseteq N$ for all L_t from ω_L and therefore $M(\omega_L) \subseteq N$.

Finally, if $r \neq 0$ but $r - pt_0 = 0$, then for all complex x

$$L_{xB_2} = L_{t_0} = L_{r/p}.$$

From this it follows that $B_2L \subseteq L$, so that $B_2 \in \mathrm{Ad}\,\mathscr{A}(L)$ which contradicts (2.3). Thus a) is proved in this case.

3) $\operatorname{codim} I(\omega_L) = 3$. The proof is similar to the proof of the previous case and uses the results of Theorems 1.2(iii) and 2.4(ii) and formulae (1.2)—(1.5).

The linear subspace $L + M$ is dense in N. Hence $\exp(tB_2)(L + M)$ is dense in $\exp(tB_2)N$, for all $t \in \mathbf{C}$. Since $\mathrm{Ad}\,\mathscr{A}(N) = \mathrm{Ad}\,\mathscr{A}(M)$, $B_2 \in \mathrm{Ad}\,\mathscr{A}(N)$ and $\exp(tB_2)(L + M) = \exp(tB_2)L + M$ is dense in $\exp(tB_2)N = N$. Therefore $\mathrm{sp}(\exp(tB_2)L + M) = N$.

If $I(\omega_L) = \mathrm{Ad}\,\mathscr{A}(L)$, then $\omega_L = \{L_{tB_2} = \exp(tB_2)L : t \in \mathbf{C}\}$ and b) holds.

If $\operatorname{codim} I(\omega_L) = 2$, then it was shown in 2) that $\exp(tB_2)L$ can be any subspace in ω_L apart perhaps from L_{t_1}, where $t_1 = r/p$ if $r \neq 0$ and $r - pt_0 \neq 0$. Thus b) holds for any subspace from ω_L apart perhaps from L_{t_1}. If $\mathrm{Ad}\,\mathscr{A}(L_{t_1}) \neq \mathrm{Ad}\,\mathscr{A}(M) = \mathrm{Ad}\,\mathscr{A}(N)$, then, since they have equal codimensions, there exists B in $\mathrm{Ad}\,\mathscr{A}(N)$, which does not belong to $\mathrm{Ad}\,\mathscr{A}(L_{t_1})$ and such that $\exp(B)L_{t_1} = L' \neq L_{t_1}$. L' belongs to ω_L and it was proved above that $\mathrm{sp}(L' + M) = N$. Then $\exp(-B)(L' + M) = \exp(-B)L' + M = L_{t_1} + M$ is dense in $\exp(-B)N = N$ and b) holds for L_{t_1} if $\mathrm{Ad}\,\mathscr{A}(L_{t_1}) \neq \mathrm{Ad}\,\mathscr{A}(M)$.

If $\operatorname{codim} I(\omega_L) = 3$, then, using (1.2)—(1.5) and Theorem 2.4(ii), one can show that $\exp(tB_2)L$ can be any subspace in ω_L apart perhaps from two subspaces L_{t_1} and L_{t_2}. Therefore b) holds for all these subspaces. Repeating the argument above we also obtain that b) holds for L_{t_i}, $i = 1, 2$, if $\mathrm{Ad}\,\mathscr{A}(L_{t_i}) \neq \mathrm{Ad}\,\mathscr{A}(M)$. The theorem is proved.

Using a similar argument one can prove the following theorem.

THEOREM 2.7. *Let subspaces L and M belong to $\mathscr{L}_1$, let* $\mathrm{Ad}\,\mathscr{A}(L) \neq \mathrm{Ad}\,\mathscr{A}(M)$ *and let $N = L \cap M \in \mathscr{L}_1$.*

(i) $\mathrm{Ad}\,\mathscr{A}(N)$ *coincides either with* $\mathrm{Ad}\,\mathscr{A}(L)$ *or with* $\mathrm{Ad}\,\mathscr{A}(M)$.

(ii) *If* $\mathrm{Ad}\,\mathscr{A}(N) = \mathrm{Ad}\,\mathscr{A}(M)$, *then*

a) *$M(\omega_N) \subseteq L$ where ω_N is the orbit which contains N;*

b) *$N = L_1 \cap M$ for any $L_1 \in \omega_L$ such that* $\mathrm{Ad}\,\mathscr{A}(L_1) \neq \mathrm{Ad}\,A(M)$.

COROLLARY 2.8. *If L and M belong to $\mathscr{L}_1$, if $L \subseteq M$ and* $\mathrm{Ad}\,\mathscr{A}(L) \neq \mathrm{Ad}\,\mathscr{A}(M)$, *then $M(\omega_L) \subseteq K(\omega_M)$.*

Proof. Since $L \cap M = L \in \mathscr{L}_1$, by Theorem 2.7(ii), $M(\omega_L) \subseteq M$. Since $K(\omega_M)$ is the largest subspace in M from $\mathscr{L}_0$, $M(\omega_L) \subseteq K(\omega_M)$.

THEOREM 2.9. *Let subspaces L and M belong to the same orbit ω in $\mathscr{L}_1$. If $2 \leqslant \operatorname{codim} I(\omega)$, then $L \cap M$ and* $\mathrm{sp}(L + M)$ *belong to $\mathscr{L}_0 \cup \mathscr{L}_2$.*

Proof. If $\operatorname{codim} I(\omega) = 1$, then for every $L \in \omega$, $\mathrm{Ad}\,\mathscr{A}(L) = I(\omega)$ and we cannot apply Theorem 2.6 and Theorem 2.7. But if $2 \leqslant \operatorname{codim} I(\omega)$, then it follows from Theorem 2.4(i) and (ii) that $\operatorname{Ad}\mathscr{A}(L) \neq \operatorname{Ad}\mathscr{A}(M)$ for any L and M from ω.

From (2.2) it follows that $L \cap M$ and $\operatorname{sp}(L + M)$ belong to $\mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2$. Suppose that $N = \operatorname{sp}(L + M) \in \mathscr{L}_1$. Then, by Theorem 2.6(i), $\operatorname{Ad}\mathscr{A}(N)$ coincides either with $\operatorname{Ad}\mathscr{A}(L)$ or with $\operatorname{Ad}\mathscr{A}(M)$. Assume that $\operatorname{Ad}\mathscr{A}(N) = \operatorname{Ad}\mathscr{A}(M)$. Then, by Theorem 2.6(ii), $M(\omega_L) \subseteq N$. On the other hand, since L and M lie on the same orbit, $\operatorname{sp}(L + M) = N \subseteq M(\omega_L)$. Therefore $N = M(\omega_L) \in \mathscr{L}_0$ which contradicts the assumption that $N \in \mathscr{L}_1$. Thus, $N \in \mathscr{L}_0 \cup \mathscr{L}_2$. In the same way, using Theorem 2.7, one can prove that $L \cap M \in \mathscr{L}_0 \cup \mathscr{L}_2$.

Definition. For any subspaces M and N in $\operatorname{Lat}\mathscr{A}$ such that $M \subseteq N$ let

$$(M, N) = \{L \in \operatorname{Lat}\mathscr{A} : M \subset L \subset N\}$$

and

$$[M, N] = M \cup N \cup (M, N).$$

We say that an orbit ω *is non-trivial* if it does not consist of only one subspace.

Definition. We say that a non-trivial orbit ω *is simple* if there exists $L \in \omega$ such that

$$(K(\omega), L) = (L, M(\omega)) = \emptyset.$$

Simple orbits exist as Examples 1 and 2 (see below) show.

Lemma 2.10. *Let ω be a simple orbit.*

(i) *For any M in ω, $(K(\omega), M) = (M, M(\omega)) = \emptyset$.*

(ii) *$K(\omega) = L \cap M$ and $M(\omega) = \operatorname{sp}(L + M)$ for any distinct L and M in ω.*

Proof. Let L in ω be such that $(K(\omega), L) = (L, M(\omega)) = \emptyset$ and let $M \in \omega$. Assume that there exists a subspace N in $\operatorname{Lat}\mathscr{A}$ for which $K(\omega) \subset N \subset M$. Then for every $B \in \operatorname{Ad}\mathscr{A}$

$$K(\omega) \subset \exp(B)N \subset \exp(B)M.$$

Since L and M belong to ω, there exists a set $\{B_i\}_{i=1}^n$ such that $L = \exp(B_1) \ldots \ldots \exp(B_n)M$. Then

$$K(\omega) \subset \exp(B_1) \ldots \exp(B_n)N \subset L$$

which contradicts the condition that $(K(\omega), L) = \emptyset$. Similarly we obtain that $(M, M(\omega)) = \emptyset$. (i) is proved.

We have that

$$K(\omega) \subseteq L \cap M \subseteq L \subseteq \mathrm{sp}(L + M) \subseteq M(\omega).$$

Since ω is simple, either $L \cap M = K(\omega)$ or $L \cap M = L$, and either $\mathrm{sp}(L + M) = M(\omega)$ or $\mathrm{sp}(L + M) = L$.

If $L \cap M = L$, then $L \subseteq M \subseteq M(\omega)$ which is only possible if $L = M$. But this contradicts the fact that L and M are different. Thus $L \cap M = K(\omega)$. In the same way one can prove that $\mathrm{sp}(L + M) = M(\omega)$. The lemma is proved.

THEOREM 2.11. *Let $\mathscr{A}$ be an operator algebra such that $\mathscr{L}_2 = \emptyset$ and let the mapping $\psi\colon \omega \subseteq \mathscr{L}_1 \mapsto I(\omega)$ be injective.*

(i) *If $M \in \omega$ and $L \in \omega_1$, for $\omega \neq \omega_1$, then either $L \cap M \in \mathscr{L}_0$ and $\mathrm{sp}(L + M) \in \mathscr{L}_0$, or $M(\omega) \subseteq K(\omega_1)$, or $M(\omega_1) \subseteq K(\omega)$.*

(ii) *If L and M lie on the same orbit ω and if $2 \leqslant \mathrm{codim}\, I(\omega)$, then $L \cap M \in \mathscr{L}_0$ and $\mathrm{sp}(L + M) \in \mathscr{L}_0$.*

(iii) *If $2 \leqslant \mathrm{codim}\, I(\omega)$, then ω is a simple orbit.*

Proof. If $\mathrm{Ad}\,\mathscr{A}(L) = \mathrm{Ad}\,\mathscr{A}(M)$, then

$$I(\omega) \subseteq \mathrm{Ad}\,\mathscr{A}(L).$$

By Lemma 2.3(i), $I(\omega) \subseteq I(\omega_1)$. In the same way we obtain that $I(\omega_1) \subseteq I(\omega)$, so that $I(\omega) = I(\omega_1)$. Since ψ is injective and since ω and ω_1 are different orbits, $I(\omega) \neq I(\omega_1)$. Therefore $\mathrm{Ad}\,\mathscr{A}(L) \neq \mathrm{Ad}\,\mathscr{A}(M)$.

Let $N = L \cap M$. If $N \notin \mathscr{L}_0$, then, by Theorem 2.7(i), $\mathrm{Ad}\,\mathscr{A}(N)$ is either equal to $\mathrm{Ad}\,\mathscr{A}(L)$ or to $\mathrm{Ad}\,\mathscr{A}(M)$. Let $\mathrm{Ad}\,\mathscr{A}(N) = \mathrm{Ad}\,\mathscr{A}(M)$. Repeating the argument above we obtain that N belongs to ω. Since $N \subseteq L$ and since $\mathrm{Ad}\,\mathscr{A}(N) \neq \mathrm{Ad}\,\mathscr{A}(L)$, it follows from Corollary 2.8 that $M(\omega) \subseteq K(\omega_1)$. In the same way, using Theorem 2.6(i) and Corollary 2.8, we obtain that if $\mathrm{sp}(L + M) \notin \mathscr{L}_0$, then either $M(\omega) \subseteq K(\omega_1)$ or $M(\omega_1) \subseteq K(\omega)$.

The proof of (ii) follows immediately from Theorem 2.9.

Finally, let $2 \leqslant \mathrm{codim}\, I(\omega)$, let $L \in \omega$ and let $N \in (K(\omega), L)$. If $\mathrm{Ad}\,\mathscr{A}(N) \neq \mathrm{Ad}\,\mathscr{A}(L)$, then, since $N \subset L$, it follows from Corollary 2.8 that $M(\omega_N) \subseteq K(\omega)$. Therefore $N \subseteq M(\omega_N) \subseteq K(\omega)$ which contradicts the fact that $N \in (K(\omega), L)$.

If $\mathrm{Ad}\,\mathscr{A}(N) = \mathrm{Ad}\,\mathscr{A}(L)$, then from the argument at the beginning of the proof it follows that N belongs to ω. But, since $2 \leqslant \mathrm{codim}\, I(\omega)$, it follows from Theorem 2.4(i) and (ii) that it is only possible if $N = L$. Therefore $(K(\omega), L) = \emptyset$. In the same way we obtain that $(L, M(\omega)) = \emptyset$ which completes the proof.

3. THE STRUCTURE OF OPERATOR ALGEBRAS WHICH HAVE SIMPLE ORBITS

In this section we consider a simple orbit ω and investigate the structure of $P\mathscr{A}P$ where P is the projection onto $M(\omega) \ominus K(\omega)$.

Halmos [4] studied subspaces K and L in a Hilbert space H in generic position, that is,

$$K \cap L = K \cap L^{\perp} = K^{\perp} \cap L = K^{\perp} \cap L^{\perp} = \{0\}.$$

In order to prove Theorem 3.1 about simple orbits we need to consider subspaces K and L in H which only satisfy two out of these four conditions.

LEMMA 3.0. *Let K be a closed subspace in a Hilbert space H, so that $H = K \oplus K^{\perp}$ and $H = \left\{\begin{pmatrix} y \\ x \end{pmatrix} : y \in K,\ x \in K^{\perp}\right\}$. If L is a closed subspace in H such that $K \cap L = \{0\}$ and $\mathrm{sp}(K + L) = H$ (the conditions $K^{\perp} \cap H^{\perp} = \{0\}$ and $\mathrm{sp}(K + L) = H$ are equivalent), then there exists a closed operator F from $K^{\perp}$ into K such that $D(F)$ is dense in $K^{\perp}$ and that $L = \left\{\begin{pmatrix} Fx \\ x \end{pmatrix} : x \in D(F)\right\}$.*

Proof. Let Q be the projection onto K. Then $1 - Q$ maps L onto a linear subspace D in $K^{\perp}$. If D is not dense in $K^{\perp}$, then there exists $x \in K^{\perp}$ such that $((1 - Q)z, x) = 0$ for all $z \in L$. Then $(z, x) = 0$ for all $z \in L$ and, obviously, $(y, x) = 0$ for $y \in K$. Therefore x is orthogonal to $K + L$ which contradicts the assumption that $\mathrm{sp}(K + L) = H$. Therefore D is dense in $K^{\perp}$.

Let $x \in D$. There exists $z \in L$ such that $(1 - Q)z = x$. If z_1 is another element in L such that $(1 - Q)z_1 = x$, then

$$z - z_1 = Q(z - z_1) + (1 - Q)(z - z_1) = Q(z - z_1).$$

Hence $z - z_1 \in K \cap L$, so that $z = z_1$. Therefore there exists a linear operator S from D onto L such that $(1 - Q)Sx = x$ for every $x \in D$ and that $S(1 - Q)z = z$ for every $z \in L$.

For every $x \in D$ set $Fx = QSx$. F is a linear operator from $K^{\perp}$ into K, $D(F) = D$ and for every $z \in L$,

$$z = \begin{pmatrix} Qz \\ (1 - Q)z \end{pmatrix} = \begin{pmatrix} Fx \\ x \end{pmatrix} \quad \text{where } x = (1 - Q)z.$$

Let $z_n = \begin{pmatrix} Fx_n \\ x_n \end{pmatrix} \in L$, let $x_n \to x$ and $Fx_n \to y$. Then $z_n = \begin{pmatrix} y \\ x \end{pmatrix}$ and, since L is closed, $\begin{pmatrix} y \\ x \end{pmatrix} \in L$. Therefore $x \in D(F)$ and $y = Fx$ so that F is closed which completes the proof.

THEOREM 3.1. *Let ω be a simple orbit in* Lat $\mathscr{A}$ *and let* $L \in \omega$. *Set*

$$H_1 = K(\omega), \quad H_2 = L \ominus K(\omega), \quad H_3 = M(\omega) \ominus L, \; H_4 = \mathscr{H} \ominus M(\omega).$$

There exist closed operators F and $G \neq 0$ from H_3 into H_2 such that every operator $A = (A_{ij}) \in \mathscr{A}$, $1 \leqslant i, j \leqslant 4$ satisfies the following conditions:

(C_1) $A_{33}D(F) \subseteq D(F)$ *and* $A_{33}D(G) \subseteq D(G)$;

(C_2) $A_{22}G \,|\, D(G) = GA_{33} \,|\, D(G)$;

(C_3) $A_{23} \,|\, D(F) = (FA_{33} - A_{22}F) \,|\, D(F)$;

(C_4) $A_{ij} = 0$ *if* $i > j$;

(C_5) *algebras* $\mathscr{B}_2 = \{A_{22} : A \in \mathscr{A}\}$ *and* $\mathscr{B}_3 = \{A_{33} : A \in \mathscr{A}\}$ *are transitive on* H_2 *and on* H_3 *correspondingly.*

Proof. We have that $\mathscr{H} = H_1 \oplus H_2 \oplus H_3 \oplus H_4$. Let $A = (A_{ij}) \in \mathscr{A}$. Then $A_{ij} = 0$, if $i > j$, since the subspaces $K(\omega)$, L and $M(\omega)$ belong to Lat $\mathscr{A}$, and (C_4) holds. Since L does not belong to $\mathscr{L}_0$, there exists $B = (B_{ij}) \in \mathrm{Ad}\,\mathscr{A}$ such that L is not invariant for B. Taking this into account and the fact that $K(\omega)$ and $M(\omega)$ belong to $\mathscr{L}_0$ we obtain that $B_{21} = B_{31} = B_{41} = B_{42} = B_{43} \neq 0$ and $B_{32} \neq 0$.Using that $A' = [B, A] \in \mathscr{A}$ for any $A \in \mathscr{A}$, we obtain

$$B_{32}A_{22} = A_{33}B_{32}. \tag{3.1}$$

It follows from (3.1) that Ker B_{32} is invariant for $\mathscr{B}_2$ and that Im B_{32} is invariant for $\mathscr{B}_3$. Therefore the subspaces $K(\omega) \oplus \mathrm{Ker}\, B_{32}$ and $L \oplus \mathrm{sp}(\mathrm{Im}\, B_{32})$ belong to Lat $\mathscr{A}$, and

$$K(\omega) \subseteq K(\omega) \oplus \mathrm{Ker}\, B_{32} \subseteq L \subseteq L \oplus \mathrm{sp}(\mathrm{Im}\, B_{32}) \subseteq M(\omega).$$

Since $B_{32} \neq 0$ and since ω is simple, it follows from Lemma 2.10(i) that Ker $B_{32} = 0$ and that $\mathrm{sp}(\mathrm{Im}\, B_{32}) = H_3$. Therefore the operator $G = B_{32}^{-1}$ is closed, $G \neq 0$ and $D(G) = \mathrm{Im}\, B_{32}$ is dense in H_3. By (3.1),

$$A_{33}D(G) \subseteq D(G) \quad \text{and} \quad A_{22}G \,|\, D(G) = GA_{33} \,|\, D(G) \tag{3.2}$$

for any $A \in \mathscr{A}$. Thus (C_2) holds. From the fact that ω is simple, in the same way as above, we obtain that $\mathscr{B}_2$ and $\mathscr{B}_3$ are transitive algebras on H_2 and on H_3 correspondingly, so that (C_5) holds.

Let L_1 be another subspace in ω. Then $L_1 = K(\omega) \oplus M_1$ and $M_1 \subseteq H_2 \oplus H_3$. By Lemma 2.10(ii), $L_1 \cap L = K(\omega)$, so that $M_1 \cap H_2 = \{0\}$, and $\mathrm{sp}(L + L_1) = M(\omega)$, so that $\mathrm{sp}(M_1 + H_2) = H_2 \oplus H_3$. By Lemma 3.0, there exists a closed operator F from H_3 into H_2 such that $D(F)$ is dense in H_3 and that

$M_1 = \left\{ \begin{pmatrix} Fx \\ x \end{pmatrix} : x \in D(F) \subseteq H_3 \right\}$. Since $L_1 \in \operatorname{Lat} \mathscr{A}$, we have

$$A_{33} D(F) \subseteq D(F) \quad \text{and} \quad A_{23} \,|\, D(F) = (FA_{33} - A_{22} F) \,|\, D(F).$$

From this it follows that (C_1) and (C_3) hold which completes the proof.

Let H and K be Hilbert spaces, let $\mathscr{K} = H \oplus K$ and let F and G be closed operators from K into H. Let

$$\mathscr{A}(F, G) = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in B(\mathscr{K}) : (C_1)\, A_{22} D(F) \subseteq D(F),\ A_{22} D(G) \subseteq D(G); \right.$$

$$\left. (C_2)\, A_{11} G \,|\, D(G) = G A_{22} \,|\, D(G);\ (C_3)\, A_{12} \,|\, D(F) = (F A_{22} - A_{11} F \,|\, D(F) \right\}.$$

Let us consider the following restrictions of F and G.

(R_1) $D^* = D(F^*) \cap D(G^*)$ is dense in H and $D = D(F) \cap D(G)$ is dense in K;

(R_2) $G \neq 0$;

(R_3) GD is dense in H and $G^* D^*$ is dense in K.

Algebras $\mathscr{A}(F, G)$ were considered in [7] and the following theorem was obtained there.

THEOREM 3.2. ([7]). *Let the operators F and G satisfy* (R_1)—(R_3).

(i) *The operators $F + tG$ and $F^* + \bar{t}G^*$ are closable for any complex t.*

(ii) *Let $S_t = \operatorname{cl}(F + tG)$ and $R_t = \operatorname{cl}(F^* + \bar{t}G^*)$. Then $S_t \subseteq R_t^*$ and $S_0 \subseteq F \subseteq \subseteq R_0^*$. The algebra $\mathscr{A}(F, G)$ is reflexive if*

a) $\bigcap_{t \in \mathbf{C}} D(S_t) = D$ *and* $\operatorname{cl}(G \,|\, D) = G$, *or*

b) $\bigcap_{t \in \mathbf{C}} D(R_t) = D^*$ *and* $\operatorname{cl}(G^* \,|\, D^*) = G^*$.

(iii) $\operatorname{Lat} \mathscr{A}(F, G)$ *consists of* $\{0\}$, *of* H, *of* $\mathscr{K}$ *and of all subspaces* $M_S = \left\{ \begin{pmatrix} Sx \\ x \end{pmatrix} : x \in D(S) \subseteq K \right\}$, *where S can be S_t, R_t^*, for $t \in \mathbf{C}$, F or any closed operator from K into H such that*

1) *$S_t \subseteq S \subseteq R_t^*$ for a certain t,*

2) *$A_{22} D(S) \subset D(S)$ for any $A \in \mathscr{A}(F, G)$.*

From Theorem 3.1 it follows that if ω is a simple orbit in $\operatorname{Lat} \mathscr{A}$, then for every $L \in \omega$ there exist closed operators $F(L)$ and $G(L)$ from $M(\omega) \ominus L$ into $L \ominus K(\omega)$ such that $P \mathscr{A} P \subseteq \mathscr{A}(F(L), G(L))$. Therefore $\operatorname{Lat} \mathscr{A}(F(L), G(L)) \subseteq [K(\omega), M(\omega)]$.

However it is not clear whether the operators $F(L)$ and $G(L)$ always satisfy restrictions (R_1) and (R_3). The following theorem can be proved.

THEOREM 3.3. *If $P\mathscr{A}P$ contains a finite rank operator, then ω lies in $\mathscr{L}_1$ and the operators $F(L)$ and $G(L)$ satisfy* (R_1) *and* (R_3), *so that Theorem* 3.2 *holds for* $\mathscr{A}(F(L), G(L))$.

REMARK. It follows from Theorem 3.2 that the structure of Lat $\mathscr{A}(F, G)$ can be quite complicated because it is difficult to describe all the operators S which satisfy conditions 1) and 2). However it is possible to prove that if the subspace H does not belong to $\mathscr{L}_0$, then the structure of Lat $\mathscr{A}(F, G)$ is much simpler. Namely, $S_t = R_t^*$ for all t except for no more than two values of t at most. Using this and the results of Theorems 3.1 and 3.3, one can describe the structure of $[K(\omega), M(\omega)]$ for a simple orbit ω in detail. It can also be proved that $\mathscr{A}(F, G)$ is reflexive if H does not belong to $\mathscr{L}_0$.

Now we shall consider two examples of reflexive operator algebras which have simple orbits and which belong to $\mathscr{R}_1$.

EXAMPLE 1. Let $K = H$ and let $G = I_H$. Reflexive algebras

$$\mathscr{A}(F) = \mathscr{A}(F, I_H) = \left\{\begin{pmatrix} A & A_F \\ 0 & A \end{pmatrix} \in B(\mathscr{K}) : 1)\ AD(F) \subseteq D(F), \right.$$
$$\left. 2)\ A_F \mid D(F) = [F, A] \mid D(F)\right\}$$

were considered in [5]. It was shown that Lat $\mathscr{A}(F)$ consists of $\mathscr{K}$, of $\{0\}$, of H and of all subspaces $M_t = \left\{\begin{pmatrix} Fx + tx \\ x \end{pmatrix} : x \in D(F)\right\}$. Put $M_\infty = H$.

(i) If F is bounded, then Lat $\mathscr{A}(F) = \mathscr{L}_0 \cup \mathscr{L}_1$ and $\dim(\operatorname{Ad}\mathscr{A}(F)/(\mathscr{A}(F)) = 3$, where $\mathscr{L}_0 = \{0\} \cup \mathscr{K}$ and $\mathscr{L}_1 = \bigcup_{t \in S^2} M_t = \omega$ consists of one orbit with respect to Ad $\mathscr{A}(F)$.

(ii) If F is unbounded, then $\dim(\operatorname{Ad}\mathscr{A}(F)/\mathscr{A}(F))$ is 1 or 2 and Lat $\mathscr{A} = \mathscr{L}_0 \cup \mathscr{L}_1$, where $\mathscr{L}_0 = \{0\} \cup M_\infty \cup \mathscr{K}$ and $\mathscr{L}_1 = \bigcup_{t \in \mathbf{C}} M_t = \omega$ consists of one orbit with respect to Ad $\mathscr{A}(F)$.

In both cases ω is simple, $K(\omega) = \{0\}$, $M(\omega) = \mathscr{K}$, $I(\omega) = \mathscr{A}(F)$, and $M_t \cap M_r = \{0\}$ and $\operatorname{sp}(M_t \dotplus M_r) = \mathscr{K}$ for all $t \neq r$ from S^2.

EXAMPLE 2. Let $\mathscr{H}$ be the direct sum of $n+1$ copies of H and let $\{F_i\}_{i=1}^n$ be closed operators on H. Let

$$\mathscr{A} = \left\{ \begin{pmatrix} A & A_{F_1} & * & \cdots & * \\ 0 & A & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ \vdots & & \ddots & \ddots & A_{F_n} \\ 0 & \cdots & \cdots & 0 & A \end{pmatrix} \in B(\mathscr{H}) : \begin{array}{l} 1)\ AD(F_i) \subseteq D(F_i) \text{ for all } i, \\ 2)\ A_{F_i} \mid D(F_i) = [F_i, A] \mid D(F_i) \text{ for all } i \end{array} \right\}.$$

By $\mathscr{H}_i$, $0 \leqslant i \leqslant n+1$, we denote subspaces of $\mathscr{H}$ which are the direct sum of i copies of H, so that

$$\{0\} = \mathscr{H}_0 \subset \mathscr{H}_1 = H \subset \ldots \subset \mathscr{H}_{n+1} = \mathscr{H}.$$

Then all $\mathscr{H}_i \in \operatorname{Lat}\mathscr{A}$. Set $S_t^i = F_i + tI$ for all $t \in \mathbf{C}$. Let

$$M_{S_t^i} = \left\{ \begin{pmatrix} S_t^i x \\ x \end{pmatrix} : x \in D(F_i) \right\} \subseteq H \oplus H \quad \text{and} \quad \mathscr{M}_{S_t^i} = \mathscr{H}_{i-1} \oplus M_{S_t^i}.$$

We can consider $\mathscr{M}_{S_t^i}$ as subspaces in $\mathscr{H}_{i+1}$. Set $D = \bigcap_{i=1}^n D(F_i)$ and $D^* = \bigcap_{i=1}^n D(F_i^*)$

In [5] the following theorem was proved.

THEOREM 3.4. ([5]). *Let D and D^* be dense in H and let for every i the closure of $F_i \mid D = F_i$ and the closure of $F_i^* \mid D^* = F_i^*$. Then $\mathscr{A}$ is a reflexive algebra,* $\operatorname{Lat}\mathscr{A} = \mathscr{L}_0 \cup \mathscr{L}_1$, $\mathscr{L}_0 = \bigcup_{i=1}^{n+1} \mathscr{H}_i$, $\mathscr{L}_1 = \bigcup_{i=1}^n \omega_i$ *and* $\omega_i = \bigcup_{t \in \mathbf{C}} \mathscr{M}_{S_t^i}$ *are orbits with respect to* $\operatorname{Ad}\mathscr{A}$. $\dim(\operatorname{Ad}\mathscr{A}/I(\omega_i))$ *is* 1 *or* 2 *for every* i.

From this theorem it follows that all orbits ω_i are simple and that $K(\omega_i) = \mathscr{H}_{i-1}$ and $M(\omega_i) = \mathscr{H}_{i+1}$. It is also easy to see that

(1) if $t \neq r$, $\mathscr{M}_{S_t^i} \cap \mathscr{M}_{S_r^i} = \mathscr{H}_{i-1}$ and $\operatorname{sp}(\mathscr{M}_{S_t^i} + \mathscr{M}_{S_r^i}) = \mathscr{H}_{i+1}$;

(2) for all t and r, $\mathscr{M}_{S_t^i} \cap \mathscr{M}_{S_r^{i+1}} = \mathscr{H}_{i-1}$ and $\operatorname{sp}(\mathscr{M}_{S_t^i} + \mathscr{M}_{S_r^{i+1}}) = \mathscr{H}_{i+2}$;

(3) $M(\omega_i) = \mathscr{H}_{i+1} \subseteq K(\omega_j) = \mathscr{H}_{j-1}$ if $i + 1 < j$,
which agrees with the results of Theorem 2.11.

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