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TENSOR PRODUCTS OF LINEAR OPERATORS IN BANACH SPACES AND TAYLOR'S JOINT SPECTRUM. II

VOLKER WROBEL

1. INTRODUCTION

In a previous paper [16] we proved that Taylor's joint spectrum for an n-tuple of tensor products of operators acting on Banach spaces is equal to the Cartesian product of their spectra provided $n = 2$ or the uniform cross-norm under consideration is associative. Though this includes the ε- and the π-tensor products there are many important uniform cross-norms which are non-associative and therefore excluded from our considerations if $n > 2$. In the present paper we extend the result mentioned above to non-associative cross-norms.

In connection with this result we investigate the problem of tensor product stability of Taylor's joint spectrum and we introduce a modified joint spectrum. For $n = 2$ we characterize this joint spectrum in terms of the invertibily of a single operator similar to Vasilescu's approach in the case of Hilbert space operators [16].

Though this is a sequel of our paper [16] we recall some definitions and facts which have to be used in explicit form in subsequent constructions. First we recall J. L. Taylor's concept of joint spectrum for an n-tuple of mutually commuting operators as given in [13] and [14].

Let $s = (s_1, \dots, s_n)$ denote an n-tuple of indeterminates and let X denote a complex Banach space. Then

$$\wedge[s; X] := \bigoplus_{p=0}^{\infty} \wedge^p [s; X]$$

denote the exterior forms over the complex numbers $\mathbf{C}$ with coefficients in X generated by the indeterminates $s_1, \dots, s_n$ and $\wedge^p[s; X]$, the space of p-forms. Exterior multiplication is denoted by $\wedge$. Given an n-tuple $(a_1, \dots, a_n) \in L(X)^n$ of mutually commuting operators, we consider the *Koszul complex* $F(X; a)$:

$$(1.1) \qquad 0 \to \wedge^0[s; X] \xrightarrow{\delta^0(a)} \wedge^1[s; X] \dots \xrightarrow{\delta^{n-1}(a)} \wedge^n[s; X] \to 0$$

where

$$\delta^p(a):\ \wedge^p[s;X]\to\wedge^{p-1}[s;X]$$

given by

$$\delta^p(a)x_{j_1\dots j_p}\otimes s_{j_1}\wedge\dots\wedge s_{j_p}:=\sum_{k=1}^n a_k(x_{j_1\dots j_p})\otimes s_k\wedge s_{j_1}\wedge\dots\wedge s_{j_p}$$

acts as a coboundary operator on $\wedge[s;X]$ since $a=(a_1,\dots,a_n)$ consists of mutually commuting operators. Note that in accordance with this definition $\delta^p(a)=0$ for $p\in\mathbf{Z}\setminus\{0,1,\dots,n-1\}$. We say that $a=(a_1,\dots,a_n)$ is *non-singular* (*singular*), if the sequence $F(X,a)$ is exact (non-exact). By definition $z\in\mathbf{C}^n$ belongs to *Taylor's joint spectrum* $\sigma(a;X)$ if $a-z:=(a_1-z_1,\dots,a_n-z_n)$ is singular.

A *cross-norm* α associates with each pair of normed spaces (X,Y) a norm $\alpha(\cdot\,;X\otimes Y)$ on the algebraic tensor product $X\otimes Y$ such that

$$\alpha(x\otimes y;X\otimes Y)=\|x\|\cdot\|y\|\quad\text{for all } x\in X,\ y\in Y.$$

Given (X,Y) we denote this normed tensor product by $X\otimes_\alpha Y$ and its completion by $X\hat{\otimes}_\alpha Y$. Moreover, once having fixed X and Y we shall write $\alpha(z)$ instead of $\alpha(z;X\otimes Y)$ for $z\in X\otimes Y$. A cross-norm α is a *quasi-uniform cross-norm* (Ichinose [8]) if for each 4-tuple (X_1,X_2,Y_1,Y_2) of normed spaces there exists a constant $c=c(X_1,X_2,Y_1,Y_2)$ such that

$$\alpha((a\otimes b)z;Y_1\otimes Y_2)\leqslant c\|a\|\cdot\|b\|\alpha(z;X_1\otimes X_2)$$

for all $a\in L(X_1,Y_1)$, $b\in L(X_2,Y_2)$.

α is called *uniform cross-norm*, if α is quasi-uniform with $c=1$ (Schatten [12]). The most prominent uniform cross-norms are the π- (the greatest uniform cross-norm) and the ε-cross-norm (the smallest uniform cross-norm). Given $a\in L(X_1,Y_1)$, $b\in L(X_2,Y_2)$ let $a\hat{\otimes}_\alpha b$ denote the unique extension of $a\otimes b$ upon $X_1\hat{\otimes}_\alpha X_2$.

If only two Banach spaces X and Y are considered, there may be a norm on $X\otimes Y$ which fulfils the assumptions of a quasi-uniform cross-norm on $X\otimes Y$, but which need not be defined for all pairs of Banach spaces. Such a norm will be called *individual* quasi-uniform cross-norm in distinguishing it from the functorial concept of a quasi-uniform cross-norm.

Observe that given an individual quasi-uniform cross-norm α on $X\otimes Y$, we can extend the definition of α upon $\left(\prod_1^n X\right)\otimes Y$ by setting

$$\alpha\left(\sum_i x_i\otimes y_i;\left(\prod_1^n X\right)\otimes Y\right):=\|(\alpha(\sum_i x_{ik}\otimes y_i))_k\|_{l_p},$$

where $\|\cdot\|_{l_p}$ denotes the l_p-norm and x_{ik} denotes the k-th component of $x_i\in\prod_1^n X$.

If X, Y, Z denote three normed spaces and α a quasi-uniform cross-norm, then $(X \otimes_\alpha Y) \otimes_\alpha Z$ and $X \otimes_\alpha (Y \otimes_\alpha Z)$ are the same sets, but in general they will represent non-isomorphic normed spaces, since α need not be associative.

In order to convince the reader that even very natural uniform cross-norms are non-associative, we present an example which is based on a cross-norm characterization of $\mathscr{L}_2$-spaces due to J. Harksen [7, 5.30]. The author is indebted to A. Defant for a conversation on this point.

1.1. EXAMPLE. Given a finite sequence $x_1, \ldots, x_k$ of elements of a Banach space X, let

$$\lambda_2((x_i)) := \sup\{(\sum_i |\varphi(x_i)|^2)^{1/2} : \varphi \in X', \|\varphi\| \leqslant 1\}.$$

Then we define Cohen's w_2-cross-norm (compare [7]) on $X \otimes Y$ by

$$w_2(z) := \inf\{\lambda_2((x_i)_i)\lambda_2((y_i)_i) : z = \sum_i x_i \otimes y_i\}.$$

w_2 is a uniform cross-norm, and by a result of J. Harksen [7, 5.30] the following are equivalent:

(1) X is an $\mathscr{L}_2$-space;

(2) For all Banach spaces Y the tensor products $X \otimes_{w_2} Y$ and $X \otimes_\varepsilon Y$ are topologically isomorphic.

Therefore let Y denote an arbitrary Banach space. If w_2 were associative, then

$$\begin{aligned} (\ell_2 \hat{\otimes}_{w_2} \ell_2) \hat{\otimes}_{w_2} Y = \ell_2 \hat{\otimes}_{w_2} (\ell_2 \hat{\otimes}_{w_2} Y) &= \\ = \ell_2 \hat{\otimes}_\varepsilon (\ell_2 \hat{\otimes}_\varepsilon Y) &= \qquad \text{as } \ell_2 \text{ is an } \mathscr{L}_2\text{-space} \\ = (\ell_2 \hat{\otimes}_\varepsilon \ell_2) \hat{\otimes}_\varepsilon Y &= \qquad \text{as } \varepsilon \text{ is associative} \\ = (\ell_2 \hat{\otimes}_{w_2} \ell_2) \hat{\otimes}_\varepsilon Y & \end{aligned}$$

and therefore $\ell_2 \hat{\otimes}_{w_2} \ell_2 = l_2 \hat{\otimes}_\varepsilon l_2$ has to be an $\mathscr{L}_2$-space, which is not true. This establishes the example.

Thus given n Banach spaces $X_1, \ldots, X_n$ and a quasi-uniform cross-norm α there are

$$k(n) := \sum_{j=1}^{n-1} k(j)k(n-j) \quad \text{with } k(1) = k(2) = 1$$

possibilities to build up completed tensor products of $X_1 \otimes X_2 \otimes \ldots \otimes X_n$. Nevertheless each of these completed tensor products has the form

$$X = Y \hat{\otimes}_\alpha Z \tag{1.2}$$

where Y and Z are completed tensor products of $X_1 \otimes \ldots \otimes X_j$, and $X_{j+1} \otimes \ldots \otimes X_n$ $(j \geqslant 1)$, respectively.

Given operators $a_i \in L(X_i)$ $(1 \leqslant i \leqslant n)$ it is easily seen that each operator $I_1 \otimes \ldots \otimes I_{i-1} \otimes a_i \otimes I_{i+1} \otimes \ldots \otimes I_n$ (I_j denoting the identity operator on X_j) on the algebraic tensor product $X_1 \otimes X_2 \otimes \ldots \otimes X_n$ has a unique continuous extension onto each of the $k(n)$ completed tensor products $X = Y \hat{\otimes}_\alpha Z$.

Once having chosen the completed tensor products X let $\tilde{a}_i$ denote the extension of $I_1 \otimes \ldots \otimes I_{i-1} \otimes a_i \otimes I_{i+1} \otimes \ldots \otimes I_n$ onto X. Our main result will say that for Taylor's joint spectrum

$$(1.3) \qquad \sigma(\tilde{a}_1, \ldots, \tilde{a}_n ; X) = \prod_{j=1}^{n} \sigma(a_i ; X_i)$$

is true for all completed tensor products X of $X_1 \otimes \ldots \otimes X_n$ with respect to α. This is a generalization of our previous result [16], where α had to be associative.

In order to prove (1.3) we shall consider a more general situation:

Given pairwise commuting operators $a_1, \ldots, a_n \in L(Y)$ and $b_1, \ldots, b_m \in L(Z)$ (Y and Z denoting arbitrary complex Banach spaces) and an individual cross-norm α, we shall prove that

$$(1.4) \qquad \begin{aligned} &\sigma(a_1, \ldots, a_n ; Y) \times \sigma(b_1, \ldots, b_m, Z) \subseteq \\ &\subseteq \sigma(a_1 \hat{\otimes}_\alpha I_Z, \ldots, a_n \hat{\otimes}_\alpha I_Z, I_Y \hat{\otimes}_\alpha b_1, \ldots, I_Y \hat{\otimes}_\alpha b_m ; Y \hat{\otimes}_\alpha Z). \end{aligned}$$

From this (1.3) will be easily deduced.

Of course it would be nice if equality holds in (1.4) as it does for Hilbert spaces and the Hilbert space tensor norm by the main result of Ceauşescu and Vasilescu [2]. As it seems in all known examples there is equality

$$(1.5) \qquad \sigma(a_1, \ldots, a_n ; Y) = \sigma(a_1 \hat{\otimes}_\alpha I_Z, \ldots, a_n \hat{\otimes}_\alpha I_Z; Y \hat{\otimes}_\alpha Z)$$

and hence equality in (1.4). We shall prove that (1.5) holds true if Y is an $\mathscr{L}_p$-space $(p = 1, 2, \infty)$ and thereby we generalize the Ceauşescu and Vasilescu result to the setting of $\mathscr{L}_p$-spaces and arbitrary individual quasi-uniform cross-norms.

In general however (1.5) for $\alpha = \varepsilon$ or $\alpha = \pi$ means that the Koszul complexes $F(X; a - z)$ are exact if and only if the dual complexes $F(X', a' - z)$ split.

This observation will be the starting point for a modified joint spectrum which we shall study in detail in the case $n = 2$.

2. JOINT SPECTRA AND TENSOR PRODUCTS

The following theorem is a generalization of one half of the main result of Ceauşescu and Vasilescu [2, 2.2] from the setting of Hilbert spaces and the Hilbert space cross-norm to the setting of general Banach spaces and arbitrary quasi-uniform cross-norms. A generalization of the other easier half of Ceauşescu's and Vasilescu's result will be discussed in the next section.

After submission of this paper, we received a preprint [5] of J. Eschmeier, where Theorem 2.1 and the corresponding result for essential spectra also has been obtained.

2.1. THEOREM. *Let X and Y denote two complex Banach spaces, and let $X \hat{\otimes}_\alpha Y$ denote the completion of the algebraic tensor product $X \otimes Y$ with respect to some individual quasi-uniform cross-norm α. Given two systems $a = (a_1, \ldots, a_n) \in L(X)^n$ and $b = (b_1, \ldots, b_m) \in L(Y)^m$ of pairwise commuting operators on X and Y, respectively, let $\tilde{a} := (\tilde{a}_1, \ldots, \tilde{a}_n)$, $\tilde{b} := (\tilde{b}_1, \ldots, \tilde{b}_m)$, $\tilde{a}_i := a_i \hat{\otimes}_\alpha I_Y$, $\tilde{b}_j := I_X \hat{\otimes}_\alpha b_j$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ denote their unique continuous extensions upon $X \hat{\otimes}_\alpha Y$. Then*

$$\sigma(a_1, \ldots, a_n; X) \times \sigma(b_1, \ldots, b_m; Y) \subset \sigma(\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_m; X \hat{\otimes}_\alpha Y).$$

Proof. By doing translations otherwise it is enough to prove that

$$(0,0) \notin \sigma(\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_m; X \hat{\otimes}_\alpha Y) \tag{2.1}$$

implies

$$(0, 0) \notin \sigma(a_1, \ldots, a_n; X) \times \sigma(b_1, \ldots, b_m; Y). \tag{2.2}$$

By definition we have

$$\delta(\tilde{a}, \tilde{b}) = \delta(a) \hat{\otimes}_\alpha I_Y + I_X \hat{\otimes}_\alpha \delta(b) := \sum_{i=1}^{n} \tilde{a}_i s_i + \sum_{j=1}^{m} \tilde{b}_j t_j$$

acting on

$$\wedge [X \hat{\otimes}_\alpha Y] := \bigoplus_{r=0}^{\infty} \wedge^r [X \hat{\otimes}_\alpha Y]$$

where

$$\wedge^r [X \hat{\otimes}_\alpha Y] := \bigoplus_{p+q=r} \wedge^p [s; X] \hat{\otimes}_\alpha \wedge^q [t; Y], \tag{2.3}$$

i.e., $\wedge^p [s; X] \hat{\otimes}_\alpha \wedge^q [t; Y]$ consists of forms of bidegree (p, q) in $s := (s_1, \ldots, s_n)$ and $t := (t_1, \ldots, t_m)$ with coefficients in $X \hat{\otimes}_\alpha Y$.

If we assume (2.1) then we especially know that

$$\delta^{n+m-1}(\tilde{a}, \tilde{b}) : \wedge^{n+m-1} [X \hat{\otimes}_\alpha Y] \to \wedge^{n+m} [X \hat{\otimes}_\alpha Y]$$

is a surjection.

If neither

$$\delta^{n-1}(a) : \Lambda^{n-1}[s; X] \to \Lambda^{n}[s; X]$$

nor

$$\delta^{m-1}(b) : \Lambda^{m-1}[t; Y] \to \Lambda^{m}[t; Y]$$

were surjections, we find functionals

$$(\varphi_k)_{k\in\mathbf{N}} \subset (\Lambda^{n}[s; X])' \quad \text{with } \|\varphi_k\| = 1 \ (k \in \mathbf{N})$$

$$(\psi_k)_{k\in\mathbf{N}} \subset (\Lambda^{m}[t; Y])' \quad \text{with } \|\psi_k\| = 1 \ (k \in \mathbf{N})$$

such that

$$\varphi_k \circ \delta^{n-1}(a) \to 0 \quad \text{and} \quad \psi_k \circ \delta^{m-1}(b) \to 0 \quad \text{as } k \to \infty.$$

Because of (2.3) this implies

$$(\varphi_k \hat{\otimes}_\alpha \psi_k) \circ \delta^{n+m-1}(\tilde{a}, \tilde{b}) \to 0 \quad \text{as } k \to \infty$$

but

$$\|\varphi_k \hat{\otimes}_\alpha \psi_k\| \geqslant 1$$

contradicting the fact that $\delta^{n+m-1}(\tilde{a}, \tilde{b})$ is a surjection.

Without loss of generality we may therefore assume that $\delta^{n-1}(a)$ is a surjection and that the sequence $F(Y, b)$ (cf. (1.1)) is singular.

Let $q \in \{0, \ldots, m\}$ be minimal such that

$$\operatorname{Ker} \delta^{q}(b) \neq \operatorname{im} \delta^{q-1}(b).$$

We distinguish two cases:

1° If $q = 0$, take $0 \neq \eta \in \operatorname{Ker} \delta^0(b)$. If $\xi \in \operatorname{Ker} \delta^0(a)$ then $\xi \otimes \eta \in \operatorname{Ker} \delta^0(\tilde{a}, \tilde{b})$ by (2.3) and thus $0 = \xi \otimes \eta$ because $\delta^0(\tilde{a}, \tilde{b})$ is a topological monomorphism. Therefore we have $\xi = 0$ and hence

$$\operatorname{Ker} \delta^0(a) = \{0\}. \tag{2.4}$$

If $\xi \in \operatorname{Ker} \delta^{p+1}(a)$, then $\xi \otimes \eta \in \operatorname{Ker} \delta^{p+1}(\tilde{a}, \tilde{b})$ is a form of bidegree $(p+1, 0)$ and thus by (2.1) and (2.3) there exists a form $z \in \Lambda^{p}[X \hat{\otimes}_\alpha Y]$ with bidegree $(z) = (p, 0)$ such that

$$\xi \otimes \eta = \delta^{p}(a) \hat{\otimes}_\alpha I(z).$$

Choosing $\psi \in (\Lambda^0[t_1, \ldots, t_m; Y])'$ such that $\psi(\eta) = 1$, we get

$$\xi = \delta^{p}(a) \circ (I \hat{\otimes}_\alpha \psi)(z)$$

thus proving

(2.5) $$\operatorname{Ker}\delta^{p+1}(a) = \operatorname{im}\delta^{p}(a), \quad \text{for all } p \in \{0, \ldots, n-1\} \text{ if } q = 0.$$

2° If $q \geqslant 1$, choose a sequence $(\psi_k)_{k\in\mathbf{N}} \subset (\operatorname{Ker}\delta^q(b))'$ such that

$$\psi_k \circ \delta^{q-1}(b) \to 0 \quad \text{as } k \to \infty \text{ and } \|\psi_k\| = 1 \ (k \in \mathbf{N}).$$

Next choose $(\eta_k)_{k\in\mathbf{N}} \subset \operatorname{Ker}\delta^q(b)$ such that $\psi_k(\eta_k) \geqslant 1/2$, $\|\eta_k\| = 1$. Let $\xi \in \operatorname{Ker}\delta^0(a)$. Then

$$\xi \otimes \eta_k \in \operatorname{Ker}\delta^q(\tilde{a}, \tilde{b}) \text{ are forms of bidegree } (0, q).$$

Thus (2.1) and (2.3) guarantee the existence of forms $z_k \in \Lambda^{q-1}[X \hat{\otimes}_\alpha Y]$ of bidegree $(0, q-1)$ such that $\{z_k : k \in \mathbf{N}\}$ is bounded and

$$\xi \otimes \eta_k = I \hat{\otimes}_\alpha \delta^{q-1}(b) z_k \quad (k \in \mathbf{N}).$$

In applying the sequence $(I \hat{\otimes} \psi_k)_{k\in\mathbf{N}}$, we obtain

$$\xi = \lim_{k\to\infty} [\psi_k(\eta_k)]^{-1}(I \hat{\otimes}_\alpha \psi_k \circ \delta^{q-1}(b))(z_k) = 0$$

and thus

(2.6) $$\operatorname{Ker}\delta^0(a) = \{0\}, \quad \text{if } q \geqslant 1.$$

In concluding the proof, assume $\delta^p(a)$ were not an open map onto $\operatorname{Ker}\delta^{p+1}(a)$. Then we find a sequence $(\varphi_k)_{k\in\mathbf{N}} \subset (\operatorname{Ker}\delta^{p+1}(a))'$ such that $\|\varphi_k\| = 1$ and

$$\varphi_k \circ \delta^p(a) \to 0 \quad \text{as } k \to \infty.$$

Next choose a sequence $(\xi_k)_{k\in\mathbf{N}} \subset \operatorname{Ker}\delta^{p+1}(a)$ such that $\|\xi_k\| = 1$, $\varphi_k(\xi_k) \geqslant 1/2$ $(k \in \mathbf{N})$. We have $\xi_k \otimes \eta_k \in \operatorname{Ker}\delta^{p+1+q}(\tilde{a}, \tilde{b})$, and $\operatorname{bidegree}(\xi_k \otimes \eta_k) = (p+1, q)$ $(k \in \mathbf{N})$. By (2.1) and (2.3) we find forms z_{1k}, z_{2k} such that $\{z_{ik} : k \in \mathbf{N}\}$ $(i = 1, 2)$ are bounded sets with $\operatorname{bidegree}(z_{1k}) = (p, q)$ and $\operatorname{bidegree}(z_{2k}) = (p+1, q-1)$ such that

$$\xi_k \otimes \eta_k = \delta^p(a) \hat{\otimes}_\alpha I(z_{1k}) + I \hat{\otimes}_\alpha \delta^{q-1}(b)(z_{2k}) \quad (k \in \mathbf{N}).$$

An application of the functionals $\varphi_k \hat{\otimes}_\alpha \psi_k$ $(k \in \mathbf{N})$ yields

$$\frac{1}{4} \leqslant \varphi_k(\xi_k)\psi_k(\eta_k) = ((\varphi_k \circ \delta^p(a)) \hat{\otimes}_\alpha \psi_k)(z_{1k}) +$$

$$+ (\varphi_k \hat{\otimes}_\alpha (\psi_k \circ \delta^{q-1}(b)))(z_{2k}) \to 0$$

as $k \to \infty$ and hence a contradiction.

Therefore

$$\text{Ker}\, \delta^{p+1}(a) = \text{im}\, \delta^p(a) \tag{2.7}$$

for all $p \in \{0, \ldots, n-1\}$ if $q \geq 1$ and (2.3) — (2.7) prove the exactness of $F(X, a)$ and thus $0 \notin \sigma(a_1, \ldots, a_n; X)$.

REMARK. 1. If $0 \notin \text{Sp}(a_1, \ldots, a_n; \{a_1, \ldots, a_n\}^c)$, then $F(X, a)$ is split exact as the construction in Taylor's proof of [13, Lemma 1.1] shows.

2. We see that the question of having equality in (1.4) and hence a full generalization of the Ceauşescu and Vasilescu result [2, 2.2] merely depends on the question of having

$$\sigma(a_1, \ldots, a_n; X) = \sigma(a_1 \hat{\otimes}_\alpha I_Y, \ldots, a_n \hat{\otimes}_\alpha I_Y; X \hat{\otimes}_\alpha Y)$$

$$\sigma(b_1, \ldots, b_m; Y) = \sigma(I_X \hat{\otimes}_\alpha b_1, \ldots, I_X \hat{\otimes}_\alpha b_m; X \hat{\otimes}_\alpha Y)$$

because there is always the trivial inclusion

$$\sigma(a_1 \hat{\otimes}_\alpha I_Y, \ldots, a_n \hat{\otimes}_\alpha I_Y, I_X \hat{\otimes}_\alpha b_1, \ldots, I_X \hat{\otimes}_\alpha b_m; X \hat{\otimes}_\alpha Y) \subseteq$$

$$\subseteq \sigma(a_1 \hat{\otimes}_\alpha I_Y, \ldots, a_n \hat{\otimes}_\alpha I_Y; X \hat{\otimes}_\alpha Y) \times \sigma(I_X \hat{\otimes}_\alpha b_1, \ldots, I_X \hat{\otimes}_\alpha b_m; X \hat{\otimes}_\alpha Y).$$

This problem of tensor stability will be investigated in the next section. At this stage we want to draw a consequence of 2.1 which generalizes our main result in [16] to arbitrary quasi-uniform cross-norms, and consequently the results of Dash and Schechter [4] for the joint bicommutant spectrum in Banach spaces and that of Ceauşescu and Vasilescu [1] for Taylor's joint spectrum in the Hilbert space setting.

2.2. COROLLARY. *Let $X_1, \ldots, X_n$ denote complex Banach spaces and let X denote one of the completed tensor products build up from the algebraic tensor product $X_1 \otimes X_2 \otimes \ldots \otimes X_n$ with respect to a quasi-uniform cross-norm α.*

Given $a_i \in L(X_i)$ let $\tilde{a}_i$ denote the unique continuous extension of $I_1 \otimes \ldots \otimes I_{i-1} \otimes \otimes a_i \otimes I_{i+1} \otimes \ldots \otimes I_n$ upon X $(1 \leq i \leq n)$. Then we have

$$\sigma(\tilde{a}_1, \ldots, \tilde{a}_n; X) = \prod_{i=1}^{n} \sigma(a_i; X_i). \tag{2.8}$$

Proof. We proceed by induction. For $n = 1$ nothing has to be shown. Assume (2.8) is true for all completed tensor products of less than $n-1$ factors X_i and operators $a_i \in L(X_i)$. Let X be a completed tensor product build up from $X_1 \otimes X_2 \otimes \ldots \otimes X_n$. Then X splits into

$$X = Y \hat{\otimes}_\alpha Z$$

where Y and Z are completed tensor products of $X_1 \otimes \ldots \otimes X_i$ and $X_{i+1} \otimes \ldots \ldots \otimes X_n$ with respect to α. Let b_j denote the restriction of $\tilde{a}_j$ to Y $(1 \leqslant j \leqslant i)$ and c_j the restriction of $\tilde{a}_j$ to Z $(i+1 \leqslant j \leqslant n)$. Then

$$\tilde{a}_j = b_j \hat{\otimes}_\alpha I_Z \quad \text{for} \quad 1 \leqslant j \leqslant i$$

$$\tilde{a}_j = I_Y \hat{\otimes}_\alpha c_j \quad \text{for} \quad i+1 \leqslant j \leqslant n.$$

Consequently we have by Theorem 2.1

$$\sigma(b_1, \ldots, b_i ; Y) \times \sigma(c_{i+1}, \ldots, c_n ; Z) \subseteq \sigma(\tilde{a}_1, \ldots, \tilde{a}_n ; X) \subseteq \prod_{j=1}^{n} \sigma(\tilde{a}_j ; X)$$

and by induction hypothesis

$$\prod_{j=1}^{n} \sigma(a_j ; X_j) \subseteq \sigma(\tilde{a}_1, \ldots, \tilde{a}_n ; X) \subseteq \prod_{j=1}^{n} \sigma(\tilde{a}_j ; X).$$

But $\lambda - \tilde{a}_j$ is invertible in X if and only if $\lambda - a_j$ is invertible in X_j and thus $\sigma(\tilde{a}_j ; X) = \sigma(a_j ; X_j)$ which proves the corollary.

3. TENSOR STABILITY OF JOINT SPECTRA

It is an almost trivial matter of fact that given Banach spaces X and Y, pairwise commuting operators $a_1, \ldots, a_n \in L(X)$, and an individual quasi-uniform cross-norm α there is always the inclusion

$$\sigma(a_1, \ldots, a_n ; X) \subseteq \sigma(a_1 \hat{\otimes} I_Y, \ldots, a_n \hat{\otimes} I_Y ; X \hat{\otimes}_\alpha Y). \tag{3.1}$$

We are going to study necessary and sufficient conditions of having equality in (3.1) at least for the ε- and the π-topology. Contrary to this situation we always have (see [16] for a definition of the following spectra)

$$\begin{aligned} &\mathrm{Sp}(a_1, \ldots, a_n ; L(X)) = \mathrm{Sp}(\tilde{a}_1, \ldots, \tilde{a}_n ; L(X \hat{\otimes}_\alpha Y)) \\ &\mathrm{Sp}(a_1, \ldots, a_n ; \{a_1, \ldots, a_n\}^c) = \mathrm{Sp}(\tilde{a}_1, \ldots, \tilde{a}_n ; \{\tilde{a}_1, \ldots, \tilde{a}_n\}^c) \\ &\mathrm{Sp}(a_1, \ldots, a_n ; \langle a_1, \ldots, a_n \rangle) = \mathrm{Sp}(\tilde{a}_1, \ldots, \tilde{a}_n ; \langle \tilde{a}_1, \ldots, \tilde{a}_n \rangle), \end{aligned} \tag{3.2}$$

where $\tilde{a}_i$ denotes the unique continuous extension of $a_i \otimes I_Y$ onto the completion $X \hat{\otimes}_\alpha Y$, and the algebras on the right hand side are considered as subalgebras of $L(X \hat{\otimes}_\alpha Y)$. The proof of (3.2) is routine observing that $a_i \circ (I_X \otimes \varphi) = (I_X \otimes \varphi)(a_i \otimes I_Y)$

($\varphi \in Y'$) and $\gamma_y \circ a_i = (a_i \otimes I_Y) \circ \gamma_y$ for all $y \in Y$, where γ_y denotes the mapping $x \mapsto x \otimes y$ from X into $X \otimes Y$.

The corresponding result for the bicommutant spectra

$$\mathrm{Sp}(a_1, \ldots, a_n ; \{a_1, \ldots, a_n\}^{cc}) = \mathrm{Sp}(\tilde{a}_1, \ldots, \tilde{a}_n ; \{\tilde{a}_1, \ldots \tilde{a}_n\}^{cc})$$

is at least true, if X or Y has the approximation property, for then by a result of Grothendieck [6, p. 15] the intersection of the kernels of $\varphi \otimes I$, $\varphi \in X'$ (resp. $I \otimes \varphi$, $\varphi \in Y'$) is trivial.

If there is equality in (3.1) for all Y and $\alpha = \pi$ we shall call $\sigma(a_1, \ldots, a_n : X)$ *tensor stable*. In order to study tensor stability we have to look for sufficient and necessary conditions when exactness of the sequence

$$\begin{aligned} &F(X; (a_1 - z_1, \ldots, a_n - z_n)) : \\ (3.3) \qquad &: 0 \to \Lambda^0 \xrightarrow{\delta_z^0} \Lambda^1 \xrightarrow{\delta_z^1} \Lambda^2 \to \ldots \to \Lambda^{n-1} \xrightarrow{\delta_z^{n-1}} \Lambda^n \to 0 \end{aligned}$$

implies exactness of the tensorized sequence

$$\begin{aligned} &F(X \hat{\otimes}_\alpha Y : (\tilde{a}_1 - z_1, \ldots, \tilde{a}_n - z_n)) : \\ (3.4) \qquad &: 0 \to \Lambda^0 \hat{\otimes}_\alpha Y \xrightarrow{\delta_z^0 \hat{\otimes}_\alpha I_Y} \Lambda^1 \hat{\otimes}_\alpha Y \to \ldots \\ &\ldots \to \Lambda^{n-1} \hat{\otimes}_\alpha Y \xrightarrow{\delta_z^{n-1} \hat{\otimes}_\alpha I_Y} \Lambda^n \hat{\otimes}_\alpha Y \to 0. \end{aligned}$$

Here we used $\delta_z^p := \delta^p(a - z)$ as convention.

A trivial sufficient condition to guarantee tensor stability is that each exact sequence (3.3) splits at each place (split exact), for then the mapping property of α guarantees exactness of (3.4). On the other hand, if X is reflexive then this sufficient condition will be shown to be also necessary. Especially if X is a Hilbert space then so is Λ^p and consequently (3.3) splits. Thus taking into account Theorem 2.1 we can improve the result of Ceaușescu and Vasilescu [2, 2.2] to the setting of two Hilbert spaces X and Y and an arbitrary quasi-uniform cross-norm α.

3.1. Definition. Call an exact sequence of Banach spaces

$$(3.5) \qquad 0 \to X_0 \xrightarrow{\delta^0} X_1 \xrightarrow{\delta^1} X_2 \xrightarrow{\delta^2} \ldots \longrightarrow X_{n-1} \xrightarrow{\delta^{n-1}} X_n \to 0$$

a $\hat{\otimes}$*-sequence*, if the tensorized sequences

$$\begin{aligned} &0 \to X_0 \hat{\otimes}_\pi Y \xrightarrow{\delta^0 \hat{\otimes} I} X_1 \hat{\otimes}_\pi Y \to \ldots \\ (3.6) \qquad &\ldots \to X_{n-1} \hat{\otimes}_\pi Y \xrightarrow{\delta^{n-1} \hat{\otimes} I} X_n \hat{\otimes}_\pi Y \to 0 \end{aligned}$$

are exact for all Banach spaces Y.

The following result characterizes $\otimes$-sequences.

3.2. THEOREM. *Given an exact sequence of Banach spaces* (3.5) *the following are equivalent:*

(1) (3.5) *is a* $\otimes$*-sequence;*

(2) *The dual sequence*

$$(3.7) \qquad 0 \leftarrow (X_0)' \xleftarrow{(\delta^0)'} (X_1)' \leftarrow \dots \leftarrow (X_{n-1})' \xleftarrow{(\delta^{n-1})'} (X_n)' \leftarrow 0$$

is split exact;

(3) *The tensorized sequences*

$$(3.8) \qquad \begin{gathered} 0 \to X_0 \hat{\otimes}_\varepsilon Y \xrightarrow{\delta^0 \hat{\otimes} I} X_1 \hat{\otimes}_\varepsilon Y \to \dots \\ \dots \to X_{n-1} \hat{\otimes}_\varepsilon Y \xrightarrow{\delta^{n-1} \hat{\otimes} I} X_n \hat{\otimes}_\varepsilon Y \to 0 \end{gathered}$$

are exact for all Banach spaces Y.

For $n = 2$ the equivalence (2) $\Leftrightarrow$ (3) is due to Grothendieck [6, p. 27] whereas (1) $\Leftrightarrow$ (3) is due to Kaballo and Vogt [10, 1.3].

Proof. Of course (3.6) or (3.8) are exact sequences if and only if the corresponding short sequences

$$(3.9) \qquad 0 \to X_0 \hat{\otimes}_\alpha Y \xrightarrow{\delta^0 \hat{\otimes} I} X_1 \hat{\otimes}_\alpha Y \xrightarrow{\delta^1 \hat{\otimes} I} \operatorname{Ker}(\delta^2 \hat{\otimes}_\alpha I) \to 0$$

$$(3.10) \qquad 0 \to \operatorname{Ker}(\delta^p \hat{\otimes}_\alpha I) \hookrightarrow X_p \hat{\otimes}_\alpha Y \xrightarrow{\delta^p \hat{\otimes} I} \operatorname{Ker}(\delta^{p+1} \hat{\otimes}_\alpha I) \to 0$$

$(1 \leqslant p \leqslant n-2)$

$$(3.11) \qquad 0 \to \operatorname{Ker}(\delta^{n-1} \hat{\otimes}_\alpha I) \hookrightarrow X_{n-1} \hat{\otimes}_\alpha Y \xrightarrow{\delta^{n-1} \hat{\otimes}_\alpha I} X_n \hat{\otimes}_\alpha Y \to 0$$

are exact for $\alpha = \pi$ or ε, respectively. Assume (3.9) — (3.11) are exact. If $\alpha = \varepsilon$, he n $(\operatorname{Ker}(\delta^p)) \hat{\otimes}_\varepsilon Y$ is always an isometric subspace of $X_p \hat{\otimes}_\varepsilon Y$ by the properties of the ε-topology.[1] Moreover $(\operatorname{Ker}(\delta^p)) \hat{\otimes}_\varepsilon Y \subset \operatorname{Ker}(\delta^p \hat{\otimes}_\varepsilon I)$ and consequently they are equal because of (3.9) and (3.10).

Thus (3.9)—(3.11) are tensorized short exact sequences for $\alpha = \varepsilon$ and all Banach spaces Y and by Grothendieck's result the dual sequences

$$(3.12) \qquad 0 \leftarrow (X_0)' \xleftarrow{(\delta^0)'} (X_1)' \xleftarrow{(\delta^1)'} (\operatorname{Ker} \delta^2)' \leftarrow 0$$

$$(3.13) \qquad 0 \leftarrow (\operatorname{Ker} \delta^p)' \leftarrow (X_p)' \xleftarrow{(\delta^p)'} (\operatorname{Ker} \delta^{p+1})' \leftarrow 0$$

$(1 \leqslant p \leqslant n-2)$

$$(3.14) \qquad 0 \leftarrow (\operatorname{Ker}(\delta^{n-1}))' \leftarrow (X_{n-1})' \xleftarrow{(\delta^{n-1})'} (X_n)' \leftarrow 0$$

split. By the Hahn-Banach theorem this gives (2), and thus (3) implies (2) and vice-versa.

But by the result [10, 1.3] of Kaballo and Vogt the splitting of (3.12) — (3.14) implies exactness of all short tensorized sequences:

$$(3.15) \qquad 0 \rightarrow X_0 \hat{\otimes}_\pi Y \xrightarrow{\delta^0 \hat{\otimes} I} X_1 \hat{\otimes}_\pi Y \rightarrow (\operatorname{Ker} \delta^2) \hat{\otimes}_\pi Y \rightarrow 0$$

$$(3.16) \qquad 0 \rightarrow (\operatorname{Ker} \delta^p) \hat{\otimes}_\pi Y \hookrightarrow X_p \hat{\otimes}_\pi Y \xrightarrow{\delta^p \hat{\otimes} I} (\operatorname{Ker} \delta^{p+1}) \hat{\otimes}_\pi Y \rightarrow 0$$

$(1 \leqslant p \leqslant n-2)$

$$(3.17) \qquad 0 \rightarrow (\operatorname{Ker} \delta^{n-1}) \hat{\otimes}_\pi Y \hookrightarrow X_{n-1} \hat{\otimes}_\pi Y \xrightarrow{\delta^{n-1} \hat{\otimes} I} X_n \hat{\otimes}_\pi Y \rightarrow 0$$

for all Banach spaces Y.

Since $\delta^p\colon X_p \rightarrow \operatorname{Ker} \delta^{p+1}$ is a topological homomorphism onto, so is

$$\delta^p \otimes_\pi I : X_p \otimes_\pi Y \rightarrow \operatorname{Ker} \delta^{p+1} \otimes_\pi Y$$

and consequently

$$\operatorname{Ker}(\delta^p \hat{\otimes}_\pi I) = \overline{\operatorname{Ker}(\delta^p \otimes I)}^{X_p \hat{\otimes}_\pi Y}$$

and

$$\overline{\operatorname{Ker}(\delta^p \otimes I)}^{X_p \hat{\otimes}_\pi Y} = (\operatorname{Ker} \delta^p)\, \hat{\otimes}_\pi Y$$

because of (3.16) and (3.17).

Thus (3.9) — (3.11) are exact for $\alpha = \pi$ which in turn gives exactness of (3.6) and hence we proved (3) $\Rightarrow$ (2) $\Rightarrow$ (1). So let us assume (1). Then (3.9) — (3.11) are exact for $\alpha = \pi$. Since $\delta^p \hat{\otimes}_\pi I(X_p \hat{\otimes}_\pi Y) = (\operatorname{Ker} \delta^{p+1}) \hat{\otimes}_\pi Y$ by the properties of the π-topology, we again obtain exactness of (3.15) — (3.17) and consequently the dual sequences (3.12) — (3.14) split by Kaballo and Vogt [10, 1.3], i.e. (1) $\Rightarrow$ (2).

Remark. It is possible to improve 3.2 in the case that $n \leqslant 3$ and that X_0, $X_1, \ldots, X_n$ possess the bounded approximation property. In this case a combination of Kaballo's theorem [9, 2.9] and Harksen's result [7, 6.9] can be used to prove that the short sequences (3.9) — (3.11) are exact for all uniform cross-norms and all Banach spaces Y.

Consequently, in this case (1) is equivalent to

$$(3)' \quad 0 \to X_0 \hat{\otimes}_\alpha Y \xrightarrow{\delta^0 \hat{\otimes}_\alpha I} X_1 \hat{\otimes}_\alpha Y \to \dots \to X_{n-1} \hat{\otimes}_\alpha Y \xrightarrow{\delta^{n-1} \hat{\otimes}_\alpha I} X_n \hat{\otimes}_\alpha Y \to 0$$

is exact for all uniform cross-norms α and Banach spaces Y.

3.3. COROLLARY. *Let* $(a_1, \dots, a_n) \in L(X)^n$ *denote an n-tuple of pairwise commuting operators. Then* $\sigma(a_1, \dots, a_n; X)$ *is tensor stable if and only if for all* $z \notin \sigma(a_1, \dots, a_n; X)$ *the dual Koszul complexes* $F(X', (a_1' - z_1, \dots, a_n' - z_n))$ *are split exact.*

Moreover, if X is reflexive, then $\sigma(a_1, \dots, a_n; X)$ *is tensor stable if and only if for all* $z \notin \sigma(a_1, \dots, a_n; X)$ *the Koszul complexes* $F(X, (a_1 - z_1, \dots, a_n - z_n))$ *are split exact. If this is true, then* $\sigma(a_1, \dots, a_n; X) = \sigma(\tilde{a}_1, \dots, \tilde{a}_n; X \hat{\otimes}_\alpha Y)$ *for* ***all*** *quasi-uniform cross-norms* α.

In generalizing results of Kaballo [9, p. 22/23] and J. Harksen [7, p. 110/111] we give sufficient conditions for having a $\otimes$-sequence

$$0 \to X_0 \xrightarrow{\delta^0} X_1 \to \dots \to X_{n-1} \xrightarrow{\delta^{n-1}} X_n \to 0.$$

Recall (cf. Lindenstrauss—Tzafriri [11, p. 198]) that a Banach space X is an $\mathscr{L}_{p,\lambda}$-space ($1 \leqslant p \leqslant \infty$, $\lambda \geqslant 1$) provided for each finite dimensional subspace N of X there exist a subspace $M \supset N$, operators $T_M : M \to \ell_p^{\dim M}$ and $S_M : \ell_p^{\dim M} \to M$ such that

$$S_M \circ T_M = I_M \quad \text{and} \quad \|S_M\| \cdot \|T_M\| \leqslant \lambda.$$

X is an $\mathscr{L}_p$-space, provided X is an $\mathscr{L}_{p,\lambda}$-space for some λ. Moreover X is an $\mathscr{L}_p$--space if and only if X' is an $\mathscr{L}_q$-space, where $1/p + 1/q = 1$ (cf. [11, p. 203]).

We shall only consider $\mathscr{L}_1$-, $\mathscr{L}_2$- and $\mathscr{L}_\infty$-spaces. For these it is easily seen that a complemented subspace of an $\mathscr{L}_p$-space is an $\mathscr{L}_p$-space (cf. [11, p. 203]).

3.4. THEOREM. *Let*

$$(3.5) \qquad 0 \to X_0 \xrightarrow{\delta^0} X_1 \xrightarrow{\delta^1} X_2 \to \dots \to X_{n-1} \xrightarrow{\delta^{n-1}} X_n \to 0$$

denote an exact sequence of Banach spaces.

Assume that either all spaces $X_0, \dots, X_{n-2}$ *are* $\mathscr{L}_\infty$*-spaces, or that* $X_0, \dots, X_{n-1}$ *are* $\mathscr{L}_2$*-spaces, or that all spaces* $X_2, \dots, X_n$ *are* $\mathscr{L}_1$*-spaces.*

Then (3.5) *is a* $\otimes$*-sequence.*

Moreover for all uniform cross-norms α *and Banach spaces* Y

$$(3.6') \quad 0 \to X_0 \hat{\otimes}_\alpha Y \xrightarrow{\delta^0 \hat{\otimes}_\alpha I} X_1 \hat{\otimes}_\alpha Y \to \dots \to X_{n-1} \hat{\otimes}_\alpha Y \xrightarrow{\delta^{n-1} \hat{\otimes}_\alpha I} X_n \hat{\otimes}_\alpha Y \to 0$$

are exact sequences.

Before going into the proof, we note the following result which generalize [2, 2.2] to the case of $\mathscr{L}_p$-spaces ($p = 1, 2, \infty$) and arbitrary uniform cross-norms.

3.5. COROLLARY. *Let X and Y denote $\mathscr{L}_p$-and $\mathscr{L}_q$-spaces ($p, q \in \{1, 2, \infty\}$) respectively and let $(a_1, \ldots, a_n) \in L(X)^n$ and $(b_1, \ldots, b_m) \in L(Y)^m$ denote systems of pairwise commuting operators. Let α denote a uniform cross-norm, and $\tilde{a}_i := a_i \hat{\otimes}_\alpha I_Y$ $(1 \leqslant i \leqslant n)$, $\tilde{b}_j := I_X \hat{\otimes}_\alpha b_j$ $(1 \leqslant j \leqslant m)$ their unique extensions upon $X \hat{\otimes}_\alpha Y$. Then*

$$\sigma(\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_m; X \hat{\otimes}_\alpha Y) = \sigma(a_1, \ldots, a_n; X) \times \sigma(b_1, \ldots, b_m : Y).$$

Proof. Observe that if X is an $\mathscr{L}_p$-space, so are the finite direct sums $\wedge^r[s_1, \ldots, s_n; X]$, and the same for Y. The result now follows from 3.4.

Proof of 3.4. Since every $\mathscr{L}_2$-space is isomorphic to a Hilbert space, each closed subspace of an $\mathscr{L}_2$-space is complemented. Thus the exactness of (3.6′) is clear in this case. Let

$$0 \to X_0 \xrightarrow{\delta^0} X_1 \xrightarrow{\delta^1} \operatorname{Ker} \delta^2 \to 0 \tag{3.18}$$

$$0 \to \operatorname{Ker} \delta^p \hookrightarrow X_p \xrightarrow{\delta^p} \operatorname{Ker} \delta^{p+1} \to 0 \quad (2 \leqslant p \leqslant n-1). \tag{3.19}$$

We consider the cases $\mathscr{L}_1$ and $\mathscr{L}_\infty$ separately.

If $X_0, \ldots, X_{n-2}$ are $\mathscr{L}_\infty$-spaces, then (3.18) is a $\otimes$-sequence by a result of Kaballo [9, p. 22]; indeed (3.18) gives rise to an exact sequence

$$0 \to X_0 \hat{\otimes}_\alpha Y \xrightarrow{\delta^0 \hat{\otimes}_\alpha I} X_1 \hat{\otimes}_\alpha Y \xrightarrow{\delta^1 \hat{\otimes}_\alpha I} (\operatorname{Ker} \delta^2) \hat{\otimes}_\alpha Y \to 0 \tag{3.20}$$

for all uniform cross-norms α and Banach spaces Y by a result of Harksen [7, p. 115].

But if (3.18) is a $\otimes$-sequence, the dual sequence splits by Grothendieck [6] and thus the dual space $(\operatorname{Ker} \delta^2)'$ is isomorphic to a complemented subspace of the $\mathscr{L}_1$-space X_1' and hence itself an $\mathscr{L}_1$-space. Consequently $\operatorname{Ker} \delta^2$ is an $\mathscr{L}_\infty$-space. In the same way we conclude that $\operatorname{Ker} \delta^p$ is an $\mathscr{L}_\infty$-space for $2 \leqslant p \leqslant n-1$. Thus we have exactness of all short sequences (3.20) and

$$0 \to (\operatorname{Ker} \delta^p) \hat{\otimes}_\alpha Y \to X_p \hat{\otimes}_\alpha Y \xrightarrow{\delta^p \hat{\otimes}_\alpha I} (\operatorname{Ker} \delta^{p+1}) \hat{\otimes}_\alpha Y \to 0 \tag{3.21}$$

$(2 \leqslant p \leqslant n-1)$.

We have

(3.22) $$(\operatorname{Ker}\delta^p)\hat{\otimes}_\alpha Y = \operatorname{Ker}(\delta^p \hat{\otimes}_\alpha I_Y) \quad (2 \leqslant p \leqslant n-1).$$

Thus (3.20), (3.21) and (3.22) prove the theorem for $\mathscr{L}_\infty$-spaces.

If $X_2, \ldots, X_n$ are $\mathscr{L}_1$-spaces, then

(3.23) $$0 \to \operatorname{Ker}\delta^{n-1} \hookrightarrow X_{n-1} \xrightarrow{\delta^{n-1}} X_n \to 0$$

is a $\otimes$-sequence by Kaballo [9, p. 23] or more generally

(3.24) $$0 \to (\operatorname{Ker}\delta^{n-1})\hat{\otimes}_\alpha Y \hookrightarrow X_{n-1}\hat{\otimes}_\alpha Y \xrightarrow{\delta^{n-1}\hat{\otimes}_\alpha I} X_n \hat{\otimes}_\alpha Y \to 0$$

is an exact sequence for all uniform cross-norms α and Banach spaces Y by Harksen [7, p. 116].

Again by Grothendieck [6] the dual sequence of (3.23) splits and consequently $(\operatorname{Ker}\delta^{n-1})'$ is isomorphic to a complemented subspace of the $\mathscr{L}_\infty$-space $(X_{n-1})'$. Thus $\operatorname{Ker}\delta^{n-1}$ is an $\mathscr{L}_1$-space. Consequently, by the same argument

$$0 \to \operatorname{Ker}\delta^{n-2} \hookrightarrow X_{n-2} \xrightarrow{\delta^{n-2}} \operatorname{Ker}\delta^{n-1} \to 0$$

and thus

$$0 \to \operatorname{Ker}\delta^{p} \hookrightarrow X_{p} \xrightarrow{\delta^{p}} \operatorname{Ker}\delta^{p+1} \to 0 \quad (2 \leqslant p \leqslant n-1)$$

are $\otimes$-sequences and $\operatorname{Ker}\delta^p$ are $\mathscr{L}_1$-spaces $(2 \leqslant p \leqslant n-1)$. More generally Harksen [7, p. 116] gives that the corresponding sequences (3.21) are exact. Then (3.22) proves the theorem.

3.6. Corollary. *Let* $n \leqslant 3$, $(a_1, \ldots, a_n) \in L(X)^n$ *as in* 3.3. *Then* $\sigma(a_1, \ldots, a_n; X)$ *is tensor stable if and only if*

(3.25) $$\operatorname{Sp}(a_1', \ldots, a_n'; L(X')) \subset \sigma(a_1, \ldots, a_n; X).$$

Proof. If (3.25) is true, then given $z \notin \sigma(a_1, \ldots, a_n; X)$, we find $b_1, \ldots, b_n$, $d_1, \ldots, d_n \in L(X')$ such that

(*) $$I_{X'} = \sum_{i=1}^n (z_i - a_i')b_i = \sum_{i=1}^n d_i(z_i - a_i').$$

Consequently $\delta^0(z_1 - a_1', \ldots, z_n - a_n')$ has a complemented range and is injective, and $\delta^{n-1}(z_1 - a_1', \ldots, z_n - a_n')$ has a complemented kernel and is surjective.

Hence the sequences

$$0 \to \Lambda^0[s_1, s_2; X'] \xrightarrow{\delta^0} \Lambda^1[s_1, s_i; X'] \xrightarrow{\delta^1} \Lambda^2[s_1, s_2; X'] \to 0$$

$$0 \to \Lambda^0[s_1, s_2, s_3; X'] \xrightarrow{\delta^0} \Lambda^1[s_1, s_2, s_3; X'] \xrightarrow{\delta^1} \Lambda^2[s_1, s_2, s_3; X'] \xrightarrow{\delta^2}$$

$$\xrightarrow{\delta^2} \Lambda^3[s_1, s_2, s_3; X'] \to 0$$

are split exact, and thus $\sigma(a_1, \ldots, a_n; X)$ is tensor stable for $n = 2, 3$.

On the other hand assume that these sequences are split exact. Then $\delta^0(z_1 - a_1', \ldots, z_n - a_n')$ has a left inverse and $\delta^{n-1}(z_1 - a_1', \ldots, z_n - a_n')$ has a right inverse. These define operators $d_1, \ldots, d_n$ and $b_1, \ldots, b_n \in L(X')$ such that (*) holds true, and consequently we have (3.25).

More generally, the proof of 3.6 shows that for an $\mathscr{L}_p$-space ($p = 1, 2, \infty$) we always have (for arbitrary $n \in \mathbf{N}$)

$$\mathrm{Sp}(a_1', \ldots, a_n'; L(X')) \subset \sigma(a_1, \ldots, a_n; X).$$

For $n = 2$ we may combine 3.6 with our previous result [17, 2.1] obtaining that under the assumptions of 3.6 we have

$$\partial \mathrm{Sp}(a_1', a_2'; L(X')) \supseteq \partial \sigma(a_1, a_2; X)$$

($\partial \cdot$ denoting the topological boundary of $\cdot$).

Thus $\sigma(a_1, a_2; X)$ is obtained from $\mathrm{Sp}(a_1', a_2'; L(X'))$ by filling in holes. There are examples where $\mathrm{Sp}(a_1', a_2'; L(X')) = \partial \mathrm{Sp}(a_1', a_2'; L(X'))$, but $\sigma(a_1, a_2; X)$ contain interior points (see below).

On the other hand it may be interesting that $\mathrm{Sp}(a_1', a_2'; L(X'))$ in general will contain more spectral subsets (i.e. relatively closed and open subsets) and hence may give rise to more spectral projection than $\sigma(a_1, a_2; X)$ does.

3.7. Example (cf. [17, 2.5]). Let X_1 denote the complete Hilbert space tensor product of two copies of $\ell_2(\mathbf{N})$. Moreover, let $S_1, S_2 \in L(\ell_2(\mathbf{N}))$ denote the left shift $S_1(x_n)_{n\in\mathbf{N}} := (x_{n+1})_{n\in\mathbf{N}}$ and the right shift $S_2(x_n)_{n\in\mathbf{N}} = (0, x_1, x_2, \ldots)$. Then it is well-known that

$$\sigma(S_i; \ell_2(\mathbf{N})) = \mathbf{D} = \{z \in \mathbf{C} : |z| \leqslant 1\} \quad (i = 1, 2).$$

Letting $a_1 := S_1 \hat{\otimes} I$, $a_2 = I \hat{\otimes} S_2$, we have

$$\sigma(a_1, a_2; X) = \sigma(S_1; \ell_2(\mathbf{N})) \times \sigma(S_2; \ell_2(\mathbf{N})) = \mathbf{D} \times \mathbf{D}.$$

On the other hand, if $\lambda = (\lambda_1, \lambda_2)$ is such that $\max\{|\lambda_1|, |\lambda_2|\} < 1$, then it is well-known that

$$(x_n)_{n\in\mathbf{N}} \mapsto (\lambda_2 x_n - x_{n-1})_{n\in\mathbf{N}} = (\lambda_2 - S_2)(x_n)_{n\in\mathbf{N}} \quad (x_0 = 0)$$

is a topological monomorphism and

$$(x_n)_{n\in\mathbf{N}} \mapsto (\lambda_1 x_n - x_{n+1})_{n\in\mathbf{N}} = (\lambda_1 - S_1)(x_n)_{n\in\mathbf{N}}$$

is a surjection, and so are the tensorized operators $\lambda_i - a_i$ $(i = 1, 2)$. Consequently $\delta^0(\lambda - a)$ is a topological monomorphism and $\delta^1(\lambda - a)$ is a surjection. Since X_1 and $\wedge^1[X_1]$ are Hilbert spaces,

$$\mathrm{Sp}(a_1, a_2; L(X_1)) = \partial(\mathbf{D}\times\mathbf{D}) = \partial\sigma(a_1, a_2; X_1).$$

Finally, let $X = X_1 \oplus X_1$ (Hilbert space direct sum),

$$b_i := a_i \oplus \frac{1}{2} a_i : X \to X \quad (i = 1, 2).$$

Then it is an easy matter of fact, that

$$\sigma(b_1, b_2; X) = \sigma(a_1, a_2; X_1) \cup \sigma\left(\frac{1}{2} a_1, \frac{1}{2} a_2; X_1\right) = \mathbf{D}\times\mathbf{D}$$

and

$$\mathrm{Sp}(b_1, b_2; L(X)) = \mathrm{Sp}(a_1, a_2; L(X_1)) \cup \mathrm{Sp}\left(\frac{1}{2} a_1, \frac{1}{2} a_2; L(X_1)\right) =$$

$$= \partial(\mathbf{D}\times\mathbf{D}) \cup \partial\left(\frac{1}{2}\mathbf{D}\times\frac{1}{2}\mathbf{D}\right).$$

Letting π denote the germ which is 1 in a neighbourhood of $\partial(\mathbf{D}\times\mathbf{D})$ and 0 in a neighbourhood of $\partial((1/2)\mathbf{D}\times(1/2)\mathbf{D})$, we see that $\pi \notin \mathcal{O}(\mathbf{D}\times\mathbf{D})$, whereas $\pi \in \mathcal{O}(\partial(\mathbf{D}\times\mathbf{D}) \cup \partial((1/2)\mathbf{D}\times(1/2)\mathbf{D}))$ and π gives rise to the spectral projection

$$\pi(b_1, b_2) := I_{X_1} \oplus 0.$$

Let us now return to the case $n \leqslant 3$ and restate that for reflexive spaces X tensor stability of $\sigma(a_1, \ldots, a_n; X)$ is equivalent with having inclusion

$$\text{(3.26)} \qquad \mathrm{Sp}(a_1, \ldots, a_n; L(X)) \subset \sigma(a_1, \ldots, a_n; X)$$

in which case we have

$$\sigma(\tilde{a}_1, \ldots, \tilde{a}_n ; X \hat{\otimes}_\alpha Y) = \sigma(a_1, \ldots, a_n ; X)$$

for all quasi-uniform cross-norms α and Banach spaces Y. It seems that there is not known any example in the literature where (3.26) is not true.

But since we were not able to decide whether (3.25) or (3.26) is always true, let

$$\sigma_{\otimes}(a_1, \ldots, a_n ; X) := \sigma(a_1, \ldots, a_n ; X) \cup \mathrm{Sp}(a_1', \ldots, a_n' ; L(X')) \quad (n \leqslant 3)$$

denote the *tensor stable Taylor spectrum*. By 3.6 and 3.5 we have $\sigma_{\otimes} = \sigma$ for $\mathscr{L}_p$-spaces ($p = 1, 2, \infty$).

In general the tensor stable Taylor spectrum $\sigma_{\otimes}(a)$ of a commuting n-tuple ($n \leqslant 3$) is nothing else than the set of points, where the dual Koszul complex is not split exact.

Obviously $\sigma_{\otimes}$ is an ordinary joint spectrum with projection property and analytic functional calculus. For $n = 2$ we are going to characterize $\sigma_{\otimes}$ in terms of classical operator theory quite analogously as Vasilescu [15] has done in the Hilbert space setting where $\sigma_{\otimes} = \sigma$.

4. A CHARACTERIZATION OF $\sigma_{\otimes}(a_1, a_2 ; X)$

In the case $n = 2$, the Koszul complex $F(X, a)$ is simply

$$0 \to X \xrightarrow{\delta^0} X \oplus X \to X \xrightarrow{\delta^1} 0 \tag{4.1}$$

with

$$\delta^0 x := a_1 x \oplus a_2 x, \quad \delta^1(x \oplus y) := a_1 y - a_2 x.$$

Recall that $\sigma(a_1, \ldots, a_n ; X) = \sigma(a_1', \ldots, a_n' ; X')$.

4.1. THEOREM. *Let $a = (a_1, a_2) \in L(X)^2$ denote a pair of commuting operators on a complex Banach space X. Then the following are equivalent:*

(1) $(0, 0) \notin \sigma_{\otimes}(a_1, a_2 ; X)$;

(2) *There exist $b_1, b_2, d_1, d_2 \in L(X')$ such that*

$$J_1 := d_1 a_1' + d_2 a_2' \quad \text{and} \quad J_2 := a_1' b_1 + a_2' b_2 \tag{4.2}$$

are topological isomorphisms on X' and $\begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}$ *is a topological isomorphism on*

$X' \oplus X'$. *Moreover,* $\begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}$ *is a topological isomorphism on* $X' \oplus X'$ *if and only if* $\begin{pmatrix} b_1 & -a_2' \\ b_2 & a_1' \end{pmatrix}$ *is.*

Proof. Suppose (4.2). Then

$$\begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}\begin{pmatrix} b_1 & -a_2' \\ b_2 & a_1' \end{pmatrix} = \begin{pmatrix} J_2 & 0 \\ -(d_2b_1 - d_1b_2) & J_1 \end{pmatrix} =: A \tag{4.3}$$

and A is a topological isomorphism on $X' \oplus X'$ with

$$A^{-1} = \begin{pmatrix} J_2^{-1} & 0 \\ J_1^{-1}(d_2b_1 - d_1b_2)J_2^{-1} & J_1^{-1} \end{pmatrix}$$

and consequently $\begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}$ is invertible if and only if $\begin{pmatrix} b_1 & -a_2' \\ b_2 & a_1' \end{pmatrix}$ is.

Let us assume this is true. Comparing

$$\delta^0 x = a_1'x \oplus a_2'x \quad \text{with} \quad \begin{pmatrix} b_1 & -a_2' \\ b_2 & a_1' \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} = (-a_2'x) \oplus a_1'x$$

and

$$\delta^1(x \oplus y) = a_1'y - a_2'x \quad \text{with} \quad \begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} a_1'y - a_2'x \\ d_2y + d_1x \end{pmatrix}$$

we find that δ^0 is a topological monomorphism and δ^1 is a surjection. Thus assume

$$a_1'x - a_2'y = 0.$$

Then

$$\begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ -d_2x - d_1y \end{pmatrix}$$

and thus

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} a_1' & a_2' \\ -d_2 & d_1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -d_2x - d_1y \end{pmatrix} = \begin{pmatrix} b_1 & -a_2' \\ b_2 & a_1' \end{pmatrix} A^{-1}\begin{pmatrix} 0 \\ -d_2x - d_1y \end{pmatrix} =$$

$$= \begin{pmatrix} +a_2' \circ J_1^{-1}(d_2x + d_1y) \\ -a_1' \circ J_1^{-1}(d_2x + d_1y) \end{pmatrix}.$$

But this means $\operatorname{Ker}\delta^1 = \operatorname{im}\delta^0$. Hence (2) implies (1).

Next assume (1). Suppose

$$\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}a_1' & a_2'\\ -d_2 & d_1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

Then by the exactness of $F(X', a')$ we find $\xi \in \mathring{X}'$ such that

$$x = a_2'\xi, \quad y = -a_1'\xi$$

and consequently

$$0 = -d_2x + d_1y = -d_2a_2'\xi - d_1a_1'\xi = -J_1\xi.$$

Thus $\xi = 0$ and hence $0 = x = y$. Thus $\begin{pmatrix}a_1' & a_2'\\ -d_2 & d_1\end{pmatrix}$ is injective. Because of equation (4.3) $\begin{pmatrix}a_1' & a_2'\\ -d_2 & d_1\end{pmatrix}$ is always a surjection and thus (1) implies (2).

REMARKS. (1) The spectrum $\sigma_{\otimes}(a_1, a_2; X)$ and its characterization 4.1 is of course not as nice as Taylor's joint spectrum, for one has to solve two operator equations, whereas the elegance and usefulness of Taylor's joint spectrum just lies in the fact that no operator equation has to be solved. In the case of $\mathscr{L}_p$-spaces ($p = 1, 2, \infty$) it turns out that we automatically obtain the solvability of operator equations outside the Taylor spectrum. On the other hand by 4.1 one gets an independent proof of the closedness of $\sigma_{\otimes}(a_1, a_2; X)$ using that the invertible operators on a Banach space Y form an open group in $L(Y)$.

(2) In order to decide whether $(0, 0) \in \sigma_{\otimes}(a_1, a_2; X)$ we make the following considerations:

(i) If we multiply equation (4.3) by $\begin{pmatrix}I_{X'} & 0\\ (d_2b_1 - d_1b_2)J_2^{-1} & I_{X'}\end{pmatrix}$ from the left, we see that d_1, d_2 may be chosen in such a way that $d_2b_1 = d_1b_2$.

(ii) Letting $\tilde{d}_i := J_1^{-1}d_i$ and $\tilde{b}_i := b_iJ_2^{-1}$ $(i = 1, 2)$ we obtain (4.3) with $J_1 = J_2 = I_{X'}$. On the other hand we have

$$\text{(4.4)} \qquad \begin{pmatrix}\tilde{b}_1 & -a_2'\\ \tilde{b}_2 & a_1'\end{pmatrix}\cdot\begin{pmatrix}a_1' & a_2'\\ -\tilde{d}_2 & \tilde{d}_1\end{pmatrix} = \begin{pmatrix}\tilde{b}_1a_1' + a_2'\tilde{d}_2 & \tilde{b}_1a_2' - a_2'\tilde{d}_1\\ \tilde{b}_2a_1' - a_1'\tilde{d}_2 & \tilde{b}_2a_2' + a_1'\tilde{d}_1\end{pmatrix}.$$

Consequently $(0, 0) \notin \sigma_{\otimes}(a_1, a_2; X)$ if and only if (4.3) and

$$\text{(4.5)} \qquad \tilde{b}_2a_1' = a_1'\tilde{d}_2 \quad \text{and} \quad \tilde{b}_1a_2' = a_2'\tilde{d}_1$$

$$\text{(4.6)} \qquad \tilde{b}_1a_1' + a_2'\tilde{d}_2 = I_{X'} = \tilde{b}_2a_2' + a_1'\tilde{d}_1$$

are fulfilled.

Assume (4.3) and (4.5). Then $(\tilde{b}_1 a_1' + a_2'\tilde{d}_2)x = 0$ implies $0 = a_1'(\tilde{b}_1 a_1' + a_2'\tilde{d}_2)x = a_1'x$ and $0 = a_2'\tilde{d}_2 x = \tilde{d}_2 a_2'\tilde{d}_2 x = \tilde{d}_2 x - \tilde{d}_1 a_1'\tilde{d}_2 x = \tilde{d}_2 x - \tilde{d}_1\tilde{b}_2 a_1' x = \tilde{d}_2 x$. In the same way $(\tilde{b}_2 a_2' + a_1'\tilde{d}_1)x = 0$ implies $a_2'x = 0 = \tilde{d}_1 x$. Thus we have the following Banach space version of a recent result of Curto [3, 3.10] to which the referee has drawn the author's attention.

4.2. THEOREM. *Under the assumptions of* 4.1 *the following are equivalent:*

(1) $(0, 0) \notin \sigma_{\otimes}(a_1, a_2; X)$.

(2) *There exist* $d_1, d_2, b_1, b_2 \in L(X')$ *such that* $d_1 b_2 = d_2 b_1$ *and*

(i) $J_1 := d_1 a_1' + d_2 a_2'$ *as well as* $J_2 := a_1' b_1 + a_2' b_2$ *are topological isomorphisms;*

(ii) $b_2 J_2^{-1} a_1' = a_1' J_1^{-1} d_2$ *and* $b_1 J_2^{-1} a_2' = a_2' J_1^{-1} d_1$*;*

(iii) $\operatorname{Ker} a_i' \cap \operatorname{Ker} d_j = \{0\}$ *for* $i \neq j$.

If X is a Hilbert space, then there are canonical choices of d_1, d_2, b_1, b_2. Namely, if $(0, 0) \notin \sigma_{\otimes}(a_1, a_2; X) = \sigma(a_1, a_2; X)$, then the positive operators

$$J_1 = a_1^* a_1 + a_2^* a_2 \quad \text{and} \quad J_2 = a_1 a_1^* + a_2 a_2^*$$

are topological monomorphisms (= topological isomorphisms), because

$$J_i x_n \to 0 \quad (i = 1, 2)$$

implies

$$(**) \qquad \begin{cases} (J_1 x_n, x_n) = (a_1 x_n, a_1 x_n) + (a_2 x_n, a_2 x_n) \to 0 \\ (J_2 x_n, x_n) = (a_1^* x_n, a_1^* x_n) + (a_2^* x_n, a_2^* x_n) \to 0 \end{cases}$$

and thus $x_n \to 0$ as $n \to \infty$, because $\delta^0(a)$ and $\delta^0(a^*)$ are topological monomorphisms.

Consequently, we may choose $d_i = b_i = a_i^*$ $(i = 1, 2)$ and Theorem 4.1 is just Vasilescu's result [15, Theorem 1.1].

Since in this case $a_2^* J_2^{-1} a_1 = a_1 J_1^{-1} a_2^*$ implies $a_1^* J_2^{-1} a_2 = a_2 J_1^{-1} a_1^*$ and viceversa, Theorem 4.2 is just Curto [3, Theorem 3.10].

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VOLKER WROBEL
Mathematisches Seminar
der Universität Kiel,
Olshausenstrasse 40,
D-2300 Kiel 1,
Federal Republic of Germany.

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