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SCATTERING MATRIX AND SPECTRAL SHIFT OF THE NUCLEAR DISSIPATIVE SCATTERING THEORY. II

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1. INTRODUCTION

The present paper continues the investigations of the scattering matrix and of the spectral shift function of a scattering theory of maximal dissipative operators.

The scattering theory of maximal dissipative operators was developed in [12, 13]. In these papers the wave operators were introduced and a definition of the completeness of the wave operators of maximal dissipative operators was given. The scattering operator was defined in [13] and the investigation of this object was started there. A detailed representation of a dissipative scattering theory can be found in [17].

An attempt to define the notions of the scattering matrix and the spectral shift function as well as to clarify their interplay was undertaken in [15—18], where a maximal dissipative operator and a selfadjoint operator which differ by a nuclear dissipative operator were considered. The aim of the present paper is to generalize these results to a pair of operators $\{H_1, H_0\}$ consisting of a maximal dissipative operator H_1 and a selfadjoint operator H_0 both defined on a separable Hilbert space $\mathfrak{H}$ such that the resolvent difference belongs to the trace class, i.e.

$$(1.1)\qquad (H_1 - \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} \in \mathscr{L}_1(\mathfrak{H}),$$

where $\mathscr{L}_1(\mathfrak{H})$ denotes the class of trace operators on $\mathfrak{H}$.

To obtain such a generalization we start with a pair $\{T_1, T_0\}$ consisting of a contraction T_1 and a unitary operator T_0 both defined on $\mathfrak{H}$ such that their difference belongs to the trace class, i.e.

$$(1.2)\qquad T_1 - T_0 \in \mathscr{L}_1(\mathfrak{H}).$$

In the first section of chapter two we introduce the wave and scattering operators for such a situation and derive a formula of the family of scattering matrices. The second section is concerned with the definition of a spectral shift function and the verification of a trace formula for the pair $\{T_1, T_0\}$. It turns out that these considerations are independent of the assumption that T_0 is unitary. In such a way we assume through this section that T_0 is a contraction on $\mathfrak{H}$, too. Returning to the case where T_0 is a unitary operator, we prove a certain Birman-Krein formula in the last section of this chapter.

In the third chapter we try to obtain similar results for a pair of operators consisting of a maximal dissipative operator and a selfadjoint operator. For this problem the main tool will be the invariance principle of wave operators and the Cayley transform. Using the invariance principle and taking into account the Cayley transform we find a formula for the family of scattering matrices and we carry over the results of the second section of chapter two to a pair of maximal dissipative operators. The Birman-Krein formula follows then directly.

An attempt in the same direction was undertaken by A. V. Rybkin [20, 21]. The results of A. V. Rybkin partially coincide with the results of [15—18]. Further, publications of H. Langer [11], R. V. Akopjan [2, 3], V. M. Adamjan and B. S. Pavlov [1] and P. Jonas [8, 9] are related to the subject of this paper.

2. CONTRACTIVE CASE

2.1. SCATTERING MATRIX. First of all we remark that the condition (1.2) implies the existence of the wave operators W_+,

$$W_+ = \operatorname*{s-lim}_{n\to+\infty} T_1^{*n}T_0^nP^{ac}(T_0), \tag{2.1}$$

and W_-,

$$W_- = \operatorname*{s-lim}_{n\to+\infty} T_1^nT_0^{*n}P^{ac}(T_0), \tag{2.2}$$

where $P^{ac}(T_0)$ denotes the orthogonal projection from $\mathfrak{H}$ onto the absolutely continuous subspace $\mathfrak{H}^{ac}(T_0)$ of the unitary operator T_0. A theorem of this contents can be nowhere found, but it is not hard to see that such a theorem should be the discrete version of Theorem 2.1 of [14]. Consequently, transforming the considerations of Theorem 2.1 of [14] into a discrete language we obtain a proof of the above mentioned existence assertions. Moreover, these considerations imply the existence of the dilation wave operators $\tilde{W}_\pm$,

$$\tilde{W}_\pm = \operatorname*{s-lim}_{n\to\pm\infty} U_1^{*n}T_0^nP^{ac}(T_0), \tag{2.3}$$

where U_1 denotes a minimal unitary dilation of T_1 defined on the dilation space $\mathscr{H}$, $\mathfrak{H} \subseteq \mathscr{H}$.

In the following if the wave operators $W_\pm$ exist, then we call the triplet $\mathcal{A} = \{T_1, T_0; I\}$ a scattering system.

The scattering operator S of the scattering system $\mathcal{A}$ is defined by

$$S = W_+^* W_-. \tag{2.4}$$

Obviously, the scattering operator S is a contraction verifying $\ker(S) \supseteq \mathfrak{H} \ominus \mathfrak{H}^{ac}(T_0)$ and $\mathrm{ima}(S) \subseteq \mathfrak{H}^{ac}(T_0)$. Moreover, the scattering operator commutes with the unitary operator T_0, i.e.

$$T_0 S = S T_0. \tag{2.5}$$

Let $E_0(\cdot)$ be the spectral measure of the unitary operator T_0, i.e.

$$T_0 = \int_0^{2\pi} e^{it} dE_0(t). \tag{2.6}$$

With T_0 we associate a selfadjoint operator A_0 given by

$$A_0 = \int_0^{2\pi} t \, dE_0(t). \tag{2.7}$$

Obviously, we have $T_0 = e^{iA_0}$ and $\mathfrak{H}^{ac}(T_0) = \mathfrak{H}(A_0)$.

In the following we consider the spectral representation of $A_0^{ac} = A_0 \upharpoonright \mathfrak{H}^{ac}(A_0)$. We denote this spectral representation by $L^2(\Delta, |\cdot|; \mathfrak{H}_t, \mathscr{S})$, where $\Delta \subseteq [0, 2\pi]$ is a spectral core of A_0^{ac}, $|\cdot|$ is the Lebesgue measure on $\mathbf{R}^1$, which is thought to be restricted to Δ, $\{\mathfrak{H}_t\}_{t\in\Delta}$ is a family of separable Hilbert spaces and S is an admissible subsystem of $\times_{t\in\Delta} \mathfrak{H}_t$. For more details of spectral representations of selfadjoint operators we refer the reader to [5].

Because of $T_0^{ac} = T_0 \upharpoonright \mathfrak{H}^{ac}(T_0) = e^{iA_0^{ac}}$ in this spectral representation the operator T_0^{ac} is represented by the multiplication operator which is induced by the function e^{it}, $t \in \Delta$. Having in mind this fact we call the spectral representation of A_0^{ac} also the spectral representation of T_0^{ac}.

The above mentioned properties of S allow to introduce the family of scattering matrices $\{S(e^{it})\}_{t\in\Delta}$ with respect to the spectral representation $L^2(\Delta, |\cdot|; \mathfrak{H}_t, \mathscr{S})$ of T_0^{ac}. The family of scattering matrices $\{S(e^{it})\}_{t\in\Delta}$ consists of contractions for a.e. $t \in \Delta \bmod |\cdot|$.

With the family of scattering matrices we associate the family of scattering amplitudes $\{T(e^{it})\}_{t\in\Delta}$ which is defined by

$$T(e^{it}) = S(e^{it}) - I_{\mathfrak{H}_t}, \tag{2.8}$$

$\in \Delta$, where $I_{\mathfrak{H}_t}$ is the identity operator on $\mathfrak{H}_t$.

It is important to note that the family of scattering amplitudes allows an analytical representation. To explain this representation it is necessary to introduce some notation. We remark that on account of Lemma 2.9 of [15] there are bounded operators $B = B^*$ and C such that the representation

$$T_1 - T_0 = BCB \tag{2.9}$$

holds. Moreover, because of (1.2) the operator B can be chosen a Hilbert-Schmidt one, i.e. $B \in \mathscr{L}_2(\mathfrak{H})$. If $B = B^* \in \mathscr{L}_2(\mathfrak{H})$, then on account of Proposition 13 of [5, p. 57] the derivative $M(t)$,

$$M(t) = \frac{\mathrm{d}}{\mathrm{d}t} BP^{ac}(T_0)E_0(t)B \tag{2.10}$$

exists for a.e. $t \in [0, 2\pi] \bmod |\cdot|$ in the trace norm. Hence we have $M(t) \in \mathscr{L}_1(\mathfrak{H}$d and $M(t) \geqslant 0$ for a.e. $t \in [0,2\pi] \bmod |\cdot|$. Further, we need the operator-value) family $\{\Gamma_\rho(e^{it})\}_{t\in[0,2\pi]}$,

$$\Gamma_\rho(e^{it}) = C - CB(T_1 - \rho e^{it})^{-1}BC, \tag{2.11}$$

$\rho > 1$. Taking into account Proposition 14 of [5, p. 57] it is not hard to see that the limit $\Gamma(e^{it}) = \lim_{\rho\downarrow 1}\Gamma_\rho(e^{it})$ exists for a.e. $t \in [0, 2\pi] \bmod |\cdot|$ in the operator norm.

THEOREM 2.1. *Let $L^2(\Delta, |\cdot|; \mathfrak{H}_t, \mathscr{L})$ be a spectral representation of T_0^{ac} and let $\{T(e^{it})\}_{t\in\Delta}$ be the family of scattering amplitudes of the scattering system $\{T_1, T_0; I\}$, which obeys* (1.2). *Then there is a family of isometries $\{V(e^{it})\}_{t\in\Delta}$, $V(e^{it})$: $(\mathrm{ima}(M(t))^- \to \mathfrak{H}_t$, $t \in \Delta$, such that the representation*

$$T(e^{it}) = 2\pi V(e^{it})\sqrt{M(t)}\, e^{-it}\Gamma(e^{it})\sqrt{M(t)}\, V(e^{it})^* \tag{2.12}$$

holds for a.e. $t \in \Delta \bmod |\cdot|$.

REMARK 2.2. It is quite possible that the set $\delta = \{t \in \Delta : M(t) = 0\}$ has positive Lebesgue measure. In this case we set $V(e^{it}) = 0$ and $T(e^{it}) = 0$ for every $t \in \delta$.

The proof essentially follows the considerations of Theorem 2.15 and Corollary 2.17 of [15]. Therefore we omit the proof.

COROLLARY 2.3. *If the assumptions of Theorem* 2.1 *are valid, then we have*

$$T(e^{it}) \in \mathscr{L}_1(\mathfrak{H}) \tag{2.13}$$

for a.e. $t \in \Delta \bmod |\cdot|$.

Proof. The relation (2.13) immediately follows from Theorem 2.1. ▨

2.2. SPECTRAL SHIFT FUNCTION. In distinction from section one, through this section T_0 will denote a contraction on $\mathfrak{H}$, too.

The aim of this section is to define a spectral shift function for a pair of contractions $\{T_1, T_0\}$. An attempt in this direction was made in [15—18] for a dissipative situation. In the language of contractions these results can be expressed as follows. Let $\mathfrak{G}_{\mathbf{T}^1}$ be a set of functions defined by the condition that their elements $\varphi(\cdot)$ admits a Fourier decomposition

$$\varphi(z) = \sum_{l=-\infty}^{+\infty} a_l z^l, \tag{2.14}$$

$z \in \mathbf{T}^1 = \{z \in \mathbf{C} : |z| = 1\}$ such that the condition

$$\sum_{l=-\infty}^{+\infty} |l a_l| < +\infty \tag{2.15}$$

is fulfilled. Introducing the functions $\varphi_+(\cdot) \in \mathfrak{G}_{\mathbf{T}^1}$,

$$\varphi_+(z) = \sum_{l=0}^{+\infty} a_l z^l, \tag{2.16}$$

and $\varphi_-(\cdot) \in \mathfrak{G}_{\mathbf{T}^1}$,

$$\varphi_-(z) = \sum_{l=-\infty}^{-1} a_l z^{-l}, \tag{2.17}$$

$z \in \mathbf{T}^1$, we decompose $\varphi(\cdot)$ into a sum of two functions

$$\varphi(z) = \varphi_+(z) + \varphi_-(\bar{z}), \tag{2.18}$$

$z \in \mathbf{T}^1$. The condition (1.2) yields

$$\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_0) - \varphi_-(T_0^*) \in \mathscr{L}_1(\mathfrak{H}) \tag{2.19}$$

for every $\varphi(\cdot) \in \mathfrak{G}_{T^1}$. If in addition to (1.2) the defect operators $D_{T_1} = \sqrt{I - T_1^* T_1}$, $D_{T_1^*} = \sqrt{I - T_1 T_1^*}$, $D_{T_0} = \sqrt{I - T_0^* T_0}$ and $D_{T_0^*} = \sqrt{I - T_0 T_0^*}$ belong to the trace class, then there is a real measurable function $\mu(\cdot) \in L^1([0, 2\pi])$ such that the trace formula

$$\text{(2.20)} \quad \begin{aligned} &\mathrm{tr}(\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_0) - \varphi_-(T_0^*)) = \\ &= \int_0^{2\pi} \mu(t) \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\mathrm{e}^{it}) \mathrm{d}t \end{aligned}$$

holds for every $\varphi(\cdot) \in \mathfrak{G}_{T^1}$. The function $\mu(\cdot)$ is called a spectral shift function of the pair $\{T_1, T_0\}$ and is defined by (2.20) up to an additive constant. The function $\mu(\cdot)$ admits the representation

$$\text{(2.21)} \quad \mu(t) = -\lim_{\rho \downarrow 1} \frac{1}{\pi} \operatorname{Im} \log \det(I + (T_1 - T_0)(T_0 - \rho \mathrm{e}^{it})^{-1}) + \text{const.}$$

for a.e. $t \in [0, 2\pi] \bmod \cdot$, where we have assumed

$$\lim_{z \to \infty} \log \det(I + (T_1 - T_0)(T_0 - z)^{-1}) = 0.$$

If the condition (1.2) is fulfilled but the defect operators do not belong to the trace class, then in accordance with [18] it is quite possible that the representation (2.21) makes sense but the spectral shift function defined by (2.20) is even not locally summable. Hence the trace formula (2.20) loses its meaning.

In the following we try to solve these problems modifying the trace formula (2.20). For this purpose we restrict the set $\mathfrak{G}_{T^1}$ to the set $\mathscr{F}_{T^1}$ which elements are characterized by

$$\text{(2.22)} \quad \sum_{l=-\infty}^{+\infty} l^2 |a_l| < +\infty.$$

Proposition 2.4. *Let $\{T_1, T_0\}$ be a pair of contractions on $\mathfrak{H}$ obeying (1.2). Then there is a real measurable function $\xi(\cdot) \in L^1([0, 2\pi])$ such that the formula*

$$\text{(2.23)} \quad \begin{aligned} &\mathrm{tr}(\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_0) - \varphi_-(T_0^*)) = \\ &= -\int_0^{2\pi} \xi(t) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \varphi(\mathrm{e}^{it}) \mathrm{d}t \end{aligned}$$

holds for every $\varphi(\cdot) \in \mathscr{F}_{T^1}$. *The function* $\xi(\cdot)$ *is defined by* (2.23) *up to an additive constant.*

Proof. Because of (1.2) we get $\|T_1^k - T_0^k\|_1 \leqslant k\|T_1 - T_0\|_1$, $k = 1, 2, \ldots,$ where $\|\cdot\|_1$ is the trace norm. Hence the sequence $\left\{\frac{1}{k^2}\operatorname{tr}(T_1^k - T_0^k)\right\}_{k=1}^{\infty}$ belongs to ℓ^2. Therefore we can define a real L^2-function $\xi(\cdot)$ by

$$\xi(t) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\operatorname{tr}(T_1^k - T_0^k)\mathrm{e}^{-\mathrm{i}tk} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\operatorname{tr}(T_1^{*k} - T_0^{*k})\mathrm{e}^{\mathrm{i}tk}, \tag{2.24}$$

$t \in [0, 2\pi]$. Obviously, we have $\xi(\cdot) \in L^1([0, 2\pi])$. Then since

$$-\int_0^{2\pi}\xi(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^{\mathrm{i}tn}\,\mathrm{d}t = n^2\int_0^{2\pi}\xi(t)\mathrm{e}^{\mathrm{i}tn}\,\mathrm{d}t = \operatorname{tr}(T_1^n - T_0^n), \tag{2.25}$$

$n = 1, 2, \ldots,$ we have

$$\operatorname{tr}(\varphi_+(T_1) - \varphi_+(T_0)) = -\int_0^{2\pi}\xi(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi_+(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \tag{2.26}$$

and similarly

$$\operatorname{tr}(\varphi_-(T_1^*) - \varphi_-(T_0^*)) = -\int_0^{2\pi}\xi(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi_-(\mathrm{e}^{-\mathrm{i}t})\,\mathrm{d}t \tag{2.27}$$

for every $\varphi(\cdot) \in \mathscr{F}_{T^1}$. But (2.18), (2.26) and (2.27) imply (2.23).

To prove the uniqueness it is sufficient to show that

$$\int_0^{2\pi}\xi(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = 0, \tag{2.28}$$

$\varphi(\cdot) \in \mathscr{F}_{T^1}$, implies $\xi(t) = \text{const.}$ for a.e. $t \in [0, 2\pi]$. But this is obvious. ▨

In the following we call a real measurable function $\xi(\cdot) \in L^1([0, 2\pi])$ verifying (2.23) an integrated spectral shift function of the pair $\{T_1, T_0\}$. We note that the integrated spectral shift function is defined up to an additive constant.

Let $P_r(\theta)$ be the Poisson kernel,

$$P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta)}, \tag{2.29}$$

$\theta \in [0, 2\pi]$, and let $\xi(\cdot) \in L^1([0, 2\pi])$. If the limit $\xi^*(s)$,

$$\xi^*(s) = \lim_{r \uparrow 1} - \int_0^{2\pi} \xi(t) \frac{\mathrm{d}}{\mathrm{d}t} P_r(t - s)\, \mathrm{d}t \tag{2.30}$$

exists at $s \in [0, 2\pi]$, then we call $\xi^*(s)$ the generalized derivative of $\xi(\cdot)$ at the point $s \in [0, 2\pi]$. It is possible to show that if the usual derivative $\xi'(s)$ of $\xi(\cdot)$ exists at s, then the generalized derivative also exists at s and both derivatives coincide, i.e. $\xi^*(s) = \xi'(s)$.

THEOREM 2.5. *Let $\{T_1, T_0\}$ be a pair of contractions on $\mathfrak{H}$ such that the condition (1.2) is fulfilled. If $\xi(\cdot)$ denotes an integrated spectral shift function of $\{T_1, T_0\}$, then for* a.e. $t \in [0, 2\pi]$ *mod* $|\cdot|$ *the generalized derivative $\xi^*(t)$ exists and we have*

$$\xi^*(t) = - \lim_{r \uparrow 1} \frac{1}{\pi} \operatorname{Im} \log \det(I + (T_1 - T_0)(T_0 - r^{-1}e^{it})^{-1}), \tag{2.31}$$

where we have fixed a branch of the logarithm by the condition $\lim_{|z| \to +\infty} \log \det(I + (T_1 - T_0)(T_0 - z)^{-1}) = 0$.

Proof. Because of (1.2) the determinant $\det(I + (T_1 - T_0)(T_0 - z)^{-1})$, $|z| > 1$, makes sense. We get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z} \log \det(I + (T_1 - T_0)(T_0 - z)^{-1}) = \\ &= - \operatorname{tr}((T_1 - z)^{-1} - (T_0 - z)^{-1}), \end{aligned} \tag{2.32}$$

$|z| > 1$. Setting $\varphi(e^{it}) = \dfrac{1}{e^{it} - z}$, $t \in [0, 2\pi]$, $|z| > 1$, we have $\varphi(\cdot) \in \mathscr{F}_{T^1}$. Using

Proposition 2.4 we get

(2.33)
$$\operatorname{tr}((T_1 - z)^{-1} - (T_0 - z)^{-1}) = \int_0^{2\pi} \xi(t) \frac{e^{it} + z}{(e^{it} - z)^3} e^{it}\, dt,$$

$|z| > 1$. But from (2.33) we obtain

(2.34)
$$\log\det(I + (T_1 - T_0)(T_0 - z)^{-1}) = -i \int_0^{2\pi} \xi(t) \frac{d}{dt} \frac{e^{it}}{(e^{it} - z)}\, dt,$$

$|z| > 1$. Hence we find

(2.35)
$$\frac{1}{\pi} \operatorname{Im} \log\det(I + (T_1 - T_0)(T_0 - r^{-1}e^{is})^{-1}) = \int_0^{2\pi} \xi(t) \frac{d}{dt} P_r(t - s)\, dt,$$

$0 < r < 1$, $s \in [0, 2\pi]$.

It remains to establish the existence of $\lim_{r \uparrow 1} \frac{1}{\pi} \operatorname{Im} \log\det(I + (T_1 - T_0)(T_0 - r^{-1}e^{is})^{-1})$ for a.e. $s \in [0, 2\pi]$ mod $|\cdot|$. To this end we use the notion of the regularized determinant $\widetilde{\det}(I + \cdot)$, which is applicable to Hilbert-Schmidt operators. For a detailed presentation of this determinant the reader is referred to [7]. Taking into account the factorization (2.9) we get

(2.36)
$$\log\det(I + (T_1 - T_0)(T_0 - r^{-1}e^{is})^{-1}) = \log\widetilde{\det}(I + CB(T_0 - r^{-1}e^{is})^{-1}B) + \operatorname{tr}(CB(T_0 - r^{-1}e^{is})^{-1}B),$$

$0 < r < 1, s \in [0, 2\pi]$. From Proposition 14 of [5, p. 57] we find that $\lim_{r\uparrow 1} CB(T_0 - r^{-1}e^{is})^{-1}B$ exists for a.e. $s \in [0, 2\pi] \bmod |\cdot|$ in the Hilbert-Schmidt norm. But the determinant $\widetilde{\det}(I + \cdot)$ is continuous in the Hilbert-Schmidt norm. Consequently, the limit $\lim_{r\uparrow 1} \widetilde{\det}(I + CB(T_0 - r^{-1}e^{is})^{-1}B)$ exists for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. For the same reason the limit $\lim_{r\uparrow 1} \widetilde{\det}(I - CB(T_1 - r^{-1}e^{is})^{-1}B)$ exist for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. Hence we obtain

$$\begin{aligned} &\lim_{r\uparrow 1} \widetilde{\det}(I + CB(T_0 - r^{-1}e^{is})^{-1}B)\cdot \\ (2.37)\qquad &\cdot \lim_{r\uparrow 1} \widetilde{\det}(I - CB(T_1 - r^{-1}e^{is})^{-1}B) = \\ &= \lim_{r\uparrow 1} \exp\{-\mathrm{tr}(CB(T_0 - r^{-1}e^{is})^{-1}BCB(T_1 - r^{-1}e^{is})^{-1}B) \neq 0 \end{aligned}$$

for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. But (2.37) implies $\lim_{r\uparrow 1} \widetilde{\det}(I + CB(T_0 - r^{-1}e^{is})^{-1}B) \neq 0$ for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. Consequently, the limit $\lim_{r\uparrow 1} \log \widetilde{\det}(I + CB(T_0 - r^{-1}e^{is})^{-1}B)$ exists for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. It follows that $\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \log \det(I + CB(T_0 - r^{-1}e^{is})^{-1}B)$ exists for a.e. $s \in [0, 2\pi] \bmod |\cdot|$.

To show the existence of $\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \mathrm{tr}(CB(T_0 - r^{-1}e^{is})^{-1}B)$ we use Theorem 2 of [4]. Considering the transformation $\mathbf{R}^1 \ni \lambda \to 2 \operatorname{arcctg} \lambda = t \in [0, 2\pi]$, Theorem 2 of [4] can be formulated as follows. If $g(\cdot) : [0, 2\pi] \to \mathbf{C}$ is a function of bounded variation, then the limit

$$(2.38)\qquad \lim_{r\uparrow 1} \int_0^{2\pi} \frac{1}{e^{it} - r^{-1}e^{is}}\, dg(t)$$

exists for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. Denoting by $F_0(\cdot)$ the spectral measure of a unitary dilation of T_0 we find

$$\begin{aligned} &\mathrm{tr}(CB(T_0 - r^{-1}e^{is})^{-1}B) = \\ (2.39)\qquad & \\ &= \int_0^{2\pi} \frac{1}{e^{it} - r^{-1}e^{is}}\, d(\mathrm{tr}(CBF_0(t)B)), \end{aligned}$$

$0 < r < 1$, $s \in [0, 2\pi]$. Setting $g(t) = \mathrm{tr}(CBF_0(t)B)$, $t \in [0, 2\pi]$, it is not hard to see that $g(\cdot)$ is of bounded variation. Hence the limit

$$\lim_{r\uparrow 1} \frac{1}{\pi} \mathrm{Im} \int_0^{2\pi} \frac{1}{e^{it} - r^{-1}e^{is}} \, d(\mathrm{tr}(CBF_0(t)B))$$

exists for a.e. $s \in [0, 2\pi] \bmod |\cdot|$. ▨

Theorem 2.5 shows us that the existence of (2.21) does not require the additional conditions D_{T_1}, $D_{T_1^*}$, D_{T_0} and $D_{T_0^*} \in \mathscr{L}_1(\mathfrak{H})$. But if these conditions are fulfilled, then normalizing the spectral shift function $\mu(\cdot)$ of $\{T_1, T_0\}$ by the condition $\int_0^{2\pi} \mu(t)dt = 0$ an integrated spectral shift function $\xi(\cdot)$ of $\{T_1, T_0\}$ can be obtained by the formula $\xi(t) = \int_0^t \mu(s)\,ds$, $t \in [0, 2\pi]$. Consequently, every integrated spectral shift function of $\{T_1, T_0\}$ is absolutely continuous in this case. Hence the usual derivative $\xi'(t)$ exists for a.e. $t \in [0, 2\pi] \bmod |\cdot|$ and the equality $\xi^*(t) = \xi'(t) = \xi(t)$ holds for a.e. $t \in [0, 2\pi] \bmod |\cdot|$.

From this point of view it seems to be quite natural to call every function $\mu(\cdot)$, which differs form $\xi^*(\cdot)$ by a real constant, a spectral shift function of the pair $\{T_1, T_0\}$, what we will do in the following.

At the end of this section we note that introducing a distribution theory of functions over $C^\infty(\mathbf{T}^1) \subseteq \mathscr{F}_{\mathbf{T}^1}$, regarding $\xi(\cdot) \in L^1([0, 2\pi])$ as a distribution ξ defined by $(\xi, \varphi) = \int_0^{2\pi} \xi(t)\varphi(t)dt$, $\varphi(\cdot) \in C^\infty(\mathbf{T}^1)$, and considering the distribution derivative ξ' given by $(\xi', \varphi) = -(\xi, \varphi')$ the generalized trace formula (2.23) can be transformed into

$$\mathrm{tr}(\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_0) - \varphi_-(T_0^*)) = (\xi', \varphi'), \tag{2.40}$$

$\varphi \in C^\infty(\mathbf{T}^1)$. In accordance with the considerations of P. Jonas [9] the distribution derivative ξ' of the integrated spectral shift function $\xi(\cdot)$ can be called the spectral shift functional or the spectral shift distribution of the pair $\{T_1, T_0\}$.

Following this line the contents of Theorem 2.5 can be understood as the possibility to localize the spectral shift distribution for a.e. point of $[0, 2\pi] \bmod |\cdot|$. Hence the spectral shift function $\xi(\cdot)$ of the pair $\{T_1, T_0\}$ is nothing else than the a.e. mod $|\cdot|$ localized spectral shift distribution of $\{T_1, T_0\}$. But it is in general impossible to restore ξ' or $\xi(\cdot)$ from the spectral shift function $\xi^*(\cdot)$.

2.3. Scattering matrix and spectral shift function. We return to the situation where T_0 is a unitary operator on $\mathfrak{H}$. Our next aim is to generalize the well-known Birman-Krein formula to our contractive situation.

To this end we remember that if $\{U_1, T_0\}$ is a pair of unitary operators on $\mathfrak{H}$ such that the condition $U_1 - T_0 \in \mathscr{L}_1(\mathfrak{H})$ is fulfilled, then a spectral shift function $\mu(\cdot)$ of the pair $\{U_1, T_0\}$ exists, which belongs to $L^1([0, 2\pi])$, and in accordance with (2.21) can be represented by

$$\text{(2.14)} \quad \mu(t) = -\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \log \det(I + (U_1 - T_0)(T_0 - r^{-1}e^{it})^{-1}) + \text{const.}$$

for a.e. $t \in [0, 2\pi] \bmod |\cdot|$. The spectral shift function is only defined up to an additive constant. Usually a certain spectral shift function $\mu^0(\cdot)$ is fixed by the condition

$$\text{(2.42)} \qquad \int_0^{2\pi} \mu^0(t)\, dt = -i \operatorname{tr}(\log(U_1 T_0^*)),$$

where by $-i \log(U_1 T_0^*)$ we denote that operator of $-i \operatorname{Log}(U_1 T_0^*)$, whose spectrum is situated in $(-\pi, \pi]$, and is called the mean value of the spectral shift functions of $\{U_1, T_0\}$. The family of scattering matrices $\{S_0(e^{it})\}_{t\in\Delta}$ of the scattering system $\Lambda_0 = \{U_1, T_0; I\}$ defined in accordance with Section 1 consists of unitary operators in this case. Taking into account Corollary 2.3 it makes sense to consider the function $\det(S_0(e^{it}))$, $t \in \Delta$. Now M. Š. Birman and M. G. Krein have established in [6], that the spectral shift function $\mu^0(\cdot)$ and the family of scattering matrices $\{S_0(e^{it})\}_{t\in\Delta}$ are related by

$$\text{(2.43)} \qquad \det(S_0(e^{it})) = e^{2\pi i \mu^0(t)}$$

for a.e. $t \in \Delta \bmod |\cdot|$.

To extend the Birman-Krein formula to our contractive situation it is necessary to introduce the characteristic function of a contraction, which was widely investigated by B. Sz.-Nagy and C. Foiaş in [22]. In the following we deal with contractions T_1 characterized by the condition $\dim \ker(T_1) = \dim \ker(T_1^*)$. Contractions of this structure allows a representation of the form $T_1 = U_1 R$, where U_1 is a unitary operator on $\mathfrak{H}$ and $R = |T_1|$. For this restricted class of contractions the characteristic function $\theta_{T_1}(\cdot)$ can be defined by

$$\text{(2.44)} \qquad \theta_{T_1}(z) = R - z\sqrt{I - R^2}\, U_1^*(I - zT_1)^{-1}\sqrt{I - R^2},$$

$z \in \mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, where the values of $\theta_{T_1}(\cdot)$ are considered as bounded linear operators acting from $(\operatorname{ima}(\sqrt{I - R^2}))^-$ into itself. It is not hard to see that the definition (2.44) is equivalent in the sense of [22, Chapter V, 2.4] to the definition of the characteristic function given in [22, Chapter V, 1.1].

The characteristic function is an analytic contractive one, which completely characterizes the contraction T_1. Asking $I - R \in \mathscr{L}_1(\mathfrak{H})$ we obtain $\theta_{T_1}(z) - I \in \mathscr{L}_1(\mathfrak{H})$ for every $z \in \mathbf{D}$. Hence it makes sense to define the complex-valued function $\delta_{T_1}(\cdot)$,

$$\delta_{T_1}(z) = \det(\theta_{T_1}(z)), \tag{2.45}$$

$z \in \mathbf{D}$. The complex-valued function $\delta_{T_1}(\cdot)$ is an analytic contractive one, too. But this fact implies that the limit $\delta_{T_1}(e^{it})$,

$$\delta_{T_1}(e^{it}) = \lim_{r \uparrow 1} \delta_{T_1}(re^{it}), \tag{2.46}$$

exists for a.e. $t \in [0, 2\pi] \bmod |\cdot|$.

THEOREM 2.6. *Let $L^2(\Delta, |\cdot|; \mathfrak{H}_t, \mathscr{S})$ be a spectral representation of T_0^{ac} and le $S(e^{it})\}_{t \in \Delta}$ be the family of scattering matrices of the scattering system $A = \{T_1, T_0; I\}$ which obeys* (1.2). *Then there is a spectral shift function $\nu^0(\cdot)$ of the pair $\{T_1, T_0\}$ such that*

$$\det(S(e^{it})) = \delta_{T_1}(e^{it})e^{2\pi i \nu^0(t)} \tag{2.47}$$

holds for a.e. $t \in \Delta \bmod |\cdot|$.

Proof. We prove this theorem in several steps.

1. On account of [10, Chapter IV, §5] and condition (1.2) we find that T_1 is a Fredholm operator with the property $\mathrm{nul}(T_1) = \mathrm{def}(T_1)$. Consequently, the operator T_1 allows the representation $T_1 = U_1 R$, where U_1 is a unitary operator on $\mathfrak{H}$ and $R = |T_1|$. Moreover, we find $T_1 - U_1 \in \mathscr{L}_1(\mathfrak{H})$ and $U_1 - T_0 \in \mathscr{L}_1(\mathfrak{H})$.

2. The considerations of the first step allows to divide the scattering system A into two new scattering systems $A_1 = \{T_1, U_1; I\}$ and $A_0 = \{U_1, T_0; I\}$ such that we have

$$\det(S(e^{it})) = \det(S_1(e^{it}))\det(S_0(e^{it})) \tag{2.48}$$

for a.e. $t \in \Delta \bmod |\cdot|$, where $\{S_1(e^{it})\}_{t \in \Delta}$ and $\{S_0(e^{it})\}_{t \in \Delta}$ are the families of scattering matrices of the scattering systems A_1 and A_0, respectively. The fact that the spectral cores of both families of scattering matrices are the same must be established but it can be easily done.

By $\mu^0(\cdot)$ we denote a spectral shift function of the pair $\{U_1, T_0\}$ normalized by (2.42). If $\xi_1(\cdot)$ denotes the integrated spectral shift function of the pair $\{T_1, U_1\}$, we choose the generalized derivative $\xi_1^*(\cdot)$ for the spectral shift function of the pair $\{T_1, U_1\}$. Setting

$$\nu^0(t) = \mu^0(t) + \xi_1^*(t), \tag{2.49}$$

$t \in [0, 2\pi]$, we obviously define a spectral shift function of the pair $\{T_1, T_0\}$. Taking into account (2.48) and (2.43) it remains to show

$$\det(S_1(e^{it})) = \delta_{T_1}(e^{it})e^{2\pi i \xi_1^*(t)} \tag{2.50}$$

for a.e. $t \in \Delta \bmod |\cdot|$.

3. We prove the relation (2.50). To this end we apply Theorem 2.1 to the scattering system $A_1 = \{T_1, U_1; I\}$. Replacing T_0 by U_1 the operator-valued function $e^{-it}M(t)$, $t \in [0, 2\pi]$, can be represented by

$$e^{-it}M(t) = \lim_{r\uparrow 1} B\,\frac{1}{2\pi}\,\frac{(1-r^2)U_1^*}{I + r^2 - rU_1^*e^{it} - rU_1e^{-it}}\,B \tag{2.51}$$

for a.e. $t \in [0, 2\pi] \bmod |\cdot|$. The limit can be taken in the trace norm. Using this representation and the property $T_1 = U_1R$ we get

$$\det(S_1(e^{it})) =$$

$$= \lim_{r\uparrow 1} \det(I + B\,\frac{(1-r^2)U_1^*}{I + r^2 - rU_1^*e^{it} - rU_1e^{-it}}\,B\cdot$$

$$\cdot\{C - CB(T_1 - r^{-1}e^{it})^{-1}BC\}) = \tag{2.52}$$

$$= \lim_{r\uparrow 1} \det(I - \sqrt{I-R}\,\frac{1-r^2}{I + r^2 - rU_1^*e^{it} - rU_1e^{-it}}\,\sqrt{I-R}\cdot$$

$$\cdot\{I + \sqrt{I-R}\,(T_1 - r^{-1}e^{it})^{-1}U_1\sqrt{I-R}\})$$

for a.e. $t \in \Delta \bmod |\cdot|$. We set

$$\pi(z) = I + \sqrt{I-R}\,(T_1 - z)^{-1}U_1\sqrt{I-R}, \tag{2.53}$$

$|z| > 1$. We find

$$[\pi(z)]^{-1} = I - \sqrt{I-R}\,(U_1 - z)^{-1}U_1\sqrt{I-R} \tag{2.54}$$

and

$$\det(\pi(z)) = [\det(I + (T_1 - U_1)(U_1 - z)^{-1})]^{-1}, \tag{2.55}$$

$|z| > 1$. Consequently, we obtain

$$\begin{aligned}\det(S_1(e^{it})) = \lim_{r\uparrow 1}\det(I - \sqrt{I-R}\,(U_1 - r^{-1}e^{it})^{-1}U_1\sqrt{I-R}\, - \\ - \sqrt{I-R}\,\frac{1-r^2}{I + r^2 - rU_1^* e^{it} - rU_1 e^{-it}}\sqrt{I-R})\det(\pi(r^{-1}e^{it})),\end{aligned} \tag{2.56}$$

for a.e. $t \in [0, 2\pi] \bmod |\cdot|$. Hence we get

$$\det(S_1(e^{it})) = \lim_{r\uparrow 1}\det(I - \sqrt{I-R}\,(I - re^{it}U_1^*)^{-1}\sqrt{I-R}\cdot\det(\pi(r^{-1}e^{it})), \tag{2.57}$$

for a.e. $t \in [0, 2\pi] \bmod |\cdot|$.

A simple calculation proves the equality

$$\begin{aligned}&\det(\theta_{T_1}(re^{it})) = \\ &= \det(I - \sqrt{I-R}(I + re^{it}U_1^*)(I - re^{it}T_1^*)^{-1}\sqrt{I-R}),\end{aligned} \tag{2.58}$$

$0 \leqslant r < 1$, $t \in [0, 2\pi]$. Using (2.58) we find

$$\begin{aligned}&\det(\theta_{T_1}(re^{it}))\,\overline{[\det(\pi(r^{-1}e^{it}))]^{-1}} = \\ &= \det(I - \sqrt{I-R}(I - re^{it}U_1^*)^{-1}\sqrt{I-R}),\end{aligned} \tag{2.59}$$

$0 < r < 1$, $t \in [0, 2\pi]$. Putting (2.59) into (2.57) we obtain

$$\det(S_1(e^{it})) = \lim_{r\uparrow 1}\det(\theta_{T_1}(re^{it}))\,\frac{\det(\pi(r^{-1}e^{it}))}{\overline{\det(\pi(r^{-1}e^{it}))}} \tag{2.60}$$

for a.e. $t \in [0, 2\pi]$. From (2.55), (2.31) and (2.46) we obtain (2.50). ▧

3. DISSIPATIVE CASE

In this chapter we transform the results of chapter two to a pair $\{H_1, H_0\}$ of operators on $\mathfrak{H}$ which consists of a maximal dissipative operator H_1 and a selfadjoint operator H_0 which verify the condition

(3.1) $$(H_1 - \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} \in \mathscr{L}_1(\mathfrak{H}).$$

We formulate the results and only sketch the proofs.

The main tool to obtain such a transform is the Cayley transform

(3.2) $$T_j = (H_j + \mathrm{i})(H_j - \mathrm{i})^{-1},$$

$j = 0, 1$. It is not hard to see that (3.1) implies the condition (1.2) for the Cayley transforms T_1 and T_0. In such a way the results of Chapter 1 hold for the scattering system $\Lambda = \{T_1, T_0; I\}$. The problem is to carry over these results to the scattering system $\Xi = \{H_1, H_0; I\}$. Notice that under (3.1) the wave operators Ω_+,

(3.3) $$\Omega_+ = \operatorname*{s-lim}_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_1^*}\mathrm{e}^{-\mathrm{i}tH_0}P^{\mathrm{ac}}(H_0),$$

and Ω_-,

(3.4) $$\Omega_- = \operatorname*{s-lim}_{t\to+\infty} \mathrm{e}^{-\mathrm{i}tH_1}\mathrm{e}^{\mathrm{i}tH_0}P^{\mathrm{ac}}(H_0),$$

exist, where $P^{\mathrm{ac}}(H_0)$ denotes the projection from $\mathfrak{H}$ onto the absolutely continuous subspace $\mathfrak{H}^{\mathrm{ac}}(H_0)$ of the selfadjoin operator H_0. In Section 2.1 it was remarked that under (1.2) beside $W_\pm$ the dilation wave operator $\widetilde{W}_\pm$ exist. The same can be said concerning the dilation wave operators of Ξ. See for instance [14]. Taking into account the invariance principle [5, Corollary 26, p. 248] we obtain that the dilation wave operators of the scattering systems Λ and Ξ coincide. But from this fact we get the equalities $W_\pm = \Omega_\pm$. Hence the scattering operators S and $\Sigma = \Omega_+^*\Omega_-$ of the scattering systems Λ and Ξ coincide. Moreover the family $\{\Sigma(\lambda)\}_{\lambda\in N}$ defined by

(3.5) $$\Sigma(\lambda) = S(\mathrm{e}^{\mathrm{i}\,2\,\mathrm{arcctg}\,\lambda})$$

$\lambda \in N = \{\lambda \in \mathbf{R}^1 : \lambda = \mathrm{ctg}(t/2),\ t \in \Delta\}$ is a family of scattering matrices of the scattering system Ξ. Obviously we have

(3.6) $$\Sigma(\lambda) - I_{\mathfrak{H}_{2\,\mathrm{arcctg}\,\lambda}} \in \mathscr{L}_1(\mathfrak{H})$$

for a.e. $\lambda \in N \bmod |\cdot|$.

By the transformation $\Psi(\lambda) = \varphi\left(\frac{\lambda + i}{\lambda - i}\right)$, $\lambda \in \mathbf{R}^1$, we obtain a new set of functions from $\mathscr{F}_{\mathbf{T}^1}$ which we denote by $\mathscr{F}_{\mathbf{R}^1}$. Similarly we introduce the functions $\Psi_{\pm}(\cdot)$. A simple calculation shows the validity of

$$\lim_{\lambda \to \pm\infty} (1 + \lambda^2)\Psi'(\lambda) = -2\frac{d}{dt}\varphi(e^{it})|_{t=0} = -2\varphi'(1). \tag{3.7}$$

Hence we get a certain subset $\mathscr{F}'_{\mathbf{R}^1}$ of $\mathscr{F}_{\mathbf{R}^1}$ setting

$$\mathscr{F}'_{\mathbf{R}^1} = \{\Psi(\cdot) \in \mathscr{F}_{\mathbf{R}^1} : \lim_{\lambda \to \pm\infty} (1 + \lambda^2)\Psi'(\lambda) = 0\}. \tag{3.8}$$

Supposing that H_0 is also a maximal dissipative operator Proposition 2.4 reads now as follows.

PROPOSITION 3.1. *Let $\{H_1, H_0\}$ be a pair of maximal dissipative operators on $\mathfrak{H}$ such that the condition* (3.1) *is fulfilled. Then for every $\Psi(\cdot) \in \mathscr{F}'_{\mathbf{R}^1}$ we have*

$$\Psi_+(H_1) + \Psi_-(-H_1^*) - \Psi_+(H_0) - \Psi_-(-H_0^*) \in \mathscr{L}_1(\mathfrak{H}). \tag{3.9}$$

Moreover there is a real measurable function $\rho(\cdot)$ belonging to $L^1(\mathbf{R}^1, (1 + \lambda^2)^{-2}d\lambda)$ such that

$$\begin{aligned} &\operatorname{tr}(\Psi_+(H_1) + \Psi_-(-H_1^*) - \Psi_+(H_0) - \Psi_-(-H_0^*)) = \\ &= -\int_{-\infty}^{+\infty} \rho(\lambda)\Psi''(\lambda)d\lambda \end{aligned} \tag{3.10}$$

holds for every $\Psi(\cdot) \in \mathscr{F}'_{\mathbf{R}^1}$. The function $\rho(\cdot)$ is defined by (3.10) *up to a linear function.*

The proof essentially follows the considerations of Chapter 3 of [19].

In the following we call a real measurable function $\rho(\cdot)$ belonging to $L^1(\mathbf{R}^1, (1+\lambda^2)^{-2}d\lambda)$ and satisfying (3.10) the integrated spectral shift function of $\{H_1, H_0\}$. We remark that the integrated spectral shift function is defined up to a linear function.

Let $\theta(\cdot)$ be a smooth function on $\mathbf{R}^1$ verifying $0 \leqslant \theta(x) \leqslant 1$, $x \in \mathbf{R}^1$, $\theta(x) = 1$ for $x \in [-1, +1]$ and $\theta(x) = 0$ for $|x| \geqslant 2$. Let $\rho(\cdot)$ be a locally summable function on $\mathbf{R}^1$. If the limit $\rho^*(\lambda)$,

$$\rho^*(\lambda) = -\lim_{y \to +0} \int_{-\infty}^{+\infty} \rho(x)\theta(x - \lambda)\frac{d}{dx}\frac{1}{\pi}\frac{y}{(x - \lambda)^2 + y^2}dx, \tag{3.11}$$

$\lambda \in \mathbf{R}^1$, exists, then we call $\rho^*(\lambda)$ the generalized derivative of $\rho^*(\cdot)$ at $\lambda \in \mathbf{R}^1$. It is not hard to show that if the usual derivative $\rho'(\lambda)$ exists at λ, then the generalized derivative $\rho^*(\lambda)$ also exists at λ and equals $\rho'(\lambda)$, i.e. $\rho'(\lambda) = \rho^*(\lambda)$. Hence the generalized derivative of a linear function exists at every point and equals a constant.

Now Theorem 2.5 implies

THEOREM 3.2. *Let $\{H_1, H_0\}$ be a pair of maximal dissipative operators on $\mathfrak{H}$ such that the condition* (3.1) *is fulfilled. If $\rho(\cdot)$ denotes an integrated spectral shift function of the pair $\{H_1, H_0\}$, then the generalized derivative $\rho^*(\lambda)$ exists for a.e. $\lambda \in \mathbf{R}^1$* mod $|\cdot|$.

The proof is based on the fact that for every integrated spectral shift function $\xi(\cdot)$ of the pair of Cayley transforms $\{T_1, T_0\}$ there is a real constant such that we have

$$\xi^*(t)\Big|_{t=2\,\mathrm{arcctg}\,\lambda} + \text{const.} = -\rho^*(\lambda) \tag{3.12}$$

for a.e. $\lambda \in \mathbf{R}^1$ mod $|\cdot|$.

In accordance with the previous chapter we call the generalized derivative of the integrated spectral shift function of $\{H_1, H_0\}$ a spectral shift function of the pair $\{H_1, H_0\}$. We note that the spectral shift function is defined up to a constant.

We return to the situation where H_0 is selfadjoint. Next we generalize the Birman-Krein formula to a pair $\{H_1, H_0\}$, where H_1 is a maximal dissipative operator and H_0 is a selfadjoint operator verifying (3.1). Let $\theta_{T_1}(\cdot)$ be the characteristic function of the Cayley transform T_1 of H_1. We call the operator-valued function $\theta_{H_1}(\cdot)$ defined by

$$\theta_{H_1}(z) = \theta_{T_1}\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right), \tag{3.13}$$

$\operatorname{Im} z < 0$, the characteristic function of the maximal dissipative operator H_1. Obviously the characteristic function $\theta_{H_1}(\cdot)$ is a contractive analytic one defined on the lower half plane. Because of (3.1) we have $\theta_{H_1}(z) - I \in \mathscr{L}_1(\mathfrak{H})$ for every z in the lower half plane. Hence it makes sense to define the complex-valued function $\delta_{H_1}(z) = \det(\theta_{H_1}(z))$, $\operatorname{Im} z < 0$, which is a contractive analytic one, too. Consequently for a.e. $\lambda \in \mathbf{R}^1$ mod $|\cdot|$ the limits $\delta_{H_1}(\lambda) = \lim_{y\to+0} \delta_{H_1}(\lambda - \mathrm{i}y)$ exist. Obviously we have

$\delta_{H_1}(\lambda) = \delta_{T_1}\left(\dfrac{\lambda+\mathrm{i}}{\lambda-\mathrm{i}}\right)$ for a.e. $\lambda \in \mathbf{R}^1$ mod $|\cdot|$.

THEOREM 3.3. *Let* $\{\Sigma(\lambda)\}_{\lambda \in N}$ *be the family of scattering matrices of the scattering system* $\Xi = \{H_1, H_0; I\}$ *verifying* (3.1). *Then there is an integrated spectral shift function* $\rho(\cdot)$ *of the pair* $\{H_1, H_0\}$ *such that*

$$\det(\Sigma(\lambda)) = \delta_{H_1}(\lambda) e^{-2\pi i \rho^*(\lambda)} \tag{3.14}$$

holds for a.e. $\lambda \in N \bmod |\cdot|$.

The proof uses Theorem 2.5 and the relation (3.12).

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