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BOUNDS ON RESONANCES FOR STARK-WANNIER AND RELATED HAMILTONIANS

ARNE JENSEN

1. INTRODUCTION

In this paper we consider the one-dimensional Stark-effect Hamiltonian

$$H = -\frac{d^2}{dx^2} + Fx + V(x), \quad F > 0,$$

on $L^2(\mathbf{R})$, where $V(x)$ is a real valued function such that $V(x) = W'(x)$, where $W(z)$ is analytic and bounded in a strip $\{z \mid |\mathrm{Im}\, z| < a\}$. This class includes some periodic and almost-periodic functions, a class of considerable current interest, in particular in connection with the study of models for disordered systems. Using the translation group we associate with H the analytic family

$$H(\zeta) = -\frac{d^2}{dx^2} + Fx + V(x - \zeta) - F\zeta.$$

This family was studied in [3, 4] for different classes of potentials. The discrete eigenvalues of $H(\zeta)$ are called resonances. A resonance z_0 satisfies $0 < -\mathrm{Im}\, z_0 < F^{-1} \cdot \mathrm{Im}\, \zeta$ and does not depend on ζ, ζ satisfying this inequality.

In Section 2 we prove that the operator $(H_0^2 + 1)^{-1/4} V (H_0^2 + 1)^{-1/4}$, where H_0 is the closure of $-\frac{d^2}{dx^2} + Fx$, and related operators belong to the trace ideal $\mathcal{I}_4$ (see [8] for the definition). Furthermore, we obtain an explicit bound on the trace norms, see (2.1).

In Section 3 we restate results on resonances, previously obtained for different classes of potentials in [3, 4]. Furthermore, we obtain $\sigma_{\mathrm{ess}}(H(\zeta)) = \mathbf{R} - - iF \cdot \mathrm{Im}\, \zeta$, which is new for our class, see [4].

In Section 4 we obtain an explicit lower bound on the width $|\mathrm{Im}\, z_0|/2$ of a resonance. As a consequence we show that for F tending to infinity the resonances move away from the real axis.

In Section 5 we characterize the resonances as the zeros of an analytically continued modified Fredholm determinant. As a consequence we obtain a perturbation result for resonances. The modified perturbation determinant was used in [1] to show the existence of resonances for $V(x)$ a trigonometric polynomial and F large. The argument in [1] seems to depend crucially on the fact that this V is a polynomial in the variable e^{ix}. We have not been able to extend the argument in [1] to our larger class of potentials.

In Section 6 we combine the results of Sections 2 and 5 to obtain a bound on the asymptotic distribution of the resonances.

2. TRACE NORM ESTIMATES

The operator

$$-\frac{d^2}{dx^2} + Fx$$

is essentially selfadjoint on $\mathscr{S}(\mathbf{R})$, the Schwartz space. Let H_0 denote its closure. The trace ideal (von Neumann-Schatten class) is denoted $\mathscr{I}_p, p > 0$. All the results on trace ideals needed here can be found in [8].

For $\operatorname{Im} z > 0$ we use the functional calculus to define the operator

$$K(z) = (H_0 - z)^{-1/2}.$$

We note that $K(z)$ is an analytic operator-valued function of z, $\operatorname{Im} z > 0$, with values in the bounded operators on $L^2(\mathbf{R})$, $\mathscr{B}(L^2(\mathbf{R}))$. Furthermore, $K(z)^2 = (H_0 - z)^{-1}$ and $\operatorname{Ker} K(z) = \{0\}$, $\operatorname{Im} z > 0$.

LEMMA 2.1. *Let W be a function on* $\mathbf{R}$ *such that* $W^{(j)} \in L^\infty(\mathbf{R})$, $j = 0, 1, 2$. *Let* $V = W'$. *Then*

(i) *The operator* $(H_0 + i)^{-1}V(H_0 + i)^{-1}$ *is compact on* $L^2(\mathbf{R})$.

(ii) *For* $\operatorname{Im} z > 0$, $K(z)VK(z) \in \mathscr{I}_4$, *and with* $\beta = \operatorname{Im} z$ *we have the estimate*

$$\|K(z)VK(z)\|_{\mathscr{I}_4} \leqslant cF^{-1/4}\beta^{-1/12}\|W'\|_\infty^{2/3} \cdot \left(\frac{1}{\beta}\|W\|_\infty + \left(\frac{2F}{3\sqrt{3}\,\beta^3} + \frac{1}{\beta^2}\right)\|W'\|_\infty + \frac{1}{2\beta^2}\|W''\|_\infty\right)^{1/3}, \tag{2.1}$$

where c is independent of F, z, and W.

Proof. Part (i) was proved in [5]. It also follows from part (ii). We prove (ii). As quadratic forms on $\mathscr{S}(\mathbf{R}) \times \mathscr{S}(\mathbf{R})$ we have

$$i[H_0, W] = 2\,W'p - iW''$$

and thus for any $\beta > 0$, writing $W' = V$, we have

$$(H_0 + i\beta)^{-1} V p (H_0 + i\beta)^{-1} =$$

$$= \frac{1}{2}(H_0 + i\beta)^{-1}\{i(H_0 + i\beta)W - iW(H_0 + i\beta) + iW''\}(H_0 + i\beta)^{-1}.$$

The right hand side is a bounded operator on $L^2(\mathbf{R})$. Thus the left hand side extends to a bounded operator on $L^2(\mathbf{R})$. We omit writing closures. Thus

$$\|(H_0 + i\beta)^{-1} V \cdot p \cdot (H_0 + i\beta)^{-1}\| \leqslant$$

$$\leqslant \frac{1}{\beta} \|W\|_\infty + \frac{1}{2\beta^2} \|W''\|_\infty.$$

Here $\|\cdot\|$ is the norm on $\mathscr{B}(L^2(\mathbf{R}))$ and $\|\cdot\|_\infty$ is the $L^\infty(\mathbf{R})$-norm. We have

$$\|(H_0 + i\beta)(H_0^2 + \beta^2)^{-1/2}\| = 1.$$

An explicit computation shows that the commutator

$$[p, (H_0^2 + \beta^2)^{-1/2}]$$

extends to a bounded operator on $L^2(\mathbf{R})$ with

$$\|[p, (H_0^2 + \beta^2)^{-1/2}]\| \leqslant \frac{2F}{3\sqrt{3}\,\beta^2}.$$

Thus one has the estimate

$$\|(H_0^2 + \beta^2)^{-1/2} V (H_0^2 + \beta^2)^{-1/2} (p^2 + 1)^{1/2}\| \leqslant$$

$$\leqslant \frac{1}{\beta} \|W\|_\infty + \left(\frac{2F}{3\sqrt{3}\,\beta^3} + \frac{1}{\beta^2}\right)\|W'\|_\infty + \frac{1}{2\beta^2} \|W''\|_\infty.$$

Complex interpolation gives for any δ, $0 \leqslant \delta \leqslant 1$,

$$\|(H_0^2 + \beta^2)^{-\delta/2} V (H_0^2 + \beta^2)^{-\delta/2} (p^2 + 1)^{\delta/2}\| \leqslant$$

$$\leqslant (\|W'\|_\infty)^{1-\delta}\left(\frac{1}{\beta} \|W\|_\infty + \left(\frac{2F}{3\sqrt{3}\,\beta^3} + \frac{1}{\beta^2}\right)\|W'\|_\infty + \frac{1}{2\beta^2} \|W''\|_\infty\right)^\delta.$$

Using $e^{-ip^3/3F} H_0 e^{ip^3/3E} = Fx$ and [8, Theorem 4.1] we get

$$(1 + p^2)^{-1/6} (H_0^2 + \beta^2)^{-1/6} \in \mathscr{I}_4$$

and

$$\|(1+p^2)^{-1/6}(H_0^2+\beta^2)^{-1/6}\|_{\mathscr{I}_4} \leqslant$$

$$\leqslant (2\pi)^{-1/4}\|(1+\xi^2)^{-1/6}\|_{L^4(\mathbf{R})}\cdot\|(\beta^2+F^2\xi^2)^{-1/6}\|_{L^4(\mathbf{R})} \leqslant$$

$$\leqslant cF^{-1/4}\beta^{-1/12}.$$

We now have

$$\|(H_0^2+\beta^2)^{-1/6}V(H_0^2+\beta^2)^{-1/3}\|_{\mathscr{I}_4} \leqslant$$

$$\leqslant \|(H_0^2+\beta^2)^{-1/6}\,V(H_0^2+\beta^2)^{-1/6}(p^2+1)^{1/6}\|\cdot$$

$$\cdot\|(p^2+1)^{-1/6}\,(H_0^2+\beta^2)^{-1/6}\|_{\mathscr{I}_4} \leqslant \tag{2.2}$$

$$\leqslant cF^{-1/4}\beta^{-1/12}(\|W'\|_\infty)^{2/3}\cdot\left(\frac{1}{\beta}\|W\|_\infty + \right.$$

$$\left. + \left(\frac{2F}{3\sqrt{3}\,\beta^3}+\frac{1}{\beta^2}\right)\|W'\|_\infty + \frac{1}{2\beta^2}\|W''\|_\infty\right)^{1/3}.$$

To complete the proof we first note that another application of complex interpolation yields

$$(H_0^2+\beta^2)^{-1/4}V(H_0^2+\beta^2)^{-1/4} \in \mathscr{I}_4$$

with an $\mathscr{I}_4$-norm satisfying the estimate (2.2). We have

$$\|(H_0^2+\beta^2)^{1/4}K(\mathrm{i}\beta)\| = 1,$$

so we get $K(\mathrm{i}\beta)VK(\mathrm{i}\beta) \in \mathscr{I}_4$ with an $\mathscr{I}_4$-norm satisfying the estimate (2.2). To get the final result we use the translation group $(U(\alpha)f)(x) = f(x-\alpha)$. We then get for $z = \alpha + \mathrm{i}\beta$, $\beta > 0$,

$$\|K(\alpha+\mathrm{i}\beta)VK(\alpha+\mathrm{i}\beta)\|_{\mathscr{I}_4} =$$

$$= \left\|U\left(-\frac{\alpha}{F}\right)K(\alpha+\mathrm{i}\beta)VK(\alpha+\mathrm{i}\beta)U\left(\frac{\alpha}{F}\right)\right\|_{\mathscr{I}_4} =$$

$$= \left\|K(\mathrm{i}\beta)V\left(\cdot+\frac{\alpha}{F}\right)K(\mathrm{i}\beta)\right\|_{\mathscr{I}_4}.$$

Since the L^∞-norm is translation invariant, the result follows.

REMARK 2.2. It follows from the proof that $K(z)VK(z) \in \mathscr{I}_p$ for any $p > 3$.

3. RESONANCES FOR STARK-EFFECT HAMILTONIANS

In this section we briefly give results on resonances for Stark-effect Hamiltonians with potentials satisfying our new assumptions.

The translation group is given by

$$(U(\alpha)f)(x) = f(x - \alpha).$$

It maps the domain $\mathscr{D}(H_0)$ into itself. We have

$$H_0(\alpha) = U(\alpha)H_0U(-\alpha) = H_0 - F\alpha.$$

This family extends to an analytic family of type A (see [6]): for $\zeta \in \mathbf{C}$,

$$H_0(\zeta) = H_0 - F\zeta, \quad \mathscr{D}(H_0(\zeta)) = \mathscr{D}(H_0)$$

with

$$\sigma(H_0(\zeta)) = \mathbf{R} - \mathrm{i}F\cdot\mathrm{Im}\,\zeta.$$

We introduce the following class of potentials:

Assumption 3.1. *$W(z)$ is analytic in $S_a = \{z \mid |\mathrm{Im}\, z| < a\}$, $a > 0$, with*

$$\sup_{z \in S_a} (|W(z)| + |W'(z)| + |W''(z)|) < \infty.$$

$W(z)$ is real valued for $\mathrm{Im}\, z = 0$. V denotes multiplication by $W'(x)$, $x \in \mathbf{R}$.

Let V satisfy Assumption 3.1. We let $V_\zeta(x) = V(x - \zeta)$ for $\zeta \in S_a$ and let V_ζ denote the multiplication operator. The map $\zeta \mapsto V_\zeta$ is analytic from S_a to $\mathscr{B}(L^2(\mathbf{R}))$. We get an analytic family of type A from

$$H(\zeta) = H_0(\zeta) + V_\zeta, \quad \zeta \in S_a, \ \mathscr{D}(H(\zeta)) = \mathscr{D}(H_0).$$

For $\zeta = 0$ we write $H = H_0 + V$. For $\zeta = \alpha$ real $H(\alpha) = U(\alpha)HU(-\alpha)$, $H = H_0 + V$, is selfadjoint.

For a closed operator T we let $\sigma_{\mathrm{d}}(T)$ denote the isolated eigenvalues of finite multiplicity. The essential spectrum is given by

$$\sigma_{\mathrm{ess}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T).$$

This definition is appropriate in our context.

Theorem 3.2. *Let V satisfy Assumption 3.1. Then for $\zeta \in S_a$*

$$\sigma_{\mathrm{ess}}(H(\zeta)) = \mathbf{R} - \mathrm{i}F\cdot\mathrm{Im}\,\zeta.$$

Proof. Let $\zeta \in S_a$. For $\operatorname{Im} z \neq -F \operatorname{Im} \zeta$ we have

$$\|V_\zeta(H_0(\zeta) - z)^{-1}\| \leqslant \|V_\zeta\| \cdot (|\operatorname{Im} z + F \cdot \operatorname{Im} \zeta|)^{-1}.$$

Thus we can find a constant $c_0 = c_0(V)$ such that for all z with $|\operatorname{Im} z| > c_0$ and all $\zeta \in S_a$, $\|V_\zeta(H_0(\zeta) - z)^{-1}\| < 1$. Writing

$$H(\zeta) - z = (1 + V_\zeta(H_0(\zeta) - z)^{-1})(H_0(\zeta) - z),$$

we conclude that $z \in \rho(H(\zeta))$ (the resolvent set) for all $\zeta \in S_a$. Let now $\zeta \in S_a$ be fixed. Let $z = \eta - F\zeta$, $|\operatorname{Im} z| > c_0$. Then

$$(H(\zeta) - z)^{-1} - (H_0(\zeta) - z)^{-1} =$$

$$= (H(\zeta) + F\zeta - \eta)^{-1} - (H_0 - \eta)^{-1} = -(H(\zeta) - z)^{-1} V_\zeta (H_0(\zeta) - z)^{-1}.$$

Since $V_\zeta(x) = W'(x - \zeta)$ satisfies the assumptions in Lemma 2.1 (i), this operator is compact. It now follows from [7; Theorem XIII.14] that

$$\sigma_{\mathrm{ess}}(H(\zeta) + F\zeta) = \sigma_{\mathrm{ess}}(H_0) = \mathbf{R}$$

from which the result follows. ▨

REMARK 3.3. We note that in [4] a larger class of translation-analytic potentials is considered. However, for this class one has obtained only $\sigma_{\mathrm{ess}}(H(\zeta)) \subseteq \mathbf{R} - iF \cdot \operatorname{Im} \zeta$.

THEOREM 3.4. *Let V satisfy Assumption* 3.1. *Let $\zeta \in S_a$, $0 < \operatorname{Im} \zeta < a$. Then any $z \in \sigma_d(H(\zeta))$ satisfies $-F \cdot \operatorname{Im} \zeta < \operatorname{Im} z < 0$. Each $z \in \sigma_d(H(\zeta))$, $\operatorname{Im} z < 0$, is independent of ζ, if $-F^{-1} \cdot \operatorname{Im} z < \operatorname{Im} \zeta < a$.*

Proof. The proof is similar to the one given for dilation-analytic potentials, see [2, 7] and the remarks in [3, 4].

DEFINITION 3.5. The S_a-translation-analytic vectors are defined by

$$\mathcal{T}_a = \{f \in L^2(\mathbf{R}) \mid f : \mathbf{R} \to \mathbf{C} \text{ has an analytic extension to}$$

$$f : S_a \to \mathbf{C} \text{ such that } \zeta \mapsto f(x - \zeta) \text{ is analytic from } S_a \text{ to } L^2(\mathbf{R})\}.$$

THEOREM 3.6. *Let $f, g \in \mathcal{T}_a$. Then*

$$z \mapsto (f, (H - z)^{-1} g)$$

has a meromorphic continuation from $\{z \mid \operatorname{Im} z > 0\}$ *to* $\{z \mid \operatorname{Im} z > -Fa\}$ *with poles at most at*

$$R_- = \bigcup_{0<\operatorname{Im}\zeta<a} \sigma_d(H(\zeta))$$

For each $z_0 \in R_-$ *there exist* $f, g \in \mathscr{T}_a$ *such that the continuation of* $(f, (H-z)^{-1}g)$ *has a pole at* z_0.

Proof. The proof is similar to the one given for dilation-analytic potentials in [2, 7] and is omitted.

DEFINITION 3.7. The set R_- is called *the set of resonances for H*. The algebraic multiplicity of $z_0 \in R_-$ is the algebraic multiplicty of the eigenvalue -1 of $V_\zeta(H_0 - z_0 - F\zeta)^{-1}$, $-F^{-1}\operatorname{Im} z_0 < \operatorname{Im}\zeta < a$.

We conclude this section with some examples of potentials satisfying our assumptions.

1) A trigonometric polynomial

$$V(x) = \sum_{n=-N}^{N} c_n e^{inx}, \quad c_{-n} = \bar{c}_n, c_0 = 0,$$

clearly satisfies Assumption 3.1 for any $a > 0$.

2) Let V be a real valued function,

$$V(x) = \int_{-\infty}^{\infty} e^{i\omega x} d\mu(\omega)$$

where the Borel measure μ satisfies

$$\int_{-\infty}^{\infty} e^{|\omega| a}(|\omega|^{-1} + |\omega|) d|\mu|(\omega) < \infty$$

for some $a > 0$. Then it is easy to see that V satisfies Assumption 3.1. Taking for μ a sum of point measures, we get a large class of almost-periodic functions.

3) V analytic in S_a, $V(x)$ real valued, V periodic with period ξ such that

$$\int_0^{\xi} V(x) dx = 0.$$

Then it is easy to see that V satisfies Assumption 3.1. In this case $H = H_0 + V$ is called a Stark-Wannier Hamiltonian.

4. A LOWER BOUND ON THE WIDTH OF RESONANCES

If $z_0 \in R_-$ is a resonance, $|\operatorname{Im} z_0|/2$ is called the width of the resonance. In this section we give a lower bound on the width of a resonance and show that the resonances move away from the real axis as F tends to infinity.

THEOREM 4.1. *Let V satisfy Assumption 3.1. Let ζ satisfy $0 < \operatorname{Im} \zeta < a$. Then for $F \geqslant 1$*

$$R_- \cap \left\{ z \,\middle|\, 0 > \operatorname{Im} z > \operatorname{Im} \zeta \left(-1 + \frac{1}{F} \sup\{|V'(\omega)| \mid |\operatorname{Im} \omega| \leqslant \operatorname{Im} \zeta\} \right) \right\} = \emptyset.$$

Proof. Let $\zeta = i\beta, 0 < \beta < a$. Let

$$c(\beta) = \sup\{|V'(\omega)| \mid |\operatorname{Im} \omega| \leqslant \beta\}.$$

Then for $F \geqslant 1$

$$\left| V\left(x - \frac{\zeta}{F} \right) - V(x) \right| \leqslant \frac{\beta}{F} c(\beta)$$

and thus

$$\| V_{\zeta/F} - V \|_{\mathscr{B}(L^2(\mathbf{R}))} \leqslant \frac{\beta}{F} c(\beta).$$

Now

$$H\left(\frac{\zeta}{F} \right) - z = (1 + (V_{\zeta/F} - V)(H - (z + \zeta))^{-1}) \cdot (H - (z + \zeta)).$$

For $\operatorname{Im} z + \beta > 0$ we have

$$\| (V_{\zeta/F} - V)(H - (z + \zeta))^{-1} \| \leqslant \frac{\beta}{F} c(\beta) (\operatorname{Im} z + \beta)^{-1}.$$

It follows that if the right hand side above is less than one then $H(\zeta/F) - z$ has a bounded inverse, so z cannot be a resonance. The proof is completed by noting that by Theorem 3.4 the resonances do not depend on ζ once they have emerged from the essential spectrum. ▨

COROLLARY 4.2. *Let V satisfy Assumption 3.1 for all $a > 0$. Then for any $z_0(F) \in R_-$ we have $\lim_{F \to \infty} \operatorname{Im} z_0(F) = -\infty$. The convergence is uniform on*

$$\{ z_0(1) \in R_- \mid -\gamma < \operatorname{Im} z_0 < 0 \}$$

for any $\gamma > 0$.

5. DETERMINANT CHARACTERIZATION OF RESONANCES: A PERTURBATION RESULT

In this section resonances are characterized as zeros of a modified Fredholm determinant. This leads to a perturbation result for resonances.

LEMMA 5.1. (i) *Let* $z_0 \in R_-$, *and let* $\zeta \in S_a, \operatorname{Im}\zeta > -(\operatorname{Im} z_0)/F$. *Then*

$$\ker(1 + K(z_0 + F\zeta)V_\zeta K(z_0 + F\zeta)) \neq \{0\}.$$

(ii) *Assume that*

$$\ker(1 + K(z_0 + F\zeta)V_\zeta K(z_0 + F\zeta)) \neq \{0\},$$

where $-aF < \operatorname{Im} z_0 < 0$ *and* $\zeta \in S_a$, $\operatorname{Im}\zeta > -(\operatorname{Im} z_0)/F$. *Then* $z_0 \in R_-$.

Proof. The proof consists of straightforward computations, using the definition of a resonance as a discrete eigenvalue of $H(\zeta)$, and the property $K(z)^2 = (H_0 - z)^{-1}$. Details are omitted.

DEFINITION 5.2. Let V satisfy Assumption 3.1. We define for $\operatorname{Im} z > 0$

$$\delta_V(z) = \det{}_4(1 + K(z)VK(z)).$$

We refer to [8] for the definition of $\det_4$ and its properties. We use Lemma 2.1 in this definition.

THEOREM 5.3. *The function* $\delta_V(z)$ *has an analytic continuation to* $\{z \mid \operatorname{Im} z > -Fa\}$. $\delta_V(z_0) = 0$ *if and only if* $z_0 \in R_-$. *The order of a zero* z_0 *of* $\delta_V(z)$ *is equal to the algebraic multiplicity of the eigenvalue* z_0 *of* $H(\zeta)$, *for any* ζ *satisfying*

$$-F^{-1}\operatorname{Im} z_0 < \operatorname{Im}\zeta < a.$$

Proof. We have for $\alpha \in \mathbf{R}$, $\operatorname{Im} z > 0$

$$U(\alpha)K(z)VK(z)U(-\alpha) = K(z + F\alpha)V_\alpha K(z + F\alpha),$$

and thus for all $\alpha \in \mathbf{R}$

$$\delta_V(z) = \det{}_4(1 + K(z + F\alpha)V_\alpha K(z + F\alpha)).$$

For a fixed z, $\operatorname{Im} z > 0$, the map

$$\alpha \mapsto K(z + F\alpha)V_\alpha K(z + F\alpha)$$

extends to an analytic map (with values in $\mathcal{I}_4$) of ζ, ζ satisfying

$$-\min\{\alpha, F^{-1}\cdot \operatorname{Im} z\} < \operatorname{Im}\zeta < a. \tag{5.1}$$

Thus for a fixed z, $\operatorname{Im} z > 0$, we get

$$\delta_V(z) = \det_4(1 + K(z + F\zeta)V_\zeta K(z + F\zeta))$$

for all ζ satisfying (5.1). But for a fixed ζ, $0 < \operatorname{Im}\zeta < a$, this gives an analytic continuation of $\delta_V(z)$ to $\{z \mid \operatorname{Im} z > -F\cdot \operatorname{Im}\zeta\}$. This proves the first part. The second part follows from properties of $\det_4$ and straightforward computations.

The properties of $\det_4$ given in [8] and the explicit trace norm estimate in Lemma 2.1 lead to the following perturbation result:

THEOREM 5.4. *Let V satisfy Assumption* 3.1 *with $V = W'$. Let $z_0 \in R_-$ be a resonance of $H = H_0 + V$. Let γ be a circle enclosing z_0 and no other resonance of H, such that $\gamma \subset S_{Fa}$. Then there exists $c_0 > 0$ such that for any $\tilde{V}$ satisfying Assumption* 3.1 *and*

$$\sup_{z\in S_a}\left(\sum_{j=0}^{2}|W^{(j)}(z) - \tilde{W}^{(j)}(z)|\right) < c_0 \tag{5.2}$$

the Hamiltonian $H_0 + \tilde{V}$ has resonances inside γ with total algebraic multiplicity equal to the algebraic multiplicity of z_0.

Proof. Let $z_0 \in R_-$. Choose $\zeta_0 \in S_a$ such that $\gamma \subset \{z \mid -F\operatorname{Im}\zeta_0 < \operatorname{Im} z < 0\}$, where we decrease the radius of γ, if necessary. Let $\tilde{V}$ satisfy Assumption 3.1. Using the construction of analytic continuation in the proof of Theorem 5.3 we have for $z \in \gamma$ from [8; Theorem 9.2]

$$\begin{aligned}
|\delta_V(z) - \delta_{\tilde{V}}(z)| &= |\det_4(1 + K(z + F\zeta_0)V_{\zeta_0}K(z + F\zeta_0)) - \\
&\quad - \det_4(1 + K(z + F\zeta_0)\tilde{V}_{\zeta_0}K(z + F\zeta_0))| \leqslant \\
&\leqslant \|K(z + F\zeta_0)(V_\zeta - \tilde{V}_\zeta)K(z + F\zeta_0)\|_{\mathcal{I}_4}\cdot \\
&\quad \cdot\exp(c(1 + \|K(z + F\zeta_0)V_{\zeta_0}K(z + F\zeta_0)\|_{\mathcal{I}_4} + \\
&\quad + \|K(z + F\zeta_0)\tilde{V}_{\zeta_0}K(z + F\zeta_0)\|_{\mathcal{I}_4})^4).
\end{aligned}$$

Using Lemma 2.1 (ii) we see that there exists c_0 such that if $\tilde{V}$ satisfies (5.2), then

$$|\delta_V(z) - \delta_{\tilde{V}}(z)| < |\delta_V(z)| \quad \text{for all } z \in \gamma.$$

The result now follows from Rouché's theorem and Theorem 5.3. ∎

REMARK 5.5. As a consequence of Theorem 5.4 we note that all the usual results in perturbation theory (see [6]) hold for the resonances of $H_0 + gV$. We do not elaborate on this point.

6. BOUNDS ON THE NUMBER OF RESONANCES

The characterization of resonances as zeros of $\delta_V(z)$ and Jensen's formula from complex analysis together give a bound on the number of resonances.

LEMMA 6.1. *Let $V = W'$ satisfy Assumption 3.1 for all $n > 0$. Then for all $z, \operatorname{Im} z < 0$, we have the estimates*

$$(6.1) \qquad |\delta_V(z)| \leqslant \exp\left(c_1\cdot(F^{-1} + F^{3/4})\cdot\left(\sum_{j=0}^{2}\left\|W^{(j)}\left(\cdot - \mathrm{i}\frac{1-\operatorname{Im} z}{F}\right)\right\|_{\infty}\right)^4\right)$$

and

$$(6.2) \qquad |\delta_V(z)| \leqslant \exp\left(c_2 F^{-8/3}\cdot\left(\sum_{j=0}^{2}\left\|W^{(j)}\left(\cdot - \mathrm{i}\left(1 - \frac{\operatorname{Im} z}{F}\right)\right)\right\|_{\infty}\right)^4\right)$$

where c_1 and c_2 are positive constants independent of F and W.

Proof. For $0 < -F^{-1}\operatorname{Im} z < \operatorname{Im}\zeta$ we have from the proof of Theorem 5.3 that the continuation of $\delta_V(z)$ is given by

$$\delta_V(z) = \det{}_4^{*}(1 + K(z + F\zeta)V_\zeta K(z + F\zeta)).$$

For the first estimate we choose $z + F\zeta = \mathrm{i}$ or $\zeta = F^{-1}(\mathrm{i} - z)$. The estimate (6.1) now follows from Lemma 2.1 (ii) and $|\det_4(1 + T)| \leqslant \exp(c\|T\|_{\mathscr{I}_4}^4)$, see [8]. The estimate (6.2) follows, if we use $z + F\zeta = F\mathrm{i}$ and Lemma 2.1 (ii). In this case $\beta = F$ (2.1). ▨

REMARK 6.2. By choosing $z + F\zeta = F^\mu\mathrm{i}$ we can get other estimates. For explicit W one may thereby improve the estimates.

We let $n(\rho)$ denote the number of resonances z_0 (counted with algebraic multiplicity) with $|z_0| < \rho$, and let

$$N(\rho) = \int_0^\rho s^{-1}n(s)\mathrm{d}s.$$

THEOREM 6.3. *Let V satisfy Assumption 3.1 for all $a > 0$. Then there exists $\rho_0 > 0$ and $c_0 > 0$ such that for $\rho \geqslant \rho_0$ we have*

$$(6.3) \qquad N(\rho) \leqslant c_0(F^{-1} + F^{4/3})\left(\sum_{j=0}^{2}\left\|W^{(j)}\left(\cdot - \mathrm{i}\frac{1+\rho}{F}\right)\right\|_{\infty}\right)^4$$

and

$$N(\rho) \leqslant c_0 F^{-8/3} \left(\sum_{j=0}^{2} \left\| W^{(j)} \left(\cdot - i\left(1 + \frac{\rho}{F}\right)\right) \right\|_{\infty} \right)^4 . \tag{6.4}$$

Proof. Jensen's formula and Theorem 5.3 imply

$$N(\rho) \leqslant \sup_{|z|=\rho} \log |\delta_V(z)| - \log |\delta_V(0)| .$$

The estimates (6.3) and (6.4) now follow from the maximum principle and the estimates (6.1), (6.2), respectively. ▨

REMARK 6.4. Using the estimate

$$n(\rho) \leqslant \frac{1}{\log 2} N(2\rho)$$

we also get results on $n(\rho)$.

We note that (6.3), (6.4) also give information on the coupling constant dependence of $n(\rho)$: For gV we have $n(\rho) \leqslant cg^4$.

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ARNE JENSEN
Matematisk Institut,
Aarhus Universitet,
Ny Munkegade, Bygning 530,
DK-8000 Aarhus C,
Denmark.

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