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NORMAL OPERATORS AND THE CLASSES $\mathbf{A}_n$

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0. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite-dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the Banach algebra of all bounded linear operators on $\mathscr{H}$. The paper [5] which solved the invariant subspace problem for subnormal operators initiated the study of dual subalgebras of $\mathscr{L}(\mathscr{H})$, which has led to interesting new results on invariant subspaces and reflexivity of more general operators in $\mathscr{L}(\mathscr{H})$. An in-depth study of these results and a detailed bibliography as of 1984 are given in [2]. Some more recent results can be found in [6], [8], and [9].

In the study of dual algebras the solution of systems of equations in the predual has played a central role (cf. [1]). In this paper we solve certain systems of equations in the predual of the dual algebra generated by a normal operator; in particular we characterize completely by spectral multiplicity of the unitary part the normal operators in the classes $\mathbf{A}_n$ ($1 \leqslant n \leqslant \aleph_0$) to be defined below. These provide new examples of operators in $\mathbf{A}_n \setminus \mathbf{A}_{n+1}$; the only other known is the unilateral shift of multiplicity n. We show also that a direct summand which is a unilateral or bilateral shift has limitations on its equation solving ability.

1. PRELIMINARIES

For $T \in \mathscr{L}(\mathscr{H})$ denote by $\sigma(T)$ the spectrum of T; recall that T is a contraction if $\|T\| \leqslant 1$. A contraction T is absolutely continuous if the unitary part of T is absolutely continuous (or acts on the space (0)). If $\mathscr{K}$ is another Hilbert space then $\mathscr{H} \oplus \mathscr{K} = \{u \oplus v : u \in \mathscr{H},\ v \in \mathscr{K}\}$ is a Hilbert space with $\|u \oplus v\|^2 = \|u\|^2 + \|v\|^2$. Moreover if $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$, then $T \oplus S \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is given by $(T \oplus S)(u \oplus v) = T(u) \oplus S(v)$. For $1 \leqslant n \leqslant \aleph_0$ let $\mathscr{H}^{(n)}$ denote $\underbrace{\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \ldots}_{n \text{ times}}$.

If $T \in \mathscr{L}(\mathscr{H})$ then $T^{(n)} \in \mathscr{L}(\mathscr{H}^{(n)})$ is the operator $\underbrace{T \oplus T \oplus T \ldots}_{n \text{ times}}$.

If $\mathcal{M}$ is a subspace of $\mathcal{H}$ then $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathcal{H}$ then $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap \mathcal{N}^{\perp}$. If $S \in \mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ is a subspace of $\mathcal{H}$ such that $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces for S and $\mathcal{M} \supseteq \mathcal{N}$ then $\mathcal{K}$ is a semi-invariant subspace for S. Moreover by $S_{\mathcal{K}} \in \mathcal{L}(\mathcal{K})$ we denote the operator $P_{\mathcal{K}}S|\mathcal{K}$. It is well-known that if $\mathcal{K}$ is semi-invariant for S then $(S^k)_{\mathcal{K}} = (S_{\mathcal{K}})^k$ for any k in $\mathbf{N}$. Also, if T is unitarily equivalent to $S_{\mathcal{K}}$ for some semi-invariant subspace $\mathcal{K}$ of S then T is called a compression of S, or equivalently, S a dilation of T.

Let $\mathbf{D}$ denote the open unit disc in $\mathbf{C}$ and $\mathbf{T}$ denote the unit circle. Let m denote Lebesgue arc-length measure on $\mathbf{T}$. Let $H^{\infty}(\mathbf{D}) = \{f \text{ analytic on } \mathbf{D} : \sup\{|f(\lambda)| : |\lambda| < 1\} < \infty\}$. If $f \in H^{\infty}(\mathbf{D})$ then $\|f\|_{\infty} = \sup\{|f(\lambda)| : \lambda \in \mathbf{D}\}$. If Γ is a Borel measurable subset of $\mathbf{T}$ and $p \geqslant 1$ then $L^p(\Gamma) = \left\{f : \int_{\Gamma} |f|^p \, dm < \infty\right\}$. Let M_{Γ} be the operator on $L^2(\Gamma)$ defined by $(M_{\Gamma}f)(z) = zf(z)$ for all f in $L^2(\Gamma)$. If $m(\Gamma) = 0$ then M_{Γ} is the zero operator operating on the space (0). The operator $M_{\mathbf{T}}$ is the usual bilateral shift.

We need some results about spectral multiplicity for absolutely continuous unitary operators. The following theorem is adapted from [10, Corollary II.9.12].

THEOREM 1.1. *Let U be an absolutely continuous unitary operator. Then there exists a decreasing sequence $\{\Delta_n\}_{n=1}^{\infty}$ of Borel subsets of $\sigma(U)$ such that U is unitarily equivalent to $M_{\Delta_1} \oplus M_{\Delta_2} \oplus M_{\Delta_3} \oplus \dots$.*

For the following definitions assume that U and $\{\Delta_n\}$ are as in Theorem 1.1.

DEFINITION 1.2. Let Γ be a subset of $\mathbf{T}$ and $n \in \mathbf{N}$. We say the spectral multiplicity of U on Γ is at least n if $m(\Gamma \setminus \Delta_n) = 0$. We say that the spectral multiplicity of U on Γ is at least $\aleph_0$ if $m\left(\Gamma \setminus \bigcap_{n=1}^{\infty} \Delta_n\right) = 0$.

DEFINITION 1.3. Let Γ be a Borel subset of $\mathbf{T}$. Then $\mathcal{m}_U(\Gamma) = \max\{k \in \mathbf{N} \cup \aleph_0 : U \text{ has spectral multiplicity at least } k \text{ on } \Gamma\}$ if this set is non-empty, and $\mathcal{m}_U(\Gamma) = 0$ otherwise.

Note that if the spectral multiplicity of U on some non-empty Borel set Γ is at least n then there must exist a reducing subspace $\mathcal{M}$ for U such that $U|\mathcal{M}$ is unitarily equivalent to $M_{\Gamma}^{(n)}$. Also note that if $\mathcal{m}_U(\Gamma) = n$ for n positive and finite then we can conclude that $m(\Gamma \setminus \Delta_n) = 0$ but that $m(\Gamma \setminus \Delta_{n+1}) > 0$. Thus the following lemma is immediate.

LEMMA 1.4. *Let U be an absolutely continuous unitary operator. Let Γ be a Borel subset of $\mathbf{T}$ with $\mathcal{m}_U(\Gamma) = n$ where n is positive and finite. Then we can write $U = U' \oplus M_{\Gamma}^{(n)}$. Moreover there exists $\Gamma' \subseteq \Gamma$ such that $m(\Gamma') > 0$ and U' has spectral measure with no mass on Γ'.*

If $V \subset \mathbf{D}$, then NTL(V) is the set of all $e^{it} \in \mathbf{T}$ such that there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset V$ with $\lambda_n \to e^{it}$ nontangentially. It is well-known that NTL(V) is a Borel subset of $\mathbf{T}$. A set $V \subset \mathbf{D}$ is called dominating for $\mathbf{T}$ if $m(\mathbf{T} \setminus \mathrm{NTL}(V)) = 0$. It is well-known that V is dominating for $\mathbf{T}$ if and only if $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in V\}$ for all f belonging to $H^\infty(\mathbf{D})$ (cf. [4, Theorem 3]).

Our work takes place in the context of dual algebras, and our notation is as in [2]. We recall nonetheless some of the notation and definitions for the convenience of the reader. The Banach algebra $\mathscr{L}(\mathscr{H})$ can be regarded as the dual of $\mathscr{C}_1(\mathscr{H})$, the trace class operators on $\mathscr{H}$, via the pairing $\langle T, L\rangle = \mathrm{tr}(TL)$, $T \in \mathscr{L}(\mathscr{H})$, $L \in \mathscr{C}_1(\mathscr{H})$. The weak* or ultraweak topology on $\mathscr{L}(\mathscr{H})$ is the topology induced by this pairing. A dual algebra $\mathscr{A}$ is a weak* closed, unital subalgebra of $\mathscr{L}(\mathscr{H})$. Let ${}^\perp\mathscr{A}$ denote the preannihilator of the dual algebra $\mathscr{A}$, that is, ${}^\perp\mathscr{A} = \{L \in \mathscr{C}_1(\mathscr{H}) : \langle A, L\rangle = 0$ for all $A \in \mathscr{A}\}$. Then $\mathscr{A}$ may be identified with the dual of the Banach space $Q_{\mathscr{A}} = \mathscr{C}_1(\mathscr{H})/{}^\perp\mathscr{A}$ via the pairing $\langle A, [L]_{\mathscr{A}}\rangle = \mathrm{tr}(AL)$, $A \in \mathscr{A}$, $L \in \mathscr{C}_1(\mathscr{H})$, where $[L]_{\mathscr{A}}$ denotes the coset of L in $Q_{\mathscr{A}}$. The weak* topology induced by this pairing on $\mathscr{A}$ coincides with the relative weak* topology on $\mathscr{A}$ (cf. [2, Proposition 1.19]). For x and y belonging to $\mathscr{H}$, $x \otimes y$ denotes the rank-one operator in $\mathscr{C}_1(\mathscr{H})$ defined by $(x \otimes y)(u) = (u, y)x$, for $u \in \mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$ then $\mathrm{tr}(T(x \otimes y)) = (Tx, y)$. If $T \in \mathscr{L}(\mathscr{H})$, then $\mathscr{A}_T$ denotes the ultraweakly closed subalgebra of $\mathscr{L}(\mathscr{H})$ generated by T and the identity. We write Q_T instead of $Q_{\mathscr{A}_T}$ and the coset of L in Q_T is written $[L]_T$. We now define as in [2] some important properties of dual algebras.

Definition 1.5. Let $\mathscr{A}$ be a dual algebra, and let n and m be cardinal numbers such that $1 \leqslant m, n \leqslant \aleph_0$. We say that $\mathscr{A}$ has property $(\mathbf{A}_{m,n})$ if for every array $\{[L_{i,j}] : 0 \leqslant i < m,\ 0 \leqslant j < n\}$ of elements of $Q_{\mathscr{A}}$ there exist sequences $\{x_i : 0 \leqslant i < m\}$ and $\{y_j : 0 \leqslant j < n\}$ such that

$$[L_{ij}] = [x_i \otimes y_j] \quad \text{for } 0 \leqslant i < m,\ 0 \leqslant j < n.$$

Property $(\mathbf{A}_{n,n})$ is usually written as property $(\mathbf{A}_n)$.

Let $L^1 = L^1(\mathbf{T})$. It is well-known that $L^\infty = L^\infty(\mathbf{T})$ is the dual space of L^1 under the pairing $\langle f, g\rangle = (2\pi)^{-1}\int_{\mathbf{T}} fg\,\mathrm{d}m$, $f \in L^\infty$, $g \in L^1$. Furthermore, $H^\infty = H^\infty(\mathbf{T})$ is a weak*-closed subspace of L^∞, and ${}^\perp(H^\infty)$ is the subspace $H_0^1 = \left\{f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}\,\mathrm{d}t = 0 \text{ for } n = 0, 1, 2, \ldots\right\}$. It follows (cf. [2, Proposition 1.19]) that H^∞ is the dual space of L^1/H_0^1, where the duality is given by the pairing: $\langle f, [g]\rangle = (2\pi)^{-1}\int_{\mathbf{T}} fg\,\mathrm{d}m$, $f \in H^\infty$, $[g] \in L^1/H_0^1$. If $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous contraction, and $f \in H^\infty$, then we can define $f(T)$ using the Sz.-Nagy—Foiaş functional calculus. Let $\Phi_T : H^\infty \to \mathscr{A}_T$ be the map given by $\Phi_T(f) = f(T)$; then there exists a bounded, linear, one-to-one map $\varphi_T : Q_T \to L^1/H_0^1$ such that $\varphi_T^* = \Phi_T$ (cf. [7, Theorem 3.2] or [16, Theorem III.1.2]).

We are now ready to define the classes $\mathbf{A}_n$.

DEFINITION 1.6. $\mathbf{A}(\mathscr{H})$ is the set of all absolutely continuous contractions $T \in \mathscr{L}(\mathscr{H})$ such that Φ_T is an isometry. We write $\mathbf{A}$ for $\mathbf{A}(\mathscr{H})$ when no confusion will result.

DEFINITION 1.7. If n and m are cardinal numbers such that $1 \leqslant n, m \leqslant \aleph_0$, then $\mathbf{A}_{m,n}(\mathscr{H})$ is the set of all absolutely continuous contractions $T \in \mathscr{L}(\mathscr{H})$ such that $T \in \mathbf{A}(\mathscr{H})$ and $\mathscr{A}_T$ has property $(\mathbf{A}_{m,n})$. We usually write $\mathbf{A}_n(\mathscr{H})$ for $\mathbf{A}_{n,n}(\mathscr{H})$. When no confusion will result we write $\mathbf{A}_{m,n}$ for $\mathbf{A}_{m,n}(\mathscr{H})$.

If $T \in \mathbf{A}$ and $\lambda \in \mathbf{D}$, then $[C_\lambda]_T = \varphi_T^{-1}([P_\lambda])$, where $[P_\lambda] \in L^1/H_0^1$ and $P_\lambda(e^{it})$ is the usual Poisson kernel function, $P_\lambda(e^{it}) = (1 - |\lambda|^2)|1 - \bar{\lambda} e^{it}|^{-2}$. It is well-known that if f belongs to $H^\infty(\mathbf{D})$, then $\langle f(T), [C_\lambda]_T \rangle = f(\lambda)$.

If $T \in \mathbf{A}(\mathscr{H})$ and $S \in \mathbf{A}(\mathscr{K})$ then it follows easily from [7, Theorem 3.2] that $T \oplus S \in \mathbf{A}(\mathscr{H} \oplus \mathscr{K})$. Moreover, the preduals Q_T, Q_S and $Q_{T\oplus S}$ are all naturally isometrically isomorphic. (For example $\varphi_{T\oplus S}^{-1} \circ \varphi_T$ is an isometric isomorphism from Q_T to $Q_{T\oplus S}$.) One may also easily conclude that

$$[C_\lambda]_{T\oplus S} = \varphi_{T\oplus S}^{-1} \circ \varphi_T([C_\lambda]_T) \tag{1.8}$$

and if $u, v \in \mathscr{H}$ and $z, w \in \mathscr{K}$ then

$$[(u \oplus 0) \otimes (v \oplus 0)]_{T\oplus S} = \varphi_{T\oplus S}^{-1} \circ \varphi_T([u \otimes v]_T) \tag{1.9}$$

and

$$[(0 \oplus z) \otimes (0 \oplus w)]_{T\oplus S} = \varphi_{T\oplus S}^{-1} \circ \varphi_S([z \otimes w]_S). \tag{1.10}$$

We will have use for the Möbius transform of an absolutely continuous contraction. Recall that if $\mu \in \mathbf{D}$, then $\psi_\mu(z) = (z - \mu)(1 - \bar{\mu}z)^{-1}$ is the usual Möbius transform. Note that $\psi_\mu \in H^\infty(\mathbf{D})$. Given an absolutely continuous contraction T let $T_\mu = \psi_\mu(T)$. It is easy to see that $\mathscr{A}_T = \mathscr{A}_{T_\mu}$, T_μ is an absolutely continuous contraction since T is and T_μ is a completely non-unitary contraction if T is. Moreover if $T \in \mathbf{A}$ and $\lambda \in \mathbf{D}$, then

$$[C_{\psi_\mu(\lambda)}]_{T_\mu} = [C_\lambda]_T. \tag{1.11}$$

Note also that in the special case where $T = M_{\mathbf{T}}^{(n)}$, then T_μ is unitarily equivalent to T; thus given $\tilde{b}_1$ and $\tilde{b}_2$ belonging to $L^2(\mathbf{T})^{(n)}$ we can find b_1 and b_2 in $L^2(\mathbf{T})^{(n)}$

such that $\|b_i\| = \|\tilde{b}_i\|$ $(i = 1, 2)$ and

$$[\tilde{b}_1 \otimes \tilde{b}_2]_{T_\mu} = [b_1 \otimes b_2]_T \quad (T = M_{\mathbf{T}}^{(n)}). \tag{1.12}$$

We now state the main theorem of the paper.

THEOREM 1.13. *Suppose N is a normal absolutely continuous contraction. Let $N = U \oplus N'$ be the canonical decomposition where U is unitary (or acts on the space (0)) and N' is completely nonunitary. Let $\Gamma = \mathbf{T} \setminus \mathrm{NTL}(\sigma(N') \cap \mathbf{D})$ and let n be a cardinal number such that $1 \leqslant n \leqslant \aleph_0$. Then the following are equivalent:*

i) *U has spectral multiplicity at least n on Γ;*
ii) *N belongs to $\mathbf{A}_n$.*

The remainder of this paper is devoted to proving Theorem 1.13. In Section 2 we prove that i) implies ii) and in Section 3 we show the reverse implication.

2. A DECOMPOSITION OF NORMAL OPERATORS IN A

The proof of the first half of Theorem 1.13 rests on a direct sum decomposition of a normal operator in $\mathbf{A}$. We start with a slight generalization of [12, Proposition 2.21] which is independent of normality.

PROPOSITION 2.1. *Suppose n is a positive integer, and $T_i \in \mathbf{A}_{1,n}(\mathscr{H}_i)$ for $1 \leqslant i \leqslant n$. Let $\mathscr{H} = \bigoplus_{i=1}^{n} \mathscr{H}_i$ and $T = \bigoplus_{i=1}^{n} T_i$; then $T \in \mathbf{A}_n(\mathscr{H})$.*

Proof. We first show that $T \in \mathbf{A}(\mathscr{H})$. If $f \in H^\infty(\mathbf{D})$, then $\|f(T)\| = \sup_i \|f(T_i)\| = \|f\|_\infty$ since $T_i \in \mathbf{A}(\mathscr{H}_i)$. To show that $\mathscr{A}_T$ has property $\mathbf{A}_n$, suppose we are given an array $\{[L_{ij}]_T : 1 \leqslant i,j \leqslant n\}$ in Q_T. Let $[M_{ij}]_{T_i} = \varphi_{T_i}^{-1}\varphi_T([L_{ij}])$ for $1 \leqslant i, j \leqslant n$. For each i, since $T_i \in \mathbf{A}_{1,n}(\mathscr{H}_i)$ we can find vectors x_i and $\{y_{ij} : 1 \leqslant j \leqslant n\}$ in $\mathscr{H}_i$ such that $[M_{ij}]_{T_i} = [x_i \otimes y_{ij}]_{T_i}$ for $1 \leqslant j \leqslant n$. Letting $w_j = \bigoplus_{i=1}^{n} y_{ij}$ we have $w_j \in \mathscr{H}$ for each j. Let $\tilde{x}_i = 0 \oplus 0 \ldots \oplus x_i \oplus \ldots \oplus 0$, where x_i appears in the i'th place. It is then easy to compute that $[L_{ij}]_T = [\tilde{x}_i \otimes w_j]_T$ $(1 \leqslant i,j \leqslant n)$.

The similar result for $n = \aleph_0$ in the above proposition is already known; in fact one can use the weaker hypothesis that for each i, $T_i \in \mathbf{A}_1(\mathscr{H}_i)$, and the same conclusion is valid (cf. [2, Proposition 5.8]). The following result is then immediate since if N, normal, is in $\mathbf{A}$ then $\mathscr{A}_N$ has property $\mathbf{A}_{1,n}$ for all finite n (cf. [12, remark last line p. 31, Theorem 2.15, and Corollary 2.6]) and if $n = \aleph_0$ the result follows from [2, Proposition 5.8].

COROLLARY 2.2. *Suppose n is a cardinal number with $1 \leqslant n \leqslant \aleph_0$ and N_i is a normal operator in $\mathbf{A}(\mathscr{H}_i)$ for $0 \leqslant i < n$. Let $\mathscr{H} = \bigoplus_{0 \leqslant i < n} \mathscr{H}_i$ and $N = \bigoplus_{0 \leqslant i < n} N_i$; then $N \in \mathbf{A}_n(\mathscr{H})$.*

The first half of Theorem 1.13 is a consequence of the following proposition.

PROPOSITION 2.3. *Suppose N is a completely non-unitary normal contraction and $1 \leqslant n \leqslant \aleph_0$. Then there exist $\{N_i : 0 \leqslant i < n\}$ such that $N = \bigoplus_i N_i$ and for each i, N_i is a completely non-unitary normal contraction and*

$$m(\{\mathrm{NTL}(\sigma(N) \cap \mathbf{D})\} \setminus \{\mathrm{NTL}(\sigma(N_i) \cap \mathbf{D})\}) = 0.$$

Proof of Theorem 1.13, (i) implies (ii). Apply Proposition 2.3 to N' to obtain $\{N_i\}_{i=1}^{n}$ such that $N' = \bigoplus_{i=1}^{n} N_i$ and $m(\{\mathrm{NTL}(\sigma(N') \cap \mathbf{D})\} \setminus \{\mathrm{NTL}(\sigma(N_i) \cap \mathbf{D})\}) = 0$ for each i.

Recall that $\Gamma = \mathbf{T} \setminus \mathrm{NTL}(\sigma(N') \cap \mathbf{D})$ and that the hypotheses of the theorem imply that U has a reducing subspace $\mathscr{M}$ such that $U|\mathscr{M}$ is unitarily equivalent to $M_\Gamma^{(n)}$. Let $\Gamma_i = \mathbf{T} \setminus \mathrm{NTL}(\sigma(N_i) \cap \mathbf{D})$. Let $M_1 = N_1 \oplus U|\mathscr{M}^\perp \oplus M_\Gamma$ and for $i \geqslant 2$, $M_i = N_i \oplus M_\Gamma$. Note that $\Gamma \subseteq \Gamma_i$ and $m(\Gamma_i \setminus \Gamma) = 0$ for all i, and thus M_{Γ_i} is unitarily equivalent to M_Γ. Since $\sigma(M_i) \cap \mathbf{D} = \sigma(N_i) \cap \mathbf{D}$, we see that $\Gamma_i = \mathbf{T} \setminus \mathrm{NTL}(\sigma(M_i) \cap \mathbf{D})$. We can now conclude using [15, Theorem 3.1] that M_i belongs to $\mathbf{A}$ for each i. Letting $M = \bigoplus_{i=1}^{n} M_i$, we now apply Corollary 2.2 to conclude that M belongs to $\mathbf{A}_n$. Since N is obviously unitarily equivalent to M the proof is complete.

In order to prove Proposition 2.3 we shall need several lemmas. For each θ, $0 \leqslant \theta < 2\pi$ and each α, $0 < \alpha < \pi$ let $T_{\theta\alpha}$ be the region contained in $\mathbf{D}$ and inside the angle with the following properties: vertex at $e^{i\theta}$, measure equal to α and bisected by the line segment from 0 to $e^{i\theta}$.

LEMMA 2.4. *Let $C \subset \mathbf{T}$ be closed, let $V \subset \mathbf{D}$, and suppose $0 < \alpha < \pi$ and each $e^{i\theta} \in C$ is a limit point of $V \cap T_{\theta\alpha}$. Let $\delta > 0$ and $0 < r_1 < 1$. Then there exist r_2 with $r_1 < r_2 < 1$ and P, Q satisfying:*

(2.5) $P, Q \subset V \cap \{z : r_1 < |z| < r_2\}$,

(2.6) $P \cap Q = \emptyset$,

(2.7) *P and Q are finite, and*

(2.8) *for each $e^{i\theta} \in C$, there exist $p \in P \cap T_{\theta\alpha}$ and $q \in Q \cap T_{\theta\alpha}$ such that $|p - e^{i\theta}| < \delta$ and $|q - e^{i\theta}| < \delta$.*

Proof. Choose for each $e^{i\theta}$ in C a λ_θ in $V \cap T_{\theta\alpha} \cap \{z : |z| > r_1\}$ such that $|\lambda_\theta - e^{i\theta}| < \delta$. Let $D_\theta = \{e^{i\nu} : |e^{i\nu} - \lambda_\theta| < \delta\}$; note that $\{D_\theta\}$ is an open cover of C.

For $\{D_{\theta_n}\}$ some finite subcover, let $P = \{\lambda_{\theta_n}\}$. Let $r_1' = \max\{|\lambda_{\theta_n}|\}$. Repeat the process replacing r_1 with r_1' and obtain Q. Choosing $r_2 = \max\{|\lambda| : \lambda \in Q\}$ finishes the construction and (2.5) — (2.8) are immediate.

The next lemma from [11] partitions a set V in **D** into disjoint sets each with the same non-tangential limit points as V, up to sets of measure zero.

LEMMA 2.9. *Given $V \subset \mathbf{D}$ there exist sets V_1 and V_2 satisfying*

(2.10) *$V_i \subset V$ and V_i is countable for $i = 1, 2$,*

(2.11) *$m\{\mathrm{NTL}(V)\setminus\mathrm{NTL}(V_i)\} = 0$ for $i = 1, 2$,*

(2.12) *$V_1 \cap V_2 = \emptyset$, and*

(2.13) *if p is in $V_1 \cup V_2$ then p is not a limit point of $(V_1 \cup V_2)$.*

Proof. For each positive integer n, let $B_n = \{e^{i\theta} \in \mathrm{NTL}(V) : e^{i\theta}$ is a limit point of $V \cap T_{\theta,\pi-1/n}\}$. By the regularity of the measure m, for each n we may choose a sequence $\{C_n^j\}_{j=1}^{\infty}$ of increasing closed sets contained in B_n such that $m(B_n\setminus C_n^j) < 1/j$ for each j. We now construct V_i for $i = 1, 2$. Let $k \to (n(k), j(k))$ be the enumeration of $\{(n,j) : n, j \geqslant 1\}$ suggested by the matrix below:

$$\begin{matrix} 1 & 2 & 4 & 7 & . & \\ 3 & 5 & 8 & . & . & \\ 6 & 9 & . & . & . & \\ 10 & . & . & . & . & . \end{matrix}$$

We now construct families $\{P_{n(k)}^{j(k)}\}_{k=1}^{\infty}$ and $\{Q_{n(k)}^{j(k)}\}_{k=1}^{\infty}$ inductively on k. For $k = 1$ apply Lemma 2.4 to V with $C = C_1^1$, $r_1 = r_1^{(1)} = 1/2$, $\delta = 1$, and $\alpha = \pi - 1/1$ to produce P_1^1, Q_1^1 and $r_2^{(1)}$. Now suppose we have chosen $\{P_{n(k)}^{j(k)}\}$, $\{Q_{n(k)}^{j(k)}\}$ and $r_2^{(k)}$ for $1 \leqslant k < l$. We now apply Lemma 2.4 to V with $C = C_{n(l)}^{j(l)}$, $\delta = 1/j(l)$, $\alpha = \pi - 1/n(l)$ and $r_1 = r_1^{(l)} = \max\{r_2^{(k)} : 1 \leqslant k < l\}$ to produce $P_{n(l)}^{j(l)}$, $Q_{n(l)}^{j(l)}$ and $r_2^{(l)}$. Let $V_1 = \bigcup_{k=1}^{\infty} P_{n(k)}^{j(k)}$ and $V_2 = \bigcup_{k=1}^{\infty} Q_{n(k)}^{j(k)}$. Clearly $V_i \subset V$ and V_i is countable for $i = 1, 2$ which proves statement (2.10).

Since $\mathrm{NTL}(V) = \bigcup_n B_n$ to show that (2.11) holds it suffices to show that

$$m(B_n\setminus\mathrm{NTL}(V_i)) = 0 \quad \text{for each } n. \tag{2.14}$$

We now proceed to establish (2.14). Choose $e^{i\theta}$ in B_n and suppose $e^{i\theta}$ belongs to $C_n^{j_0}$, which implies that $e^{i\theta}$ belongs to C_n^j for all $j \geqslant j_0$ since $\{C_n^j\}_{j=1}^{\infty}$ is increasing. We now show that $e^{i\theta}$ belongs to $\mathrm{NTL}(V_i)$ by showing that $e^{i\theta}$ is a limit point of $V_i \cap T_{\theta,\pi-1/n}$. Given $\varepsilon > 0$, choose k sufficiently large that $n(k) = n$, $j(k) \geqslant j_0$, and $1/j(k) < \varepsilon$. We now use property (2.8) of Lemma 2.4 to find p and q uch that $p \in P^{j(k)}_{n(k)} \cap T_{\theta,\pi-1/n}$, $|p - e^{i\theta}| < 1/j(k) < \varepsilon$, $q \in Q^{j(k)}_{n(k)} \cap T_{\theta,\pi-1/n}$, and $|q - e^{i\theta}| < 1/j(k) < \varepsilon$. Hence, $e^{i\theta} \in \mathrm{NTL}(V_i)$ for $i = 1, 2$. We conclude that if $e^{i\theta}$ belongs to B_n, then either $e^{i\theta}$ belongs to $\mathrm{NTL}(V_i)$ or to $(B_n \setminus \bigcup_j C^n_j)$. However, since $m(B_n \setminus \bigcup_j C^n_j) = 0$ we see that (2.14) is verified which establishes (2.11).

Note that (2.5)—(2.7) and the choice of $r^{(k)}_1$ assure that (2.12) and (2.13) will be true.

We now state a lemma whose proof is elementary and will be omitted. If $\lambda \in \mathbf{C}$ and $r > 0$ then $B(\lambda, r)$ denotes the open ball centered at λ with radius r.

LEMMA 2.15. *Suppose N in $\mathscr{L}(\mathscr{H})$ is normal with spectral measure $E(\cdot)$ and λ belongs to $\sigma(N)$. Then for any $r > 0$, λ belongs to $\sigma(N|E(B(\lambda, r))\mathscr{H})$.*

We may now prove Proposition 2.3 and thus finish the proof of the first half of Theorem 1.13.

Proof of Proposition 2.3. We prove the case $n = 2$, as the rest follow easily by an inductive argument. Apply Lemma 2.9 with $V = \sigma(N) \cap \mathbf{D}$ to obtain V_1 and V_2 satisfying (2.10)—(2.13). Since each V_i is countable write $V_i = \{p^{(i)}_j\}^{\infty}_{j=1}$. Using (2.13) we can find sequences of positive numbers $\{r^{(i)}_j\}^{\infty}_{j=1}$ such that $B(p^{(i)}_j, r^{(i)}_j) \subset \mathbf{D}$ and $B(p^{(i)}_j, r^{(i)}_j) \cap B(p^{(l)}_k, r^{(l)}_k) = \emptyset$ if $i \neq l$ or $j \neq k$. Let $E(\cdot)$ denote the spectral measure of N. Let $\mathscr{M}_i = \bigoplus_{j=1}^{\infty} E(B(p^{(i)}_j, r^{(i)}_j))\mathscr{H}$ for $i = 1, 2$. We see that $\mathscr{M}_1$ is orthogonal to $\mathscr{M}_2$ and each is reducing for N. Let $M_i = N|\mathscr{M}_i$. Then Lemma 2.15 tells us that $\sigma(M_i) \supset V_i$, which implies that $m\{\mathrm{NTL}(\sigma(N) \cap \mathbf{D}) \setminus \mathrm{NTL}(\sigma(M_i) \cap \mathbf{D})\} = 0$. Letting $N_1 = M_1 \oplus N|(\mathscr{M}_1 \oplus \mathscr{M}_2)^{\perp}$ and $N_2 = M_2$ the proof is complete.

3. A UNITARY DIRECT SUMMAND AND EQUATIONS IN THE PREDUAL

The following result will yield the second half of Theorem 1.13.

THEOREM 3.1. *Let N be a normal operator in $\mathbf{A}(\mathscr{H})$. Let $N = U \oplus N'$ be the canonical decomposition where U is unitary (or acts on the space (0)) and N' is completely nonunitary. Let $\Gamma = \mathbf{T} \setminus \mathrm{NTL}(\sigma(N') \cap \mathbf{D})$. Suppose $m(\Gamma) > 0$. Let $n = m_U(\Gamma)$. If n is finite then $N \notin \mathbf{A}_{n+1}$.*

We postpone the proof of Theorem 3.1 to the end of this section.

Proof of Theorem 1.13, ii) implies i). If $m(\Gamma) = 0$ then the theorem is trivially true. Assume $m(\Gamma)$ is positive. We first treat the case where $N \in \mathbf{A}_{\aleph_0}$. Let $k = m_U(\Gamma)$. From [15, Theorem 3.1] we know that $k \geqslant 1$. If k is finite then Theorem 3.1 tells us that $N \notin \mathbf{A}_{k+1}$. However, $\mathbf{A}_{\aleph_0} \subseteq \mathbf{A}_{k+1}$, so k must be infinite, and the theorem is therefore true in this case. We now treat the case $N \in \mathbf{A}_n$ where n is finite. Again let

$k = m_U(\Gamma)$. As before $k \geqslant 1$ and if k is infinite then the theorem is true. Moreover if k is finite Theorem 3.1 tells us that $N \notin \mathbf{A}_{k+1}$. We can then easily see that $n \leqslant k$ which means that U has multiplicity at least n on Γ.

The proof of Theorem 3.1 will require conversion of information about systems of equations in Q_T into information about systems in $L^1 = L^1(\mathbf{T})$. Given a contraction $T \in \mathscr{L}(\mathscr{H})$ and $x, y \in \mathscr{H}$, denote by $h^T_{x,y}$ (or just $h_{x,y}$ when no confusion will result) the function on $\mathbf{T}$ whose Fourier coefficients are

$$(3.2)\qquad \begin{aligned} h^T_{x,y}(-n) &= \langle T^n, [x \otimes y]_T \rangle = (T^n x, y)_{\mathscr{H}} \quad (n \geqslant 0) \\ h^T_{x,y}(n) &= \overline{\langle T^n, [y \otimes x]_T \rangle} = \overline{(T^n y, x)_{\mathscr{H}}} \quad (n > 0). \end{aligned}$$

In our applications it will be clear that $h_{x,y}$ is in fact an element of L^1 (though for a general result and a thorough discussion see [3]). If $T = R \oplus S$, $x = u \oplus w$, and $y = v \oplus z$ it follows easily from (1.9) and (1.10) that

$$(3.3)\qquad h^T_{x,y} = h^R_{u,v} + h^S_{w,z}.$$

Also if $T = M^{(n)}_{\mathbf{T}}$, $u = (u^1, \ldots, u^n)$, and $v = (v^1, \ldots, v^n)$ where u^i, v^i belong to $L^2 = L^2(\mathbf{T})$ for each i, then

$$(3.4)\qquad h^T_{u,v} = u^1\overline{v^1} + \ldots + u^n\overline{v^n}.$$

Proof of Theorem 3.1. Let $n = m_U(\Gamma)$ and note that since $N \in \mathbf{A}$ it must be the case that $n \geqslant 1$ (cf. [15, Theorem 3.1]). Recall that if S is an absolutely continuous contraction, $T \in \mathbf{A}$ and $S \oplus T \notin \mathbf{A}_m$, then $T \notin \mathbf{A}_m$ (cf. [2, Proposition 4.11] or [1, Proposition 3.2]). So without loss of generality we may assume that a restriction of N to an invariant subspace is unitarily equivalent to $M^{(n)}_{\mathbf{T}}$, since if not we replace N by $N \oplus (M_{\mathbf{T}\setminus\Gamma})^{(n)}$. Let $U = U' \oplus M^{(n)}_{\mathbf{T}}$. Since $m_U(\Gamma) = n$ and n is finite, using Lemma 1.4 we can find $\Gamma' \subseteq \Gamma$ such that $m(\Gamma') > 0$ and U' has spectral measure with no mass on Γ'. From now on let $B = M_{\mathbf{T}}$. Let $N'' = N' \oplus U'$, so $N = N'' \oplus B^{(n)}$, and from now on we will write vectors in $\mathscr{H}$ in the form $v \oplus b$ with respect to the decomposition of $\mathscr{H}$ induced by $N = N'' \oplus B^{(n)}$.

We claim that there exists an infinite sequence $\{\lambda_k : k \in \mathbf{N}\} \subset \mathbf{D}$ such that for each k

$$(3.5)\qquad \inf_{\|v\|=1} \|[C_{\lambda_k}]_N - [(v \oplus 0) \otimes (v \oplus 0)]_N\| \geqslant 1 - 1/k.$$

If the claim is false there exists k_0 such that $\sup_{\lambda \in \mathbf{D}} \inf_{\|v\|=1} \|[C_\lambda]_N - [(v \oplus 0) \otimes (v \oplus 0)]_N\| \leqslant 1 - 1/k_0$. One may now easily show using (1.8) and (1.9) that $\Phi_{N''}$ is bounded below and hence an isometry (as in [2, Chapter 7]), which would imply that $N'' \in \mathbf{A}$.

Since $N'' = N' \oplus U'$ and the spectral measure of U' has no mass on $\Gamma' \subseteq \subseteq \mathbf{T} \setminus \mathrm{NTL}(\sigma(N'') \cap \mathbf{D})$, this would contradict [15, Theorem 3.1]. Let $\{\lambda_k\}$ be a sequence which satisfies (3.5). It follows easily that for each k

$$\text{(3.6)} \qquad \text{if } [C_{\lambda_k}]_N = [(v \oplus b) \otimes (v \oplus b)]_N \text{ then } \|b\| \geq \sqrt{1 - 1/k}.$$

Suppose now that $N \in \mathbf{A}_{n+1}$ in order to obtain a contradiction. It is known that if $N \in \mathbf{A}_{n+1}$ then for every $\lambda \in \mathbf{D}$, N has a compression unitarily equivalent to the operator λI_{n+1} on $\mathbf{C}^{n+1}$ (cf. [2, Corollary 4.14] or [1, Corollary 3.6]). Letting $\lambda = \lambda_k$, this implies that for each k in $\mathbf{N}$ there exists a sequence of vectors $\{x_i : 1 \leq \leq i \leq n+1\} \subset \mathscr{H}$ such that

$$\text{(3.7)} \qquad \begin{aligned} &[x_i \otimes x_i]_N = [C_{\lambda_k}]_N \quad (1 \leq i \leq n+1), \text{ and} \\ &[x_i \otimes x_j]_N = [0]_N \quad (i \neq j,\ 1 \leq i,j \leq n+1). \end{aligned}$$

Note that the sequence $\{x_i\}$ depends on λ_k.

Let $N_k = N_{\lambda_k} = \psi_{\lambda_k}(N)$ and define $B_k^{(n)}, N_k'', N_k'$, and U_k similarly. Now using (1.11) the following is a direct consequence of (3.7):

$$\text{(3.8)} \qquad \begin{aligned} &[x_i \otimes x_i]_{N_k} = [C_0]_{N_k} \quad (1 \leq i \leq n+1), \text{ and} \\ &[x_i \otimes x_j]_{N_k} = [0]_{N_k} \quad (i \neq j,\ 1 \leq i,j \leq n+1). \end{aligned}$$

We now let $h_{ij} = h^{N_k}_{x_i, x_j}$ defined as in (3.2). Recalling the definition of $[C_0]_{N_k}$ and (3.8) a simple computation yields

$$\text{(3.9)} \qquad \begin{aligned} &h_{ii} = P_0 \quad (1 \leq i \leq n+1), \text{ and} \\ &h_{ij} = 0 \quad (i \neq j,\ 1 \leq i,j \leq n+1). \end{aligned}$$

(Recall that P_0 is the usual Poisson kernel function.) Now let $x_i = v_i \oplus \tilde{b}_i$. Define b_i relative to $\tilde{b}_i$ as in (1.12) where $\mu = \lambda_k$ and $T = B^{(n)}$. Then $b_i = (b_i^1, \ldots, b_i^n)$ where $b_i^m \in L^2$ for $1 \leq i \leq n+1$, $1 \leq m \leq n$. We now compute using (3.1) and (3.2)

$$\text{(3.10)} \qquad \begin{aligned} h_{ij} &= h^{N_k}_{v_i \oplus 0, v_j \oplus 0} + h^{N_k}_{0 \oplus \tilde{b}_i, 0 \oplus \tilde{b}_j} = h^{N_k''}_{v_i, v_j} + h^{B_k^{(n)}}_{\tilde{b}_i, \tilde{b}_j} = \\ &= h^{N_k''}_{v_i, v_j} + h^{B^{(n)}}_{b_i, b_j} = h^{N_k''}_{v_i, v_j} + \sum_{m=1}^{n} b_i^m \overline{b_j^m} \quad (1 \leq i,j \leq n+1). \end{aligned}$$

Since 0, P_0, and $\sum_{m=1}^{n} b_i^m \bar{b}_j^m$ are in L^1 we see that $h_{v_i,v_j}^{N_k''}$ belongs to L^1 for all i, j using (3.9).

We next claim that

$$\|h_{v_i,v_j}^{N_k''}\|_1 \leqslant 2/k \quad (1 \leqslant i, j \leqslant n+1). \tag{3.11}$$

Recall that $N'' = N' \oplus U$ so $N_k'' = N_k' \oplus U_k$. Decompose $v_i = w_i \oplus u_i$, and observe that from (3.6) we have $\|v_i\| \leqslant 1/\sqrt{k}$. Also, $\|h_{v_i,v_j}^{N_k''}\|_1 \leqslant \|h_{w_i,w_j}^{N_k'}\|_1 + \|h_{u_i,u_j}^{U_k}\|_1$. Since U and U_k are unitary and $\|u_i\| \leqslant 1/\sqrt{k}$ it is easy to check that $\|h_{u_i,u_j}^{U_k}\|_1 \leqslant 1/k$. Since N' is a completely non-unitary contraction, so is N_k' and we may view the latter as a Sz.-Nagy—Foiaş functional model. Since $\|w_i\| \leqslant 1/\sqrt{k}$, we have: $\|h_{w_i,w_j}^{N_k'}\|_1 \leqslant 1/k$ by [3, Lemma 1.1], whose $w_i \cdot w_j$ is our $h_{w_i,w_j}^{N_k'}$. Thus we have (3.11).

Now (3.9), (3.10), and (3.11) yield

$$\begin{aligned} &\left\| P_0 - \sum_{m=1}^{n} b_i^m \bar{b}_i^m \right\|_1 \leqslant 2/k, \quad (1 \leqslant i \leqslant n+1) \text{ and} \\ &\left\| \sum_{m=1}^{n} b_i^m \overline{b_j^m} \right\|_1 \leqslant 2/k \quad (i \neq j,\ 1 \leqslant i, j \leqslant n+1). \end{aligned} \tag{3.12}$$

Recall that (3.12) holds for any $k \in \mathbf{N}$ (where the b_i^m in fact depend on k). Since P_0 is the function identically 1 on $\mathbf{T}$, by taking k sufficiently large we deduce that the following hold pointwise on a set $\Delta_\varepsilon \subseteq \mathbf{T}$ of positive measure

$$\begin{aligned} &\left| 1 - \sum_{m=1}^{n} |b_i^m(e^{it})|^2 \right| \leqslant \varepsilon \quad (1 \leqslant i \leqslant n+1) \text{ and} \\ &\left| \sum_{m=1}^{n} b_i^m(e^{it}) \overline{b_j^m}(e^{it}) \right| \leqslant \varepsilon \quad (i \neq j,\ 1 \leqslant i, j \leqslant n+1). \end{aligned} \tag{3.13}$$

These equations are clearly inconsistent for ε sufficiently small (they yield an "almost orthonormal" family of $n+1$ vectors in $\mathbf{C}^n$, namely:

$$\{(b_i^1(e^{it}), b_i^2(e^{it}), \ldots b_i^n(e^{it})) : 1 \leqslant i \leqslant n+1\}).$$

This is a contradiction. Therefore $N \notin \mathbf{A}_{n+1}$.

4. REMARKS

We observe that in the proof of Theorem 3.1 we deduce that $N \notin \mathbf{A}_{n+1}$ by showing that there is a $\lambda \in \mathbf{D}$ for which N is not a dilation of the operator λI_{n+1} on $\mathbf{C}^{n+1}$. It is not known whether for any $n \geqslant 2$ there is an operator $T \in \mathbf{A} \setminus \mathbf{A}_n$ such that for each $\lambda \in \mathbf{D}$, T dilates the operator λI_n on $\mathbf{C}^n$. It is known that if for even a dominating set $\Lambda \subseteq \mathbf{D}$, T dilates $\lambda I_{\mathscr{K}}$ for each $\lambda \in \Lambda$, then $T \in \mathbf{A}_{\aleph_0}$ (cf. [2, proof of Proposition 6.1]).

We note that Theorem 1.13 and the proof of Theorem 3.1 yield as a special case the results of [14] concerning "hole-filling" for a cyclic normal operator (which may be taken to be in $\mathbf{A}$). If $N \in \mathbf{A}$ is cyclic and has an "outer hole" then $\mathrm{NTL}(\sigma(N) \cap \cap \mathbf{D}) \neq \mathbf{T}$, so $N \in \mathbf{A}_1$ by Theorem 1.13 and the proof of Theorem 3.1 shows there is a $\lambda \in \mathbf{D}$ such that N does not dilate λI_2. Clearly then N has no pure subnormal restriction S with $\mathrm{i}(S - \lambda) < -1$ (where $\mathrm{i}(\cdot)$ is the semi-Fredholm index). If $N \in \mathbf{A}$ is cyclic with no outer hole in its spectrum, then $\mathrm{NTL}(\sigma(N) \cap \mathbf{D}) = \mathbf{T}$ and $N \in \mathbf{A}_{\aleph_0}$. Then for any $\lambda \in \mathbf{D}$ and $n \in \mathbf{N} \cup \{\aleph_0\}$ one may easily produce subnormal restrictions S with $\mathrm{i}(S - \lambda) = -n$ which may be taken to be pure if $\lambda \notin \sigma(N)$.

We observe that the proof of Theorem 3.1, which in effect shows some limitation on the power of a unitary direct summand of finite multiplicity in solving systems of equations, has in fact consequences for non-normal operators. Further, since the unilateral shift S is a restriction of the bilateral shift $B = M_{\mathbf{T}}$, we gain information about operators of the form $T \oplus S^{(n)}$ as well. An examination of the proof of Theorem 3.1 shows that the first assertion below holds, and it easily implies the second.

COROLLARY 4.1. *Suppose $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous contraction and $n \in \mathbf{N}$. Then*

(4.2) $T \oplus B^{(n)} \in \mathbf{A}_{n+1}$ *implies* $T \in \mathbf{A}$, *and*

(4.3) $T \oplus S^{(n)} \in \mathbf{A}_{n+1}$ *implies* $T \in \mathbf{A}$.

Similar techniques yield the following result which has also been obtained by B. Chevreau (unpublished).

PROPOSITION 4.4. *Suppose $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous contraction and $j \in \mathbf{N}$. Then*

(4.5) $T \oplus B^{(j)} \in \mathbf{A}_{\aleph_0}$ *implies* $T \in \mathbf{A}_{\aleph_0}$ *and*

(4.6) $T \oplus S^{(j)} \in \mathbf{A}_{\aleph_0}$ *implies* $T \in \mathbf{A}_{\aleph_0}$.

Proof. Since $T \oplus S^{(j)} \in \mathbf{A}_{\aleph_0}$ implies $T \oplus B^{(j)} \in \mathbf{A}_{\aleph_0}$, we prove (4.5) holds. Let $B^{(j)}$ act on the space $\mathscr{K}$. It is known that if $T \oplus B^{(j)}$ is in $\mathbf{A}_{\aleph_0}$ then for each $\lambda \in \mathbf{D}$ there exists a sequence $\{x_n(\lambda)\}_{n=1}^{\infty}$ in the unit ball of $\mathscr{H} \oplus \mathscr{K}$ satisfying

(4.7) $$\overline{\lim_n} \|[C_\lambda]_{T \oplus B^{(j)}} - [x_n(\lambda) \otimes x_n(\lambda)]\| = 0,$$

(4.8) $\lim_n [z \otimes x_n(\lambda)] = 0$ for all $z \in \mathscr{H} \oplus \mathscr{K}$, and

(4.9) $\lim_n [x_n(\lambda) \otimes z] = 0$ for all $z \in \mathscr{H} \oplus \mathscr{K}$

(cf. [2, Chapter 6]).

Write $x \in \mathscr{H} \oplus \mathscr{K}$ in the obvious decomposition $u \oplus v$. Suppose that for some $\lambda \in \mathbf{D}$ and sequence $\{x_n(\lambda)\} = \{u_n(\lambda) \oplus v_n(\lambda)\}$ satisfying (4.7)—(4.9) we hav

$$\lim_n \|v_n(\lambda)\| \geqslant c > 0.$$

It is then easy to show using Möbius transforms and

$$[(u_1 \oplus v_1) \otimes (u_2 \oplus v_2)]_{T \oplus B^{(j)}} = [(u_1 \oplus 0) \otimes (u_2 \oplus 0)]_{T \oplus B^{(j)}} +$$

$$+ [(0 \oplus v_1) \otimes (0 \oplus v_2)]_{T \oplus B^{(j)}}$$

that for each ξ in $\mathbf{D}$ there exists a sequence $\{0 \oplus v_n(\xi)\}$ satisfying

$$\lim_n \|[C_\xi] - [(0 \oplus v_n(\xi)) \otimes (0 \oplus v_n(\xi))]\| \leqslant \sqrt{1-c^2},$$

$$\lim_n [(0 \oplus v_n) \otimes z] = 0 \quad \text{for all } z \in \mathscr{H} \oplus \mathscr{K},$$

and

$$\lim_n [z \otimes (0 \oplus v_n)] = 0 \quad \text{for all } z \in \mathscr{H} \oplus \mathscr{K}.$$

Transfering these equations to $Q_{B^{(j)}}$ using (1.8) and (1.10) we deduce from [2, Theorem 6.3] that $B^{(j)} \in \mathbf{A}_{\aleph_0}$, which contradicts $B \in \mathbf{A}_1 \setminus \mathbf{A}_2$ and [2, Theorem 3.8].

Thus for each $\lambda \in \mathbf{D}$ and $\{x_n(\lambda)\} = \{u_n(\lambda) \oplus v_n(\lambda)\}$ satisfying (4.7)—(4.9) we have

$$\overline{\lim_n} \|u_n(\lambda)\| = 1. \tag{4.10}$$

We may then deduce as in the proof of Theorem 3.1 that $T \in \mathbf{A}$ and an argument from (4.7)—(4.10) yields

$$\overline{\lim_n} \|[C_\lambda]_T - [u_n(\lambda) \otimes u_n(\lambda)]\| = 0,$$

$$\lim_n [u_n(\lambda) \otimes z] = 0 \quad \text{for all } z \in \mathscr{H},$$

and

$$\lim_n [z \otimes u_n(\lambda)] = 0 \quad \text{for all } z \in \mathscr{H}.$$

Citing [2, Theorem 6.3] again, we have $T \in \mathbf{A}_{\aleph_0}$.

Finally we remark that an improvement of Proposition 4.4 to deduce from $T \oplus B^{(j)} \in \mathbf{A}_{j+1}$ that $T \in \mathbf{A}_1$ would clarify greatly the role of unitary direct summands in the solution of systems of equations.

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