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THE DIFFERENTIAL AND INTEGRAL STRUCTURE OF REPRESENTATIONS OF LIE GROUPS

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0. INTRODUCTION

The work of Nelson [11] and Goodman [5], [6] established that the differential and integral structure of a unitary representation U of a Lie group $\mathfrak{G}$ can be characterized by the Laplacian associated with a basis of the Lie algebra. The corresponding situation for non-unitary representations or representations on Banach space is less well understood and our first aim, in Section 1, is to clarify the situation by characterizing those representations for which the Laplacian does determine the differential structure.

Subsequently, in Section 2, we prove a commutator theorem which gives criteria for a dissipative operator to generate a contraction semigroup. This section is completely independent of Lie groups and Lie algebras but it provides the key element for the discussion of the integration problem for isometric representations given in Section 3. Specifically we consider a representation of a Lie algebra $\mathfrak{g}$ on the Banach space $\mathscr{B}$ by operators $V(x)$, $x \in \mathfrak{g}$, such that $\pm V(x)$ are dissipative. We then give necessary and sufficient conditions, in terms of the Laplacian $\overline{\Delta}$ associated with a basis $x_1, \ldots, x_d$ of $\mathfrak{g}$, for the representation of $\mathfrak{g}$ to be integrable to a representation of the corresponding Lie group satisfying the criteria of Section 1. Basically we require that $\overline{\Delta}$ generates a contraction semigroup but conditions on the C^2- and C^3-structure are also essential. Although our main integrability result contains Nelson's characterization of unitary representations we do not need analytic techniques; analytic elements play no role in our proofs.

1. REPRESENTATIONS OF LIE GROUPS

Let $(\mathscr{B}, \mathfrak{G}, U)$ denote a strongly continuous representation of a Lie group $\mathfrak{G}$ by linear operators $U(g)$, $g \in \mathfrak{G}$, acting on a Banach space $\mathscr{B}$. Fix a basis $x_1, \ldots, x_d$ of the Lie algebra $\mathfrak{g}$ associated with $\mathfrak{G}$ and define X_i as the infinitesimal generator of the one-parameter subgroup $t \in \mathbf{R} \to U(\exp(tx_i))$. Then for each $n \geqslant 1$ introduce the subspace

$$\mathscr{B}_n = \mathscr{B}_n(U) = \bigcap_{i_1, \ldots, i_n = 1}^{d} D(X_{i_1} \ldots X_{i_n})$$

and define norms $\|\cdot\|_n$ on the $\mathscr{B}_n$ by setting $\|\cdot\|_0 = \|\cdot\|$ and

$$\|a\|_n = \|a\| + \sup_{1 \leqslant i \leqslant d} \|X_i a\|_{n-1}, \quad a \in \mathscr{B}_n.$$

Since the X_i are closed, $\mathscr{B}_n$ is a Banach space with respect to the norm $\|\cdot\|_n$, and $\mathscr{B}_n$ is continuously embedded in $\mathscr{B}_{n-1}$, because $\|a\|_{n-1} \leqslant \|a\|_n$, $a \in \mathscr{B}_n$. Moreover it follows from the group structure that $U\mathscr{B}_n = \mathscr{B}_n$, $n \geqslant 1$, and that $U_n = U|\mathscr{B}_n$ is $\|\cdot\|_n$-continuous. Thus one obtains a family $(\mathscr{B}_n, \mathfrak{G}, U_n)$ of continuous representations of $\mathfrak{G}$.

Next define the C^∞-elements of the representation by

$$\mathscr{B}_{\mathrm{d}U} = \mathscr{B}_\infty(U) = \bigcap_{n \geqslant 1} \mathscr{B}_n(U)$$

and if X is the generator of $t \to U(e^{tx})$ define $\mathrm{d}U(x) = X|\mathscr{B}_{\mathrm{d}U}$. It follows that

1. $\mathscr{B}_{\mathrm{d}U}$ is norm-dense and invariant under $\mathrm{d}U$, i.e. for each $x \in \mathfrak{g}$ one has $\mathrm{d}U(x)\mathscr{B}_{\mathrm{d}U} \subseteq \mathscr{B}_{\mathrm{d}U}$;

2. $(\mathrm{ad}\, \mathrm{d}U(x))(\mathrm{d}U(y))a = \mathrm{d}U((\mathrm{ad}\, x)(y))a$, $x, y \in \mathfrak{g}$, $a \in \mathscr{B}_{\mathrm{d}U}$.

Thus one obtains a representation $(\mathscr{B}_{\mathrm{d}U}, \mathfrak{g}, \mathrm{d}U)$ of the Lie algebra $\mathfrak{g}$.

If $\mathscr{B}^*$ denotes the dual of $\mathscr{B}$ then there exists a weak* continuous representation $(\mathscr{B}^*, \mathfrak{G}, U_*)$ of $\mathfrak{G}$ on $\mathscr{B}^*$ defined by $U_*(g) = U(g^{-1})^*$, $g \in \mathfrak{G}$. Now let X_i denote the weak* generator of $t \to U_*(\exp(tx_i))$ and define $\mathscr{B}_n^* = \mathscr{B}_n^*(U_*)$ etc. by repetition of the above procedures. Then one obtains a second representation $(\mathscr{B}_{\mathrm{d}U}^*, \mathfrak{g}, \mathrm{d}U_*)$ of $\mathfrak{g}$ but in this representation $\mathscr{B}_{\mathrm{d}U}^*$ is only weak* dense. The two representations satisfy the duality property $\mathrm{d}U_*(x) \subseteq -\mathrm{d}U(x)^*$, $x \in \mathfrak{g}$, and $-\mathrm{d}U(x)^*$ is the weak* closure of $\mathrm{d}U_*(x)$.

Note that in the sequel all statements concerning operators and representations on $\mathscr{B}$ will refer to the norm, or $\sigma(\mathscr{B}, \mathscr{B}^*)$-, topology but statements concerning operators and representations on $\mathscr{B}^*$ will refer to the weak*, or $\sigma(\mathscr{B}^*, \mathscr{B})$-, topology. For example, if X is an operator on $\mathscr{B}$ then $\bar{X}$ will denote its norm closure but if X acts on $\mathscr{B}^*$ then $\bar{X}$ denotes its weak* closure.

Now define the Laplacians Δ and Δ_* associated with the basis $x_1, \ldots, x_d$ of $\mathfrak{g}$ by

$$\Delta = -\sum_{i=1}^{d} \mathrm{d}U(x_i)^2, \quad \Delta_* = -\sum_{i=1}^{d} \mathrm{d}U_*(x_i)^2.$$

Because $dU_*(x) \subseteq -dU(x)^*$ it follows that $\Delta_* \subseteq \Delta^*$ and in particular Δ_* is closable. Then by duality Δ is contained in the adjoint of $\bar{\Delta}_*$ and hence Δ is closable. Note also if $\rho = (\rho_{ij})$ is an orthogonal transformation of $\mathbf{R}^d$ and one defines a new basis x^ρ by

$$x_i^\rho = \sum_{j=1}^{d} \rho_{ij} x_j$$

then

$$\Delta = -\sum_{i=1}^{d} dU(x_i^\rho)^2, \quad \Delta_* = -\sum_{i=1}^{d} dU_*(x_i^\rho)^2.$$

But in general the definition of Δ is basis dependent.

Now we consider representations satisfying special regularity properties.

THEOREM 1.1. *Let $(\mathscr{B}, \mathfrak{G}, U)$ be a strongly continuous representation of the Lie group $\mathfrak{G}$ and for a fixed basis $x_1, \ldots, x_d$ of the associated Lie algebra $\mathfrak{g}$ define the Laplacian Δ by*

$$\Delta = -\sum_{i=1}^{d} dU(x_i)^2.$$

Then Δ is closable, its closure $\bar{\Delta}$ generates a strongly continuous semigroup holomorphic in the open right half-plane, and $\mathscr{B}_2(U) \subseteq D(\bar{\Delta})$.

The following conditions are equivalent

1. $\mathscr{B}_2(U) = D(\bar{\Delta})$;

2. $$\bar{\Delta} = -\sum_{i=1}^{d} \overline{dU(x_i^\rho)}^2$$

for each orthogonal transformation ρ of $\mathbf{R}^d$;

3. *there is a $K \geqslant 1$ such that*

$$\|\overline{dU(x_i)}\, \overline{dU(x_j)}a\| \leqslant K(\|\bar{\Delta}a\| + \|a\|), \quad a \in \mathscr{B}_2(U),$$

for all $i, j = 1, \ldots, d$;

4. *there is an $\varepsilon_0 > 0$ such that $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_2$ onto $\mathscr{B}$ for all $\varepsilon \in (0, \varepsilon_0]$.*

Moreover, if these conditions are satisfied then $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_{n+2}$ onto $\mathscr{B}_n$ for all $\varepsilon \in (0, \varepsilon_0]$ and $n \geqslant 1$.

Proof. The general properties of Δ given in the first statement follow from combination of results of Nelson and Stinespring [12] and Langlands [8], [9]. Nelson and Stinespring consider the Laplacians as defined above but Langlands examines their extensions

$$\Delta_1 = -\sum_{i=1}^{d} \overline{V(x_i)^2}, \quad \Delta_{1*} = -\sum_{i=1}^{d} \overline{V_*(x_i)^2}$$

where we have set $V = \mathrm{d}U$ for simplicity.

Now Theorem 3.1 of [12] establishes that $\overline{\Delta}$ generates a strongly continuous semigroup and Theorem 8 of [8] proves that $\overline{\Delta}_1$ generates a similar semigroup which is holomorphic in the open right half plane (see also [9], Theorem 2). But a generator cannot have a proper generator extension and hence $\overline{\Delta} = \overline{\Delta}_1$. Therefore $\overline{\Delta} \supseteq \Delta_1$ and $D(\overline{\Delta}) \supseteq D(\Delta_1) \supseteq \mathscr{B}_2$. Since these arguments apply equally well to the dual representations one also has $\overline{\Delta}_{1*} = \overline{\Delta}_*$ and $D(\overline{\Delta}_*) \supseteq \mathscr{B}_2^*$. Moreover, it follows from Theorem 7 of [8] (see Theorem 1 of [9]) that $(\overline{\Delta}_1)^* = \overline{\Delta}_{1*}$ and then by combination of these conclusions one has $(\overline{\Delta})^* = \overline{\Delta}_*$.

Now consider the four conditions.

1 $\Rightarrow$ 2. It follows from the above considerations that

$$D(\overline{\Delta}) \supseteq D(\Delta_1) = \bigcap_{i=1}^{d} D(\overline{V(x_i)^2}) \supseteq \mathscr{B}_2$$

and hence Condition 1 implies that $D(\overline{\Delta}) = D(\Delta_1) = \mathscr{B}_2$. Since $\overline{\Delta} \supseteq \Delta_1$ this means $\overline{\Delta} = \Delta_1$. But the definition of Δ is invariant under orthogonal transformations ρ of $\mathbf{R}^d$ and hence this argument can be applied to the basis x^ρ. This establishes Condition 2.

The proof that 2 $\Rightarrow$ 3 relies upon the following lemma.

LEMMA 1.2. *For each $\varepsilon > 0$ there is a C_ε such that*

$$\|\overline{\mathrm{d}U(x_i)}a\| \leqslant \varepsilon \|\overline{\mathrm{d}U(x_i)^2}a\| + C_\varepsilon \|a\|, \quad a \in D(\overline{\mathrm{d}U(x_i)^2}).$$

Proof. Since U is continuous there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that

$$\|U(\exp(tx_i))\| \leqslant M\exp(\omega|t|), \quad t \in \mathbf{R}.$$

Again set $V = \mathrm{d}U$. Then one has

$$\|(I + \alpha\overline{V(x_i)})a\| \geqslant M^{-1}(I - \alpha\omega)\|a\|, \quad a \in D(\overline{V(x_i)}),$$

for all $\alpha \in \mathbf{R}$ such that $|\alpha|\omega < 1$. Therefore if $1 > \alpha\omega > 0$ then

$$\alpha\|\overline{V(x_i)}a\| \leqslant \|(I - \alpha\overline{V(x_i)})a\| + \|a\| \leqslant$$

$$\leqslant M(1 - \alpha\omega)^{-1}\|(I - \alpha^2\overline{V(x_i)^2})a\| + \|a\|$$

for all $a \in D(\overline{V(x_i)}^2)$. Hence for each $\varepsilon > 0$

$$\|\overline{V(x_i)}a\| \leqslant \varepsilon\|\overline{V(x_i)}^2a\| + (2(M+\varepsilon\omega)^2/\varepsilon)\|a\|.$$

Now consider the implication 2 ⇒ 3 in Theorem 1.1. It follows from Condition 2 that

$$D(\bar{\Delta}) = \bigcap_{i=1}^{d} D(\overline{V(x_i)}^2).$$

Now equip $D(\bar{\Delta})$ with the two norms

$$\|a\|' = \|a\| + \|\bar{\Delta}a\|, \quad \|a\|'' = \|a\| + \sum_{i=1}^{d} \|\overline{V(x_i)}^2a\|.$$

Then $D(\bar{\Delta})$ is complete with respect to $\|\cdot\|'$. But it follows from Lemma 1.2 that $D(\bar{\Delta})$ is also complete with respect to $\|\cdot\|''$. Moreover $\|a\|' \leqslant \|a\|''$ for all $a \in D(\bar{\Delta})$ by Condition 2. Hence by the closed graph theorem (see, for example, [16], Chapter 4, Proposition 11) there exists a $K \geqslant 1$ such that $\|a\|'' \leqslant K\|a\|'$, $a \in D(\bar{\Delta})$. In particular

$$\sum_{i=1}^{d} \|\overline{V(x_i)}^2a\| \leqslant K(\|\bar{\Delta}a\| + \|a\|), \quad a \in \mathscr{B}_2 \subseteq D(\bar{\Delta}).$$

Now applying this argument to the basis x_1^ρ obtained by applying an orthogonal transformation ρ to x_i and using the invariance of Δ under such transformations one concludes that there is a $K_\rho \geqslant 1$ such that

$$\sum_{i=1}^{d} \|\overline{V(x_i^\rho)}^2a\| \leqslant K_\rho(\|\bar{\Delta}a\| + \|a\|), \quad a \in \mathscr{B}_2. \tag{1.1}$$

But for $a \in \mathscr{B}_1$, $x, y \in \mathfrak{g}$, and $\lambda, \mu \in \mathbf{R}$, one has

$$\overline{V(\lambda x + \mu y)}a = \lambda\overline{V(x)}a + \mu\overline{V(y)}a \tag{1.2}$$

and for $a \in \mathscr{B}_2$

$$(\operatorname{ad}\overline{V(x)})(\overline{V(y)})a = \overline{V((\operatorname{ad}x)(y))}a \tag{1.3}$$

because of the group representation properties. Then for $a \in \mathscr{B}_2$

$$4\overline{V(x_i)}\,\overline{V(x_j)}a = \overline{V((x_i + x_j)/\sqrt{2})^2}a - \overline{V((x_i - x_j)/\sqrt{2})^2}a - 2\overline{V((\operatorname{ad}x_i)(x_j))}a.$$

Now Condition 3 follows from (1.1), (1.4), Lemma 1.2, and the structure relations for $\mathfrak{g}$.

$3 \Rightarrow 1$. If $a \in D(\bar{\Delta})$ one can choose a sequence $a_n \in D(\Delta) \subseteq \mathscr{B}_2$ such that $a_n \to a$ and $\Delta a_n \to \bar{\Delta} a$. It then follows from Condition 3 and Lemma 1.2 that $V(x_j)a_n \to \overline{V(x_j)}a$ and $V(x_i)\, V(x_j)a_n \to \overline{V(x_i)}\,\overline{V(x_j)}a$. Hence $D(\bar{\Delta}) \subseteq \mathscr{B}_2$ and one must have $D(\bar{\Delta}) = \mathscr{B}_2$.

$3 \Rightarrow 4$. We have just argued that Condition 3 is equivalent to Condition 1 and 2. Therefore $D(\bar{\Delta}) = \mathscr{B}_2$ and

$$\|(I + \varepsilon\bar{\Delta})a\| \leqslant \|a\| + \varepsilon \sum_{i=1}^{d} \|\overline{V(x_i)}^2 a\| \leqslant (1 + d\varepsilon)\|a\|_2, \quad a \in \mathscr{B}_2, \tag{1.4}$$

for all $\varepsilon > 0$. Next since $\bar{\Delta}$ is the generator of a continuous semigroup there exist $M \geqslant 1$ and $\omega \geqslant 1$ such that $(I + \varepsilon\bar{\Delta})^{-1}$ exists if $0 < \varepsilon\omega < 1$ and

$$\|(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant M(1 - \varepsilon\omega)^{-1}\|a\|. \tag{1.5}$$

Then from Condition 3 extended to the closures by the argument used to prove $3 \Rightarrow 1$ one has

$$\begin{aligned} \|\overline{V(x_i)}\,\overline{V(x_j)}(I + \varepsilon\bar{\Delta})^{-1}a\| &\leqslant K(\|\bar{\Delta}(I + \varepsilon\bar{\Delta})^{-1}a\| + \|(I + \varepsilon\bar{\Delta})^{-1}a\|) \leqslant \\ &\leqslant K(\|a\|/\varepsilon + (1 + 1/\varepsilon)\|(I + \varepsilon\bar{\Delta})^{-1}a\|). \end{aligned} \tag{1.6}$$

Now it follows from (1.5), (1.6), and Lemma 1.2 that there exists a $K_\varepsilon \geqslant 1$ such that $\|(I + \varepsilon\bar{\Delta})^{-1}a\|_2 \leqslant K_\varepsilon\|a\|$ for $0 < \varepsilon\omega < 1$. Hence $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_2$ onto $\mathscr{B}$ whenever $0 < \varepsilon\omega < 1$.

$4 \Rightarrow 3$. If Condition 4 is valid then for each $\varepsilon \in (0, \varepsilon_0]$ there is a $K_\varepsilon > 0$ such that $\|(I + \varepsilon\bar{\Delta})^{-1}b\|_2 \leqslant K_\varepsilon\|b\|$. Consequently

$$\|\overline{V(x_i)}\,\overline{V(x_j)}(I + \varepsilon\bar{\Delta})^{-1}b\| \leqslant K_\varepsilon\|b\|, \quad b \in \mathscr{B}.$$

But if $a \in \mathscr{B}_2 \subseteq D(\bar{\Delta})$ then there is a $b \in \mathscr{B}$ such that $a = (I + \varepsilon\bar{\Delta})^{-1}b$. Therefore

$$\|\overline{V(x_i)}\,\overline{V(x_j)}a\| \leqslant K_\varepsilon\|(I + \varepsilon\bar{\Delta})a\|, \quad a \in \mathscr{B}_2,$$

and Condition 3 holds.

Now consider the last statement of the theorem.

First we prove that $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 = \mathscr{B}_1$ or, equivalently, $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 = \mathscr{B}_3$. Now it follows from Condition 2 of the theorem that $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 \subseteq \mathscr{B}_1$ and hence it suffices to prove that $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 \subseteq \mathscr{B}_3$. But it follows from Condition 1 of the theorem that

$$\mathscr{B}_3 = \bigcap_{k=1}^{d} D(\bar{\Delta}\overline{V(x_k)}).$$

Moreover, if $a \in \mathscr{B}_1$ then

$$\varepsilon\bar{\Delta}(I + \varepsilon\bar{\Delta})^{-1}a = a - (I + \varepsilon\bar{\Delta})^{-1}a \in \mathscr{B}_1$$

because $D(\bar{\Delta}) = \mathscr{B}_2 \subseteq \mathscr{B}_1$ and hence

$$(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 \subseteq \bigcap_{k=1}^{d} D(\overline{V(x_k)}\bar{\Delta}).$$

Consequently it suffices to prove that

$$\bigcap_{k=1}^{d} D(\overline{V(x_k)}\bar{\Delta}) \subseteq \bigcap_{k=1}^{d} D(\bar{\Delta}\overline{V(x_k)}). \tag{1.7}$$

Now if $\omega \in \mathscr{B}^*_{V_*}$, where $V_* = dU_*$, then

$$(\mathrm{ad}\,\Delta_*)(V_*(x))\omega = -\sum_{i=1}^{d} (V_*(x_i)V_*(y_i) + V_*(y_i)V_*(x_i))\omega$$

where $y_i = (\mathrm{ad}\,x_i)(x)$. Hence if $a \in D(\overline{V(x)}\bar{\Delta}) \subseteq D(\bar{\Delta}) = \mathscr{B}_2$ then

$$(\Delta_*\omega)(V(x)a) = \omega(\overline{V(x)}\,\bar{\Delta}a) - \sum_{i=1}^{d} \omega((\overline{V(x_i)}\,\overline{V(y_i)} + \overline{V(y_i)}\,\overline{V(x_i)})a)$$

by duality. But the right hand side is continuous in ω and $\Delta^* = \bar{\Delta}_*$ by the Nelson-Stinespring-Langlands results quoted at the beginning of the proof and hence $\overline{V(x)}\,a \in D(\bar{\Delta})$. Therefore (1.7) is satisfied and hence $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 = \mathscr{B}_1$.

Second, we argue that $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_3$ onto $\mathscr{B}_1$. Now it follows from Condition 2 of the theorem that

$$\|(I + \varepsilon\bar{\Delta})a\|_1 \leqslant (1 + \varepsilon d)\|a\|_3,$$

i.e., $(I + \varepsilon\bar{\Delta})$ is a bounded map from $\mathscr{B}_3$ onto $\mathscr{B}_1$, and it remains to examine the boundedness properties of the inverse map $(I + \varepsilon\bar{\Delta})^{-1}$. Now it follows from (1.5), (1.6), and Lemma 1.2 that there exist $L^{(1)}_\varepsilon$, $L^{(2)}_\varepsilon$ such that

$$\|\overline{V(x_i)}(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant L^{(1)}_\varepsilon\|a\|, \quad a \in \mathscr{B}, \tag{1.8}$$

$$\|\overline{V(x_i)}\,\overline{V(x_j)}\,(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant L^{(2)}_\varepsilon\|a\|, \quad a \in \mathscr{B}. \tag{1.9}$$

for $\varepsilon \in (0, \varepsilon_0]$. But for $a \in \mathscr{B}_1$

$$\overline{V(x_k)}(I + \varepsilon\bar{\Delta})^{-1}a = (I + \varepsilon\bar{\Delta})^{-1}\overline{V(x_k)}a + \varepsilon(I + \varepsilon\bar{\Delta})^{-1}(\mathrm{ad}\,\bar{\Delta})(\overline{V(x_k)})(I + \varepsilon\bar{\Delta})^{-1}a.$$

Hence if C_{ij}^k denote the structure constants of $\mathfrak{g}$ and one sets

$$C = \sup_k \sum_{i,j=1}^{d} |C_{ik}^j|$$

then

$$\|\overline{V(x_i)}\,\overline{V(x_j)}\,\overline{V(x_k)}\,(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant L_\varepsilon^{(2)}(\|\overline{V(x_k)}a\| + \varepsilon C L_\varepsilon^{(2)}\|a\|). \tag{1.10}$$

Therefore by (1.5), (1.8), (1.9), and (1.10) there exists a $K_\varepsilon^{(3)}$ such that

$$\|(I + \varepsilon\bar{\Delta})^{-1}a\|_3 \leqslant K_\varepsilon^{(3)}\|a\|_1, \quad a \in \mathscr{B}_1.$$

Thus $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_3$ onto $\mathscr{B}_1$ for $\varepsilon \in (0, \varepsilon_0]$.

Third, we prove that $(I + \varepsilon\bar{\Delta})\mathscr{B}_{n+2} \supseteq \mathscr{B}_n$ for $n \geqslant 1$. In fact we prove $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_n \subseteq \subseteq \mathscr{B}_{n+2}$, by induction. Suppose $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_n \subseteq \mathscr{B}_{n+2}$ for $n = 1, \ldots, m$ where $m \geqslant 1$. Next, for brevity, adopt the schematic notation V^m for a monomial of order m in the $\overline{V(x_i)}$ and $\sum CV^m$ for a linear combination of such monomials. Then for $a \in \mathscr{B}_{m+1}$ one has

$$V^m(I + \varepsilon\bar{\Delta})^{-1}a = (I + \varepsilon\bar{\Delta})^{-1}V^m a + \varepsilon(I + \varepsilon\bar{\Delta})^{-1}(\mathrm{ad}\,\bar{\Delta})(V^m)(I + \varepsilon\bar{\Delta})^{-1}a. \tag{1.11}$$

This commutation rule is justified because $\bar{\Delta}V^m = \sum CV^{m+2}$, $V^m\bar{\Delta} = \sum CV^{m+2}$, and $(I + \bar{\Delta})^{-1}a \in \mathscr{B}_{m+2}$, by Condition 2 of the theorem and the induction hypothesis. Then since $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B} \subseteq \mathscr{B}_2$ one concludes that $(I + \varepsilon\bar{\Delta})^{-1}a \in \mathscr{B}_{m+2}$. Therefore $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_{m+1} \subseteq \mathscr{B}_{m+2}$. Next it follows from the structure relations of $\mathfrak{g}$ that $(\mathrm{ad}\,\bar{\Delta})(V^m) = \sum CV^{m+1}$ and consequently (1.11) gives

$$V^m(I + \varepsilon\bar{\Delta})^{-1}a = (I + \varepsilon\bar{\Delta})^{-1}V^m a + \varepsilon(I + \varepsilon\bar{\Delta})^{-1}\sum CV^{m+1}(I + \varepsilon\bar{\Delta})^{-1}a.$$

Hence multiplying by another V one obtains

$$\begin{aligned} V^{m+1}(I + \varepsilon\bar{\Delta})^{-1}a &= (I + \varepsilon\bar{\Delta})^{-1}(V^{m+1}a + \varepsilon\sum CV^{m+2}(I + \varepsilon\bar{\Delta})^{-1}a) + \\ &+ \varepsilon(I + \varepsilon\bar{\Delta})^{-1}(\mathrm{ad}\,\bar{\Delta})(V)(I + \varepsilon\bar{\Delta})^{-1}(V^m a + \varepsilon\sum CV^{m+1}(I + \varepsilon\bar{\Delta})^{-1}a). \end{aligned} \tag{1.12}$$

This commutation relation is justified because $a \in \mathscr{B}_{m+1}$, by assumption, $(I + \varepsilon\Delta)^{-1}a \in \in \mathscr{B}_{m+2}$, by the previous argument, and hence $V^m a + \varepsilon\sum CV^{m+1}(I + \varepsilon\bar{\Delta})^{-1}a \in \mathscr{B}_1$.

But $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 \subseteq \mathscr{B}_3$ and $(\mathrm{ad}\,\bar{\Delta})(V) = \sum CV^2$. Finally $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B} \subseteq \mathscr{B}_2$ and thus it follows from (1.12) that $(I + \varepsilon\bar{\Delta})^{-1}a \in \mathscr{B}_{m+3}$. Thus $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_{m+1} \subseteq \mathscr{B}_{m+3}$ and the induction is complete. Note that it follows from Condition 2 of the theorem that $(I + \varepsilon\bar{\Delta})\mathscr{B}_{m+3} \subseteq \mathscr{B}_{m+1}$ and hence we have established that $(I + \varepsilon\bar{\Delta})$ maps $\mathscr{B}_{m+3}$ onto $\mathscr{B}_{m+1}$ for all $m \geqslant 0$.

Finally we prove that the mapping $(I + \varepsilon\bar{\Delta})$ from $\mathscr{B}_{m+2}$ onto $\mathscr{B}_m$ is an isomorphism. First if $a \in \mathscr{B}_{m+2}$

$$\|(I + \varepsilon\bar{\Delta})a\|_m \leqslant (1 + \varepsilon d)\|a\|_{m+2}$$

and we now estimate the crossnorm of the inverse.

Set $\rho_0(a) = \|a\|$ and $\rho_m(a) = \sup \|V^m a\|$ for $m \geqslant 1$ where the supremum is over all possible monomials V^m. Moreover set $K_\varepsilon = \|(I + \varepsilon\bar{\Delta})^{-1}\|_2$. Then K_ε is finite, for $\varepsilon \in (0, \varepsilon_0]$, by assumption. Now it follows from (1.11) that there exists an $L \geqslant 0$ such that

$$\rho_{m+2}((I + \varepsilon\bar{\Delta})^{-1}a) \leqslant K_\varepsilon(\rho_m(a) + \varepsilon mL\rho_{m+1}((I + \varepsilon\bar{\Delta})^{-1}a))$$

for all $a \in \mathscr{B}_m$. Then by iteration, and use of the obvious bounds $\rho_p(a) \leqslant \|a\|_m$, $p \leqslant m$, one concludes that there is an $M_\varepsilon \geqslant 1$ independent of m such that

$$\rho_{m+2}((I + \varepsilon\bar{\Delta})^{-1}a) \leqslant (m + 1)!\, M_\varepsilon^m\|a\|_m.$$

Therefore

$$\|(I + \varepsilon\bar{\Delta})^{-1}a\|_{m+2} \leqslant (m + 2)!\, M_\varepsilon^m\|a\|_m, \quad a \in \mathscr{B}_m$$

and $(I + \varepsilon\bar{\Delta})^{-1}$ is a bounded map from $\mathscr{B}_m$ onto $\mathscr{B}_{m+2}$. This establishes that $(I + \varepsilon\bar{\Delta})^{-1}$ is an isomorphism from $\mathscr{B}_m$ onto $\mathscr{B}_{m+2}$ for all $m \geqslant 0$ and all $\varepsilon \in (0, \varepsilon_0]$.

Note that the last statement of the theorem establishes that the norm $\|\cdot\|_{2n}$ on $\mathscr{B}_{2n}$ is equivalent to the norm $a \in \mathscr{B}_{2n} \to \|(I + \varepsilon\bar{\Delta})^n a\|$ and the norm $\|\cdot\|_{2n+1}$ on $\mathscr{B}_{2n+1}$ is equivalent to the norm $a \in \mathscr{B}_{2n+1} \to \|(I + \varepsilon\bar{\Delta})^n a\|_1$. Thus the differentiable structure of the representations satisfying the conditions of the theorem is essentially determined by the Laplacian $\bar{\Delta}$. The last statement of the theorem can also be rephrased in terms of the representations $(\mathscr{B}_n, \mathfrak{G}, U_n)$.

COROLLARY 1.3. *If the four equivalent conditions of Theorem* 1.1 *are satisfied or the representation* $(\mathscr{B}, \mathfrak{G}, U)$ *then they are satisfied for all the representations* $(\mathscr{B}_n, \mathfrak{G}, U_n)$.

Proof. We have shown that the conditions of Theorem 1.1 imply that there is a $K_n \geqslant 0$ such that

$$\|(I + \varepsilon\bar{\Delta})^{-1}a\|_{n+2} \leqslant K_n\|a\|_n, \quad a \in \mathscr{B}_n$$

for small $\varepsilon > 0$. Consequently

$$\|\overline{V(x_i)}\,\overline{V(x_j)}a\|_n \leqslant K_n\|(I + \varepsilon\bar{\Delta})a\|_n, \quad a \in \mathscr{B}_{n+2}.$$

The conditions of Theorem 1.1 also have a tendency to be stable under duality.

PROPOSITION 1.4. *Let* $(\mathscr{B}, \mathfrak{G}, U)$ *be a continuous representation with dual* $(\mathscr{B}^*, \mathfrak{G}, U_*)$. *Then the following conditions are equivalent:*

1) *There is a* $K \geqslant 1$ *such that*

$$\|dU(x_i)dU(x_j)a\| \leqslant K(\|\bar{\Delta}a\| + \|a\|), \quad a \in \mathscr{B}_{dU};$$

2. $D(\bar{\Delta}_*) \subseteq \mathscr{B}_2^*$ *and there is a* $K_* \geqslant 1$ *such that*

$$\|\overline{dU_*(x_i)}\,\overline{dU_*(x_j)}a\| \leqslant K_*(\|\bar{\Delta}_*a\| + \|a\|), \quad a \in D(\bar{\Delta}_*).$$

Proof. For simplicity we set $V = dU$ and $V_* = dU_*$.

$1 \Rightarrow 2$. It follows from Condition 1 by closure that $D(\bar{\Delta}) \subseteq \mathscr{B}_2$, and hence $D(\bar{\Delta}) = \mathscr{B}_2$, and

$$\|\overline{V(x_i)}\,\overline{V(x_j)}a\| \leqslant K(\|\bar{\Delta}a\| + \|a\|), \quad a \in \mathscr{B}_2. \tag{1.13}$$

Then from these bounds and Lemma 1.2 one deduces that K can be chosen such that

$$\|\overline{V(x_i)}a\| \leqslant K(\|\bar{\Delta}a\| + \|a\|).$$

Thus for small $\varepsilon > 0$ both $\overline{V(x_i)}(I + \varepsilon\bar{\Delta})^{-1}$ and $\overline{V(x_i)}\,\overline{V(x_j)}(I + \varepsilon\bar{\Delta})^{-1}$ are bounded in norm, uniformly in i and j. Moreover since $(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 = \mathscr{B}_3$ there is a K' such that

$$\|(\mathrm{ad}\,\bar{\Delta})(\overline{V(x_i)})(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant K'\|a\|, \quad a \in \mathscr{B}_1.$$

Then for $a \in \mathscr{B}_1$ and $\omega \in \mathscr{B}^*$

$$|\omega((I + \varepsilon\bar{\Delta})^{-1}\,\overline{V(x_i)}a)| \leqslant K''\|\omega\| \cdot \|a\|.$$

But $\Delta^* = \bar{\Delta}_*$ by the Nelson-Stinespring-Langlands analysis and therefore this estimate implies $D(\bar{\Delta}_*) \subseteq \mathscr{B}_1^*$ and

$$\|\overline{V^*(x_i)}\omega\| \leqslant K''\|(I + \varepsilon\bar{\Delta}_*)\omega\|, \quad \omega \in D(\bar{\Delta}_*).$$

Next $\overline{\overline{V(x_i)}\,\overline{V(x_j)}(I+\varepsilon\bar{\Delta})^{-1}}$ is bounded in norm, uniformly in i and j, and $(I+\varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 = \mathscr{B}_3$ for small $\varepsilon > 0$. Therefore if $a \in \mathscr{B}_2 \subseteq \mathscr{B}_1$ and $\omega \in \mathscr{B}^*$ one has

$$|\omega((I+\varepsilon\bar{\Delta})^{-1}\overline{V(x_i)}\,\overline{V(x_j)}a)| \leqslant |\omega(\overline{V(x_i)}\,(\overline{V(x_j)}(I+\varepsilon\bar{\Delta})^{-1}a)| +$$

$$+\ \varepsilon|\omega((I+\varepsilon\bar{\Delta})^{-1}(\mathrm{ad}\,\bar{\Delta})\,(\overline{V(x_i)}\,\overline{V(x_j)})\,(I+\varepsilon\bar{\Delta})^{-1}a)| +$$

$$+\ \varepsilon|\omega((I+\varepsilon\bar{\Delta})^{-1}\,\overline{V(x_i)}\,(\mathrm{ad}\,\bar{\Delta})\,(\overline{V(x_j)})(I+\varepsilon\bar{\Delta})^{-1}a)| \leqslant K'''\|\omega\|\cdot\|a\|.$$

To obtain the last estimate we have used the structure relations of $\mathfrak{g}$ and the fact, just established, that $\|\overline{V_*(x_i)}(I+\varepsilon\bar{\Delta})^{-1}\|$ is bounded uniformly in i for small $\varepsilon > 0$. But this estimate proves that $D(\bar{\Delta}_*) \subseteq \mathscr{B}_2^*$ and

$$\|\overline{V_*(x_i)}\,\overline{V^*(x_j)}\omega\| \leqslant K'''\|(I+\varepsilon\bar{\Delta}_*)\omega\|, \quad \omega \in D(\bar{\Delta}_*),$$

i.e., Condition 2 is verified.

$2 \Rightarrow 1$. The proof is similar.

Note that there is an asymmetry in the two conditions of Proposition 1.4. It suffices that Condition 1 is satisfied on $\mathscr{B}_{dU}$ because it then extends by continuity to $D(\bar{\Delta})$. But it is generally insufficient to assume Condition 2 on $\mathscr{B}^*_{dU}$ because this subspace is only a core of $\bar{\Delta}_*$ in the weak* topology. For the same reason it is probably also insufficient to assume the condition on $\mathscr{B}_2^*$. Of course this asymmetry does not arise if $\mathscr{B}$ is reflexive.

Corollary 1.5. *If $\mathscr{B}$ is reflexive then the four equivalent conditions of Theorem 1.1 are self-dual, i.e., they hold for $(\mathscr{B}, \mathfrak{G}, U)$ if, and only if, they hold for $(\mathscr{B}^*, \mathfrak{G}, U_*)$.*

There is one curious aspect of the four conditions in Theorem 1.1. They are formulated in terms of one basis of $\mathfrak{g}$ and since Δ is invariant under orthogonal transformations of the basis the conditions are invariant under such transformations. But it is not evident that they are independent of the original choice of basis. This would follow if they were also stable under scaling transformations $x_i \to x_i^\lambda = \lambda_i x_i$ where $\lambda_i \in \mathbf{R} \setminus \{0\}$. This is the case in many examples, as we discuss below, and it also follows if a seemingly stronger version of Condition 3 of the theorem is satisfied.

Proposition 1.6. *Let $(\mathscr{B}, \mathfrak{G}, U)$ be a strongly continuous representation of the Lie group $\mathfrak{G}$ and for a fixed basis $x_1, \ldots, x_d$ of the associated Lie algebra $\mathfrak{g}$ define the Laplacian Δ by*

$$\Delta = -\sum_{i=1}^{d} dU(x_i)^2.$$

Assume that for each $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$\|dU(x_i)dU(x_j)a\| \leqslant (1+\varepsilon)\|\Delta a\| + C_\varepsilon\|a\|, \quad a \in \mathscr{B}_{dU}, \tag{1.14}$$

then the four equivalent conditions of Theorem 1.1 *are satisfied for every basis of* $\mathfrak{g}$.

Proof. Again set $V = dU$. Now $\mathscr{B}_2 \subseteq D(\bar{\Delta})$ and it follows by closure, using Lemma 1.2, that

$$\|\overline{V(x_i)}\,\overline{V(x_j)}a\| \leqslant (1+\varepsilon)\|\bar{\Delta} a\| + C_\varepsilon\|a\|, \quad a \in \mathscr{B}_2.$$

In particular Condition 3 of Theorem 1.1 is satisfied for the basis $x_1, \ldots, x_d$. Since the definition of Δ is invariant under orthogonal transformations the condition is also satisfied for any basis obtained from $x_1, \ldots, x_d$ by such a transformation.

Next consider the scaling transformation $x_i \to x_i^\lambda = \lambda_i x_i$ where $\lambda_i \in \mathbf{R} \setminus \{0\}$. Define Δ_λ on $\mathscr{B}_V$ by

$$\Delta_\lambda = -\sum_{i=1}^{d} \lambda_i^2 V(x_i)^2.$$

Then by the Nelson-Stinespring-Langlands results $D(\bar{\Delta}_\lambda) \supseteq \bigcap_{i=1}^{d} D(\overline{V(x_i)}^2) \supseteq \mathscr{B}_2$ and

$$\bar{\Delta}_\lambda a = -\sum_{i=1}^{d} \lambda_i^2 \overline{V(x_i)}^2 a, \quad a \in \mathscr{B}_2.$$

Therefore, for each $\rho > 0$,

$$\|\bar{\Delta} a\| \leqslant \|\bar{\Delta}_\lambda a\|/\rho^2 + \sum_{i=1}^{d} |1 - \lambda_i^2/\rho^2|\,\|\overline{V(x_i)}^2 a\| \leqslant$$

$$\leqslant \|\bar{\Delta}_\lambda a\|/\rho^2 + \sum_{i=1}^{p} |1 - \lambda_i^2/\rho^2|((1+\varepsilon)\|\bar{\Delta} a\| + C_\varepsilon\|a\|), \quad a \in \mathscr{B}_2.$$

Hence if λ is in the set $C_{\rho,\varepsilon}$, where

$$C_{\rho,\varepsilon} = \left\{\lambda \in \mathbf{R}^d;\ \sum_{i=1}^{d} |1 - \lambda_i^2/\rho^2| < 1/(1+\varepsilon)\right\},$$

one has a bound of the form

$$\|\bar{\Delta} a\| \leqslant K(\lambda, \rho, \varepsilon)(\|\bar{\Delta}_\lambda a\| + \|a\|), \quad a \in \mathscr{B}_2.$$

Since $D(\bar{\Delta}) = \mathscr{B}_2$ it follows that $\bar{\Delta}_\lambda$ is closed on $\mathscr{B}_2$ and consequently $D(\bar{\Delta}_\lambda) = \mathscr{B}_2$ But if $\lambda \in \mathbf{R}^d$ and $\lambda_i \neq \{0\}$, $i = 1, \ldots, d$ then one can choose ρ and ε such that

$\lambda \in C_{\rho,\varepsilon}$. Thus $D(\bar{\Delta}_\lambda) = \mathscr{B}_2$ for all λ with $\lambda_i \neq 0$, i.e. Condition 1 of Theorem 1.1 is invariant under all such scaling transformations.

Finally since each non-singular transformation of $\mathbf{R}^d$ can be decomposed in the form $O_1 D O_2$ where O_1, O_2 are orthogonal transformations and D is a non-singular scaling transformation, Condition 1 of Theorem 1.1 is valid for all possible bases of $\mathfrak{g}$.

We conclude with the discussion of some common examples of representations satisfying the criteria of Theorem 1.1. The simplest general class of examples is provided by unitary representations on Hilbert space. Then the conditions are satisfied for all bases. This is a consequence of the following lemma, which improves Lemma 6.2 of Nelson [11].

LEMMA 1.7. *Let $(\mathscr{H}, \mathfrak{g}, V)$ denote a representation of $\mathfrak{g}$ by skew-symmetric operators on the Hilbert space $\mathscr{H}$. Fix a basis $x_1, \ldots, x_d$ of $\mathfrak{g}$ and define*

$$\Delta = -\sum_{i=1}^{d} V(x_i)^2;$$

then for each $\varepsilon > 0$

$$\|V(x_i)a\| \leqslant \varepsilon\|\Delta a\| + (1/2\varepsilon)\|a\|,$$

$$\|V(x_i)V(x_j)a\| \leqslant (1+\varepsilon)\|\Delta a\| + (2C^2/\varepsilon)\|a\|, \quad a \in \mathscr{H}_V,$$

for all $i, j = 1, \ldots, d$ where C is defined in terms of the structure constants C_{ij}^k of $\mathfrak{g}$ by

$$C = \sup_{1 \leqslant i \leqslant d} \sum_{j,k=1}^{d} |C_{ij}^k|.$$

Proof. First, by positivity

$$\|V(x_i)a\|^2 = -(a, V(x_i)^2 a) \leqslant$$

$$\leqslant (a, \Delta a) \leqslant \|(\varepsilon\sqrt{2}\Delta + 1/\varepsilon\sqrt{2})a\|^2/2$$

which immediately yields the first bound. Second, by positivity and the structure relations

$$\|V(x_i)V(x_j)a\|^2 \leqslant (V(x_j)a, \Delta V(x_j)a) =$$

$$= -(V(x_j)^2 a, \Delta a) + \sum_{i,k=1}^{d} C_{ij}^k (V(x_j)a, (V(x_i)V(x_k) + V(x_k)V(x_i))a).$$

Hence if

$$k = \sup_{1 \leqslant i,j \leqslant d} \|V(x_i)V(x_j)a\|$$

one finds by the Cauchy-Schwarz inequality and the first bound that

$$k^2 \leqslant k\|\Delta a\| + 2kC(\delta\|\Delta a\| + (1/2\delta)\|a\|)$$

for all $\delta > 0$. Choosing $\delta = \varepsilon/2C$ one obtains the second bound.

Now if $(\mathscr{H}, \mathfrak{G}, U)$ is a continuous unitary representation of $\mathfrak{G}$ then the representatives $dU(x)$ of $\mathfrak{g}$ are automatically skew-symmetric. Thus Lemma 1.7 applies and hence the conditions of Theorem 1.1 are satisfied for every basis of $\mathfrak{g}$.

Other examples, and counterexamples, occur in the theory of partial differental equations. Let $\mathscr{B}$ be one of the standard function spaces over $\mathbf{R}^d$ and consider the group $\mathbf{R}^d$ acting by translation. Then with the basis $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ one has $V(x_j) = i\partial/\partial x_j = D_j$ and Condition 3 of Theorem 1.1 can often be verified by the use of Riesz transforms.

If $\hat{f}$ denotes the Fourier transform of f then $(D_j\hat{f})(p) = p_j\hat{f}(p)$, $(\Delta\hat{f})(p) = p^2\hat{f}(p)$, and the Riesz transforms R_j are defined as the operators with action $(R_j\hat{f})(p) = (p_j/|p|)\hat{f}(p)$. Thus $D_iD_jf = R_iR_j\Delta f$ and one has the formal bounds

$$\|D_iD_jf\| \leqslant \|R_i\| \cdot \|R_j\| \cdot \|\Delta f\|.$$

Therefore this method works for those function spaces for which the R_i are bounded operators. This class includes the spaces $L^q(\mathbf{R}^d)$ if $1 < q < \infty$ (see, for example, [17], pages 59 and 77) but also includes many other Besov-Hölder-Sobolev spaces. On these spaces one can also establish that the bounds are valid for any choice of basis. Consider, for example, the spaces $L^q(\mathbf{R}^d)$. The group of rotations of $\mathbf{R}^d$ acts isometrically on $L^q(\mathbf{R}^d)$ and there is an action of the group of dilations $x \to \lambda x = (\lambda_1 x_1, \ldots, \lambda_d x_d)$, $\lambda_i \in \mathbf{R} \setminus \{0\}$, given by $(U_\lambda f)(x) = f(\lambda x)$ and one has $\|U_\lambda\|_q = \|U_{-\lambda}\|_q^{-1}$. Therefore $\|U_\lambda R_i U_{-\lambda}\|_q \leqslant \|R_i\|_q$ and then by the group property $\|U_\lambda R_i U_{-\lambda}\|_q = \|R_i\|_q$. Since the norm $\|R_i\|$ is also invariant under rotations one then has

$$\|D_i^{(\rho,\lambda)}D_j^{(\rho,\lambda)}f\|_q \leqslant R^2\|\Delta_\lambda f\|$$

where $D_j^{(\rho,\lambda)}$ denotes the partial differential operators with respect to the rotated and dilated basis, Δ_λ is the Laplacian with respect to the dilated basis, and $R = \|R_i\|$ is independent of the choice of basis.

It is remarkable that there are no such bounds for $L^1(\mathbf{R}^d)$ or $L^\infty(\mathbf{R}^d)$, if $d \geqslant 2$. De Leeuw and Mirkil [3] proved that Condition 3 of Theorem 1.1 fails for translations on $L^\infty(\mathbf{R}^d)$ and Ornstein [13] established the same result on $L^1(\mathbf{R}^d)$. In fact the L^1 result follows from the L^∞ result by Proposition 1.4. It also follows from a simple scaling and translation invariance argument that the bounds fail for the L^1 and L^∞ norms on $C^\infty(\Lambda)$, for each open subset Λ of $\mathbf{R}^d$. Hence the bounds fail for $L^1(\mathbf{T}^d)$ and $L^\infty(\mathbf{T}^d)$, for $d \geqslant 2$. They also fail on $C_0(\mathbf{R}^d)$ or $C_0^u(\mathbf{R}^d)$ (for further details see [19], especially Section 2.2.3).

The Riesz transform method sketched above can be generalized to compact Lie groups $\mathfrak{G}$ acting by translations on $L^q(\mathfrak{G})$, $1 < q < \infty$, (see [18], pages 57 and 131) but the general situation is unclear. It would seem worthwhile to investigate representations of specific classes of simply connected groups, e.g. compact or semi-simple groups, and particular types of representation, e.g. isometric representations on reflexive spaces.

Our main aim, however, is to consider the integration problem associated with representations satisfying the criteria of Theorem 1.1. This will be discussed in Section 3 with the aid of a semigroup generator theorem which we prove in the next section.

2. COMMUTATOR THEORY

In this section we derive a sufficient condition for a dissipative operator K to generate a contraction semigroup. The main theorem is similar to the results of [1], [15] insofar it compares K with a known generator H and the key condition is an estimate on the commutator of H with K of the type $(\mathrm{ad}\,H)(K) = O(H)$. But the present result differs from the earlier results in two respects. First, in [1] and [15] it was assumed that H generates a C_0-group of isometries but now we only suppose that H generates a C_0-semigroup of contractions. On the other hand in [1] it was essentially assumed that $K = O(H)$, or in [15] that $K = O(H^n)$ for some $n \geqslant 1$, but here we make the stronger assumption that K is infinitesimally small with respect to H. Nevertheless the discussion is again based on the singular perturbation technique developed in [14] and elaborated in [1], [15].

As a preliminary recall that an operator K on the Banach space $\mathcal{B}$ is dissipative in the sense of Lumer and Phillips [10] if for each $a \in \mathrm{D}(K)$ there is an $\omega \in \mathcal{B}^*$ such that $\omega(a) = \|\omega\|\cdot\|a\|$ and $\mathrm{Re}\,\omega(Ka) \geqslant 0$. Equivalently, K is dissipative if, and only if,

$$\|(I + \alpha K)a\| \geqslant \|a\|, \quad a \in \mathrm{D}(K),$$

for all small $\alpha > 0$. (For a proof of equivalence see, for example, [1], Theorem 2.1.1.) Moreover, if K is norm-densely defined then it is dissipative if, and only if, for each $a \in \mathrm{D}(K)$ and each $\omega \in \mathcal{B}^*$ such that $\omega(a) = \|\omega\|\cdot\|a\|$ one has $\mathrm{Re}\,\omega(Ka) \geqslant 0$. In particular this last criterion implies that the the sum of two norm-densely defined dissipative operators is dissipative.

THEOREM 2.1. *Let S be a C_0-semigroup of contractions on the Banach space $\mathcal{B}$ with generator H and let $a \in \mathrm{D}(H) \to \|a\|_1 = \|Ha\| + \|a\|$ denote the graph norm* *on the domain* D(H) *of* H. *Next let* K *be an operator with the following properties:*

1. K *is closed and dissipative,*
2. $\mathrm{D}(H) \subseteq \mathrm{D}(K)$ *and for each* $\varepsilon > 0$ *there is* C_ε *such that*

$$\|Ka\| \leqslant \varepsilon\|Ha\| + C_\varepsilon\|a\|, \quad a \in \mathrm{D}(H),$$

3. *for some* $k \geqslant 0$ *and* $\delta > 0$

$$\|(\mathrm{ad}\, S_t)(K)a\| \leqslant kt\|a\|_1, \quad t \in [0, \delta], \; a \in \mathrm{D}(H).$$

It follows that K *generates a* C_0*-semigroup of contractions* T.

Moreover if $K\mathscr{D} \subseteq \mathrm{D}(H)$ *for some dense* S*-invariant subspace* $\mathscr{D} \subseteq \mathrm{D}(H^2)$ *then*

1. $T_t\mathrm{D}(H) \subseteq \mathrm{D}(H), \quad t \geqslant 0,$
2. $T|\mathrm{D}(H)$ *is* $\|\cdot\|_1$*-continuous,*
3. $\|T_t a\|_1 \leqslant e^{kt}\|a\|_1, \quad t \geqslant 0.$

Proof. It follows from assumptions 1 and 2 that the family of operators $K_\varepsilon = \varepsilon H + K$, $\varepsilon > 0$, generate C_0-semigroups of contractions (see, for example, [2], Theorem 3.1.32). In particular the resolvents $(I + \alpha K_\varepsilon)^{-1}$ are contractions for each $\alpha > 0$, and $\varepsilon > 0$, and $(I + \alpha K_\varepsilon)^{-1}\mathscr{B} = \mathrm{D}(H)$.

Next since K is closed, densely defined, and dissipative, it generates a C_0-semigroup of contractions if, and only if, the range condition $\mathrm{R}(I + \alpha K) = \mathscr{B}$ is satisfied for all small $\alpha > 0$. Now assume there is an $\omega \in \mathscr{B}^*$ such that

$$\omega((I + \alpha K)a) = 0$$

for some small positive α and all $a \in \mathrm{D}(H)$. Then

$$\omega((I + \alpha K)(I + \alpha K_\varepsilon)^{-1}a) = 0, \quad \varepsilon > 0, \; a \in \mathscr{B}.$$

But this gives the identity

$$\omega(a) = \alpha\varepsilon\omega(H(I + \alpha K_\varepsilon)^{-1}a), \quad \varepsilon > 0, \; a \in \mathscr{B}. \tag{2.1}$$

Now we need an estimate, similar to Lemma 2.2 of [1], for $H(I + \alpha K_\varepsilon)^{-1}a$.

LEMMA 2.2. *If* $a \in \mathrm{D}(H)$ *and* $\alpha k < 1$ *then*

$$\|H(I + \alpha K_\varepsilon)^{-1}a\| \leqslant (1 - \alpha k)^{-1}\|a\|_1.$$

Proof. First one has the identity

$$S_t(I + \alpha K_\varepsilon)^{-1}a = (I + \alpha K_\varepsilon)^{-1}S_t a + (\mathrm{ad}\, S_t)((I + \alpha K_\varepsilon)^{-1})a =$$

$$= (I + \alpha K_\varepsilon)^{-1}(S_t a + \alpha(\mathrm{ad}\, S_t)(K)(I + \alpha K_\varepsilon)^{-1}a).$$

Therefore, using Condition 3 of the theorem, one estimates that

$$\|(I - S_t)(I + \alpha K_\varepsilon)^{-1}a\| \leqslant \|(I - S_t)a\| + \alpha kt(\|H(I + \alpha K_\varepsilon)^{-1}a\| + \|a\|).$$

Dividing by t and taking the limit $t \to 0$ then gives

$$\|H(I + \alpha K_\varepsilon)^{-1}a\| \leqslant \|Ha\| + \alpha k(\|H(I + \alpha K_\varepsilon)^{-1}a\| + \|a\|)$$

and a simple rearrangement using $\alpha k < 1$ gives the desired result.

Now, returning to the proof of the theorem, we see from (2.1) and Lemma 2.2 that

$$|\omega(a)| \leqslant \alpha\varepsilon(1 - \alpha k)^{-1}\|\omega\|(\|Ha\| + \|a\|)$$

for all $a \in \mathrm{D}(H)$ and $\varepsilon > 0$, whenever $\alpha k < 1$. Therefore, taking the limit $\varepsilon \to 0$, one concludes that $\omega(a) = 0$ for all $a \in \mathrm{D}(H)$. Since $\mathrm{D}(H)$ is norm dense it follows that $\omega = 0$. Hence $\mathrm{R}(I + \alpha K) = \mathscr{B}$ for $\alpha k < 1$, and K is a generator.

The proof of the last statement of the theorem is now just a repetition of the argument used to prove a similar result in [1], Theorem 2.1. First, since $\mathscr{D} \subseteq \mathrm{D}(H)$ is norm-dense and S-invariant it follows that it is $\|\cdot\|_1$-dense in $\mathscr{B}_1$. Second, one argues that $K + kI$ is $\|\cdot\|_1$-dissipative on $\mathscr{D}$. This follows because dissipativity of K gives

$$\|(I + \alpha(K + kI))a\| \geqslant (I + \alpha k)\|a\|, \quad a \in \mathrm{D}(H) \tag{2.2}$$

and hence

$$\|(I - S_t)(I + \alpha(K + kI))a\| \geqslant (1 + \alpha k)\|(I - S_t)a\| - \alpha\|(\mathrm{ad}\, S_t)(K)a\|.$$

But if $a \in \mathscr{D}$ then $Ka \in \mathrm{D}(H)$ and one deduces from this last inequality, by use of assumption 3 of the theorem, that

$$\|H(I + \alpha(K + kI))a\| \geqslant \|Ha\| - \alpha k\|a\|, \quad a \in \mathscr{D}. \tag{2.3}$$

Thus by addition of (2.2) and (2.3) one obtains

$$\|(I + \alpha(K + kI))a\|_1 \geqslant \|a\|_1, \quad a \in \mathscr{D}.$$

Third, one argues that $(I + \alpha(K + kI))\mathscr{D}$ is $\|\cdot\|_1$-dense in $\mathrm{D}(H)$. For this we note that if $a \in \mathscr{D}$ then $Ka \in \mathrm{D}(H)$ and

$$\|(\operatorname{ad} H)(K)a\| \leqslant k\|a\|_1 \tag{2.4}$$

by use of Condition 3. Now if $\mathscr{B}_1 = \mathrm{D}(H)$ equipped with the norm $\|\cdot\|_1$ and if there is an $\omega \in \mathscr{B}_1^*$ such that

$$\omega((I + \alpha(K + kI))a) = 0, \quad a \in \mathscr{D},$$

then by standard approximation theory, using (2.4) and S-invariance of $\mathscr{D}$, one deduces that

$$\omega((I + \alpha(K + kI))Ra) = 0, \quad a \in \mathscr{D}, \tag{2.5}$$

where $R = (I + H)^{-1}$. But R is a bounded map of $\mathscr{B}$ into $\mathscr{B}_1$. Hence the adjoint R^* is a bounded map from $\mathscr{B}_1^*$ onto $\mathscr{B}^*$ and to prove $\omega = 0$ it suffices to prove that $R^*\omega = 0$ on $\mathscr{B}$. Now using (2.4) and (2.5) one has

$$(R^*\omega)((I + \alpha K)a) = -\alpha k(R^*\omega)(a) + \alpha\omega((\operatorname{ad} R)(K)a) \leqslant$$

$$\leqslant \alpha k\|R^*\omega\|\cdot\|a\| + \alpha\omega(R(\operatorname{ad} K)(H)Ra) \leqslant 3\alpha k\|R^*\omega\|\cdot\|a\| \leqslant$$

$$\leqslant 3\alpha k\|R^*\omega\|\cdot\|(I + \alpha K)a\|, \quad a \in \mathscr{D},$$

where the last step uses dissipativity of K. Since $(I + \alpha K)\mathscr{D}$ is dense in $\mathscr{B}$ it follows that $(1 - 3\alpha k)\|R^*\omega\| \leqslant 0$ and choosing α such that $3\alpha k < 1$ one concludes that $R^*\omega = 0$.

Finally it follows from the $\|\cdot\|_1$-density of $\mathscr{D}$, the $\|\cdot\|_1$-dissipativity of $K + kI$, and the range condition, that $K + kI$ generates a C_0-semigroup of contractions on $(\mathscr{B}_1, \|\cdot\|_1)$ and hence K generates a $\|\cdot\|_1$-continuous semigroup T on $\mathscr{B}$, satisfying

$$\|T_t a\|_1 \leqslant e^{kt}\|a\|_1, \quad a \in \mathscr{B}_1.$$

Note that this theorem proves that every dissipative operator K which commutes with a contraction semigroup S and is dominated by its generator in the perturbation sense, i.e. $\mathrm{D}(H) \subseteq \mathrm{D}(K)$ and

$$\|Ka\| \leqslant \varepsilon\|Ha\| + C_\varepsilon\|a\|, \quad a \in \mathrm{D}(H),\ \varepsilon > 0,$$

is automatically a semigroup generator. This allows, for example, a simple proof that the fractional powers of H generate contraction semigroups.

If $\lambda \in (0, 1)$ then H^λ can be defined by

$$H^\lambda a = \lim_{\delta\to 0} \int_\delta^\infty \frac{\mathrm{d}t}{t}(I - S_t)a/t^\lambda \tag{2.6}$$

with $\mathrm{D}(H^\lambda)$ the set of a for which the limit exists (see, for example, [20], Section IX.11). But if $a \in \mathrm{D}(H^\lambda)$, $\omega \in \mathscr{B}^*$, and $\omega(a) = \|\omega\|\cdot\|a\|$ then $|\omega(S_t a)| \leqslant \|\omega\|\cdot\|a\| = \omega(a)$ and

$$\operatorname{Re}\omega(H^\lambda a) = \lim_{\delta\to 0} \int_\delta^\infty \frac{\mathrm{d}t}{t} \operatorname{Re}\omega((I - S_t)a/t^\lambda) \geqslant 0,$$

i.e. H^λ is dissipative. Next if $\varepsilon > 0$ choose δ such that

$$\int_0^\delta \frac{\mathrm{d}t}{t^\lambda} = \varepsilon$$

and define C_ε by

$$C_\varepsilon = 2\int_\delta^\infty \frac{\mathrm{d}t}{t^{1+\lambda}}.$$

Then for $a \in \mathrm{D}(H)$ one readily estimates from (2.6) that

$$\|H^\lambda a\| \leqslant \varepsilon\|Ha\| + C_\varepsilon\|a\|.$$

Thus one deduces from Theorem 2.1 the well known result that H^λ generates a contraction semigroup.

The perturbation bound in Theorem 2.1 also follows if $\mathrm{D}(K^2) \subseteq \mathrm{D}(H)$ and one has a bound $\|K^2 a\| \leqslant k_0\|Ha\|$, or even if $\mathrm{D}(K^n) \subseteq \mathrm{D}(H)$ and $\|K^n a\| \leqslant k_n\|Ha\|$ for some $n \geqslant 2$ and $k_n \geqslant 0$. This is a direct consequence of the following simple observation.

LEMMA 2.3. *If K is dissipative and $\varepsilon > 0$ then*

$$\|Ka\| \leqslant \varepsilon\|K^2 a\| + (2/\varepsilon)\|a\|, \quad a \in \mathrm{D}(K^2). \tag{2.7}$$

Proof. One has

$$\varepsilon\|Ka\| \leqslant \|(I - \varepsilon K)a\| + \|a\| \leqslant$$

$$\leqslant \|(I - \varepsilon^2 K^2)a\| + \|a\| \leqslant \varepsilon^2\|K^2 a\| + 2\|a\|$$

where the second inequality follows from dissipativity.

Once one has the bound (2.7) for all $\varepsilon > 0$ it then follows by a simple induction-iteration argument that for each pair of integers $n > m \geqslant 1$, and each $\varepsilon > 0$, there is a $C_{n,m}(\varepsilon)$ such that

$$\|K^m a\| \leqslant \varepsilon \|K^n a\| + C_{n,m}(\varepsilon)\|a\|.$$

Thus these bounds always hold for dissipative operators.

3. INTEGRATION OF LIE ALGEBRAS

The aim of this section is to establish an integration theorem for isometric representations of the type characterized by Theorem 1.1 for a general Lie group $\mathfrak{G}$. Hence we begin with a representation $(\mathscr{B}_V, \mathfrak{g}, V)$ of the Lie algebra $\mathfrak{g}$ by operators $V(x)$, $x \in \mathfrak{g}$, acting on the dense invariant subspace $\mathscr{B}_V$ of $\mathscr{B}$ and look for conditions which ensure the existence of a unique representation $(\mathscr{B}, \mathfrak{G}, U)$ of the simply connected Lie group $\mathfrak{G}$ having $\mathfrak{g}$ as Lie algebra such that $\mathscr{B}_V \subseteq \mathscr{B}_{dU}$ and $V(x) \subseteq dU(x)$, for all $x \in \mathfrak{g}$. Now if U is isometric then each of the semigroups $t \geqslant 0 \to U(\exp(\pm tx))$, $x \in \mathfrak{g}$, is contractive. Hence the operators $dU(x)$, $x \in \mathfrak{g}$, are conservative, i.e. both $\pm dU(x)$ are dissipative. Therefore we will assume the $V(x)$, $x \in \mathfrak{g}$, are conservative, and, in particular, closable.

Now let $x_1, \ldots, x_d$ be a basis of $\mathfrak{g}$ and Δ the corresponding Laplacian,

$$\Delta = -\sum_{i=1}^{d} V(x_i)^2.$$

Our general strategy is to apply Theorem 2.1 with $H = \overline{\Delta}$ and $K = \pm\overline{V(x)}$, $x \in \mathfrak{g}$. But for this it is necessary to verify the hypotheses of Theorem 2.1 and this requires some assumptions on the action of Δ. These assumptions are phrased in terms of the subspaces $\mathscr{B}_n(V)$ where

$$\mathscr{B}_n = \mathscr{B}_n(V) = \bigcap_{i_1,\ldots,i_n=1}^{d} D(\overline{V(x_{i_1})} \ldots \overline{V(x_{i_n})}),$$

equipped with the norms $\|\cdot\|_n$ defined by $\|\cdot\|_0 = \|\cdot\|$ and

$$\|a\|_n = \|a\| + \sup_{1 \leqslant i \leqslant d} \|\overline{V(x_i)}a\|_{n-1}.$$

Since $\overline{V(x_i)}$ are closed, $\mathscr{B}_n$ is a Banach space with respect to the norm $\|\cdot\|_n$, and $\mathscr{B}_n$ is continuously embedded in $\mathscr{B}_{n-1}$, for each $n \geqslant 0$. Moreover, we define $\mathscr{B}_\infty$ as the intersection of the $\mathscr{B}_n$. Now the integration theorem is as follows.

THEOREM 3.1. *Let $(\mathscr{B}_V, \mathfrak{g}, V)$ be a representation of the Lie algebra $\mathfrak{g}$ by conservative operators $V(x), x \in \mathfrak{g}$. Further let Δ denote the Laplacian associated with a fixed basis $x_1, \ldots, x_d$ of $\mathfrak{g}$.*

It follows that Δ is dissipative, hence closable, and the following conditions are equivalent:

1. $\mathscr{B}_2 \subseteq \mathrm{D}(\bar{\Delta})$ *and* $(I + \varepsilon\bar{\Delta})\mathscr{B}_2 = \mathscr{B}$, $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 \supseteq \mathscr{B}_1$ *for all (small)* $\varepsilon > 0$;
2. *there is a* $K \geqslant 1$ *such that*

$$\|V(x_i)V(x_j)a\| \leqslant K(\|\Delta a\| + \|a\|), \quad a \in \mathscr{B}_V,\ i, j = 1, \ldots, d,$$

$\mathscr{B}_3 \subseteq \mathrm{D}(\bar{\Delta})$ *and* $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 \supseteq \mathscr{B}_1$ *for all (small)* $\varepsilon > 0$;

3. $\bar{\Delta}$ *generates a continuous semigroup*, $\mathrm{D}(\bar{\Delta}) \subseteq \mathscr{B}_2$, *and*

$$\mathrm{D}(\bar{\Delta}^2) \subseteq \bigcap_{i=1}^{d} \mathrm{D}(\bar{\Delta}, \overline{V(x_i)});$$

4. $\bar{\Delta}$ *generates a continuous semigroup*, $\mathrm{D}(\bar{\Delta}) \subseteq \mathscr{B}_2$, *and* $\mathrm{D}(\bar{\Delta}^2) \subseteq \mathscr{B}_3$;
5. $\bar{\Delta}$ *generates a continuous semigroup*, $\mathrm{D}(\bar{\Delta}) \subseteq \mathscr{B}_2$, *and* $\mathscr{B}_\infty$ *is a core of* $\bar{\Delta}^2$;
6. *there exists a (unique) continuous isometric representation* $(\mathscr{B}, \mathfrak{G}, U)$ *of the simply connected Lie group* $\mathfrak{G}$ *having* $\mathfrak{g}$ *as its Lie algebra such that* $\mathscr{B}_V \subseteq \mathscr{B}_{\mathrm{d}U}$, $V(x) \subseteq \mathrm{d}U(x), x \in \mathfrak{g}$, $U\mathrm{D}(\bar{\Delta}) = \mathrm{D}(\bar{\Delta})$, *and* $(\mathscr{B}, \mathfrak{G}, U)$ *satisfies the equivalent condition of Theorem* 1.1.

Moreover, if these conditions are satisfied then $\overline{V(x)} = \overline{\mathrm{d}U(x)}$, $x \in \mathfrak{g}$, *and* $\mathrm{D}(\bar{\Delta}^n) = \mathscr{B}_{2n}$ *for* $n = 1, 2, \ldots$.

REMARK 3.2. Each of the first five conditions states, either explicitly or implicitly, that $\bar{\Delta}$ generates a continuous semigroup and hence $\mathscr{B}_2 \subseteq \mathrm{D}(\bar{\Delta})$. In addition the conditions require the converse inclusion $\mathrm{D}(\bar{\Delta}) \subseteq \mathscr{B}_2$, and this ensures the range condition $(I + \varepsilon\bar{\Delta})\mathscr{B}_2 = \mathscr{B}$. Finally the range condition $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 = \mathscr{B}_1$, or a domain condition for $\bar{\Delta}^2$, also appears essential, but in special contexts such as skew-symmetric operators on Hilbert space these follow from the other requirements.

Proof. The preliminary statements concerning Δ are a consequence of a number of standard results and the following simple observation.

LEMMA 3.3. *If X is conservative then $-X^2$ is dissipative.*

Proof. Since X is conservative $\|(I \pm \alpha X)a\| \geqslant \|a\|$ for all small $\alpha > 0$. Therefore

$$\|(I - \alpha^2 X^2)a\| = \|(I - \alpha X)(I + \alpha X)a\| \geqslant \|a\|, \quad a \in \mathrm{D}(X^2)$$

for all small $\alpha > 0$. Thus $-X^2$ is dissipative.

Now each $V(x_i)$ is conservative, by assumption. Hence $-V(x_i)^2$ is dissipative, by Lemma 3.3. But the sum of two norm-densely defined dissipative operators is dissipative. Hence Δ is dissipative. But each norm-densely defined dissipative operator is automatically closable and its closure is dissipative (see, for example [2], Proposition 3.1.15). Therefore Δ is closable and $\bar{\Delta}$ is dissipative.

Next, to prove the equivalence of the first three conditions in Theorem 3.1 we need the following 'infinitesimal' version of Theorem 1.1.

PROPOSITION 3.4. *Let $(\mathscr{B}_V, \mathfrak{g}, V)$ be a representation for which the $V(x)$, $x \in \mathfrak{g}$ are conservative. Assume that the closure $\bar{\Delta}$ of the Laplacian Δ is the generator of a strongly continuous semigroup. Then $\mathscr{B}_2 \subseteq D(\bar{\Delta})$ and the following conditions are equivalent:*

1. $\mathscr{B}_2 = D(\bar{\Delta})$;
2. $\bar{\Delta} = -\sum_{i=1}^{d} \overline{V(x_i^\rho)^2}$

for each orthogonal transformation ρ of $\mathbf{R}_d$;

3. *there is a $K \geqslant 1$ such that*

$$\|V(x_i)V(x_j)a\| \leqslant K(\|\bar{\Delta}a\| + \|a\|), \quad a \in \mathscr{B}_V,$$

for all $i, j = 1, \ldots, d$;

4. $(I + \varepsilon\bar{\Delta})$ *is an isomorphism from $\mathscr{B}_2$ onto $\mathscr{B}$, for all $\varepsilon > 0$.*

Proof. Consider the operator

$$\Delta_1 = -\sum_{i=1}^{d} \overline{V(x_i)^2}.$$

Then Δ_1 is dissipative by the arguments used to deduce that $\bar{\Delta}$ is dissipative. But $\Delta_1 \supseteq \Delta$ and since a generator has no proper dissipative extension it follows that $\bar{\Delta}_1 = \bar{\Delta}$. Therefore $\bar{\Delta} \supseteq \Delta_1$. In particular $D(\bar{\Delta}) \supseteq D(\Delta_1) \supseteq \mathscr{B}_2$.

Now the proof of equivalence of the four conditions is essentially identical to the proof of the similar result in Theorem 1.1 but Lemma 2.3, applied to $\overline{V(x_i)}$, replaces Lemma 1.2. The estimates are also simplified by noting that the semigroup generated by $\bar{\Delta}$ is contractive, because $\bar{\Delta}$ is dissipative. In particular $(I + \varepsilon\bar{\Delta})^{-1}$ is well-defined, and contractive, for all $\varepsilon > 0$.

Next we consider some implications of the conditions of Proposition 3.4. If $x = \sum_{i=1}^{d} \lambda_i x_i \in \mathfrak{g}$ and $y = \sum_{i=1}^{d} \mu_i y_i \in \mathfrak{g}$ we define $|x| = \sum_{i=1}^{d} |\lambda_i|$ and $|y| = \sum_{i=1}^{d} |\mu_i|$.

LEMMA 3.5. *Assume the* $V(x), x \in \mathfrak{g}$, *are conservative and there is a* $K \geqslant 1$ *such that*

$$\|V(x_i)V(x_j)a\| \leqslant K(\|\Delta a\| + \|a\|), \quad a \in \mathscr{B}_V, i, j = 1, \ldots, d. \tag{3.1}$$

Then $D(\bar{\Delta}) \subseteq \mathscr{B}_2$ *and*

1. $\|\overline{V(x)}a\| \leqslant \delta(\|\bar{\Delta}a\| + \|a\|) + (2K|x|^2/\delta)\|a\|, \quad \delta > 0,$

2. $\|\overline{V(x)}\,\overline{V(y)}a\| \leqslant |x| \cdot |y| K(\|\bar{\Delta}a\| + \|a\|),$

3. $(\mathrm{ad}\,\overline{V(x)})(\overline{V(y)})a = \overline{V((\mathrm{ad}\,x)(y))}a,$

4. $\overline{V(\lambda x + \mu y)}a = \lambda\overline{V(x)}a + \mu\overline{V(y)}a, \quad \lambda, \mu \in \mathbf{R},$

for all $a \in D(\bar{\Delta})$ *and* $x, y \in \mathfrak{g}$.

Proof. Conditions 1 and 2 follow by continuity from (3.1) and Lemma 2.3. Then Conditions 3 and 4 follow by closure, using Conditions 1 and 2.

The main difference between Proposition 3.4 and Theorem 1.1 is that the conditions of the proposition do not necessarily imply that $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_3$ onto $\mathscr{B}_1$. But the next result gives criteria for this to be the case.

PROPOSITION 3.6. *Adopt the assumptions of Proposition* 3.4 *and further assume the four equivalent conditions of the proposition are satisfied. Then for each* $\varepsilon > 0$ *the following conditions are equivalent:*

1. $\mathscr{B}_1 = (I + \varepsilon\bar{\Delta})\mathscr{B}_3$;

2. $\mathscr{B}_3 = \bigcap_{i=1}^{d} D(\overline{V(x_i)}\bar{\Delta})$;

3. $\bigcap_{i=1}^{d} D(\bar{\Delta}\overline{V(x_i)}) = \bigcap_{i=1}^{d} D(\overline{V(x_i)}\bar{\Delta})$;

4. $(I + \varepsilon\bar{\Delta})$ *is an isomorphism from* $\mathscr{B}_3$ *onto* $\mathscr{B}_1$.

Moreover, if these conditions are satisfied then $D(\bar{\Delta}^2) \subseteq \mathscr{B}_3$ *and there is a* $K_1 \geqslant 0$ *such that*

$$\|(\mathrm{ad}\,\bar{\Delta})(\overline{V(x)})a\| \leqslant K_1(\|\bar{\Delta}a\| + \|a\|), \quad a \in D(\bar{\Delta}^2).$$

Proof. If $a \in \mathscr{B}_1$ then

$$\varepsilon\bar{\Delta}(I + \varepsilon\bar{\Delta})^{-1}a = a - (I + \varepsilon\bar{\Delta})^{-1}a \in \mathscr{B}_1$$

because $D(\bar{\Delta}) = \mathscr{B}_2 \subseteq \mathscr{B}_1$. Hence

$$(I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1 \subseteq \bigcap_{i=1}^{d} D(\overline{V(x_i)}\Delta).$$

Conversely if a is an element of the right hand set then $a = (I + \varepsilon\bar{\Delta})^{-1}b$, for some $b \in \mathscr{B}$, and $\bar{\Delta}a \in \mathscr{B}_1$. But

$$b = \varepsilon\bar{\Delta}a + (I + \varepsilon\bar{\Delta})^{-1}b \in \mathscr{B}_1$$

and hence

$$\bigcap_{i=1}^{d} D(\overline{V(x_i)}\bar{\Delta}) \subseteq (I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1.$$

Combining these conclusions gives

$$\bigcap_{i=1}^{d} D(\overline{V(x_i)}\bar{\Delta}) = (I + \varepsilon\bar{\Delta})^{-1}\mathscr{B}_1.$$

and this establishes that $1 \Leftrightarrow 2$.

$2 \Leftrightarrow 3$. Since $D(\bar{\Delta}) = \mathscr{B}_2$ one has

$$\bigcap_{k=1}^{d} D(\bar{\Delta}\overline{V(x_k)}) = \bigcap_{i,j,k=1}^{d} D(\overline{V(x_i)}\,\overline{V(x_j)}\,\overline{V(x_k)}) = \mathscr{B}_3$$

and therefore $2 \Leftrightarrow 3$.

$1 \Rightarrow 4$. It follows from Condition 3 of Proposition 3.4 and Condition 1 of Lemma 3.5 that

$$\|\overline{V(x_i)}a\| \leqslant \|(I + \varepsilon\bar{\Delta})a\| + \|a\|(1 + \varepsilon + 2K/\varepsilon), \quad a \in D(\bar{\Delta}),$$

and hence

$$\|\overline{V(x_i)}(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant (2 + \varepsilon + 2K/\varepsilon)\|a\|, \quad a \in \mathscr{B}. \tag{3.2}$$

Moreover, by Condition 2 of Lemma 3.5

$$\|\overline{V(x_i)}\,\overline{V(x_j)}a\| \leqslant K(\|(I + \varepsilon\bar{\Delta})a\|/\varepsilon + \|a\|(1 + 1/\varepsilon)), \quad a \in D(\bar{\Delta}),$$

and hence

$$\|\overline{V(x_i)}\,\overline{V(x_j)}(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant (2 + 1/\varepsilon)K\|a\|, \quad a \in \mathscr{B}. \tag{3.3}$$

Now $(I + \varepsilon\bar{\Delta})^{-1}$ maps $\mathscr{B}_1$ onto $\mathscr{B}_3$, by assumption. But if $a \in \mathscr{B}_1$ then

$$\overline{V(x_k)}(I + \varepsilon\bar{\Delta})^{-1}a = (I + \varepsilon\bar{\Delta})^{-1}\overline{V(x_k)}a + \varepsilon(I + \varepsilon\bar{\Delta})^{-1}(\mathrm{ad}\,\bar{\Delta})(\overline{V(x_k)})(I+\varepsilon\bar{\Delta})^{-1}a.$$

Hence if C_{ij}^k denote the structure constants of $\mathfrak{g}$ and one sets

$$C = \sup_k \sum_{i,j=1}^{d} |C_{ik}^j|$$

then

$$(3.4)\quad \|\overline{V(x_i)}\,\overline{V(x_j)}\,\overline{V(x_k)}(I + \varepsilon\bar{\Delta})^{-1}a\| \leqslant (2 + 1/\varepsilon)K(\|\overline{V(x_k)}a\| + 2\varepsilon C(2 + 1/\varepsilon)\|a\|.$$

Therefore, by (3.2), (3.3), and (3.4), there is a $K_\varepsilon \geqslant 1$ such that

$$\|(I + \varepsilon\bar{\Delta})^{-1}a\|_3 \leqslant K_\varepsilon\|a\|_1, \quad a \in \mathscr{B}_1.$$

Conversely if $a \in \mathscr{B}_3 \subseteq \mathscr{B}_2 = \mathrm{D}(\bar{\Delta})$ then

$$\|(I + \varepsilon\bar{\Delta})a\|_1 \leqslant (1 + \varepsilon d)\|a\|_3$$

by use of Condition 2 of Proposition 3.4. Therefore $(I + \varepsilon\bar{\Delta})^{-1}$ is an isomorphism from $\mathscr{B}_1$ onto $\mathscr{B}_3$.

$4 \Rightarrow 1$. This is evident.

Finally if the conditions are satisfied then

$$\mathrm{D}(\bar{\Delta}^2) \subseteq \bigcap_{k=1}^{d} \mathrm{D}(\overline{V(x_k)\bar{\Delta}}) = \mathscr{B}_3.$$

Moreover $(\mathrm{ad}\,\bar{\Delta})(\overline{V(x_i)})$ is defined on $\mathscr{B}_3$ and by Lemma 3.5 one has

$$(\mathrm{ad}\,\bar{\Delta})(\overline{V(x_i)})a = \sum_{j=1}^{d} (\overline{V(x_j)}\,\overline{V(x_{ij})} + \overline{V(x_{ij})}\,\overline{V(x_j)})a, \quad a \in \mathscr{B}_3,$$

where $x_{ij} = (\mathrm{ad}\,x_j)(x_i)$. Therefore the estimate on the commutator follows from Lemma 3.5.

Now we are prepared for the principal part of the proof of Theorem 3.1.

$1 \Rightarrow 3$. Since $\bar{\Delta}$ is dissipative and $(I + \varepsilon\bar{\Delta})\mathscr{B}_2 = \mathscr{B}$ it follows that the restriction of $\bar{\Delta}$ to $\mathscr{B}_2$ generates a continuous contraction semigroup, by the Hille-Yosida theorem. But since a generator has no strict dissipative extensions it follows that $\bar{\Delta}$ must equal its restriction to $\mathscr{B}_2$, i.e. $\mathrm{D}(\bar{\Delta}) = \mathscr{B}_2$.

Next $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 \supseteq \mathscr{B}_1$ by assumption. But since $\bar{\Delta}$ is a generator and $D(\bar{\Delta}) = \mathscr{B}_2$ it follows that

$$\bar{\Delta} = -\sum_{i=1}^{d} \overline{V(x_i)}^2$$

by Proposition 3.4. Therefore $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 \subseteq \mathscr{B}_1$, which implies $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 = \mathscr{B}_1$. But then

$$D(\bar{\Delta}^2) \subseteq \mathscr{B}_3 = \bigcap_{i=1}^{d} D(\overline{V(x_i)}\bar{\Delta})$$

by Proposition 3.6.

$1 \Rightarrow 2$. Condition 1 implies that $D(\bar{\Delta}) = \mathscr{B}_2$ and $\bar{\Delta}$ is a generator, by the above argument. Therefore the inequalities of Condition 2 follow from Proposition 3.4.

$2 \Rightarrow 1$. First, $\bar{\Delta}$ is dissipative and hence $R(I + \varepsilon\bar{\Delta})$ is closed. But $R(I + \varepsilon\bar{\Delta}) \supseteq \mathscr{B}_1 \supseteq \mathscr{B}_V$ which is dense. Therefore $R(I + \varepsilon\bar{\Delta}) = \mathscr{B}$ and $\bar{\Delta}$ generates a continuous contraction semigroup, by the Hille-Yosida theorem. Then $\mathscr{B}_2 = D(\bar{\Delta})$, by Proposition 3.4, and hence $(I + \varepsilon\bar{\Delta})\mathscr{B}_2 = \mathscr{B}$.

$3 \Rightarrow 4$. Since $\bar{\Delta}$ generates a continuous semigroup $\mathscr{B}_2 \subseteq D(\bar{\Delta})$, by Proposition 3.4. Hence $\mathscr{B}_2 = D(\bar{\Delta})$ and

$$\bigcap_{k=1}^{d} D(\bar{\Delta}\overline{V(x_k)}) = \bigcap_{i,j,k=1}^{d} D(\overline{V(x_i)}\,\overline{V(x_j)}\,\overline{V(x_k)}) = \mathscr{B}_3.$$

$4 \Rightarrow 6$. First we argue that the assumptions of Theorem 2.1 are fulfilled with $H = \bar{\Delta}$ and $K = \pm\overline{V(x)}$, for each $x \in \mathfrak{g}$.

Now it follows from Condition 4 and Proposition 3.4 that $\mathscr{B}_2 = D(\bar{\Delta})$, and the inequalities of Condition 2 are satisfied. Hence for each $\varepsilon > 0$ there is a C_ε such that

$$\|\overline{V(x)}a\| \leqslant \varepsilon\|\bar{\Delta}a\| + C_\varepsilon\|a\|, \quad a \in D(\bar{\Delta}),$$

by Lemma 3.5. Moreover since $D(\bar{\Delta}^2) \subseteq \mathscr{B}_3$, by assumption, it then follows from the last statement of Proposition 3.6 that there is a $K_1 \geqslant 0$ such that

$$\|(\mathrm{ad}\,\bar{\Delta})(\overline{V(x)})a\| \leqslant K_1(\|\bar{\Delta}a\| + \|a\|), \quad a \in D(\bar{\Delta}^2).$$

But if S denotes the semigroup generated by $\bar{\Delta}$ one has $S_t D(\bar{\Delta}^2) \subseteq D(\bar{\Delta}^2)$ for all $t \geqslant 0$ and hence

$$\|(\mathrm{ad}\,S_t)(\overline{V(x)})a\| = -\int_0^t ds\; S_{t-s}(\mathrm{ad}\,\bar{\Delta})(\overline{V(x)})S_s a, \quad a \in D(\bar{\Delta}^2).$$

Consequently

$$\|(\operatorname{ad} S_t)(\overline{V(x)})a\| \leqslant K_1 t(\|\bar{\Delta}a\| + \|a\|)$$

for all $a \in D(\bar{\Delta}^2)$ and $t > 0$. Then this bound extends to all $a \in D(\bar{\Delta})$ by continuity. Thus the three initial hypotheses of Theorem 2.1 are valid with $K = \pm\overline{V(x)}$ and $H = \bar{\Delta}$. Hence it follows from this theorem that each $\overline{V(x)}$, $x \in \mathfrak{g}$, generates a strongly continuous group of isometries $V_t^x = \exp\{-t\,\overline{V(x)}\}$ of $\mathscr{B}$.

Next since the equivalent conditions of Proposition 3.6 are fulfilled $\overline{V(x)}\mathscr{B}_3 \subseteq \subseteq D(\bar{\Delta})$ and in particular $\overline{V(x)}D(\bar{\Delta}^2) \subseteq D(\bar{\Delta})$. Hence the last conclusions of Theorem 2.1 are valid. Thus $V_t^x D(\bar{\Delta}) \subseteq D(\bar{\Delta})$ and $V^x|D(\bar{\Delta})$ is continuous with respect to the graph norm

$$a \in D(\bar{\Delta}) \to \|a\|_\Delta = \|\bar{\Delta}a\| + \|a\|.$$

Note that since $D(\bar{\Delta}) = \mathscr{B}_2$ and $(I + \varepsilon\bar{\Delta})$ is an isomorphism from $\mathscr{B}_2$ onto $\mathscr{B}$, by Conditions 1, 2 and 3, and Proposition 3.4, the norm $\|\cdot\|_\Delta$ is in fact equivalent to $\|\cdot\|_2$.

It remains to prove that the groups V^x, $x \in \mathfrak{g}$, define a representation of $\mathfrak{G}$. This is achieved by a standard line of reasoning (see, for example, [7], Chapters 8 and 9 or [4], Section 5) based on the following lemma.

LEMMA 3.7. *If $x, y \in \mathfrak{g}$ define $e_t(x; y) \in \mathfrak{g}$ by*

$$e_t(x; y) = \sum_{n \geqslant 0} \frac{(-t)^n}{n!} (\operatorname{ad} x)^n (y).$$

Then

$$V_t^x \overline{V(y)} V_{-t}^x a = \overline{V(e_t(x, y))}a, \quad a \in D(\bar{\Delta}).$$

Proof. For $t > 0$ define

$$D_t^x = (1 - V_t^x)/t - \overline{V(x)} = \overline{V(x)} \frac{1}{t} \int_0^t ds (V_s^x - 1).$$

Then for $a \in D(\bar{\Delta})$ it follows from Lemma 3.5 that

$$\|\overline{V(y)} D_t^x a\| \leqslant \frac{1}{t} \int_0^t ds \|\overline{V(x)}\,\overline{V(y)}(V_s^x - I)a\| \leqslant$$

$$\leqslant K|x| \cdot |y| \sup_{0 < s \leqslant t} \|(V_s^x - I)a\|_2.$$

Since V^x is continuous with respect to the norm $\|\cdot\|_2$ by the foregoing one concludes that

$$\lim_{t\to 0} \overline{V(y)}(I - V_t^x)a/t = \overline{V(y)}a, \quad a \in D(\bar{\Delta}).$$

Similarly

$$\lim_{t\to 0} (I - V_t^x)\overline{V(y)}a/t = \overline{V(x)}\,\overline{V(y)}a, \quad a \in D(\bar{\Delta}).$$

It then follows from these observations, the invariance of $D(\bar{\Delta})$ under V^x, and Lemma 3.5 that

$$\frac{d}{dt} V_t^x \overline{V(y)} V_{-t}^x a = - V_t^x \overline{V((\operatorname{ad} x)(y))} V_t^x a, \quad a \in D(\bar{\Delta}),$$

by yet another application of Lemma 3.5.

Finally define

$$F_t(s) = V_{t-s}^x \overline{V(e_s(x, y))} V_{-(t-s)}^x a$$

where $a \in D(\bar{\Delta})$. Then it follows from the foregoing that $dF_t(s)/ds = 0$. Therefore $F_t(0) = F_t(t)$. But this is exactly the statement of the lemma.

Now let $\mathcal{N}$ be an open neighbourhood of the origin in $\mathfrak{g}$ which is mapped differeomorphically onto an open neighbourhood $\mathcal{M}$ of the identity in $\mathfrak{G}$ by the exponential map. Thus for each $g \in \mathcal{M}$ there exists $x = \sum_{i=1}^{d} \lambda_i x_i \in \mathcal{N}$ such that $g = \exp\{x\}$. We then define $U(g) = V_1^x = \exp\{-\overline{V(x)}\}$. Our aim is to prove that this defines a representation of $\mathfrak{G}$. Thus if g, h, and gh, are in $\mathcal{M}$ we must prove that $U(g)U(h) = U(gh)$. Now we may assume that $\exp\{tx\} \in \mathcal{M}$ and $\exp\{tx\}h \in \mathcal{M}$ for all $t \in [0, 1]$. Then

$$\exp\{tx\}h = \exp\left\{\sum_{i=1}^{d} \lambda_i(t)x_i\right\} = \exp\{x(t)\}$$

where the coefficients λ_i are continuously differentiable in a neighbourhood of $[0, 1]$. Next for $a \in D(\bar{\Delta})$ define F by

$$F(t) = V_1^{x(t)}a = \exp\{-\overline{V(x(t))}\}a.$$

Since $V_s^{x(t)}D(\bar{\Delta}) = D(\bar{\Delta}) \subseteq D(\overline{V(y)})$ for all $y \in \mathfrak{g}$ it follows that

$$F(t) - F(s) = \int_0^1 d\lambda\; V_\lambda^{x(t)} \overline{V(x(s) - x(t))} V_{1-\lambda}^{x(s)} a$$

and hence

$$\mathrm{d}F(t)/\mathrm{d}t = -\int_0^1 \mathrm{d}\lambda\, V_\lambda^{x(t)}\overline{V(x'(t))}V_{-\lambda}^{x(t)}F(t) =$$

$$= -V\left(\left(\int_0^1 \mathrm{d}\lambda\, e_\lambda(x(t); x'(t))\right)\right) F(t),$$

by use of Lemmas 3.5 and 3.7. But it is a straightforward calculation, within $\mathfrak{g}$, to show that

$$\int_0^1 \mathrm{d}\lambda\, e_\lambda(x(t); x'(t)) = x.$$

Consequently $\mathrm{d}F(t)/\mathrm{d}t = \overline{V(x)}F(t)$. But this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(V_{-t}^x V_1^{x(t)}a) = 0, \quad a \in \mathrm{D}(\overline{\Delta}),$$

or, by integration from 0 to 1,

$$V^{x(1)}a = V_1^x V_1^{x(0)}a, \quad a \in \mathrm{D}(\overline{\Delta}).$$

Now in terms of U this gives

$$U(gh)a = U(g)U(h)a, \quad a \in \mathrm{D}(\overline{\Delta})$$

and hence, by continuity $U(g)U(h) = U(gh)$. Thus U is a representation.

Next strong continuity of U is evident and $V(x) \subseteq \mathrm{d}U(x)$, $x \in \mathfrak{g}$, by construction. In fact $\overline{V(x)} = \overline{\mathrm{d}U(x)}$. Moreover the construction also gives $U\mathrm{D}(\overline{\Delta}) = \mathrm{D}(\overline{\Delta})$ because $\mathrm{D}(\overline{\Delta}) = \mathscr{B}_2(V) = \mathscr{B}_2(U)$. But if W is a second representation with the property that $\mathrm{d}W(x) \supseteq V(x)$ then the generator of the one-parameter subgroup $t \to W(\mathrm{e}^{tx})$ must be a conservative extension of $\overline{V(x)}$. But $\overline{V(x)}$ is a generator and hence has no proper conservative extensions. Thus $W(\mathrm{e}^{tx}) = V_t^x = U(\mathrm{e}^{tx})$ and the representation U is unique.

5 $\Rightarrow$ 4. It follows from Proposition 3.4 that Δ is dissipative, $\mathscr{B}_2 = \mathrm{D}(\overline{\Delta})$, and the inequalities of Condition 2 are valid. Moreover

$$\|\overline{\Delta}a\| \leqslant \|\overline{\Delta}^2 a\| + 2\|a\|, \quad a \in \mathscr{B}_\infty,$$

by Lemma 2.3. Hence one calculates from these inequalities and Lemma 3.5 that there is an $L \geqslant 1$ such that

$$\|a\|_4 \leqslant L(\|\bar{\Delta}^2 a\| + \|a\|), \quad a \in \mathscr{B}_\infty .$$

Then since $\mathscr{B}_\infty$ is a core for $\bar{\Delta}^2$ one deduces that $D(\bar{\Delta}^2) \subseteq \mathscr{B}_4 \subseteq \mathscr{B}_3$.

$6 \Rightarrow 1$. First, let $\bar{\Delta}_{dU}$ denote the Laplacian defined on $\mathscr{B}_\infty(U)$ and $\bar{\Delta}_V$ the restriction of this operator to $\mathscr{B}_V$. Then the invariance property in Condition 6 states that $UD(\bar{\Delta}_V) = D(\bar{\Delta}_V)$. But $(\mathscr{B}, \mathfrak{G}, U)$ satisfies the equivalent conditions of Theorem 1.1. Hence applying Condition 3 of the theorem to $V \subseteq dU$ it follows that $D(\bar{\Delta}_V) \subseteq \subseteq \mathscr{B}_2(V) \subseteq \mathscr{B}_1(V)$ by Lemma 3.5. Thus for each $x \in \mathfrak{g}$ one has $D(\bar{\Delta}_V) \subseteq D(\overline{V(x)}) \subseteq \subseteq D(\overline{dU(x)})$. Next since $UD(\bar{\Delta}_V) = D(\bar{\Delta}_V)$ it follows that $D(\bar{\Delta}_V)$ is a core of each of the generators $\overline{dU(x)}$, $x \in \mathfrak{g}$. But $\overline{dU(x)} = \overline{V(x)}$ on $D(\bar{\Delta}_V)$. Consequently $\overline{dU(x)} = = \overline{V(x)}$, $x \in \mathfrak{g}$.

Our next aim is to prove that $\bar{\Delta}_V = \bar{\Delta}_{dU}$ and then Condition 1 follows from Theorem 1.1.

Let S denote the contraction semigroup generated by $\bar{\Delta}_{dU}$. It follows that

$$S_t = \int_G dg\, p_t(g) U(g)$$

where dg is the left invariant Haar measure and p a convolution semigroup satisfying the heat equation on $\mathfrak{G}$ (see [11], Section 8 for details). Therefore

$$(I + \bar{\Delta}_{dU})^{-1} = \int_G dg\, r(g) U(g)$$

where r is the Laplace transform of p,

$$r(g) = \int_0^\infty dt\, e^{-t} p_t(g).$$

Now since $UD(\bar{\Delta}_V) = D(\bar{\Delta}_V)$ it follows by a Riemann approximant argument (see, for example, [2], proof of Corollary 3.1.7) that $(I + \bar{\Delta}_{dU})^{-1} D(\bar{\Delta}_V) \subseteq D(\bar{\Delta}_V)$. But $D(\bar{\Delta}_V)$ is norm-dense, since it contains $\mathscr{B}_V$, and therefore $(I + \bar{\Delta}_{dU})^{-1} D(\bar{\Delta}_V)$ is a core for $\bar{\Delta}_{dU}$. Consequently $D(\bar{\Delta}_V)$ is a core for $\bar{\Delta}_{dU}$ and one concludes that $\bar{\Delta}_V = \bar{\Delta}_{dU}$. Thus $6 \Rightarrow 1$.

$6 \Rightarrow 5$. It follows as above that $\overline{dU(x)} = \overline{V(x)}$, $x \in \mathfrak{g}$ and $\bar{\Delta}_V = \bar{\Delta}_{dU}$. Then Condition 5 follows from Theorem 1.1 and the fact that $\mathscr{B}_{dU} = \mathscr{B}_\infty(U)$ is $\|\cdot\|_4$-dense in $\mathscr{B}_4(U) = D(\bar{\Delta}_{dU}^2)$ by a standard regularization argument.

The last statements of the theorem follow from the above considerations and Theorem 1.1.

Note that the argument occurring at the end of the proof of $6 \Rightarrow 1$ establishes the following. If $(\mathscr{B}, \mathfrak{G}, U)$ is a continuous representation of $\mathfrak{G}$ and Δ the Laplacian associated with some basis of $\mathfrak{g}$ then each norm-dense, U-invariant, subspace of $D(\bar{\Delta})$ is a core for $\bar{\Delta}$. It should be emphasized that U-invariance does not necessarily imply invariance under the semigroup generated by $\bar{\Delta}$.

Theorem 3.1 contains Nelson's Hilbert space result, [11], Theorem 5, as a special case.

COROLLARY 3.8. *Let $(\mathscr{H}_V, \mathfrak{g}, V)$ denote a representation of $\mathfrak{g}$ by skew-symmetric operators $V(x)$, $x \in \mathfrak{g}$, on the Hilbert space $\mathscr{H}$ and let Δ denote the symmetric Laplacian associated with a fixed basis of $\mathfrak{g}$.*

Then the following conditions are equivalent:

1. *Δ is essentially self-adjoint;*

2. *there exists a (unique) continuous unitary representation $(\mathscr{H}, \mathfrak{G}, U)$ of the simply connected Lie group $\mathfrak{G}$ having $\mathfrak{g}$ as its Lie algebra such that $\mathscr{H}_V \subseteq \mathscr{H}_{dU}$, $V(x) \subseteq dU(x)$, $x \in \mathfrak{g}$, and $UD(\bar{\Delta}) = D(\bar{\Delta})$.*

Proof. $1 \Rightarrow 2$. First, skew symmetry implies the $V(x)$, $x \in \mathfrak{g}$, are conservative. Second, self-adjointness and positivity of $\bar{\Delta}$ implies that it generates a continuous contraction semigroup. Third, by Lemma 1.7, Condition 3 of Proposition 3.4 is satisfied and hence $D(\bar{\Delta}) = \mathscr{H}_2$. Fourth, since isometry of a group representation on Hilbert space is equivalent to unitarity, the desired implication follows from $2 \Rightarrow 6$ of Theorem 3.1 once it is established that $(I + \varepsilon\bar{\Delta})\mathscr{H}_3 = \mathscr{H}_1$, $\varepsilon > 0$. But for this it suffices by Proposition 3.4 to verify the condition

$$\bigcap_{k=1}^{d} D(\overline{\overline{V(x_k)}\bar{\Delta}}) = \bigcap_{k=1}^{d} D(\bar{\Delta}(\overline{Vx_k})).$$

Now since $D(\bar{\Delta}) = \mathscr{H}_2$ the right hand set is equal to $\mathscr{H}_3$ and the left hand set contains $\mathscr{H}_3$. Hence for equality of the sets it suffices to prove the inclusion

$$\bigcap_{k=1}^{d} D(\overline{\overline{V(x_k)}\bar{\Delta}}) \subseteq \bigcap_{k=1}^{d} D(\bar{\Delta}\overline{V(x_k)}). \tag{3.5}$$

But this is achieved by the duality argument used to verify (1.7). Now one uses self--duality of $\mathscr{H}$ and sets $V_*(x) = V(x)$. Then one has $\Delta_* = \Delta$ and one replaces the Nelson-Stinespring-Langlands result by the self-adjointness property $\Delta^* = \bar{\Delta}$.

$2 \Rightarrow 1$. Since U is unitary the $dU(x)$, $x \in \mathfrak{g}$, are skew-symmetric and Lemma 1.7 establishes that the conditions of Theorem 1.1 are satisfied. Hence the desired result follows from the implication $6 \Rightarrow 3$ in Theorem 3.1.

The duality argument used to verify (1.7) and (3.5) can be used as the basis of an alternative integration result. We define the representation $(\mathscr{B}_V, \mathfrak{g}, V)$ to have a dual $(\mathscr{B}_V^*, \mathfrak{g}, V_*)$ if there exists a representation of $\mathfrak{g}$ by operators $V_*(x)$, $x \in \mathfrak{g}$, on a weak* dense invariant subspace $\mathscr{B}_V^*$ of the dual $\mathscr{B}^*$ satisfying the duality condition $V_*(x) \subseteq -V(x)^*$, $x \in \mathfrak{g}$. Then for a fixed basis $x_1, \ldots, x_d$ of $\mathfrak{g}$ one can define Laplacians Δ on $\mathscr{B}_V$ and Δ_* on $\mathscr{B}_V^*$ by

$$\Delta = -\sum_{i=1}^{d} V(x_i)^2, \quad \Delta_* = -\sum_{i=1}^{d} V_*(x_i)^2.$$

Since $V_*(x) \subseteq -V(x)^*$ it follows that $V_*(x)^2 \subseteq (V(x)^2)^*$ and consequently $\Delta_* \subseteq \Delta^*$.

COROLLARY 3.9. *Let $(\mathscr{B}_V, \mathfrak{g}, V)$ be a representation with a dual $(\mathscr{B}_V^*, \mathfrak{g}, V_*)$ and assume the $V(x)$, $x \in \mathfrak{g}$, are conservative. Further let Δ and Δ_* denote the Laplacians associated with a fixed basis $x_1, \ldots, x_d$ of $\mathfrak{g}$.*

Then the following conditions are equivalent:

1. $\mathscr{B}_2 \subseteq D(\bar{\Delta})$, $(I + \varepsilon\bar{\Delta})\mathscr{B}_2 = \mathscr{B}$ *for all (small)* $\varepsilon > 0$, *and* $\Delta^* = \bar{\Delta}_*$;

2. $\bar{\Delta}$ *generates a continuous semigroup,* $\Delta^* = \bar{\Delta}_*$, *and there is a* $K \geq 1$ *such that*

$$\|V(x_i)V(x_j)a\| \leq K(\|\Delta a\| + \|a\|), \quad a \in \mathscr{B}_V,\ i, j = 1, \ldots, d;$$

3. *there exists a unique isometric representation $(\mathscr{B}, \mathfrak{G}, U)$ of the simply connected Lie group having $\mathfrak{g}$ as its Lie algebra such that $\mathscr{B}_V \subseteq \mathscr{B}_{dU}$, $V(x) \subseteq dU(x)$, $x \in \mathfrak{g}$, $UD(\bar{\Delta}) = D(\bar{\Delta})$, and $(\mathscr{B}, \mathfrak{G}, U)$ satisfies the equivalent conditions of Theorem 1.1.*

Proof. $1 \Rightarrow 2$. It follows as in the proof of $1 \Rightarrow 3$ in Theorem 3.1 that $D(\bar{\Delta}) = \mathscr{B}_2$ and $\bar{\Delta}$ generates a continuous contraction semigroup. Hence the desired implication follows from Proposition 3.4.

$2 \Rightarrow 1$. It follows from Condition 2, by Proposition 3.4, that $D(\bar{\Delta}) = \mathscr{B}_2$. Hence $(I + \varepsilon\bar{\Delta})\mathscr{B}_2 = \mathscr{B}$ by the Hille-Yosida theorem.

$1 \Rightarrow 3$. It suffices by Theorem 3.1 to prove that $(I + \varepsilon\bar{\Delta})\mathscr{B}_3 \supseteq \mathscr{B}_1$ for all small $\varepsilon > 0$. But we have just argued that $D(\bar{\Delta}) = \mathscr{B}_2$ and $\bar{\Delta}$ generates a continuous semigroup and hence by Proposition 3.6, and the reasoning used to deduce $1 \Rightarrow 2$ in Corollary 3.8, it suffices to verify (3.5). But this is achieved by the same duality argument used to verify (1.7) using the hypothesis $\Delta^* = \bar{\Delta}_*$.

$3 \Rightarrow 1$. This follows from $6 \Rightarrow 1$ in Theorem 3.1 and the Nelson-Stinespring-Langlands results quoted at the beginning of the proof of Theorem 1.1.

Stronger duality results can of course be obtained if $\mathscr{B}$ is reflexive.

CORALLARY 3.10. *Let* $(\mathscr{B}_V, \mathfrak{g}, V)$ *be a representation with a dual* $(\mathscr{B}_V^*, \mathfrak{g}, V_*)$. *Assume that* $\mathscr{B}$ *is reflexive and that* $V(x)$ *and* $V_*(x)$, $x \in \mathfrak{g}$, *are conservative. Further let* Δ *and* Δ_* *denote the Laplacians associated with a fixed basis* $x_1, \ldots, x_d$ *of* $\mathfrak{g}$.
Then the following conditions are equivalent:

1'. $\Delta^* = \bar{\Delta}_*$ *and* $D(\bar{\Delta}) \subseteq \mathscr{B}_2$;

2'. $\Delta^* = \bar{\Delta}_*$ *and there is a* $K \geqslant 1$ *such that*

$$\|V(x_i)V(x_j)a\| \leqslant K(\|\Delta a\| + \|a\|), \quad a \in \mathscr{B}_V, \ i, j = 1, \ldots, d;$$

3. *there exists a (unique) isometric representation* $(\mathscr{B}, \mathfrak{G}, U)$ *of the simply connected Lie group having* $\mathfrak{g}$ *as its Lie algebra such that* $\mathscr{B}_V \subseteq \mathscr{B}_{dU}$, $V(x) \subseteq dU(x)$, $x \in \mathfrak{g}$, $UD(\bar{\Delta}) = D(\bar{\Delta})$, *and* $(\mathscr{B}, \mathfrak{G}, U)$ *satisfies the equivalent conditions of Theorem* 1.1.

Proof. Since both $V(x)$ and $V_*(x)$, $x \in \mathfrak{g}$, are conservative both $\bar{\Delta}$ and $\bar{\Delta}^*$ are dissipative by the initial argument in the proof of Theorem 3.1. But then by reflexivity of $\mathscr{B}$, the identification $\Delta_* = \bar{\Delta}_*$, and the Hille-Yosida theorem, it follows that $\bar{\Delta}$ generates a continuous contraction semigroup, and $\bar{\Delta}_*$ generates the dual semigroup. Then by the initial observation of Proposition 3.4 one has $\mathscr{B}_2 \subseteq D(\bar{\Delta})$ and the equivalence of Conditions 1 and 2 follows from Proposition 3.4. Moreover the equivalence of Conditions 2 and 3 follows from Corollary 3.9.

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