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PERTURBATION INEQUALITIES FOR THE ABSOLUTE VALUE MAP IN NORM IDEALS OF OPERATORS

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1. INTRODUCTION AND STATEMENTS OF RESULTS

Let $\mathscr{B}(\mathscr{H})$ be the space of bounded linear operators on a Hilbert space $\mathscr{H}$. For an element A of $\mathscr{B}(\mathscr{H})$ the *absolute value* of A is the positive operator $|A| = (A^*A)^{1/2}$. Estimates of the distance between $|A|$ and $|B|$ for two operators A and B have been obtained by several mathematicians and physicists. (See the papers by T. Kato [10], H. Araki and S. Yamagami [5], H. Borchers [8], H. Kosaki [12] and F. Kittaneh and H. Kosaki [11].)

In particular, improving upon the result of Borchers [8], Kosaki proved in [12] that if A, B belong to the ideal $\mathscr{I}_1$ of trace class operators (see [9], [17] or [18] for definitions of norm ideals of $\mathscr{B}(\mathscr{H})$ and their properties) then

$$\| |A| - |B| \|_1 \leqslant 2^{1/2}(\|A + B\|_1\|A - B\|_1)^{1/2}. \tag{1}$$

(Actually, both Borchers and Kosaki studied the continuity of the map $A \to |A|$ in the predual of a W^*-algebra. The above result is a special case.) Recently Kittaneh and Kosaki have proved that for A, B lying in the ideal $\mathscr{I}_p$, $p \geqslant 2$, we have

$$\| |A| - |B| \|_p \leqslant (\|A + B\|_p\|A - B\|_p)^{1/2}. \tag{2}$$

Our first theorem stated below completes these results by proving the analogous inequality for $1 \leqslant p \leqslant 2$.

THEOREM 1. *Let A, B be two operators lying in the ideal $\mathscr{I}_p$ of $\mathscr{B}(\mathscr{H})$, for any index $1 \leqslant p \leqslant 2$. Then*

$$\| |A| - |B| \|_p \leqslant 2^{1/p-1/2}(\|A + B\|_p\|A - B\|_p)^{1/2}. \tag{3}$$

This inequality is best possible for each index $1 \leqslant p \leqslant 2$.

Now recall that the Schatten p-norms $\|\cdot\|_p$ are members of the larger class of *unitarily-invariant* or *symmetric norms* [9], [17], [18]. Each such norm $|||\cdot|||$ is defined on an ideal $\mathcal{I}_{|||\cdot|||}$ of $\mathcal{B}(\mathcal{H})$ which is called *the norm ideal associated with* $|||\cdot|||$. Such a norm is unitarily invariant, in the sense that $|||UAV||| = |||A|||$ for all A and for all unitary operators U and V.

Our next result provides an analogous inequality for these norm ideals.

THEOREM 2. *Let A, B be two operators lying in the norm ideal $\mathcal{I}_{|||\cdot|||}$ associated with any symmetric norm $|||\cdot|||$. Then*

$$(4)\qquad |||\,|A|-|B|\,||| \leqslant 2^{1/2}(|||A+B|||\,|||A-B|||)^{1/2}.$$

Notice that while the inequality (1) is included in (4), the inequalities (2) and (3) for $p > 1$ are not. However, since the constant $2^{1/2}$ occuring in (1) can not be improved, the inequality (4) also can not be improved if all the symmetric norms are simultaneously involved. This raises the question of identifying interesting classes of symmetric norms where one can do better.

Call a symmetric norm a *Q-norm* and denote it by $\|\cdot\|_Q$ if there exists another symmetric norm $\|\cdot\|'_Q$ such that

$$(5)\qquad \|A\|_Q = (\|A^*A\|'_Q)^{1/2}.$$

One can see that all the Schatten p-norms, when $p \geqslant 2$ are Q-norms since for each $p \geqslant 1$ we have $\|A^*A\|_p^{1/2} = \|A\|_{2p}$. On the other hand when $1 \leqslant p < 2$ then the p-norms are not Q-norms. Examples of Q-norms which are not included in the Schatten p-class have been provided in [6]. In some recent papers [4], [6], [7], we have found that it is this quadratic character of the p-norms, for $p \geqslant 2$ which is crucial in deriving several inequalities concerning them. Our next theorem is a generalisation of (2) on these lines:

THEOREM 3. *Let A, B be any two operators belonging to the norm ideal associated with any Q-norm $\|\cdot\|_Q$. Then*

$$(6)\qquad \|\,|A|-|B|\,\|_Q \leqslant (\|A+B\|_Q\|A-B\|_Q)^{1/2}.$$

2. SOME PRELIMINARIES

Given any compact operator X denote by $s_1(X) \geqslant s_2(X) \geqslant \ldots$ the *singular values* of X, i.e. the eigenvalues of $|X|$. Any symmetric norm is a *symmetric gauge function* of the s_j, i.e. given such a norm $|||\cdot|||$ we can find a symmetric gauge function Φ defined on sequences of positive real numbers such that $|||X||| = \Phi(\{s_j(X)\})$.

For $0 < p \leqslant \infty$ define

$$(7)\qquad \|X\|_p = (\sum s_j^p(X))^{1/p},$$

where, by convention, $\|X\|_\infty = s_1(X)$. For $1 \leqslant p \leqslant \infty$, $\|X\|_p$ are the Schatten p-norms. For $0 < p < 1$, (7) does not define a norm but it is a unitarily invariant functional which is a "quasi-norm", in the sense that instead of the triangle inequality we have

$$(8) \qquad \|A + B\|_p \leqslant 2^{1/p-1}(\|A\|_p + \|B\|_p), \quad 0 < p < 1.$$

Further, we have the Hölder inequality

$$(9) \qquad \|AB\|_r \leqslant \|A\|_p \|B\|_q,$$

for $0 < p, q, r \leqslant \infty$ with $1/p + 1/q = 1/r$. (See e.g., Kosaki [12] or McCarthy [14].)

Note that for $t, p > 0$ we have

$$(10) \qquad \||X|^t\|_p = \|X\|_{tp}^t.$$

We will use the notion of *majorisation*, as in [2], [13] or [18]. If x and y are two vectors, finite or infinite dimensional, with nonnegative coordinates arranged in decreasing order, we say that x is *weakly majorised* by y, in symbols $x \prec_w y$ if for $k = 1, 2, \ldots$, $\sum_{j=1}^{k} x_j \leqslant \sum_{j=1}^{k} y_j$. Further, if $\sum x_j = \sum y_j$, in addition to the above inequalities, then we say that x is *majorised* by y, in symbols $x \prec y$.

Every symmetric gauge function is monotone with respect to weak majorisation, i.e. if $x \prec_w y$ then $\Phi(x) \leqslant \Phi(y)$. In particular, if $x_j \leqslant y_j$ for all j then $\Phi(x) \leqslant \Phi(y)$ for all symmetric gauge functions.

Using this last fact, the subadditivity of Φ and the ordinary arithmetic mean — — geometric mean inequality one can easily derive a Cauchy inequality for symmetric gauge functions:

$$(11) \qquad \Phi(x_1 y_1, x_2 y_2, \ldots) \leqslant (\Phi(x_1^2, x_2^2, \ldots)\Phi(y_1^2, y_2^2, \ldots))^{1/2}.$$

We now establish some analogues of (8) and (9) which we shall use in proving our results.

Given any unitarily invariant norm $|||\cdot|||$ define

$$(12) \qquad |||A|||_{1/2} = |||\,|A|^{1/2}\,|||^2.$$

This defines a unitarily invariant functional which is a quasi-norm in the sense:

PROPOSITION 4. *For every unitarily invariant norm* $|||\cdot|||$ *we have*

$$(13) \qquad |||A + B|||_{1/2} \leqslant 2(|||A|||_{1/2} + |||B|||_{1/2}).$$

Proof. The function $f(t) = t^{1/2}$ is a monotonically increasing concave function on $[0, \infty)$ with $f(0) = 0$. Hence by a result of S. Rotfel'd [16] (see also R. C. Thompson [19], T. Ando [2])

$$\{s_j^{1/2}(A + B)\} \prec_w \{s_j^{1/2}(A) + s_j^{1/2}(B)\}.$$

Hence, by its monotonicity with respect to weak majorisation and its subadditivity, every symmetric gauge function Φ satisfies the inequality

$$\Phi(\{s_j^{1/2}(A + B)\}) \leqslant \Phi(\{s_j^{1/2}(A)\}) + \Phi(\{s_j^{1/2}(B)\}).$$

Hence, using the convexity of the function $g(t) = t^2$ we get

$$(\Phi(\{s_j^{1/2}(A + B)\}))^2 \leqslant 2[(\Phi(\{s_j^{1/2}(A)\}))^2 + (\Phi(\{s_j^{1/2}(B)\}))^2]. \tag{14}$$

Since every unitarily invariant norm arises as a symmetric gauge function of singular values, the inequality (13) follows from (12) and (14). ▨

REMARK. Using (10) one can see that if the norm in question is $\|\cdot\|_p$ then the procedure (12) associates with it $\|\cdot\|_{p/2}$. When $p \geqslant 2$ the inequality (13) follows from the triangle inequality for $\|\cdot\|_{p/2}$. For $1 \leqslant p \leqslant 2$ the inequality (13) follows from (8). In either case the inequality (13) is weaker than these inequalities in these special cases.

We will also need the following noncommutative version of the Cauchy--Schwarz Inequality.

PROPOSITION 5. *For every unitarily invariant norm* $|||\cdot|||$ *and any two operators* A *and* B *we have*

$$|||\,|AB|^{1/2}|||\leqslant (|||A|||\;|||B|||)^{1/2}. \tag{15}$$

Proof. Given an operator X denote by $\Lambda^k X$ its kth exterior power (antisymmetric tensor power) for $k = 1, 2, \ldots$. If X has singular values $s_j(X)$ the singular values of $\Lambda^k X$ are the products $s_{i_1}(X) \ldots s_{i_k}(X)$ where $i_1, \ldots, i_k$ are any k distinct indices. Recall that $\|X\| = s_1(X)$. Hence for $0 < r < \infty$

$$\prod_{j=1}^{k} s_j^r(AB) = \|\Lambda^k(AB)\|^r \leqslant (\|\Lambda^k(A)\|\,\|\Lambda^k(B)\|)^r = \prod_{j=1}^{k} s_j^r(A)s_j^r(B). \tag{16}$$

In particular, choosing $r = 1/2$ and applying a frequently used lemma of Weyl and Pólya ([20], [15]) we get the majorisation

$$\{s_j^{1/2}(AB)\} \prec_w \{s_j^{1/2}(A)s_j^{1/2}(B)\}. \tag{17}$$

Once again, using the monotony of symmetric gauge functions with respect to weak majorisation, and then applying (11) we get

$$\Phi(\{s_j^{1/2}(AB)\}) \leqslant (\Phi(\{s_j(A)\})\Phi(\{s_j(B)\}))^{1/2}$$

for every symmetric gauge function Φ. Using the correspondence between unitarily invariant norms and symmetric gauge functions we get (15) from this. ▨

We should remark that the relations (16) and (17) are well known, going back to the work of Weyl [20].

3. PROOFS OF THE THEOREMS

We will use a result of Birman, Koplienko and Solomyak [21], also proved in Ando [3], according to which for any two positive operators A, B, for every $0 < t \leqslant 1$ and for every symmetric norm $|||\cdot|||$ we have

$$|||A^t - B^t||| \leqslant |||\,|A - B|^t\,|||. \tag{18}$$

As in Kosaki [12] we will make repeatedly use of the identity

$$A^*A - B^*B = \frac{1}{2}\{(A+B)^*(A-B) + (A-B)^*(A+B)\}, \tag{19}$$

valid for any two operators A and B.

Proof of Theorem 1. Use the inequality (18). For the special case of p-norms we get from this, using (10), for positive operators A, B

$$\|A^t - B^t\|_p \leqslant \|A - B\|_{t p}^t \quad \text{for} \quad 0 < t \leqslant 1, \quad 1 \leqslant p \leqslant \infty. \tag{20}$$

In particular, for positive operators A, B we have

$$\|A^{1/2} - B^{1/2}\|_p^2 \leqslant \|A - B\|_{p/2} \quad \text{for } 1 \leqslant p \leqslant \infty.$$

Hence, for any two operators A, B

$$\|\,|A| - |B|\,\|_p^2 \leqslant \|A^*A - B^*B\|_{p/2} \quad \text{for } 1 \leqslant p \leqslant \infty. \tag{21}$$

Using the identity (19) and the inequalities (8) and (9) we get

$$\|A^*A - B^*B\|_{p/2} \leqslant 2^{2/p-1}\|A + B\|_p\|A - B\|_p, \tag{22}$$

for $1 \leqslant p \leqslant 2$. The inequality (3) of Theorem 1 now follows from (21) and (22).

The example given by Kosaki [12], namely

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

also shows that the constant $2^{1/p-1/2}$ in (3) cannot be improved upon. ▨

To prove Theorem 2 we will use the "triangle inequality" for the absolute value: Given A, B there exist isometries U and V such that

$$(23) \qquad |A + B| \leqslant U|A|U^* + V|B|V^*.$$

For finite dimensions this result was proved by Thompson [19] and for infinite dimensions by Akemann, Anderson and Pedersen [1]. See also Kosaki [12].

Proof of Theorem 2. Once again using the inequality (18) we get for any two operators A, B

$$(24) \qquad \||A| - |B|\| \leqslant \||A^*A - B^*B|^{1/2}\|.$$

Using the identity (19) and the "triangle inequality" (23) we can find isometries U and V such that

$$(25) \qquad |A^*A - B^*B| \leqslant U\left|\frac{(A+B)^*(A-B)}{2}\right|U^* + V\left|\frac{(A-B)^*(A+B)}{2}\right|V^*.$$

Now recall that the function $t \to t^{1/2}$ is operator monotone on $[0, \infty)$, i.e. if $X \geqslant \geqslant Y \geqslant 0$ then $X^{1/2} \geqslant Y^{1/2} \geqslant 0$. Also every unitarily invariant norm is monotone in the sense that if $X \geqslant Y \geqslant 0$ then $\|X\| \geqslant \|Y\|$. (See, e.g. [13] for these facts). Hence, (25) leads to the inequality

$$(26) \qquad \begin{aligned} \||A^*A - B^*B|^{1/2}\|^2 \leqslant \Bigg\|\Bigg[& U\left|\frac{(A+B)^*(A-B)}{2}\right|U^* + \\ & + V\left|\frac{(A-B)^*(A+B)}{2}\right|V^*\Bigg]^{1/2}\Bigg\|^2. \end{aligned}$$

Using Propositions 4 and 5, and the fact that $\|TXS\| \leqslant \|T\| \|X\| \|S\|$ for any three operators T, X and S (see [9]) we get from (26)

$$(27) \qquad \begin{aligned} \||A^*A - B^*B|^{1/2}\|^2 & \leqslant 2\left\{\left\|\left|\frac{(A+B)^*(A-B)}{2}\right|^{1/2}\right\|^2 + \right. \\ & \left. + \left\|\left|\frac{(A-B)^*(A+B)}{2}\right|^{1/2}\right\|^2\right\} = \||(A+B)^*(A-B)|^{1/2}\|^2 + \\ & + \||(A-B)^*(A+B)|^{1/2}\|^2 \leqslant 2\|A+B\| \|A-B\|. \end{aligned}$$

The inequality (4) of Theorem 2 now follows from (24) and (27). ▧

Proof of Theorem 3. Let $\|\cdot\|_Q$ be any Q-norm and let $\|\cdot\|'_Q$ be the norm associated with it according to the relation (5). Specialising inequality (24) in this case we get using (5):

$$\| |A| - |B| \|_Q^2 \leqslant \|A^*A - B^*B\|'_Q. \tag{28}$$

Now using the identity (19) we get

$$\|A^*A - B^*B\|'_Q \leqslant \frac{1}{2}\{\|(A+B)^*(A-B)\|'_Q + \|(A-B)^*(A+B)\|'_Q\}. \tag{29}$$

Now note that for any two operators T and S we have, using (5) and Proposition 5

$$\|TS\|'_Q = \| |TS| \|'_Q = \| |TS|^{1/2} \|_Q^2 \leqslant \|T\|_Q \|S\|_Q.$$

Using this we get from (29)

$$\|A^*A - B^*B\|'_Q \leqslant \|A + B\|_Q \|A - B\|_Q. \tag{30}$$

The inequalities (28) and (30) together give the inequality (6) of Theorem 3. ▨

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Note added in proof. After this paper was submitted for publication the author learnt that Theorem 1 of this paper (and its generalisation to von Neumann algebras) has also been proved by H. Kosaki.