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INVERSE LIMITS OF C^*-ALGEBRAS

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INTRODUCTION

The purpose of this paper is to develop certain properties of inverse limits of C^*-algebras which are needed for the development of their representable K-theory in [30]. These algebras were first systematically studied in [17] as a generalization of C^*-algebras, and were called locally C^*-algebras. (Also see [48].) They have since been studied, under various names, in [36], [12], [13], [14], and elsewhere. Voiculescu introduced essentially equivalent objects, called pro-C^*-algebras, in [41], where he applied them to the construction of noncommutative analogs of various classical Lie groups. (An example of what is meant here appears in Example 1.3(8) of this paper.) Countable inverse limits were introduced in [8] under the name F^*-algebras. They were reintroduced by Arveson in [4] as σ-C^*-algebras, and were used there for the construction of the tangent algebra of a C^*-algebra.

We will follow Voiculescu (approximately) and Arveson, and call the objects we study pro-C^*-algebras and, in the case of countable inverse limits, σ-C^*-algebras. Our interest in them stems from the fact that the category of σ-C^*-algebras contains both C^*-algebras and objects corresponding to classifying spaces of compact Lie groups. It is also possible that the noncommutative analogs of loop spaces will be found among the pro-C^*-algebras.

The topics that we treat here are chosen because they are needed for the following application. In [30] and [31] we define representable K-theory for σ-C^*-algebras, and generalize the Atiyah-Segal completion theorem [6] to C^*-algebras. This theorem asserts that, if G is a compact Lie group, X is a compact G-space, and the equivariant K-theory $\mathrm{K}_G^*(X)$ (defined in [37]) is finitely generated over the representation ring $\mathrm{R}(G)$, then a certain completion $\mathrm{K}_G^*(X)^\wedge$ is naturally isomorphic to the representable K-theory $\mathrm{RK}^*((X \times EG)/G)$. Here EG is a contractible space on which G acts freely, and it cannot be replaced by the algebra of continuous functions vanishing at infinity on any locally compact space. However, a substitute for EG can be chosen in such a way that the analog of the functor $X \mapsto (X \times EG)/G$ sends

C^*-algebras to σ-C^*-algebras. Thus, we need enough information about σ-C^*-algebras to be able to define their representable K-theory.

Our original purpose for generalizing the Atiyah-Segal completion theorem was to obtain the following corollary, not involving σ-C^*-algebras, which will be proved in [31]: if $t \mapsto \alpha^{(t)}$ is a homotopy of actions of a compact Lie group G on a C^*-algebra A, and if $K_*(C^*(G, A, \alpha^{(0)}))$ and $K_*(C^*(G, A, \alpha^{(1)}))$ are both finitely generated as $R(G)$-modules, then for appropriate completions there is an isomorphism $K_*(C^*(G, A, \alpha^{(0)}))^\wedge \cong K_*(C^*(G, A, \alpha^{(1)}))^\wedge$. Here $C^*(G, A, \alpha^{(t)})$ is the crossed product C^*-algebra and the $R(G)$-module structure is as defined in Section 2.7 of [29]. (The result is false without the completions, as will be shown in [31].) Our proof makes essential use of the representable K-theory of certain σ-C^*-algebras. (One can obtain a weaker result without explicitly using representable K-theory or σ-C^*-algebras, but the proof is artificial and the result is not strong enough to prove, for instance, the nonexistence of homotopies of actions.) Thus, even in a problem only involving C^*-algebras we are led to introduce σ-C^*-algebras and their representable K-theory.

This paper is organized as follows. In Section 1 we present some basic definiti ons and propositions, and some examples. There is some overlap with the material of [17] and [36]. For completeness we state all of the results, but we give proofs only when they are shorter or when we improve the results. In Section 2 we give a new characterization of the commutative unital pro-C^*-algebras, and give counterexamples to several plausible conjectures related to this characterization. Section 3 is devoted to tensor products, limits, approximate identities, and multiplier algebras. Most of the material has not previously appeared, although an extensive treatment of tensor products from a different point of view is given in [13], and approximate identities are shown to exist in [17]. (Our proof is much shorter.) In Section 4 we take up Hilbert modules over inverse limits of C^*-algebras. These have not previously appeared in the literature, and the proofs are not quite as straightforward as those in Section 3. Finally, in Section 5 we restrict ourselves to σ-C^*-algebras, and prove for them several results, such as a stabilization theorem for countably generated Hilbert modules, which we were unable to prove for more general inverse limits.

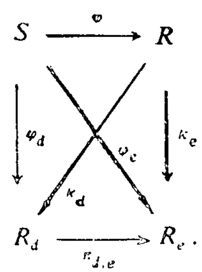
Since writing the first version of this paper, I have received a copy of the thesis of Jens Weidner [47]. Its first chapter contains a treatment of inverse limits of C^*-algebras quite similar to ours. Weidner goes on to define KK-theory for pro-C^*-algebras in a manner rather different from our approach in [30]. I am also grateful to Jens Weidner for detecting an error in our original description of the commutative pro-C^*-algebras.

1. BASIC PROPERTIES OF PRO-C^*-ALGEBRAS

Recall that an inverse system of rings consists of a directed set D, a ring R_d for each $d \in D$, and homomorphisms

$$\pi_{d,e} : R_d \to R_e$$

for all pairs $(d, e) \in D \times D$ such that $d \geqslant e$. These homomorphisms are required to satisfy $\pi_{d,d} = \mathrm{id}_{R_d}$ and $\pi_{e,f} \circ \pi_{d,e} = \pi_{d,f}$ for $d \geqslant e \geqslant f$. The inverse limit of this inverse system is a ring R equipped with homomorphisms $\varkappa_d : R \to R_d$ such that $\pi_{d,e} \circ \varkappa_d = \varkappa_e$ whenever $d, e \in D$ with $d \geqslant e$, and satisfying the following universal property in the category of rings. Given any ring S and homomorphisms $\varphi_d : S \to R_d$ satisfying $\pi_{d,e} \circ \varphi_d = \varphi_e$ for $d \geqslant e$, there exists a unique homomorphism $\varphi : S \to R$ magkin the following diagrams commute for $d \geqslant e$:



The inverse limit R, denoted by $\varprojlim R_d$, can be conveniently obtained as

$$R = \left\{ r \in \prod_{d \in D} R_d : \pi_{d,e}(r(d)) = r(e) \text{ for all } d, e \in D \text{ such that } d \geqslant e \right\}.$$

With this identification, $\varkappa_d$ simply becomes the restriction to R of the projection from $\prod_{e \in D} R_e$ to R_d. Observe that if each R_d is a topological ring, and if the maps $\pi_{d,e}$ are all continuous, then R is also a topological ring, with the restriction of the product topology, and the maps $\varkappa_d$ are continuous. In fact, this topology on R is the weakest such that the maps $\varkappa_d$ are all continuous, and R is the inverse limit of the system $\{R_d\}$ in the category of topological rings. We will refer to elements of R as coherent sequences $\{r_d\}$ (where $r_d \in R_d$ for $d \in D$) whenever it is convenient to do so.

We will occasionally also take inverse limits of modules, vector spaces, and abelian groups. Thus, we note that the results just stated for rings are also valid in these other categories. Furthermore, if $\{R_d\}$ is an inverse system of rings as above, and if $\{M_d\}$ is an inverse system of abelian groups such that each M_d is an R_d-module, and such that the maps $\sigma_{d,e} : M_d \to M_e$ satisfy $\sigma_{d,e}(rm) = \pi_{d,e}(r)\sigma_{d,e}(m)$ for $r \in R_d$ and $m \in M_d$, then $\varprojlim M_d$ is a $(\varprojlim R_d)$-module in a natural way, and the action is continuous.

The following definition is a way of singling out the inverse limits of C^*-algebras without specifying a particular system. (The inverse limit is unchanged, for example, if the directed set is replaced by a cofinal subset.)

1.1. Definition. A *pro-C^*-algebra* is a complete Hausdorff topological $*$-algebra over $\mathbf{C}$ whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 if and only if $p(a_\lambda) \to 0$ for every continuous C^*-seminorm p on A.

These objects are called locally C^*-algebras in [17] and LMC*-algebras in [36]. If the topology is determined by only countably many C^*-seminorms, then we have the σ-C^*-algebras of [4]. Closely related objects were called pro-C^*-algebras in [41]; the exact relation will be explained after Corollary 1.13.

We will use the following notation throughout this paper. If A is a pro-C^*-algebra, then $S(A)$ denotes the set of all continuous C^*-seminorms on A. For $p \in S(A)$, we let $\mathrm{Ker}(p)$ be the set $\{a \in A : p(a) = 0\}$, which is a closed ideal in A. (This notation is not quite standard because p is not a homomorphism.) We also let A_p be the completion of $A/\mathrm{Ker}(p)$ in the norm given by p, so that A_p is a C^*-algebra. (We will see in Corollary 1.12 that $A/\mathrm{Ker}(p)$ is in fact already complete.) For $a \in A$, we denote its image in A_p by a_p.

1.2. Proposition. *A topological $*$-algebra A is a pro-C^*-algebra if and only if it is the inverse limit, in the sense above, of an inverse system of C^*-algebras and $*$-homomorphisms. In this case, we have* $A \cong \varprojlim_{p \in S(A)} A_p$.

For the proof, see the remarks following Satz 1.1 in [36]. Note that $S(A)$ is directed with the order $p \leqslant q$ if $p(x) \leqslant q(x)$ for all x, and that there is a canonical surjective map $A_q \to A_p$ whenever $p \leqslant q$. One of the most useful consequences of this proposition is that every coherent sequence in $\{A_p : p \in S(A)\}$ determines an element of A.

The homomorphisms of pro-C^*-algebras are of course the continuous $*$-homomorphisms. Since $*$-homomorphisms need not be continuous (see Example 2.11), we adopt the convention throughout this paper that, unless otherwise specified, "homomorphism" means "continuous $*$-homomorphism".

1.3. Examples. (1) Any C^*-algebra is a pro-C^*-algebra.

(2) A closed $*$-subalgebra of a pro-C^*-algebra is again a pro-C^*-algebra.

(3) If X is a compactly generated space ([43], Section I.4), then $C(X)$, the set of all continuous complex-valued functions on X with the topology of uniform convergence on compact subsets, is a pro-C^*-algebra. (We should point out here that

Example 2.1(3) of [17] is wrong, since the algebras considered there need not be complete. See Example 2.12.)

(4) A product of C^*-algebras, with the product topology, is a pro-C^*-algebra.

(5) A σ-C^*-algebra ([4], page 255) is a pro-C^*-algebra. In particular, the tangent algebra defined there is a pro-C^*-algebra.

(6) Given any sets G of generators and R of relations, as in [7], satisfying the consistency condition but not necessarily the boundedness condition, there is a universal pro-C^*-algebra, which by abuse of notation we write $C^*(G, R)$, generated by the elements of G subject to the relations R. To construct it, let $F(G)$ be the free associative $*$-algebra on the set G. For any function $\rho : G \to L(H)$, where $L(H)$ is the algebra of bounded operators on some Hilbert space H, we also let ρ denote the extension to a $*$-homomorphism from $F(G)$ to $L(H)$. Then $C^*(G, R)$ is the Hausdorff completion of $F(G)$ for the topology given by the C^*-seminorms $a \mapsto \|\rho(a)\|$ as ρ runs through all functions from G to $L(H)$ such that the elements $\rho(g)$ satisfy the relations R in $L(H)$. This procedure can be shown to work for much more general relations than the ones considered in [7]. See [33] for more details.

(7) Associated to every pro-C^*-algebra as in [41] there is an inverse limit of C^*-algebras, and thus a pro-C^*-algebra in our sense. Thus, the category of pro-C^*-algebras contains various dual group algebras.

(8) We consider a specific example similar to but not the same as the examples in [41], namely the noncommutative infinite unitary group $U_{\mathrm{nc}}(\infty)$. It is the noncommutative analog of $\varinjlim U(n)$. Let $U_{\mathrm{nc}}(n)$ be the universal unital C^*-algebra generated by $\{x_{ij}\}_{i,j=1}^n$, subject to the relation that (x_{ij}) is a unitary element of $M_n(U_{\mathrm{nc}}(n))$. (These algebras were first introduced in [9].) Define a map $\pi_n : U_{\mathrm{nc}}(n+1) \to U_{\mathrm{nc}}(n)$ by $x_{ij} \mapsto x_{ij}$ for $1 \leqslant i, j \leqslant n$, $x_{n+1,n+1} \mapsto 1$, and $x_{ij} \mapsto 0$ when $i = n+1$ or $j = n+1$ but not both. Then $U_{\mathrm{nc}}(\infty)$ is defined to be $\varprojlim U_{\mathrm{nc}}(n)$.

(9) The multiplier algebra of the Pedersen ideal of a C^*-algebra (see [22]) is a pro-C^*-algebra. See [32] for details.

Our next goal is to define functional calculus in pro-C^*-algebras. For this, we need the unitization and the spectrum.

1.4. Definition. ([17], Theorem 2.3). Let A be a pro-C^*-algebra. Then its *unitization* A^+ is the vector space $A \oplus \mathbf{C}$, topologized as the direct sum and with adjoint and multiplication defined as for the unitization of a C^*-algebra. Note that A^+ is a pro-C^*-algebra, since $A^+ = \varprojlim A_p^+$.

1.5. Definition. Let A be a unital pro-C^*-algebra and let $a \in A$. Then the spectrum $\mathrm{sp}(a)$ of $a \in A$ is the set $\{\lambda \in \mathbf{C} : \lambda - a$ is not invertible$\}$. If A is not unital, then the spectrum is taken with respect to A^+.

Unlike in a C^*-algebra, the spectrum need be neither closed nor bounded. Indeed, if $S \subset \mathbf{C}$ is any nonempty subset, then $C(S)$ is a pro-C^*-algebra. (Note that S is metrizable, hence compactly generated by [43], I.4.3.) The identity function

$z : S \to \mathbf{C}$ is an element of $C(S)$ whose spectrum is S. However, the spectrum is always nonempty. Indeed, by examining coherent sequences, one obtains the following:

1.6. LEMMA. ([25], Corollary 5.3). *Let* $A = \varprojlim A_d$, *and suppose the maps* $\pi_{d,e} : A_d \to A_e$ *are all unital. Then for* $a \in A$, *we have* $\mathrm{sp}(a) = \bigcup_d \mathrm{sp}(\varkappa_d(a))$, *where* $\varkappa_d : A \to A_d$ *is the canonical map.*

A stronger result is found in Theorem 7.1 of [2]. In the case of a countable inverse limit, a further generalization is given in Theorem 4.2 of [3].

1.7. COROLLARY ([17], Corollary 2.2 and Proposition 2.1; also see [48], Theorem 3.1). *Let A be a pro-C^*-algebra, and let* $a \in A$. *Then:*

(1) *If a is selfadjoint, then* $\mathrm{sp}(a) \subset \mathbf{R}$.
(2) *If a has the form* b^*b, *then* $\mathrm{sp}(a) \subset [0, \infty)$.
(3) *If a is unitary, then* $\mathrm{sp}(a) \subset \{\lambda \in \mathbf{C} : |\lambda| = 1\}$.

Proof. This follows immediately from the lemma and the corresponding facts in C^*-algebras. Q.E.D.

1.8. PROPOSITION. ([17], Theorems 2.4 and 2.5; [48], Theorem 3.4). *Let A be a pro-C^*-algebra, and let* $a \in A$ *be normal, that is,* $a^*a = aa^*$. *Then there is a unique homomorphism from the pro-C^*-algebra* $\{f \in C(\mathrm{sp}(a)) : f(0) = 0\}$ *to A which sends the identity function to a. If A is unital, then this map extends uniquely to a homomorphism from* $C(\mathrm{sp}(a))$ *to A which sends* 1 *to* 1.

Proof. The required map is the one sending f to the coherent sequence

$$\{f(a_p) : p \in S(A)\}.$$

The proof that it satisfies the required properties is easy. Q.E.D.

We write, of course, $f(a)$ for the image of f under this map.

In the same manner, we obtain holomorphic functional calculus for arbitrary elements of a pro-C^*-algebra. For convenience, we state only the unital case. If $U \subset \mathbf{C}$ is open, then we let $H(U)$ denote the set of all holomorphic functions from U to $\mathbf{C}$, with the topology of uniform convergence on compact subsets.

1.9. PROPOSITION. *Let A be a unital pro-C^*-algebra, let* $a \in A$, *and let* $U \subset \mathbf{C}$ *be an open set containing* $\mathrm{sp}(a)$. *Then there exists a unique continuous unital homomorphism* $f \mapsto f(a)$ *from* $H(U)$ *to A sending the identity function to a. This homomorphism satisfies* $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$.

Of course, in this situation, $f \mapsto f(a)$ is not a $*$-homomorphism. Also, it is perfectly permissible to take $U = \mathrm{sp}(a)$ if $\mathrm{sp}(a)$ happens to be open.

1.10. DEFINITION. (Compare [36], Satz 3.1). Let A be a pro-C^*-algebra. Then the set of *bounded elements* of A is the set

$$\mathrm{b}(A) = \{a \in A : \|a\|_\infty = \sup\{p(a) : p \in S(A)\} < \infty\}.$$

1.11. PROPOSITION. *Let A be a pro-C^*-algebra. Then:*

(1) $\mathrm{b}(A)$ *is a C^*-algebra in the norm* $\| \ \|_\infty$.

(2) *If $a \in A$ is normal and $f \in C(\mathrm{sp}(a))$ is bounded, then $f(a) \in \mathrm{b}(A)$.*

(3) *If $a \in A$ is normal then $a \in \mathrm{b}(A)$ if and only if $\mathrm{sp}(a)$ is bounded.*

(4) $\mathrm{b}(A)$ *is dense in A.*

(5) *For $a \in \mathrm{b}(A)$, we have* $\mathrm{sp}_{\mathrm{b}(A)}(a) = \overline{\mathrm{sp}_A(a)}$.

(6) *If $q \in S(A)$, then the map from $\mathrm{b}(A)$ to A_q is surjective.*

Proof. (1) See [36], Satz 3.1.

(2) We have $p(f(a)) \leqslant \sup_{\lambda \in \mathrm{sp}(a)} |f(\lambda)|$ for all $p \in S(A)$.

(3) We have $\|a\|_\infty = \sup_{p \in S(A)} p(a) = \sup_{\lambda \in \mathrm{sp}(A)} |\lambda|$ because $\mathrm{sp}(a) = \bigcup_{p \in S(A)} \mathrm{sp}(a_p)$ and each a_p is normal.

(4) This is [36], Satz 3.6. However, a shorter proof is as follows. By considering the decomposition into real and imaginary parts, it is enough to prove that the selfadjoint part of $\mathrm{b}(A)$ is dense in the selfadjoint part of A. In [36], it is proved that for $x \in A$ selfadjoint, there is a net $\{x_\alpha\}$ in $\mathrm{b}(A) \cap \{x\}''$ (second commutant) converging to x. We produce a *sequence* $\{x_n\}$ in $\mathrm{b}(A) \cap \{x\}''$ which converges to x, by setting $x_n = f_n(x)$, where

$$f_n(\lambda) = \begin{cases} -n & \lambda \leqslant -n \\ \lambda & -n < \lambda < n \\ n & n \leqslant \lambda. \end{cases}$$

(5) Since $\mathrm{sp}_{\mathrm{b}(A)}(a)$ is closed and contains $\mathrm{sp}_A(a)$, the inclusion $\overline{\mathrm{sp}_A(a)} \subset \mathrm{sp}_{\mathrm{b}(A)}(a)$ is immediate. For the reverse inclusion, note that if the distance from λ to $\mathrm{sp}_A(a)$ is $\varepsilon > 0$, then $p((\lambda - a)^{-1}) \leqslant 1/\varepsilon$ for all $p \in S(A)$.

(6) This follows immediately from (4), as is shown in [36] in the remark after Folgerung 5.4. (Note that there A_p means the algebra $A/\mathrm{Ker}(p)$ before being completed.) Q.E.D.

1.12. COROLLARY. ([36], Folgerung 5.4). *For $p \in S(A)$, the map $A \to A_p$ is surjective, that is, $A/\mathrm{Ker}(p)$ is complete.*

1.13. COROLLARY. (Compare [36], Folgerung 3.3). *Let $\varphi : A \to B$ be a $*$-homomorphism (not necessarily continuous) between pro-C^*-algebras. Then φ defines a homomorphism from $\mathrm{b}(A)$ to $\mathrm{b}(B)$.*

Proof. Taking unitizations, we may assume that φ is unital. Then for any $a \in A$ we have $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$. If a is selfadjoint, then so is $\varphi(a)$, so $\varphi(a)$ is bounded by Proposition 1.11 (3). Now use the decomposition into real and imaginary parts. Q.E.D.

We note that this result cannot be used to prove that every homomorphism is continuous. In fact, in Example 2.11 below, we will produce a discontinuous homomorphism by exhibiting a pro-C^*-algebra A such that $\mathrm{b}(A) = A$ as sets but the topologies are different.

We can now explain how our pro-C^*-algebras are equivalent to those of [41]. If A is one of our pro-C^*-algebras, then for any cofinal subset D of $S(A)$, the pair $(\mathrm{b}(A), D)$ is a pro-C^*-algebra as in [41], while if (B, D) is a pro-C^*-algebra as in [41], D being a directed system of C^*-seminorms on B whose supremum is the norm on B, then

$$A = \varprojlim_{p \in D} B/\mathrm{Ker}(p)$$

is a pro-C^*-algebra in our sense, and satisfies $\mathrm{b}(A) = B$. Also, note that if $\{A_d\}$ is an inverse system of C^*-algebras, then $\mathrm{b}(\varprojlim A_d)$ is the inverse limit of $\{A_d\}$ in the category of C^*-algebras (as opposed to the inverse limit in the category of topological algebras, which is what we have designated $\varprojlim A_d$).

We also note that the term "bounded elements" is justified by looking at $\mathrm{b}(A)$ for some of the examples consider earlier. For example, if X is compactly generated, then $\mathrm{b}(C(X))$ is the algebra $C_{\mathrm{b}}(X)$ of all bounded continuous functions on X. If A is a product $\prod_{i \in I} A_i$, then $\mathrm{b}(A)$ is the ℓ^∞ sum of the A_i, consisting of all $a \in \prod_{i \in I} A_i$ such that $\sup\{\|a_i\| : i \in I\} < \infty$.

Recall that a unital topological algebra is called a Q-algebra if its group of invertible elements is open, and that a nonunital topological algebra is a Q-algebra if its unitization is a Q-algebra. In some papers on pro-C^*-algebras (notably [23]), it is frequently assumed that the pro-C^*-algebras in question are also Q-algebras. Therefore we include the following proposition. (This result has already been noticed by Mallios — see [15].)

1.14. PROPOSITION. *A pro-C^*-algebra A is a Q-algebra if and only if it is isomorphic, as a topological $*$-algebra, to a C^*-algebra.*

Proof. It is well known that Banach algebras are Q-algebras. So let A be a pro-C^*-algebra which is also a Q-algebra. We may assume that A is unital. Since the group of invertible elements is open, there is $p \in S(A)$ and $\varepsilon > 0$ such that the set $U = \{a \in A : p(a - 1) < \varepsilon\}$ consists entirely of invertible elements. Let $a \in \mathrm{Ker}(p)$, and suppose that $a \neq 0$. Then there is $q \in S(A)$ such that $a_q \neq 0$, whence $a_q^* a_q \neq 0$. Using Lemma 1.6, we see that there is a positive real number $\lambda \in \mathrm{sp}(a^*a)$

Therefore $1 - \lambda^{-1}a^*a$ is not invertible. However, $1 - \lambda^{-1}a^*a \in U$ since $p(\lambda^{-1}a^*a) = 0$. This is a contradiction and it follows that $\mathrm{Ker}(p) = \{0\}$.

Now let $q \in S(A)$, and suppose that $q \geqslant p$. Then there is a surjective map $A_q \to A_p$. Since $A \to A_q$ is surjective (by Corollary 1.12), while $A \to A_p$ is injective (because $\mathrm{Ker}(p) = \{0\}$), we see that $A_q \to A_p$ is injective as well. Therefore it is an isometry (because A_q and A_p are C^*-algebras), whence $q = p$. Since $S(A)$ is directed, we conclude that $p \geqslant q$ for all $q \in S(A)$. Consequently the map $A \to A_p$, which is already known to be continuous and bijective, has a continuous inverse. So A is isomorphic, as a topological $*$-algebra, to the C^*-algebra A_p. Q.E.D.

It follows that the "complete locally m-convex QC^*-algebras" of [23] and the "Waelbroeck C^*-algebras" of [24] are exactly the C^*-algebras.

2. COMMUTATIVE PRO-C^*-ALGEBRAS

In this section, we consider the commutative unital pro-C^*-algebras. The results in [17] (Section 4) and [36] (Satz 1.1) are useful representations of commutative pro-C^*-algebras, but they give us no convenient way of determining what all of the commutative pro-C^*-algebras are. Using the notion of a quasitopological space, due to Spanier [46], we obtain a much more satisfactory result, namely that a certain functor analogous to $X \mapsto C(X)$ is a contravariant category equivalence. We begin by recalling the definition.

2.1. DEFINITION. ([46]). A *quasitopology* on a set X is an assignment to each compact Hausdorff space K of a set $Q(K, X)$ of functions from K to X such that the following conditions hold:

(1) $Q(K, X)$ contains all constant functions from K to X.

(2) If $f : K_1 \to K_2$ is continuous, and $g \in Q(K_2, X)$, then $g \circ f \in Q(K_1, X)$.

(3) If K is the disjoint union of compact Hausdorff spaces K_1 and K_2, then $f \in Q(K, X)$ whenever $f|K_i \in Q(K_i, X)$ for $i = 1, 2$.

(4) If $f : K_1 \to K_2$ is continuous and surjective, and if $g : K_2 \to X$ is a function such that $g \circ f \in Q(K_1, X)$, then $g \in Q(K_2, X)$.

If X and Y are quasitopological spaces, that is, sets equipped with quasitopologies, then a function $h : X \to Y$ is said to be *quasicontinuous* if for every compact Hausdorff space K and every $f \in Q(K, X)$, the function $h \circ f$ is in $Q(K, Y)$.

Any topological space X can be made into a quasitopological space by letting $Q(K, X)$ be the set of all continuous functions from K to X. Thus, it makes sense to speak of a quasicontinuous function from, for example, a quasitopological space X to a topological space Y. We remark that, as observed in Section 11 of [38], the compactly generated spaces then become a full subcategory of the category of quasitopological spaces and quasicontinuous functions.

The spaces relevant to the study of pro-C^*-algebras are given in the following definition:

2.2. DEFINITION. A (quasi-)topological space X is called *completely Hausdorff* if for any two distinct points $x, y \in X$ there is a (quasi-)continuous function $f : X \to$ $\to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

This condition is stronger than being Hausdorff and weaker than complete regularity, even among the compactly generated topological spaces — see Examples 2.13 and 2.14.

2.3. DEFINITION. Let X be a quasitopological space. Then $C(X)$ denotes the $*$-algebra of all quasicontinuous functions $f : X \to \mathbf{C}$, with the topology determined by the seminorms $\|f\|_{K,g} = \|f \circ g\|_\infty$ for K compact Hausdorff and $g \in Q(K, X)$.

2.4. LEMMA. *If X is a quasitopological space then $C(X)$ is a pro-C^*-algebra.*

Proof. The only issue is completeness. So let $\{f_\alpha\}$ be a Cauchy net in $C(X)$. For each $x \in X$ the inclusion $\{x\} \to X$ is in $Q(\{x\}, X)$, whence f_α converges pointwise to a function $f : X \to \mathbf{C}$. If now $g \in Q(K, X)$, then $f_\alpha \circ g$ must converge uniformly to some $f^{(g)} \in C(K)$, and clearly $f^{(g)} = f \circ g$. It follows that f is quasicontinuous, and that $f_\alpha \to f$ in $C(X)$. Q.E.D.

Our main result is that $X \mapsto C(X)$, restricted to the full subcategory of completely Hausdorff quasitopological spaces, defines a contravariant category equivalence. (The claim made in an earlier version of this paper, that one could restrict to completely Hausdorff compactly generated spaces, is false, as was pointed out to us by Jens Weidner. See Example 2.11.) In proving our result, it is useful to introduce the following category of compactly generated spaces with distinguished families of compact subsets. As a byproduct of our proof, we then obtain a more concrete description of the completely Hausdorff quasitopological spaces.

2.5. DEFINITION. Let X be a topological space. A *distinguished family of compact subsets of X* is a set F of compact subsets of X satisfying the following properties:

(1) Every one point subset of X is in F.

(2) A compact subset of an element of F is in F.

(3) The union of two elements of F is in F.

(4) The family F determines the topology of X, that is, a subset $C \subset X$ is closed if and only if $C \cap K$ is closed for all $K \in F$.

If (X_1, F_1) and (X_2, F_2) are topological spaces with distinguished families of compact subsets, then a *morphism* from (X_1, F_1) to (X_2, F_2) is a continuous function $f : X_1 \to X_2$ such that $f(K) \in F_2$ for every $K \in F_1$.

2.6. PROPOSITION. *The category of completely Hausdorff spaces with distinguished families of compact subsets is equivalent to the category of completely Haus-*

dorff quasitopological spaces, via the functor assigning to (X, F) *the quasitopology given by*

$$Q_F(K, X) = \{f : K \to X : f \text{ is continuous and } f(K) \in F\}.$$

Furthermore, under the correspondence of this functor, a function from X *to a topological space is continuous if and only if it is quasicontinuous.*

Proof. We first observe that the statement of the theorem defines a functor. The sets $Q_F(K, X)$ satisfy conditions (1) through (3) of Definition 2.1 by the corresponding conditions of Definition 2.5, and they satisfy (4) for the same reason that the quasitopology defined by a topology satisfies (4). If $f : X_1 \to X_2$ is continuous and $f(K) \in F_2$ for $K \in F_1$, then it is immediate that f is quasicontinuous.

We now construct an inverse to this functor. Let X be a completely Hausdorff quasitopological space. Define a topology on X by declaring $U \subset X$ to be open if for every compact Hausdorff space K and every $g \in Q(K, X)$, the set $g^{-1}(U)$ is open in K. It is immediate that this does in fact define a topology on X, and that each $Q(K, X)$ consists of functions which are continuous with respect to this topology. Furthermore, it is easily verified that if $f : X \to Y$ is any function to a topological space Y, then f is continuous if and only if f is quasicontinuous. In particular, X is completely Hausdorff in this topology.

We now define F_X to be the set of all ranges of elements of the sets $Q(K, X)$. These ranges are all compact because the elements of $Q(K, X)$ are continuous. Conditions (1) and (3) of Definition 2.5 follow from the corresponding conditions of Definition 2.1, and 2.5 (2) is obtained by using the fact that compact subsets of X are closed and applying 2.1 (2) to an appropriate inclusion map. To verify 2.5 (4), let $C \subset X$, and suppose that $C \cap g(K)$ is closed whenever $g \in Q(K, X)$. Then $g^{-1}(C)$ is closed whenever $g \in Q(K, X)$, whence C is closed by the definition of the topology on X. This completes the verification that F is a distinguished family of compact subsets of X.

To complete the definition of the inverse functor, we look at morphisms. Thus let $f : X_1 \to X_2$ be quasicontinuous. Then for $g \in Q(K, X_1)$, the function $f \circ g$ is in $Q(K, X_2)$ and is hence continuous. It follows that f is quasicontinuous when X_1 is regarded as a quasitopological space and X_2 as a topological space. Therefore f is continuous. It is obvious that f sends ranges of elements of $Q(K, X_1)$ to ranges of elements of $Q(K, X_2)$.

It remains to prove that our two functors really are inverse to each other. If one starts with a space X with a distinguished family of compact subsets, then the topology and family of compact subsets obtained from the associated quasitopology are the same as the original topology and distinguished family, using condition 2.5 (4) (for the topology) and the fact that X is Hausdorff (for the distinguished family).

For the composition in the other order, it is necessary to show that if X is a quasitopological space, then $Q_{F_X}(K, X) = Q(K, X)$ for every compact Hausdorff space K. It is only necessary to prove that $Q_{F_X}(K, X) \subset Q(K, X)$. So let $g \in Q_{F_X}(K, X)$. Then there is a compact Hausdorff space L and a function $f \in Q(L, X)$ such that $g(K) = f(L)$. Writing $g = i \circ g_0$, where i is the inclusion of $g(K)$ in X and $g_0 : K \to g(K)$ is the obvious surjection, we see by 2.1 (2) that it suffices to show that $i \in Q(g(K), X)$. Thus, we may assume that g is surjective, and in fact a homeomorphism onto its image. Therefore $h = g^{-1} \circ f : L \to K$ is a continuous surjective map such that $g \circ h \in Q(L, X)$. So $g \in Q(K, X)$ by 2.1 (4), as desired. Q.E.D.

As in the preceding proof, we will write (X, F_X) for the topological space with distinguished family of compact subsets determined by the quasitopological space X.

2.7. THEOREM. *The functor $X \mapsto C(X)$ is a contravariant category equivalence from the category of completely Hausdorff quasitopological spaces to the category of commutative unital pro-C^*-algebras and unital homomorphisms.*

Of course, if $f : X_1 \to X_2$ is quasicontinuous, then $C(f) : C(X_2) \to C(X_1)$ is the homomorphism given by $C(f)(h) = h \circ f$.

For the proof of this theorem we need a lemma.

2.8. LEMMA. *Let X be a completely Hausdorff topological space, and let F be a family of compact subsets of X satisfying the first three conditions of Definition 2.5. Then for any compact set $L \notin F$, there exists a net of continuous functions on X which converges uniformly to 0 on the members of F and does not converge uniformly on L.*

Proof. Let $K \in F$, and choose a point $x \in L - K$. Because X is completely Hausdorff, there is for every $y \in K$ a continuous function $f_y : X \to [0, 1]$ such that $f_y(x) = 1$ and $f_y(y) = 0$. Composing f_y with a continuous function from $[0, 1]$ to $[0, 1]$ which sends 1 to 1 and vanishes on a neighborhood of 0, we may assume that f_y vanishes on a neighborhood of y. Since K is compact, the infimum of an appropriate finite subcollection of the functions f_y will be a continuous function $h_K : X \to [0, 1]$ which vanishes on K and is equal to 1 at $x \in L$. The set F is directed with respect to inclusion (by 2.5 (3)), so $\{h_K\}_{K \in F}$ is the required net. Q.E.D.

Proof of Theorem 2.7. Here also we need an inverse functor. It assigns to a commutative unital pro-C^*-algebra A the space $\Phi(A)$ of all (continuous) homomorphisms from A to $\mathbb{C}$. If K is a compact Hausdoff space, then $Q(K, \Phi(A))$ is taken to be the set of all functions $g : K \to \Phi(A)$ such that the formula $\varphi_g(a)(x) = g(x)(a)$, for $a \in A$ and $x \in K$, defines a (continuous) homomorphism from A to $C(K)$. Properties 2.1 (1) through 2.1 (3) of a quasitopology are immediate. For property 2.1 (4), let $f : K_1 \to K_2$ and $g : K_2 \to \Phi(A)$ be as in 2.1 (4). Let $\psi : C(K_2) \to C(K_1)$ be given by $\psi(h) = h \circ f$. Then ψ is a homeomorphism onto its image.

The relations $\varphi_{g\circ f} = \psi \circ \varphi_g$ and $g\circ f \in Q(K_1, X)$ now imply that $g \in Q(K_2, X)$, as desired. Also note that for every $a \in A$, the function $x \mapsto x(a)$ from $\Phi(A)$ to $\mathbf{C}$ is quasicontinuous. It follows that $\Phi(A)$ is completely Hausdorff. To complete the construction, observe that if $\psi : A_1 \to A_2$ is a (continuous) unital homomorphism, then the function $x \mapsto x \circ \psi$, from $\Phi(A_2)$ to $\Phi(A_1)$, is quasicontinuous.

We now prove that these two functors are inverses of each other. This will be done using Proposition 2.6. Let A be a commutative unital pro-C^*-algebra. For each continuous C^*-seminorm p on A, let $\Phi(A_p)$ have the usual weak* topology and identify it in the obvious way with a subset of $\Phi(A)$. Then $\Phi(A) = \bigcup_{p\in S(A)} \Phi(A_p)$. Let $\Phi(A)$ have the direct limit topology, and set $F = \{\Phi(A_p) : p \in S(A)\}$. Then it is easy to show that $(\Phi(A), F) = (\Phi(A), F_{\Phi(A)})$. We must therefore prove that the obvious map from A to the continuous functions on $\Phi(A)$, with the topology of uniform convergence on members of F, is an isomorphism of pro-C^*-algebras. This is equivalent to the assertion that $A \cong \varprojlim C(\Phi(A_p))$ via the obvious map, and follows from the natural isomorphism $C(\Phi(A_p)) \cong A_p$ and Proposition 1.2. This proves that the composite of our functors in one order is the identity.

For the other order, we let X be a completely Hausdorff quasitopological space. Topologize $\Phi(C(X))$ in the manner of the previous paragraph, and let F be the corresponding distinguished family of compact sets. We must show that the map sending x to the evaluation ev_x at x determines an isomorphism from (X, F_X) to $(\Phi(C(X)), F)$. The injectivity of $x \mapsto \mathrm{ev}_x$ follows from the fact that X is completely Hausdorff. For surjectivity, let $\alpha : C(X) \to \mathbf{C}$ be a homomorphism. Then there is K and $g \in Q(K, X)$ such that $|\alpha(h)| \leqslant \|h\|_{K,g} = \|h\circ g\|_\infty$ for all $h \in C(X)$. It follows that α defines a homomorphism from $C(K)$ to $\mathbf{C}$, which must be ev_y for some $y \in K$. Then $\alpha = \mathrm{ev}_{g(y)}$.

For $g \in Q(K, X)$, we clearly have $\{\mathrm{ev}_x : x \in g(K)\} \in F$. Conversely, if $L \subset X$ is a compact set not in F_X, then it follows from Lemma 2.8 that $\{\mathrm{ev}_x : x \in L\} \notin F$. Furthermore, if $K \in F_X$ then the relative topology from X is the same as the relative topology from the identification of K with a subset of $\Phi(C(X))$. (K is compact in both topologies.) It follows from condition 2.5 (4) and the definition of the topology on $\Phi(C(X))$ that $x \mapsto \mathrm{ev}_x$ is a homeomorphism. This completes the proof that $(X, F_X) \cong (\Phi(C(X)), F)$. Q.E.D.

2.9. COROLLARY (of the proof). *Let X be a completely Hausdorff quasitopological space. Then* $C(X) \cong \varprojlim_{K\in F_X} C(K)$.

2.10. REMARK. It follows from Proposition 2.6 and Theorem 2.7 that the category of commutative unital pro-C^*-algebras is contravariantly equivalent to the category of completely Hausdorff topological spaces with distinguished families of compact subsets, as given in Definition 2.5. Note that such a space is necessarily compactly generated, by 2.5 (4), and that every compactly generated completely

Hausdorff space can occur (with the family of all compact subsets). We have two reasons for not using completely regular spaces here. First, in Example 2.12 below, we show that there is a completely regular space X such that $C(X)$ is not a pro-C^*-algebra in any topology whatever. Secondly, condition 2.5 (4) cannot be dropped (for example, $C([0, 1])$ is not complete in the topology of uniform convergence on finite sets, that is, the topology of pointwise convergence), and it forces us to allow all compactly generated completely Hausdorff spaces. In Example 2.13 below, we show that these need not be completely regular.

The remainder of this section is devoted to counterexamples.

2.11. EXAMPLE (Weidner). We will produce a commutative unital pro-C^*-algebra A which is not isomorphic, as a pro-C^*-algebra, to $C(X)$ for any completely Hausdorff topological space X. Thus, one cannot avoid using quasitopologies or distinguished families of compact subsets, at least if one insists that the continuous functions separate the points.

Let F be the set of countable closed subsets of [0, 1] posessing only finitely many cluster points. Then F satisfies the conditions of Definition 2.5 relative to the usual topology on [0, 1]. (For 2.5 (4), note that the sets of the form $\{x_n\} \cup \{x\}$, where $x_n \to x$, already determine the topology.) Now let A be $C([0, 1])$ with the topology of uniform convergence on the members of F. It follows from Lemma 2.4 that A is a pro-C^*-algebra.

Suppose that X is a completely Hausdorff topological space such that there is an isomorphism $\varphi : A \to C(X)$. Then for each $x \in X$, the homomorphism $\mathrm{ev}_x \circ \varphi$ must be evaluation at some $f(x) \in [0, 1]$. (This follows from the proof of Theorem 2.7.) Clearly $\varphi(h) = h \circ f$ for every $h \in A$. The function f is injective because X is completely Hausdorff, and continuous because the usual topology on [0, 1] is the weak topology determined by A. Also, f must have dense range because φ is injective. If now $t \notin f(X)$, then the function $h(x) = (t - f(x))^{-1}$ is in $C(X)$ but not in the range of φ. Thus f is in fact bijective.

It follows from Lemma 2.8 that $f^{-1}(K)$ is not compact for $K \notin F$. Therefore f is not a homeomorphism, so that there is a set $C \subset [0, 1]$ which has a limit point $t \notin C$, but such that $f^{-1}(C)$ is closed. Let $\{t_n\}$ be a sequence in C which converges to t, and let $T = \{t\} \cup \{t_n : n \in \mathbf{Z}^+\}$. Then $\{f^{-1}(t_n)\} = f^{-1}(T) \cap f^{-1}(C)$ is a closed subset of X, and it is not compact because its image $T \setminus \{t\}$ in [0, 1] is not compact. Consequently $f^{-1}(T)$ is not compact. This contradicts the assumption that φ is a homeomorphism, by Lemma 2.8. Therefore the space X cannot exist.

We also point that Lemma 2.8 shows that the identity map from A to $C([0, 1])$ (with its usual topology) is a discontinuous bijective $*$-homomorphism from a pro-C^*-algebra to a C^*-algebra. Furthermore, note that every element of A is bounded, even though A is not a C^*-algebra. Actually, these phenomena can occur even for $A = C(X)$ for an appropriate compactly generated completely Hausdorff space X, for example the set of countable ordinals. (See Proposition 12.2 and the remark following it in [25].)

2.12. EXAMPLE. We will produce a completely regular space X such that $C(X)$ is not algebraically isomorphic to any pro-C^*-algebra. Let $\mathbf{Z}^+$ be the set of positive integers, let $\beta\mathbf{Z}^+$ be its Stone-Čech compactification, choose $x_0 \in \beta\mathbf{Z}^+ \setminus \setminus \mathbf{Z}^+$, and let $X = \mathbf{Z}^+ \cup \{x_0\}$. Then X is completely regular, since it is a subset of βZ^+, and it is realcompact ([16], Chapter 8), since it is countable. (See [16], 8.2.) Suppose that $C(X)$ is algebraically isomorphic to a pro-C^*-algebra. Then we must have $C(X) \cong C(Y)$ algebraically for some compactly generated completely Hausdorff space Y. By [16], 3.9, there is a completely regular space Z and a continuous surjective function $f : Y \to Z$ such that the corresponding map $C(Z) \to C(Y)$ is an algebraic isomorphism. Since Y is completely Hausdorff, f must also be injective. Let W be the realcompactification of Z ([16], 8.4 and 8.5), so that in particular $C(W) \cong \cong C(Z)$ algebraically. Therefore $C(W) \cong C(X)$ algebraically, so, by [16], 8.3, we have $W \cong X$. Since Z is a subspace of W, this homeomorphism implies that Z is countable and hence already realcompact, that is, $W = Z \cong X$. We thus have a continuous bijective map $f : Y \to X$ such that $h \mapsto h \circ f$ is an algebraic isomorphism from $C(X)$ to $C(Y)$. By [26], Example 12.5, every compact subset of X is finite. Therefore every compact subset of Y is finite, and, since Y is compactly generated, Y must be discrete. Since there are no discontinuous functions on Y, but there are discontinuous functions on X (for example, $h = 0$ on $\mathbf{Z}^+$ and $h(x_0) = 1$), we obtain a contradiction. Thus, there is no topology on $C(X)$ in which it is a pro-C^*-algebra.

2.13. EXAMPLE. We will produce a completely Hausdorff compactly generated space X which is not completely regular (in fact, not regular). Thus, the topology on X is not the weak topology determined by $C(X)$, and hence differs from the topology used in [17], Section 4 and in [36], Satz 1.1. Also, one cannot require the spaces in Definition 2.5 to be completely regular. Let Ω be the first uncountable ordinal, let ω be the first infinite ordinal, set $Y_1 = \{\varkappa : \varkappa \leqslant \Omega\}$ and $Y_2 = \{\varkappa : \varkappa \leqslant \omega\}$, and let $T = Y_1 \times Y_2 \setminus \{(\Omega, \omega)\}$. Then it is well known (see [20], Problem 4F) that T is not normal, and in fact that the closed subsets $A = \{\Omega\} \times \{\varkappa : \varkappa < \omega\}$ and $B = \{\varkappa : \varkappa < \Omega\} \times \{\omega\}$ do not have disjoint neighborhoods.

Let X be the space T with the subset A collapsed to a point, with the quotient topology. This is a space of the sort shown in [20], Problem 4G to be Hausdorff but not regular. (The point A and the closed set B do not have disjoint neighborhoods.) Now $Y_1 \times Y_2$ is compact, so that T is locally compact and hence compactly generated ([43], I.4.1). It now follows from [38], 2.6, that X is compactly generated. Furthermore, X is completely Hausdorff: let $x, y \in T$ be two points whose images in X are distinct. Then at least one of them, say x, is not in A. Since T is completely regular (being a subspace of the normal space $Y_1 \times Y_2$), there is a continuous function $f : T \to [0, 1]$ such that $f(x) = 0$ and $f = 1$ on $\{y\} \cup A$. This function defines a continuous function from X to $[0, 1]$ taking the values 0 and 1 on the images of x and y respectively.

2.14. EXAMPLE. We will produce a regular compactly generated space Y which is not completely Hausdorff. As a consequence, we obtain an inverse system $\{A_d\}$ such that the maps $A_d \to A_e$ are all surjective but the maps $\varprojlim A_d \to A_e$ are not all surjective. Indeed, we have $C(Y) = \varprojlim C(K)$ as K runs through all compact subsets of Y, and each restriction map $C(K) \to C(L)$ is surjective, but the maps $C(Y) \to C(K)$ are not all surjective. (Take $K = \{a, b\}$ where $a, b \in Y$ cannot be separated by a continuous function.)

The space Y is the space of Example 3 in Section VII.7 of [11]. It is shown there that Y is regular and not completely Hausdorff, so we need only show that Y is compactly generated. This fact was pointed out to us by Mladen Bestvina.

Let T be as in the previous example, and let $X = \mathbf{Z} \times T \cup \{a, b\}$, where $\mathbf{Z} \times T$ is given the product topology, a neighborhood base at a consists of the sets $[n, \infty) \times T \cup \{a\}$, and a neighborhood base at b consists of the sets $(-\infty, n] \times T \cup \{b\}$. (The intervals are to be interpreted in $\mathbf{Z}$.) Then the space Y is an identification space of X, from which it follows ([38], 2.6) that it is sufficient to prove that X is compactly generated. This is easily seen to follow from the fact that $\mathbf{Z} \times T$ is locally compact, and hence compactly generated, together with the fact that a and b have countable neighborhood bases.

3. TENSOR PRODUCTS, LIMITS, AND MULTIPLIER ALGEBRAS

In this section, we generalize to pro-C^*-algebras two standard constructions on C^*-algebras, namely tensor products and multiplier algebras. We also consider direct and inverse limits, and approximate identities. Tensor products have previously been studied (from a different point of view) in [13], but there is very little overlap between that paper and our discussion. Approximate identities are shown to exist in [17], but our proof is much shorter. Otherwise, our results are new.

We begin with tensor products. Unless otherwise specified, all tensor products of C^*-algebras are maximal C^* tensor products. (See [39], Section IV.4 for general information on tensor products of C^*-algebras.) The topology in the following definition appears in Section 3 of [13], where it is called the projective tensorial l.m.c. C^*-topology.

3.1. DEFINITION. Let A and B be pro-C^*-algebras. Their *maximal tensor product* $A \otimes B$ is the pro-C^*-algebra obtained by completing the algebraic tensor product of A and B for the family of greatest C^*-cross-seminorms $p \otimes q$ determined by p and q, as p runs through $S(A)$ and q runs through $S(B)$.

As an immediate corollary of the definition, we obtain:

3.2. PROPOSITION. *If* $A = \varprojlim_{d \in D} A_d$ *and* $B = \varprojlim_{e \in E} B_e$, *then* $A \otimes B \cong \varprojlim_{(d,e) \in D \times E} A_d \otimes B_e$.

Of course, in $D \times E$ we have $(d_1, e_1) \leqslant (d_2, e_2)$ exactly when $d_1 \leqslant d_2$ and $e_1 \leqslant e_2$.

Proof of Proposition 3.2. The only nontrivial point is to ensure that if $(d_1, e_1) \leqslant (d_2, e_2)$, then there is in fact an extension of the obvious homormophism of the algebraic tensor products to a homomorphism $A_{d_2} \otimes B_{e_2} \to A_{d_1} \otimes B_{e_1}$. This follows from [39]. Proposition IV.4.7. Q.E.D.

We then obtain the usual universal property.

3.3. PROPOSITION. *Let* A, B, *and* C *be pro-*C^**-algebras, and let* $\varphi : A \to C$ *and* $\psi : B \to C$ *be homomorphisms whose ranges commute. Then there is a unique homomorphism* $\eta : A \otimes B \to C$ *such that* $\eta(a \otimes b) = \varphi(a)\psi(b)$ *for all* $a \in A$, $b \in B$.

Proof. Since the algebraic tensor product is dense in $A \otimes B$, the homomorphism η is unique if it exists. For existence, it suffices to find continuous homomorphisms $\eta_r : A \otimes B \to C_r$ for $r \in S(C)$ which are coherent in the obvious sense. To define η_r, use the continuity of φ and ψ to find $p \in S(A)$ and $q \in S(B)$ such that $r \geqslant p \circ \varphi$, $q \circ \psi$. Then take η_r to be the composite $A \otimes B \to A_p \otimes B_p \to C_r$; the first map is continuous by the definition of $A \otimes B$ and the second one exists by the corresponding universal property for C^*-algebras. It is easily seen that η_r does not depend on the choice of p and q. Q.E.D.

The minimal tensor product can be defined in the same way, using the injective tensorial l.m.c. C^*-topology as in Section 3 of [13]. Minimal tensor products are also functorial, as can be seen from the corresponding result for C^*-algebras, [39] Proposition IV.4.2. See [13] for more in this direction.

For the applications we have in mind, however, at least one of the factors, say A, will be nuclear in the sense that A_p is nuclear for every $p \in S(A)$. In this case, the minimal and maximal tensor products will agree. (This remark generalizes the comments about type I algebras on page 126 of [13].) Note that any commutative pro-C^*-algebra (unital or not) is nuclear, and that any nuclear C^*-algebra is nuclear as a pro-C^*-algebra.

We now show that the tensor product of a pro-C^*-algebra A with an algebra of the form $C(X)$ is what one expects. If X is a quasitopological space (Definition 2.1), then we let $C(X, A)$ be the $*$-algebra of all quasicontinuous functions from X to A, with the topology determined by the C^*-seminorms $\|f\|_{K,g,p} = \sup_{x \in K} p(f \circ g(x))$ for K compact Hausdorff, $g \in Q(K, X)$, and $p \in S(A)$. Equivalently (using Proposition 2.6), $C(X, A)$ is the algebra of all continuous functions from X to A with the topology of uniform convergence on each element of F_X in each continuous C^*-seminorm on A.

3.4. PROPOSITION. *Let* X *be a completely Hausdorff quasitopological space. Then the obvious map from* $C(X) \otimes A$ *to* $C(X, A)$ *is an isomorphism.*

Proof. Write $A = \varprojlim A_p$ and $C(X) = \varprojlim_{K \in F_X} C(K)$. Now apply Proposition 3.2, using the fact that the C^*-seminorms $f \mapsto \sup_{x \in K} p(f(x))$ which define the topology on $C(X, A)$ are exactly the cross-norms $p \otimes \| \ \|_K$, where $\|f\|_K = \sup_{x \in K} |f(x)|$. Q.E.D.

A similar result holds when $C(X)$ is replaced by the C^*-algebra $C_0(X)$ of continuous complex-valued functions vanishing at infinity on the locally compact space X. Thus, given a pro-C^*-algebra A, we let $C_0(X, A)$ be the set of all continuous functions $f : X \to A$ which vanish at infinity in the sense that $p \circ f$ vanishes at infinity for every $p \in S(A)$.

3.5. PROPOSITION. *Let A be a pro-C^*-algebra and let X be locally compact. Then $C_0(X) \otimes A \cong C_0(X, A)$ via the obvious map.*

Proof. By the reasoning of the previous proof, we must show that the obvious map from $\varprojlim_{p \in S(A)} C_0(X, A_p)$ to $C_0(X, A)$ is an isomorphism. This is essentially trivial. Q.E.D.

3.6. REMARK. Tensor products do not behave well with respect to the functor b introduced in the previous section. For example, if X and Y are locally compact then it is easily shown that $C(X) \otimes C(Y) \cong C(X \times Y)$. Now $\mathrm{b}(C(X)) = C_b(X) \cong C(\beta X)$, where βX is the Stone-Čech compactification of X. Since $\beta(X \times Y)$ is in general larger than $\beta X \times \beta Y$, even when one of X and Y is compact (see Chapter 8 of [42]), we do not in general have $\mathrm{b}(C(X) \otimes C(Y)) = \mathrm{b}(C(X)) \otimes \mathrm{b}(C(Y))$.

One might hope that if A is a simple C^*-algebra then $\mathrm{b}(A \otimes B) = A \otimes \mathrm{b}(B)$ However, even that is false. Let $A = K$, the algebra of compact operators on a separable infinite dimensional Hilbert space H, and let $B = C(\mathbf{Z}^+)$. Choose a basis for H, and let e_n be the projection on the space spanned by its first n elements. Define $a \in C(\mathbf{Z}^+, K)$ by $a(n) = e_n$. Then $a \in \mathrm{b}(K \otimes C(\mathbf{Z}^+))$. However, $\{e_n \otimes 1\}$ is an approximate identity for $K \otimes C_b(\mathbf{Z}^+)$ and $\|(e_n \otimes 1)a - e_n \otimes 1\|$ does not converge to 0, so $a \notin K \otimes C_b(\mathbf{Z}^+)$.

We next turn to limits. The following result is sufficiently obvious that we omit its proof.

3.7. PROPOSITION. *Inverse limits exist in the category of pro-C^*-algebras.*

Slightly trickier is:

3.8. PROPOSITION. *Direct limits exist in the category of pro-C^*-algebras.*

If $\{A_\alpha\}_{\alpha \in I}$ is a direct system of pro-C^*-algebras, with homomorphisms $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ for $\alpha \leqslant \beta$, then the direct limit is constructed as follows. Let

$$D = \left\{ p \in \prod_{\alpha \in I} S(A_\alpha) : p_\beta \circ \varphi_{\alpha\beta} \leqslant p_\alpha \text{ for } \alpha \leqslant \beta \right\},$$

ordered by $p \leqslant q$ if $p_\alpha \leqslant q_\alpha$ for all α. Then D is a directed set. For $p \in D$, set $B_p = \varinjlim (A_\alpha)_{p_\alpha}$, and set $B = \varprojlim B_p$. Then $\varinjlim A_\alpha$ is the closure of the union of the images of the A_α in B. We omit the details of the proof because direct limits are sufficiently badly behaved that they do not seem to be of much use. Indeed, in the following example, we produce a countable direct system in which every map is injective and no algebra is zero, but for which the direct limit is zero. Also, we show in Example 5.10 that a countable direct limit of σ-C^*-algebras is usually not a σ-C^*-algebra.

3.9. EXAMPLE. Write $\mathbf{Q} = \{x_1, x_2, \ldots\}$, and set $X_n = \mathbf{Q} \setminus \{x_1, \ldots, x_n\}$. Set $A_n = C(X_n)$, and let $\varphi_n : A_n \to A_{n+1}$ be the restriction map. Note that φ_n is injective, since X_{n+1} is dense in X_n. We claim that $\varinjlim A_n = 0$. It suffices to show that, for any sequence $p_1, p_2, \ldots$ of continuous C^*-seminorms on $A_1, A_2, \ldots$ satisfying $p_{n+1} \circ \varphi_n \leqslant p_n$ for all n, we have $\varinjlim (A_n)_{p_n} = 0$. For each n there is a compact set $K_n \subset X_n$ such that $p_n(f) = \sup_{x \in K_n} |f(x)|$ for all $f \in A_n$. The condition $p_{n+1} \circ \varphi_n \leqslant p_n$ is equivalent to $K_{n+1} \subset K_n$. Since $\bigcap_n X_n = \emptyset$, we have $\bigcap_n K_n = \emptyset$, whence $K_m = \emptyset$ for some m. So $p_m = 0$ and $\varinjlim (A_n)_{p_n} = 0$, as desired.

Before turning to multiplier algebras, we need a lemma to the effect that pro-C^*--algebras have approximate identities. Following [28], 1.4.1, we use the following strong definition of an approximate identity.

3.10. DEFINITION. Let A be a pro-C^*-algebra. Then an *approximate identity* for A is an increasing net $\{e_\lambda\}$ of positive elements of A such that $\|e_\lambda\|_\infty \leqslant 1$ for all λ and, for all $a \in A$, we have $e_\lambda a \to a$ and $a e_\lambda \to a$. Here, of course, x is positive if it has the form $y^* y$ for some $y \in A$; equivalently, x is normal and $\mathrm{sp}(x) \subset [0, \infty)$.

3.11. PROPOSITION. *Every approximate identity for* $\mathrm{b}(A)$ *is an approximate identity for* A.

Proof. By definition, an increasing net $\{e_\lambda\}$ of positive elements, bounded by 1, is an approximate identity for A if $p(e_\lambda a - a) \to 0$ and $p(a e_\lambda - a) \to 0$ for all $a \in A$ and $p \in S(A)$. The result now follows from the fact (Proposition 1.11 (5)) that the map from $\mathrm{b}(A)$ to A_p is surjective. Q.E.D.

3.12. COROLLARY. (Compare [17], Theorem 2.6). *Every pro-C^*-algebra A has an approximate identity which is also an approximate identity for* $\mathrm{b}(A)$.

3.13. DEFINITION. Let A be a pro-C^*-algebra. Then the *multiplier algebra* of A is the set $M(A)$ of all pairs (l, r) of continuous linear maps from A to A such that l and r are respectively left and right A-module homomorphisms, and $r(a)b = = al(b)$ for all $a, b \in A$. Such a pair is called a *multiplier*. (Compare [28], 3.12.1, where such objects are called double centralizers. Since we have no reason to think that such maps are automatically continuous, we simply assume it.) Addition is defined as usual, multiplication is $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_2 r_1)$, and adjoint is $(l, r)^* = (r^*, l^*)$, where $r^*(a) = r(a^*)^*$ and similarly for l^*. For each $p \in S(A)$, we define a C^*-seminorm by $\|(l, r)\|_p = \sup\{p(l(a)) : p(a) \leq 1\}$, and a family of seminorms, indexed by $a \in A$, by $\|(l, r)\|_{p,a} = p(l(a)) + p(r(a))$. (It will be proved in the next theorem that $\|\ \|_p$ is in fact a C^*-seminorm.) The *seminorm topology* on $M(A)$ is the one generated by the seminorms $\|\cdot\|_p$ for $p \in S(A)$, and is the analog of the norm topology on the multiplier algebra of a C^*-algebra. The *strict topology* on $M(A)$ is the one generated by the seminorms $\|\cdot\|_{p,a}$ for $p \in S(A)$ and $a \in A$. Finally, we define a map from A to $M(A)$ by $a \mapsto (l_a, r_a)$, where $l_a(b) = ab$ and $r_a(b) = ba$ for $a, b \in A$.

3.14. THEOREM. *Let A be a pro-C^*-algebra. Then:*

(1) *If $A \cong \varprojlim_{d \in D} A_d$, and the maps $\varkappa_d : A \to A_d$ are all surjective, then $M(A)$, with its seminorm topology, is isomorphic to $\varprojlim M(A_d)$.*

(2) *The isomorphism of (1) identifies the strict topology on $M(A)$ with the topology on $\varprojlim M(A_d)$ obtained by taking the inverse limit for the strict topologies on the $M(A_d)$.*

(3) *$M(A)$ is a pro-C^*-algebra in its seminorm topology.*

(4) *$M(A)$ is complete in the strict topology.*

(5) *The map $a \mapsto (l_a, r_a)$ is a homeomorphism of A onto a closed (in the seminorm topology) ideal of A.*

(6) *The image of A under the map of (5) is dense in $M(A)$ for the strict topology.*

Proof. (1) Since $\varkappa_d : A \to A_d$ is surjective for all d, the maps $A_d \to A_e$ are also all surjective. Therefore we have maps $M(A_d) \to M(A_e)$ defined as in Theorem 4.2 of [1]. (They need not be surjective—see the example following that theorem.) Furthermore, if $p_d \in S(A)$ is defined by $p_d(a) = \|\varkappa_d(a)\|$, then we have $A_d \cong A_{p_d}$. Therefore the inverse system $\{A_d : d \in D\}$ is a cofinal subsystem of the inverse system $\{A_p : p \in S(A)\}$. Consequently the inverse systems $\{M(A_d) : d \in D\}$ and $\{M(A_p) : p \in S(A)\}$ have the same inverse limit, and it is enough to prove the result for $D = S(A)$.

It is clear that every element of $\varprojlim_{p \in S(A)} M(A_p)$ defines a multiplier of A, and that the resulting map to $M(A)$ is a homeomorphism onto the set of elements $x \in M(A)$ such that $\|x\|_p < \infty$ for all p. So we have to prove that if $(l, r) \in M(A)$ then $\|(l, r)\|_p < \infty$. This will follow if we can show that (l, r) defines a multiplier of A_p, since multipliers of C^*-algebras are automatically bounded ([28], 3.12.2). So let $a \in \mathrm{Ker}(p)$; we have to show that $l(a), r(a) \in \mathrm{Ker}(p)$. Since $\mathrm{Ker}(p)$ is a closed subalgebra of A, it is a pro-C^*-algebra and therefore has an approximate identity $\{e_\lambda\}$. Then $r(a) = \lim_\lambda r(e_\lambda a) = \lim_\lambda e_\lambda r(a) \in \mathrm{Ker}(p)$, since r is continuous. Similarly $l(a) \in \mathrm{Ker}(p)$. So (l, r) defines an element of $M(A_p)$.

(2) For the same reason as in (1), it is enough to consider the particular inverse system $\{A_p : p \in S(A)\}$. (Note that if $B \to C$ is a surjective map of C^*-algebras, then $M(B) \to M(C)$ is strictly continuous.) The statement to be proved is now immediate.

(3) This follows from (1) because there is always at least one inverse system $\{A_d\}$ with inverse limit A such that the maps $A \to A_d$ are surjective, namely $\{A_p : p \in \in S(A)\}$.

(4) $M(A_p)$ is complete in the strict topology by [10], Proposition 3.6, and inverse limits of complete spaces are complete. Now use (2).

(5) This follows immediately from the equation $\|(l_a, r_a)\|_p = p(a)$.

(6) Let $\{e_\lambda\}$ be an approximate identity for A, and let $(l, r) \in M(A)$. We claim that $(l_{r(e_\lambda)}, r_{r(e_\lambda)}) \to (l, r)$ strictly. Now the algebraic properties of multipliers and the definition of $\|\cdot\|_{p,a}$ give

$$\|(l, r) - (l_{r(e_\lambda)}, r_{r(e_\lambda)})\|_{p,a} = p(l(a) - e_\lambda l(a)) + p(r(a - ae_\lambda)).$$

Since $\{e_\lambda\}$ is an approximate identity and r is continuous, both terms on the right converge to 0. Q.E.D.

Using (5) of the previous theorem, we will identify A with the obvious closed subalgebra of $M(A)$.

Multiplier algebras of pro-C^*-algebras have the same kind of functoriality as for ordinary C^*-algebras:

3.15. PROPOSITION. (1) *Let $\varphi : A \to B$ be a homomorphism of pro-C^*-algebras which has dense range. Then φ determines a canonical homomorphism $M(A) \to M(B)$.*

(2) *Let B be a pro-C^*-algebra and let A be a closed subalgebra of B containing an approximate identity for B. Then $M(A)$ can be canonically identified with a subalgebra of $M(B)$.*

Proof. (1) It is enough to produce a consistent family of maps from $M(A)$ to $M(B_q)$ for $q \in S(B)$. So fix q, and note that $q \circ \varphi \in S(A)$. Furthermore, the obvious map from $A_{q\circ\varphi}$ to B_q is a homomorphism of C^*-algebras which has dense range and is therefore surjective. The required map is then the composite of $M(A) \to M(A_{q\circ\varphi})$ and the map $M(A_{q\circ\varphi}) \to M(B_q)$ defined in [1], Theorem 4.2.

(2) For $p \in S(B)$, the restriction $p|A$ is in $S(A)$, and A_p is a C^*-subalgebra of B_p containing an approximate identity for B_p. So $M(A_p) \subset M(B_p)$ by [1], Proposition 2.6. Since A is closed in B, we have $A = \varprojlim_{p \in S(B)} A_p$. Now use the easily verified fact that the inverse limit of injective maps is injective. Q.E.D.

For use in [30], we prove here the analogs of two other well known facts about multiplier algebras of C^*-algebras. For the purposes of the next lemma, a subset S of a pro-C^*-algebra A is *bounded* if for all $p \in S(A)$ there is a constant $M(p)$ such that $p(a) \leq M(p)$ for all $a \in S$. (This is the usual notion of boundedness in topological vector spaces. Note that any subset of $b(A)$ which is bounded for $\|\cdot\|_\infty$ is bounded in A, but of course not conversely.)

3.16. PROPOSITION. *Multiplication is jointly strictly continuous on bounded subsets of $M(A)$, for any pro-C^*-algebra A.*

Proof. Let $S, T \subset M(A)$ be bounded, let $p \in S(A)$, and let $M(p)$ be a bound for the values of $\|\cdot\|_p$ on S and T. Let $\{x_\lambda\}$ and $\{y_\lambda\}$ be nets in S and T converging to x and y respectively. Then, for all $a \in A$, we have

$$p(x_\lambda y_\lambda a - xya) \leqslant M(p)\, p(y_\lambda a - ya) + p(x_\lambda ya - xya) \to 0.$$

Similarly $p(ax_\lambda y_\lambda - axy) \to 0$. Q.E.D.

3.17. PROPOSITION. *Let X be a completely Hausdorff quasitopological space and let A be a pro-C^*-algebra. Then $M(C(X) \otimes A)$ can be canonically identified with the set of all strictly continuous functions from X to $M(A)$.*

Proof. This is true for X compact and A a C^*-algebra by [1], Corollary 3.4. The result of the proposition is obtained by writing $C(X) = \varprojlim_{K \in F_X} C(K)$, and taking inverse limits, using Proposition 3.4 and Theorem 3.14. Q.E.D.

4. HILBERT MODULES

We now define Hilbert modules over pro-C^*-algebras. As in the previous section, the results are the obvious generalization of the known results over C^*-algebras, and can be made to follow from them. The proofs, however, are not quite as straightforward. Hilbert modules over pro-C^*-algebras do not seem to have previously appeared in the literature, except in [23], where the special case of finitely generated projective modules is treated in Sections 1 and 2, and where the Hilbert space $\ell^2(A)$ over A, in the special case in which A is also a Q-algebra, is discussed in Sections 7 and 8. (This special case is useless for our applications -- see Proposition 1.14.)

We refer to Section 2 of [18] and Section 2.8 of [19] for the standard definitions and results which we generalize below. (See also Section 5 of [34].) We state all the definitions first, and then prove that they make sense afterwards.

4.1. DEFINITION. Let A be a pro-C^*-algebra, and let E be a complex vector space which is also a right A-module, compatibly with the complex algebra structure. Then E is a *pre-Hilbert A-module* if it is equipped with an A-valued inner product $\langle\ ,\ \rangle : E \times E \to A$ which is $\mathbf{C}$- and A-linear in its second variable, is conjugate $\mathbf{C}$- and A-linear in its first variable, satisfies $\langle \xi, \eta\rangle^* = \langle \eta, \xi\rangle$ for $\xi, \eta \in E$, and is positive ($\langle \xi, \xi\rangle \geqslant 0$ in A for all ξ, and $\langle \xi, \xi\rangle = 0$ only if $\xi = 0$). We say that E is a *Hilbert A-module* if E is complete in the family of seminorms $\|\xi\|_p = p(\langle \xi, \xi\rangle)^{1/2}$ for $p \in S(A)$.

If E is a Hilbert A-module, and $\varphi : A \to B$ is a homomorphism of pro-C^*-algebras, then we construct a Hilbert B-module $\varphi_*(E)$ as follows. First, form the algebraic tensor product $E \otimes_A B$, which is a right B-module in the obvious way. (Of course, we identify $\lambda\xi \otimes b$ and $\xi \otimes \lambda b$ for $\xi \in E$, $b \in B$, and $\lambda \in \mathbf{C}$.) Then define a B-valued pre-inner product by $\langle \xi \otimes b, \eta \otimes c \rangle = b^* \varphi(\langle \xi, \eta \rangle) c$. The Hilbert B-module $\varphi_*(E)$ is then the Hausdorff completion of $E \otimes_A B$ for the family of seminorms obtained by composing the above inner product with the C^*-seminorms in $S(B)$. Note that if $\psi : B \to C$ is another homomorphism of pro-C^*-algebras, then $\psi_*(\varphi_*(E))$ is canonically isomorphic to $(\psi \circ \varphi)_*(E)$.

If E and F are Hilbert A-modules, then we denote by $L(E, F)$ the space of all continuous adjointable A-module homomorphisms from E to F. We write $L(E)$ for the $*$-algebra $L(E, E)$. With $\varphi : A \to B$ as above, define $\varphi_* : L(E, F) \to L(\varphi_*(E), \varphi_*(F))$ by $\varphi_*(t)(\xi \otimes b) = t\xi \otimes b$. We topologize $L(E, F)$ via the seminorms $\|t\|_p = \|(\varkappa_p)_*(t)\|$ as p runs through $S(A)$, where $\varkappa_p : A \to A_p$ is the quotient map. For $\xi \in F$ and $\eta \in E$, we define the rank one module homomorphism $\theta_{\xi,\eta} \in L(E, F)$ by $\theta_{\xi,\eta}(\lambda) = \xi \langle \eta, \lambda \rangle$ for $\lambda \in E$. Then the space of *compact* module homomorphisms $K(E, F)$ is defined to be the closed linear span of $\{\theta_{\xi,\eta} : \xi \in F, \eta \in E\}$ in $L(E, F)$. We write $K(E)$ for the $*$-algebra $K(E, E)$.

The first three parts of the following theorem contain the statements needed to ensure that this definition makes sense. The other three statements are also analogs of standard results in the C^*-algebra case.

4.2. THEOREM. (1) *The functions* $\|\ \|_p$ *of the previous definition are seminorms.*

(2) *The pre-inner product defined on* $E \otimes_A B$ *satisfies all of the properties of an inner product except that* $\langle \eta, \eta \rangle$ *may be zero for nonzero* $\eta \in E \otimes_A B$.

(3) *The map* $\varphi_* : L(E, F) \to L(\varphi_*(E), \varphi_*(F))$ *is well defined.*

(4) $\varphi_*(K(E, F)) \subset K(\varphi_*(E), \varphi_*(F))$.

(5) $L(E)$ *and* $K(E)$ *are pro-C^*-algebras.*

(6) $L(E) \cong M(K(E))$ *canonically.*

Since this theorem needs to be proved in stages, we will carry out the proof as a sequence of lemmas. We also need the inverse limit description of Hilbert A-modules.

4.3. LEMMA. *Let* $\varphi : A \to B$ *be a homomorphism of pro-C^*-algebras, and let* E *be a pre-Hilbert A-module (except that we do not require that* $\langle \xi, \xi \rangle = 0$ *imply* $\xi = 0$*). Then* $\{\xi \in E : \varphi(\langle \xi, \xi \rangle) = 0\}$ *is a submodule of* E. *If* B *is a C^*-algebra, then the function* $\xi \to \|\varphi(\langle \xi, \xi \rangle)\|^{1/2}$ *is a seminorm on* E.

Proof. We first observe that it is enough to prove the first statement in the case of a C^*-algebra. Indeed, with $\varkappa_q : B \to B_q$ being the quotient map for $q \in S(B)$, we have

$$\{\xi \in E : \varphi(\langle \xi, \xi \rangle) = 0\} = \bigcup_{q \in S(B)} \{\xi \in E : \varkappa_q \circ \varphi(\langle \xi, \xi \rangle) = 0\},$$

and the union is increasing. Next, replacing B by $\varphi(A)$, we can assume that $B = A_p$, where $p(a) = \|\varphi(a)\|$.

Let $E_0 = E \cdot \mathrm{Ker}(\varphi)$, the linear span of all products ξa for $\xi \in E$ and $a \in \mathrm{Ker}(\varphi)$. Then E/E_0 is a B-module with $(\xi + E_0)b = \xi a + E_0$ where $\varphi(a) = b$, and has a B-valued pre-inner product given by $\langle \xi + E_0, \eta + E_0 \rangle = \varphi(\langle \xi, \eta \rangle)$. It now follows from the C^*-algebra case that $\xi + E_0 \mapsto \|\langle \xi + E_0, \xi + E_0 \rangle\|^{1/2}$ is a seminorm on E/E_0, whence $\xi \mapsto \|\varphi(\langle \xi, \xi \rangle)\|^{1/2}$ is a seminorm on E. In particular, $\{\xi \in E : \varphi(\langle \xi, \xi \rangle) = 0\}$ is a vector subspace of E, which is readily seen to be a submodule. Q.E.D.

If A is a pro-C^*-algebra, $p \in S(A)$, and E is a pre-Hilbert A-module, then we write E_p for the Hilbert A_p-module obtained by completing the pre-Hilbert A_p-module $E/\{\xi \in E : p(\langle \xi, \xi \rangle) = 0\}$ as in the proof of the above lemma. Note that the result of the lemma ensures that this makes sense. Also note that, with $\varkappa_p : A \to A_p$ being the quotient map, we have $(\varkappa_p)_*(E) \cong E_p$, via the map $\xi \otimes b \mapsto \bar{\xi} a$, where $\varkappa_p(a) = b$, and bars denote images in E_p of elements of E. In particular $(\varkappa_p)_*(E)$ is a Hilbert A_p-module. Similarly, for $p \geqslant q$ and $\pi_{pq} : A_p \to A_q$, we have a canonical isomorphism $E_q \cong (\pi_{pq})_*(E_p)$.

For the purposes of the next proposition, observe that if $\varphi : A \to B$ is a homomorphism of C^*-algebras and E is a Hilbert A-module, then there is a norm-reducing homomorphism σ from E to $\varphi_*(E)$ over φ, given by $\sigma(\xi) = \lim_\lambda \xi \otimes e_\lambda$ where $\{e_\lambda\}$ is an approximate identity for B. (Note that this net is Cauchy, and its limit does not depend on which approximate identity is chosen. In this case, $\varphi_*(E)$ is already known to be a Hilbert B-module by [34], Theorem 5.9.)

4.4. PROPOSITION. *Let $A = \varprojlim A_d$, with maps $\pi_{d,e} : A_d \to A_e$ and $\varkappa_d : A \to A_d$. If the $\varkappa_d$ are all surjective, then each Hilbert A-module E is the inverse limit $\varprojlim (\varkappa_d)_*(E)$ of a system of A_d-modules. Conversely (without assuming surjectivity of the $\varkappa_d$), given Hilbert A_d-modules E_d and a coherent family of isomorphisms $E_e \cong (\pi_{d,e})_*(E_d)$, the inverse limit $E = \varprojlim E_d$ is a Hilbert A-module such that $(\varkappa_d)_*(E)$ is canonically identified with a closed submodule of E_d.*

Proof. We do the second part first. The isomorphisms $E_e \cong (\pi_{d,e})_*(E_d)$ yield coherent module maps $\sigma_{d,e} : E_d \to E_e$ over $\pi_{d,e}$ satisfying $\langle \sigma_{d,e}(\xi), \sigma_{d,e}(\eta) \rangle = \pi_{d,e}(\langle \xi, \eta \rangle)$, so it is clear how to make $\varprojlim E_d$ into a pre-Hilbert $\varprojlim A_d$-module Completeness and the statement about $(\varkappa_d)_*(E)$ are immediate.

For the first part, it is enough to prove that $E \cong \varprojlim_{p \in S(A)} E_p$. There is an obvious isometry (in the sense of the A-valued inner products) from E to $\varprojlim E_p$. Since the image of E in each E_p is dense, so is the image of E in $\varprojlim E_p$. Since E is complete, we have $E \cong \varprojlim E_p$. Q.E.D.

4.5. LEMMA. *Let A be a pro-C^*-algebra, let E be a Hilbert A-module, and let $p \in S(A)$. Then the map $E \to E_p$ is surjective.*

Proof. We let $\mathrm{b}(E)$ be the set of bounded elements of E, where $\xi \in E$ is bounded if $\langle \xi, \xi \rangle$ is a bounded element of A. Then $\mathrm{b}(E)$ is a complex vector space and a right $\mathrm{b}(A)$-module because, when E is identified with $\varprojlim E_p$, we see that $\mathrm{b}(E)$ corresponds to the set of bounded coherent sequences. The Cauchy-Schwarz inequality, applied to the Hilbert modules E_p over the C^*-algebras A_p, yields, for $\xi, \eta \in \mathrm{b}(E)$, the inequality $\|\langle \xi, \eta \rangle\|_\infty^2 \leqslant \|\langle \xi, \xi \rangle\|_\infty \|\langle \eta, \eta \rangle\|_\infty$, so that the restriction to $b(E)$ of the A-valued inner product on E is a $\mathrm{b}(A)$-valued inner product on $\mathrm{b}(E)$. The proof of completeness in [36], Satz 3.1, also applies here (compare with Proposition 1.11 (1)), and shows that $\mathrm{b}(E)$ is complete for the norm $\|\xi\|_\infty = \|\langle \xi, \xi \rangle\|_\infty^{1/2}$. Therefore $\mathrm{b}(E)$ is a Hilbert $\mathrm{b}(A)$-module.

Since $\varphi : \mathrm{b}(A) \to A_p$ is a surjective map of C^*-algebras (Proposition 1.11 (5)), and since clearly $\varphi_*(\mathrm{b}(E)) \cong E_p$, the lemma will follow if we can show the following: whenever $\varphi : A \to B$ is a surjective map of C^*-algebras, and E is a Hilbert A-module, then the canonical map $\sigma : E \to \varphi_*(E)$ is surjective. Now in this case $\varphi_*(E)$ is the completion of E/E_0, where $E_0 = \{\xi \in E : \varphi(\langle \xi, \xi \rangle) = 0\}$, in its obvious pre-Hilbert B-module structure, as in the proof of Lemma 4.3. So it is enough to show that E/E_0 is already complete, and this will follow if we can show that its norm $\|\xi + E_0\| = \|\varphi(\langle \xi, \xi \rangle)\|^{1/2}$ is just the quotient norm from E. (We know that E is complete.) Thus, we have to show that, for $\xi \in E$, we have $\|\varphi(\langle \xi, \xi \rangle)\|^{1/2} = \inf_{\eta \in E_0} \|\xi + \eta\|$.

For one direction, we observe that if $\xi \in E$ and $\eta \in E_0$, then

$$\|\xi + \eta\|^2 = \|\langle \xi + \eta, \xi + \eta \rangle\| \geqslant \|\varphi(\langle \xi + \eta, \xi + \eta \rangle)\| = \|\varphi(\langle \xi, \xi \rangle)\|,$$

where $\varphi(\langle \xi, \eta \rangle) = 0$ because $\varphi(\langle \eta, \eta \rangle) = 0$, by the Cauchy-Schwarz inequality in the form $\langle \xi, \eta \rangle^* \langle \xi, \eta \rangle \leqslant \|\langle \xi, \xi \rangle\| \langle \eta, \eta \rangle$ ([34], Proposition 2.9). For the other direction, let $\xi \in E$ and choose an approximate identity $\{e_\lambda\}$ for $\mathrm{Ker}(\varphi)$. Then $\xi e_\lambda \in E_0$ for all λ, and we have

$$\lim_\lambda \|\xi - \xi e_\lambda\|^2 = \lim_\lambda \|(1 - e_\lambda)\langle \xi, \xi \rangle (1 - e_\lambda)\| = \lim_\lambda \|(1 - e_\lambda)\langle \xi, \xi \rangle^{1/2}\|^2 =$$

$$= \inf_{x \in \mathrm{Ker}(\varphi)} \|\langle \xi, \xi \rangle^{1/2} + x\|^2 = \|\varphi(\langle \xi, \xi \rangle)\|,$$

where the second last equality is [28], 1.5.4. This shows that $\inf_{\eta \in E_0} \|\xi + \eta\| \leqslant \|\varphi(\langle \xi, \xi \rangle)\|^{1/2}$, as needed. Q.E.D.

4.6. LEMMA. *Let $\varphi : A \to B$ be a homomorphism from a pro-C^*-algebra to a C^*-algebra. Then for each Hilbert A-module E, the module $\varphi_*(E)$ is a Hilbert*

B-module, and φ_* defines a map from $L(E, F)$ to $L(\varphi_*(E), \varphi_*(F))$ which sends $K(E, F)$ to $K(\varphi_*(E), \varphi_*(F))$.

We note that the existence of the map from $L(E, F)$ to $L(\varphi_*(E), \varphi_*(F))$ is exactly what is needed to define the topology on $L(E, F)$ under which $K(E, F)$ is the closure of the finite rank module homomorphisms.

Proof. Let $p \in S(A)$ be $p(a) = \|\varphi(a)\|$, and let $\psi : A_p \to B$ be the obvious map of C^*-algebras. Then $\varphi_*(E) = \psi_*(E_p)$, which is a Hilbert module by [34], Theorem 5.9.

Now let $t \in L(E, F)$. Choose an approximate identity $\{e_\lambda\}$ for $\mathrm{Ker}(\varphi)$, and observe that, for $\xi \in E$ with $\varphi(\langle \xi, \xi \rangle) = 0$, we have

$$\lim_\lambda \langle \xi - \xi e_\lambda, \xi - \xi e_\lambda \rangle = \lim_\lambda (\langle \xi, \xi \rangle - \langle \xi, \xi \rangle e_\lambda) - \lim_\lambda e_\lambda(\langle \xi, \xi \rangle - \langle \xi, \xi \rangle e_\lambda) = 0,$$

since $p(e_\lambda) \leqslant 1$ for all $p \in S(A)$. So $\xi e_\lambda \to \xi$. Therefore

$$\langle t\xi, t\xi \rangle = \lim_\lambda \langle t\xi, t(\xi e_\lambda) \rangle = \lim_\lambda \langle t\xi, t\xi \rangle e_\lambda \in \mathrm{Ker}(\varphi).$$

That is, $\langle \xi, \xi \rangle = 0$ implies $\langle t\xi, t\xi \rangle = 0$. So we obtain a map from $E/\{\xi \in E : \varphi(\langle \xi, \xi \rangle) = 0\}$ to $F/\{\xi \in F : \varphi(\langle \xi, \xi \rangle) = 0\}$ which is easily seen to be adjointable and a B-module homomorphism. By the previous lemma, this map is actually an adjointable module homomorphism from E_p to F_p, and hence an element t_p of $L(E_p, F_p)$. (The map t_p is automatically continuous, by Lemma 2 of [18].) Applying ψ_* and using the relations $\psi_*(E_p) = \varphi_*(E)$ and $\psi_*(t_p) = \varphi_*(t)$, we see that $\psi_*(t) \in L(\varphi_*(E), \varphi_*(F))$ is in fact well defined. Obviously φ_* is a homomorphism.

It remains to verify that φ_* sends $K(E, F)$ to $K(\varphi_*(E), \varphi_*(F))$. Since φ_* is continuous ($t \mapsto t_p$ is continuous by definition, and φ_* is continuous because it comes from a map of C^*-algebras), it is enough to show that $\varphi_*(\theta_{\xi,\eta})$ is a compact module homomorphism for $\xi \in F$ and $\eta \in E$. Now $(\theta_{\xi,\eta})_p$ is the rank one module homomorphism determined by the images of ξ and η in F_p and E_p respectively. Therefore $\varphi_*(\theta_{\xi,\eta}) = \psi_*((\theta_{\xi,\eta})_p) \in K(\psi_*(E), \psi_*(F))$ by [19], Section 2.8. Q.E.D.

4.7. Proposition. *Let A be a pro-C^*-algebra, and let E and F be Hilbert A-modules. Then we have canonical isomorphisms $L(E, F) \cong \varprojlim L(E_p, F_p)$ and $K(E, F) \cong \varprojlim K(E_p, F_p)$.*

Proof. That an element of $L(E, F)$ defines a coherent sequence of elements of $L(E_p, F_p)$ follows from the previous lemma, and similarly for $K(E, F)$ and $K(E_p, F_p)$. The converse for $L(E, F)$ is easily shown by using Proposition 4.4 to write $E = \varprojlim E_p$ and $F = \varprojlim F_p$. That the resulting map is a homeomorphism is essentially the definition of the topology on $L(E, F)$.

Now let $\{k_p\}$ be a coherent sequence of elements of $K(E_p, F_p)$. We have to show that the corresponding operator $k \in L(E, F)$ is actually in $K(E, F)$. For $p \in S(A)$

and $\varepsilon > 0$ choose $\bar{\xi}_1, \ldots, \bar{\xi}_n \in F_p$ and $\bar{\eta}_1, \ldots, \bar{\eta}_n \in E_p$ such that $\|\sum \theta_{\bar{\xi}_i, \bar{\eta}_i} - k_p\| < \varepsilon$. Using Lemma 4.5, choose $\xi_1, \ldots, \xi_n \in F$ and $\eta_1, \ldots, \eta_n \in E$ whose images in F_p and E_p are $\bar{\xi}_1, \ldots, \bar{\xi}_n$ and $\bar{\eta}_1, \ldots, \bar{\eta}_n$. Then set $l_{p,\varepsilon} = \sum \theta_{\xi_i, \eta_i} \in K(E, F)$. We have $l_{p,\varepsilon} \to k$ as $(p, \varepsilon) \to \infty$ (that is, as p increases and $\varepsilon \to 0$), so $k \in K(E, F)$ as desired. Q.E.D.

We are now able to prove Theorem 4.2.

Proof of Theorem 4.2. (1) This is Lemma 4.3.

(2) For $\varphi : A \to B$ and $q \in S(B)$, let φ_q be the obvious homomorphism from A to B_q. Then, for a Hilbert A-module E, we have $\varphi_*(E) \cong \varprojlim (\varphi_q)_*(E)$. The modules $(\varphi_q)_*(E)$ are Hilbert B_q-modules by Lemma 4.6, and the inverse limit is a Hilbert B-module by Proposition 4.4. The statement now follows.

(3) This follows from Lemma 4.6, Proposition 4.7, and the expression of $\varphi_*(E)$ as $\varprojlim (\varphi_q)_*(E)$ in the proof of part (2).

(4) This follows in the same way as (3).

(5) This is immediate from Proposition 4.7.

(6) It follows from the argument used in the proof of Proposition 4.7 that the map $K(E) \to K(E_p)$ has dense range. By Corollary 1.12, it must be surjective. It now follows from Proposition 4.7 and Theorem 3.14 that $M(K(E)) \cong \varprojlim M(K(E_p))$. Since $M(K(E_p)) = L(E_p)$ by [18], Theorem 1, we obtain $M(K(E)) = L(E)$ by another application of Proposition 4.7. Q.E.D.

4.8. Remark. The standard examples of Hilbert modules over pro-C^*-algebras are the same as over ordinary C^*-algebras. If A is an inverse limit of C^*-algebras, then A^n with the inner product $\langle \xi, \eta \rangle = \sum_{k=1}^{n} \xi_k^* \eta_k$, and

$$\ell^2(A) = \left\{ \xi \in \prod_{k=1}^{\infty} A : \sum_{k=1}^{\infty} \xi_k^* \xi_k \text{ converges in } A \right\}$$

with the inner product $\langle \xi, \eta \rangle = \sum_{k=1}^{\infty} \xi_k^* \eta_k$, are Hilbert A-modules. We have $K(A^n) \cong$ $\cong M_n \otimes A$ (where M_n is the set of $n \times n$ matrices over $\mathbf{C}$), $L(A^n) \cong M_n \otimes M(A)$, and $K(\ell^2(A)) \cong K(\ell^2) \otimes A$. If E is a Hilbert module which is finitely generated and projective over the unital pro-C^*-algebra A, then we obtain $K(E) = L(E)$ as usual. However, the converse, which would be the analog of the proposition in [35], is false for pro-C^*-algebras. (The proof breaks down because the group of invertible elements in a pro-C^*-algebra need not be open. Concrete examples of this phenomenon will be given in [30]. See also Proposition 1.14.)

4.9. Example. Let $A = C(\mathbf{Z})^+$, which is just $\prod_{n=1}^{\infty} \mathbf{C}$, and let $E = \prod_{n=1}^{\infty} \mathbf{C}^n$. We make E into a Hilbert A-module via $(\xi a)_n = \xi_n a_n$ and $\langle \xi, \eta \rangle_n = \langle \xi_n, \eta_n \rangle$, where the right hand side is the usual $\mathbf{C}$ valued inner product on $\mathbf{C}^n$. Let A_n be the product of the first n factors of A, and let E_n be the product of the first n factors of E. Then $A_n = A_{p_n}$, where $p_n(a) = \sup\{\|a_k\| : k \leqslant n\}$, and $E_n = E_{p_n}$. So $A = \varprojlim A_n$, $E = \varprojlim E_n$, and $L(E) = \varprojlim L(E_n) = \varprojlim K(E_n) = K(E)$ using Proposition 4.7 However, E is not finitely generated as an A-module.

In the next section, we will prove a stabilization theorem for countably generated Hilbert modules over σ-C^*-algebras. T heproof uses induction over the directed set, and we do not know if the result is true over general pro-C^*-algebras.

5. HOMOMORPHISMS AND σ-C^*-ALGEBRAS

In this section, we restrict ourselves to the σ-C^*-algebras of Arveson, which are the inverse limits of C^*-algebras whose topology is determined by countably many C^*-seminorms. Equivalently, they are the inverse limits of countable inverse systems of C^*-algebras. We do this because, in certain ways, the category of σ-C^*--algebras is much more manageable than the category of pro-C^*-algebras. In particular, we have no useful condition for the inverse limit of exact sequences of C^*-algebras to be exact, or for the maps $\varprojlim A_d \to A_d$ to be surjective. We have also been unable to show that the quotient of a pro-C^*-algebra by a closed ideal is again a pro--C^*-algebra. (The issue here is completeness. It is known that in general the quotient of a complete topological vector space need not be complete -- see [21], 23.5 or 31.6.) However, we do have the corresponding results for σ-C^*-algebras. Our proofs use induction over the directed sets of our inverse systems.

In this section, we discuss homomorphisms, ideals, and quotients of σ-C^*-algebras. We then give the σ-C^*-algebra versions of the important results from the previous section in those cases in which they differ, and prove two additional results related to the earlier sections for which we need to begin with σ-C^*-algebras. Throughout this section, we will assume that the countable directed set is always $\mathbf{Z}^+$. This can always be arranged, since any countable directed set has a cofinal subset isomorphic to $\mathbf{Z}^+$ (or else has a largest element), and limits are unchanged when the directed set is replaced by a cofinal subset. We will also always assume that the maps $A_{n+1} \to A_n$ are all surjective; this can always be arranged by replacing each A_n by the intersection of the images of the A_m for $m \geqslant n$. Note that an inverse system indexed by $\mathbf{Z}^+$ is determined by the maps $A_{n+1} \to A_n$, and that they can be arbitrary. Finally, we assume that all ideals are closed, selfadjoint, and two-sided. (It is shown in Theorem 2.7 of [17] that a closed two-sided ideal in an arbitrary pro-C^*-algebra is necessarily selfadjoint.)

5.1. LEMMA. *Let $A = \varprojlim A_n$ be a σ-C^*-algebra (with all maps $A_{n+1} \to A_n$ surjective). Then $A \to A_n$ is surjective.*

Proof. We assume $n = 1$. (The proof is the same for all n.) Given $a_1 \in A_1$, construct inductively a sequence $\{a_n\}$ such that $a_n \in A_n$ and the image of a_{n+1} in A_n is a_n. Then $\{a_n\}$ defines an element of A whose image in A_1 is a_1. Q.E.D.

5.2. THEOREM. *Let A be a σ-C^*-algebra, let B be a pro-C^*-algebra, and let $\varphi : A \to B$ be a $*$-homomorphism. Then φ is automatically continuous.*

Proof. It is enough to prove that for $p \in S(B)$ the maps $A \to B_p$, determined by φ, are continuous. Thus we reduce to the case in which B is a C^*-algebra. Taking unitizations, we may assume that A, B, and φ are unital. Now represent B faithfully on a Hilbert space and use Lemma 3.1 of [8]. Q.E.D.

We note, however, that a homomorphism of σ-C^*-algebras need not have closed range. (Consider the inclusion of $\mathrm{b}(A)$ in A for any σ-C^*-algebra A, for instance $C(\mathbf{R})$, for which $\mathrm{b}(A) \neq A$.)

A sequence

$$0 \to I \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0$$

of σ-C^*-algebras and homomorphisms will be called *exact* if it is algebraically exact, α is a homeomorphism onto its image, and β defines a homeomorphism of $A/\mathrm{Ker}(\beta)$ with B. We will see below (Corollary 5.5) that the topological conditions are redundant.

The following proposition requires our assumption of surjectivity on the homomorphisms of the inverse system; otherwise part (2) fails — see Proposition 10.2 of [5].

5.3. PROPOSITION. (1) *A homomorphism $\varphi : A \to B$ of σ-C^*-algebras has closed range and is a homeomorphism onto its image if and only if it is an inverse limit of injective maps of C^*-algebras.*

(2) *The sequence of σ-C^*-algebras*

$$(*) \qquad 0 \to I \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0$$

is exact if and only if it is an inverse limit (with surjective maps) of exact sequences of C^*-algebras.

Proof. For the "if" parts, the algebraic statements follow from Proposition 10.2 of [5], and the topological statements are easily verified; we omit the details.

For the "only if" part of (1), write $B = \varprojlim B_n$ with maps $\lambda_n : B \to B_n$ and C^*-seminorms $q_n(b) = \|\lambda_n(b)\|$. Let A_n be the image of A in B_n. Then $A_n \cong A_{q_n}$, and is hence a closed subalgebra of B_n. We clearly have φ equal to the inverse limit of the inclusions of A_n in B_n.

Now we do (2). Using (1), write α as the inverse limit of maps $\alpha_n : I_n \to A_n$. Then I_n is an ideal in A_n, and the sequence (*) is easily seen to be algebraically the inverse limit of the sequences

$$0 \to I_n \to A_n \to A_n/I_n \to 0.$$

To show that the identification is also topological, use Theorem 5.2. Q.E.D.

For general inverse systems, we know of no good criterion for the surjectivity of the last map in the inverse limit of a system of exact sequences. In particular, if A is a general inverse limit of C^*-algebras and I is an ideal in A, we have an obvious map from A/I to $\varprojlim_{p \in S(A)} A_p/I_p$, but we do not know whether it is surjective in general.

The first part of the following corollary has already been observed in [17] and [41].

5.4. Corollary. *Let A be a σ-C^*-algebra and let I be an ideal in A. Then A/I is a σ-C^*-algebra, and every homomorphism $\varphi : A \to B$ of σ-C^*-algebras such that $\varphi|I = 0$ factors through A/I.*

Proof. It essentially follows from the proof of the previous proposition that with $A = \varprojlim A_n$ and I_n being the image of I in A_n, we have $A/I \cong \varprojlim A_n/I_n$. The last statement follows from the definition of the quotient of topological vector spaces. Q.E.D.

The categorical role played by A/I is presumably played in the category of pro-C^*-algebras by the closure of the image of A in $\varprojlim A_p/I_p$.

5.5. Corollary. *For the sequence of σ-C^*-algebras and $*$-homomorphisms*

$$0 \to I \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0$$

to be exact, it is sufficient that it be algebraically exact.

Proof. Use the previous corollary (once) and Theorem 5.2 (several times). Q.E.D.

5.6. Proposition. *Let A be a σ-C^*-algebra, and let I and J be ideals in A. Then $I + J$ is a (closed) ideal in A.*

Proof. Write $A = \varprojlim A_n$, and let I_n and J_n be the images of I and J in A_n. Then we have $I = \varprojlim I_n$, $J = \varprojlim J_n$, and $I + J = \varprojlim (I_n + J_n)$. (For the last statement, one needs the fact that $I_{n+1} \cap J_{n+1} \to I_n \cap J_n$ is surjective.) Since $I_n + J_n$ is closed ([28], 1.5.8), so is $I + J$. The remaining properties are obvious. Q.E.D.

We now identify the commutative σ-C^*-algebras. We will say that a topological space X is *countably compactly generated* if there is a countable family $\{K_n\}$ of compact subsets of X such that a set $C \subset X$ is closed if and only if $C \cap K_n$ is closed for all n. Obviously we may require that $K_1 \subset K_2 \subset \dots$. Thus, X is countably compactly generated if and only if it is a countable direct limit of compact spaces. (This is not the same as being σ-compact and compactly generated, as we will see in Example 5.8.)

5.7. PROPOSITION. *The category of commutative unital σ-C^*-algebras is contravariantly equivalent to the category of countably compactly generated Hausdorff spaces.*

Proof. We must prove two things: that a countably compactly generated Hausdorff space is completely Hausdorff, and that every σ-C^*-algebra is isomorphic to $C(X)$ for some countably compactly generated Hausdorff space.

For the first part, observe that X is in fact completely regular. Indeed, given $C \subset X$ closed, $x \notin C$, and $X = \varinjlim K_n$, we construct inductively, using the Tietze extension theorem and normality of the K_n, a sequence of continuous functions $f_n : K_n \to [0, 1]$ such that $f_n(x) = 1$ if $x \in K_n$, and $f_n = 0$ on $C \cap K_n$. Then define f by $f|K_n = f_n$.

For the second part, it is by Remark 2.10 sufficient to show that if X is a topological space with a distinguished family F of compact subsets (Definition 2.5) which has a countable cofinal subset, then X is countably compactly generated and F is equal to the set of all compact subsets of X. Let $\{K_n : n \in \mathbf{Z}^+\}$ be an increasing countable cofinal subset of F. It is immediate that $\{K_n : n \in \mathbf{Z}^+\}$ determines the topology on X. If there is a compact set $L \subset X$ with $L \notin F$, then for each n we can choose $x_n \in L \setminus K_n$. The set $T = \{x_n : n \in \mathbf{Z}^+\}$ is closed because $T \cap K_n$ is finite for all n; similarly $T \setminus \{x_n\}$ is closed for each fixed n. Therefore T is a closed infinite discrete subset of the compact set L, a contradiction. Q.E.D.

We now give an example of something that looks like a σ-C^*-algebra but is not.

5.8. EXAMPLE. $C(\mathbf{Q})$ is not a σ-C^*-algebra. (Note that $C(\mathbf{Q})$ *is* a pro-C^*-algebra, because metric spaces are compactly generated by [43], I.4.3.) To prove this, suppose that $C(\mathbf{Q})$ is a σ-C^*-algebra. By the previous proposition, we then have $C(\mathbf{Q}) \cong C(X)$, where X is a countably compactly generated space. Both $\mathbf{Q}$ and X are σ-compact, hence Lindelöf, hence realcompact by [16], Theorem 8.2. Therefore [16], Theorem 10.6 implies that $\mathbf{Q}$ and X are homeomorphic. So it is enough to prove that $\mathbf{Q}$, in spite of being both countable and compactly generated, is not countably compactly generated.

The following argument was suggested by Bob Edwards. Let $K_1 \subset K_2 \subset \dots$ be compact subsets of $\mathbf{Q}$ whose union is $\mathbf{Q}$. Each K_n is nowhere dense, so that there is $x_n \in \mathbf{Q} \setminus K_n$ with $0 < x_n < 1/n$. Then $x_n \to 0$ in $\mathbf{Q}$, but $\{x_n\}$ does not converge in the direct limit topology on $\varinjlim K_n$. (The only possible limit would be 0, which is not in $\{x_n\}$. But $\{x_n\}$ is closed since $K_m \cap \{x_n\}$ is finite for all m.) Thus $C(\mathbf{Q})$ is not a σ-C^*-algebra. In fact, it cannot be a σ-C^*-algebra for any topology on $C(\mathbf{Q})$.

We next specialize some of the results of Sections 3 and 4 to σ-C^*-algebras.

5.9. PROPOSITION. (1) *The tensor product of two σ-C^*-algebra. In fact,* $(\varprojlim A_n) \otimes (\varprojlim B_n) \cong \varprojlim (A_n \otimes B_n)$.

(2) *A countable inverse limit of σ-C^*-algebras is a σ-C^*-algebra.*

(3) *The multiplier algebra of a σ-C^*-algebra is a σ-C^*-algebra is a σ-C^*-algebra. In fact, $M(\varprojlim A_n) \cong \varprojlim M(A_n)$. (Recall that $A_{n+1} \to A_n$ is assumed surjective. However, $M(A_{n+1}) \to M(A_n)$ need not be surjective.)*

(4) *If A is a σ-C^*-algebra and E is a Hilbert A-module, then $K(E)$ and $L(E)$ are σ-C^*-algebras.*

The proofs are trivial and are omitted. An uncountable inverse limit of σ-C^*--algebras obviously need not be a σ-C^*-algebra. And, as we now show, even a countable direct limit of σ-C^*-algebras need not be a σ-C^*-algebra.

5.10. EXAMPLE. Let A be any σ-C^*-algebra which is not a C^*-algebra, and write $A = \varprojlim A_n$ with maps $\varkappa_n : A \to A_n$ and seminorms given by $p_n(a) = \|\varkappa_n(a)\|$. We can clearly arrange to have $p_n < p_m$ for $n < m$. Let B_n be the direct sum of m copies of A, and define $\varphi_m : B_m \to B_{m+1}$ by $\varphi_m(a_1, \ldots, a_m) = (a_1, \ldots, a_m, 0)$. From the discussion following Proposition 3.8, we see that $\varinjlim B_n$ can be identified with the set B of all elements $a \in \prod_{m=1}^{\infty} A$ such that, for every function $s : \mathbf{Z}^+ \to \mathbf{Z}^+$, we have $\lim_{m\to\infty} p_{s(m)}(a_m) = 0$. The topology on B is given by the C^*-seminorms $q_s(a) = \sup\{p_{s(m)}(a_m) : m \in \mathbf{Z}^+\}$. To show that B is not a σ-C^*-algebra, it is enough to show that there is no countable cofinal subset of the set of seminorms q_s. Notice that $q_s \leqslant q_t$ if and only if $s \leqslant t$. So suppose we had a cofinal subset $\{s_k\}$ of the set of all functions $s : \mathbf{Z}^+ \to \mathbf{Z}^+$. Defining $t(m) = 1 + s_m(m)$, we immediately obtain a contradiction. Thus $\varinjlim B_n$ is not a σ-C^*-algebra.

We close this section by proving two results for σ-C^*-algebras for which we have been unable to prove analogous results for general pro-C^*-algebras. Note that the multiplier algebra of a σ-C^*-algebra is again a σ-C^*-algebra. We also point out that, by the corollary to Theorem 14 of [44], multipliers (double centralizers) of a σ-C^*-algebra are automatically continuous.

5.11. THEOREM. *Let $\varphi : A \to B$ be a surjective homomorphism of σ-C^*-algebras, and assume that A has a countable approximate identity. Then the map $M(A) \to M(B)$ is surjective.*

Proof. If A and B are C^*-algebras, this is Theorem 10 of [45]. In the general case, let $I = \mathrm{Ker}(\varphi)$. Using Proposition 5.3 (2), write the exact sequence

$$0 \to I \to A \xrightarrow{\varphi} B \to 0$$

as the inverse limit of exact sequences of C^*-algebras

$$0 \to I_n \to A_n \xrightarrow{\varphi_n} B_n \to 0,$$

with, of course, all the maps in the inverse systems being surjective. Let $\mu_n : I_{n+1} \to \to I_n$, $\pi_n : A_{n+1} \to A_n$, and $\sigma_n : B_{n+1} \to B_n$ be the maps of the inverse systems. Let J_n be the kernel of the obvious map $\bar{\varphi}_n : M(A_n) \to M(B_n)$. Since A has a countable approximate identity, so does each A_n and each B_n. Therefore $\bar{\varphi}_n$ is surjective, as are the maps $\bar{\pi}_n : M(A_{n+1}) \to M(A_n)$ and $\bar{\sigma}_n : M(B_{n+1}) \to M(B_n)$. We thus have an inverse system of exact sequences

$$(*) \qquad 0 \to J_n \to M(A_n) \xrightarrow{\bar{\varphi}_n} M(B_n) \to 0,$$

in which the maps of the systems $\{M(A_n)\}$ and $\{M(B_n)\}$ are all surjective. Let $\bar{\mu}$ be the map from J_{n+1} to J_n.

Consider the commutative diagram with exact rows and surjective vertical maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{n+1} & \longrightarrow & A_{n+1} & \xrightarrow{\varphi_{n+1}} & B_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \sigma_n} & & \\ 0 & \longrightarrow & I_n & \longrightarrow & A_n & \xrightarrow{\varphi_n} & B_n & \longrightarrow & 0. \end{array}$$

Set $Q = \{(a, b) \in A_n \oplus B_{n+1} : \varphi_n(a) = \sigma_n(b)\}$. Then there is a homomorphism

$$\psi : A_{n+1} \to Q$$

given $\psi(a) = (\pi_n(a), \varphi_{n+1}(a))$. A diagram chase shows that ψ is surjective. Therefore $\bar{\psi} : M(A_{n+1}) \to M(Q)$ is surjective (since A_{n+1} has a countable approximate identity). The projections from Q to A_n and to B_{n+1} are aso surjective (since π_n and φ_{n+1} are), and it is then easy to show that

$$M(Q) = \{(a, b) \in M(A_n) \oplus M(B_{n+1}) : \bar{\varphi}_n(a) = \bar{\sigma}_n(b)\}.$$

In the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_{n+1} & \longrightarrow & M(A_{n+1}) & \overset{\bar{\varphi}_{n+1}}{\longrightarrow} & M(B_{n+1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\mu}_n} & & \downarrow{\scriptstyle \bar{\pi}_n} & & \downarrow{\scriptstyle \bar{\sigma}_n} & & \\
0 & \longrightarrow & J_n & \longrightarrow & M(A_n) & \overset{\bar{\varphi}_n}{\longrightarrow} & M(B_n) & \longrightarrow & 0
\end{array}$$

with exact rows and in which $\bar{\pi}_n$ and $\bar{\sigma}_n$ are surjective, the surjectivity of

$$\bar{\psi} : M(A_{n+1}) \to M(Q)$$

now implies that $\bar{\mu}_n$ is surjective. Therefore we can use Proposition 5.3 (2) to take inverse limits in (*). In particular, $\varprojlim M(A_n) \to \varprojlim M(B_n)$ is surjective. By Theorem 3.14 (1), this is the same as saying that $M(A) \to M(B)$ is surjective. Q.E.D.

For applications of this theorem, it should be pointed out that any separable σ-C^*-algebra A has a countable approximate identity: if $\{e_\lambda\}$ is an approximate identity for $b(A)$ and $\{a_k\}$ is a countable dense subset of A, choose an increasing subsequence $\{x_n\}$ of $\{e_\lambda\}$ such that, with $\{p_n\}$ being a cofinal sequence in $S(A)$, we have $p_n(x_n a_k - a_k) + p_n(a_k x_n - a_k) < 1/n$ for $1 \leqslant k \leqslant n$. Note that the separability of A is equivalent to A being the countable inverse limit of separable C^*-algebras. However, $b(A)$ can fail to be separable when A is separable: consider $A = C(\mathbb{R})$.

Our final result is the stabilization theorem promised at the end of the previous section.

5.12. THEOREM. *Let A be a σ-C^*-algebra with a countable approximate identity, and let E be a countably generated (in the topological sense) Hilbert A-module. Then $E \oplus \ell^2(A) \cong \ell^2(A)$.*

Proof. Write $A = \varprojlim A_n$ (with surjective maps $\pi_n : A_{n+1} \to A_n$), and correspondingly write $E = \varprojlim E_n$ with $(\pi_n)_*(E_{n+1}) = E_n$. Then each A_n has a countable approximate identity, and each E_n is countably generated.

We will construct, by induction on n, isomorphisms $u_n : E_n \oplus \ell^2(A_n)^n \to \ell^2(A_n)^n$ such that $(\pi_n)_*(u_{n+1}) = u_n \oplus 1$ as maps from $E_n \oplus \ell^2(A_n)^{n+1}$ to $\ell^2(A_n)^{n+1}$. We obtain u_1 from the stabilization theorem for Hilbert modules over C^*-algebras, [18], Theorem 2.

Given u_n, construct u_{n+1} as follows. First, use the stabilization theorem to choose an isomorphism $v : \ell^2(A_{n+1})^n \to E_{n+1} \oplus \ell^2(A_{n+1})^n$. Then $u_n(\pi_n)_*(v)$ is a unitary element of $L(\ell^2(A_n)^n)$, which we identify with $M(K(H) \otimes A_n)$, where $H = \ell^2(\mathbb{C})$. Since $H \cong \bigoplus_{k=1}^{\infty} H^n$, we see from Proposition 2.2 of [27] that $u_n(\pi_n)_*(v) \oplus 1$ is in the connected component of the identity in the unitary group of $M(K(H^{n+1})\otimes A_n)$. Since $K(H^{n+1})\otimes A_{n+1}$ has a countable approximate identity, the map

$$M(K(H^{n+1})\otimes A_{n+1}) \to M(K(H^{n+1}) \otimes A_n)$$

is surjective by Theorem 10 of [45]. By Proposition 4.8 of [40], there is therefore an invertible element w of $M(K(H^{n+1})\otimes A_{n+1})$ whose image in $M(K(H^{n+1}) \otimes A_n)$ is $u_n(\pi_n)_*(v) \oplus 1$. Replacing w by $w(w^*w)^{-1/2}$, we may assume that w is unitary. Now regard w as an element of $L(\ell^2(A_{n+1})^{n+1})$ and set $u_{n+1} = w(v \oplus 1)^{-1}$. Then $(\pi_n)_*(u_{n+1}) = u_n \oplus 1$, as desired.

We now let x_n be the direct sum of u_n and the identity on $\bigoplus_{k=n+1}^{\infty} \ell^2(A_n)$. Writing $\ell^2(A_n)^\infty$ for the direct sum $\bigoplus_{k=1}^{\infty} \ell^2(A_n)$, we see that $\{x_n\}$ is a coherent sequence of isomorphisms in $L(E_n \oplus \ell^2(A_n)^\infty, \ell^2(A_n)^\infty)$. Therefore $\{x_n\}$ defines an isomorphism $x : E \oplus \ell^2(A)^\infty \to \ell^2(A)^\infty$. Since $\ell^2(A)^\infty \cong \ell^2(A)$, this completes the proof. Q.E.D.

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Added in proofs. Since this paper was submitted, two relevant items from [47] have been brought to my attention. They are that adjointable morphisms of Hilbert modules are necessarily continuous ([47], Lemma 1.9), and that quotients of pro-C^*-algebras by closed ideals can in fact fail to be complete ([47], page 83).