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TWO C^*-ALGEBRAS RELATED TO THE DISCRETE HEISENBERG GROUP

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§ 1

In this paper we study two classes of C^*-algebras related to the discrete Heisenberg group, H. The first class consists of crossed product C^*-algebras of the group C^*-algebra of H, $C^*(H)$, with the group of integers, $\mathbf{Z}$, under $*$-automorphisms, α_θ, determined by (one-dimensional) characters, θ, of H. The second class consists of twisted group C^*-algebras, $C^*(H, \omega)$, determined by 2-cocycles, ω, of H. We want to classify these two classes of C^*-algebras up to $*$-isomorphism. Although the second class has been classified by J. A. Packer [3], our method is different from hers, and seems simpler, we feel.

Our basic tools are two K-theoretical invariants. The first one, called the trace invariant and denoted $T(\cdot)$, is defined for unital C^*-algebras as follows (cf. [7]):

$$T(A) = \bigcap_{\varphi} \exp \circ \varphi_*(\mathrm{K}_0(A)), \tag{1}$$

where φ runs through all tracial states of the C^*-algebra A, and exp is the exponential map $\mathbf{R} \to \mathrm{T}$, $t \to e^{2\pi i t}$. If A does not have any tracial state, we just define $T(A)$ to be $\mathbf{T}$, the whole unit circle. When the C^*-algebra A has unique tracial state, or when all tracial states of A agree at $\mathrm{K}_0(A)$, e.g., A is a noncommutative torus [1] there is no need to take the intersection over all tracial states in (1). In this case, the trace invariant was previously studied by Rieffel [5], Pimsner and Voiculescu [4], and Elliott [1]. The above definition, however, applies to more general situations. It was proved in [7] that for any discrete group G and any character θ of G, we always have

$$T(C^*(G) \times_{\alpha_\theta} \mathbf{Z}) = \theta(G). \tag{2}$$

We will need this result in the sequel.

Our second invariant, called the twist and denoted $t(\cdot)$, is defined for a smaller family of C^*-algebras. Let A be a unital C^*-algebra satisfying:

(i) A has tracial states τ and all tracial states agree on $K_0(A)$; and

(ii) [1] generates a free direct summand of $K_0(A)$.

Then we define (cf. [6], [7])

$$t(A) = \begin{cases} \text{dist}(\tau_*(e), \mathbf{Z}), & \text{if } \{x \in K_0(A) : \tau_*(x) \in \mathbf{Q}\} \simeq \mathbf{Z}^2, \\ 0, & \text{otherwise}. \end{cases}$$

where e is the other generator of the group $\{x \in K_0(A): \tau_*(x) \in \mathbf{Q}\}$ and dist denotes the usual distance on the real line $\mathbf{R}$. The twist $t(A)$ is a rational number in the interval $[0, 0.5]$. It was shown in [7] that $t(A)$ does not depend on the choice of e and that it is a $*$-isomorphism invariant of A.

The two invariants $T(\cdot)$ and $t(\cdot)$ taken together completely classify the crossed product C^*-algebras $C_r^*(F_n)\times_\theta \mathbf{Z}$ for all characters θ of the free groups F_n [6], [7]. This generalizes the classification of irrational and rational rotation C^*-algebras of [5], [4], [2]. In the present paper, we will show that these two invariants also classify $C^*(H, \omega)$ for all 2-cocycles ω (§ 3 below). For the C^*-algebras $C^*(H)\times_\theta \mathbf{Z}$, the twist does not give desired information directly. However, we can apply it to an appropriate quotient which is a noncommutative torus of dimension 3, and thus obtain the classification (§ 2 below).

The following notations and conventions will be used throughout the paper. The discrete Heisenberg group H is the group generated by two elements a, b, with the relation that $aba^{-1}b^{-1} = c$ is in the centre of H. Given any character θ of H (by a character we always mean a one-dimensional one), there is a unique $*$-automorphism α_θ of $C^*(H)$ such that $\alpha_\theta(U_g) = \theta(g)U_g$ for the canonical generators $\{U_g : g \in H\}$ of $C^*(H)$. When no confusion will result, we simply use θ to denote α_θ. For instance, we have the C^*-crossed products $C^*(H)\times_\theta \mathbf{Z}$. We use H_{ab} to denote the abelianization of H, which is isomorphic to $\mathbf{Z}^2$. It is well-known that the natural homomorphism $\text{Aut}(H) \to \text{Aut}(H_{\text{ab}}) \simeq GL(2, \mathbf{Z})$ of automorphism groups is onto. Hence we will not distinguish automorphisms φ on H_{ab} and its lifting on H. Thus we will speak of the determinant of φ in $\text{Aut}(H)$. Note that $\varphi(c) = c^{\det \varphi}$.

§ 2

In this section we classify $C^*(H)\times_\theta \mathbf{Z}$ up to $*$-isomorphism for all characters θ. We need some preliminary results.

Given a discrete group G and a $\varphi \in \text{Aut}(G)$, there exists a unique $*$-automorphism α_φ of the reduced group C^*-algebra $C_r^*(G)$ satisfying $\alpha_\varphi(U_g) = U_{\varphi(g)}$, $g \in G$.

LEMMA 1. *Any eigenvalue λ of α_φ is a root of unity. Moreover, if $x \in C_r^*(G)$ is a nonzero eigenvector of α_φ associated to λ and if $g \in G$ is in the support of x, then the order of λ is no greater than the cardinality of the φ-orbit of g.*

Proof. Assume $\alpha_\varphi(x) = \lambda x$, where $\lambda \in \mathbf{C}$ and $0 \neq x \in C^*_r(G)$. It is well-known that we can view x as an element of the Hilbert space $\ell^2(G)$, that is, $x = \sum_{g\in G} C_g f_g$, where $\{f_g : g \in G\}$ is the canonical orthonormal basis of $\ell^2(G)$, $C_g \in \mathbf{C}$ and $\sum_g |C_g|^2 < \infty$. Since $\alpha_\varphi(f_g) = f_{\varphi(g)}$, we obtain

$$\sum_g C_g f_{\varphi(g)} = \lambda \sum_g C_g f_g = \lambda \sum_g C_{\varphi(g)} f_{\varphi(g)}.$$

Therefore

$$\lambda C_{\varphi(g)} = C_g, \quad g \in G. \tag{3}$$

In particular, $|C_{\varphi(g)}| = |C_g|$ since $|\lambda| = 1$. The condition that $\sum_g |C_g|^2 < \infty$ then implies that x is supported on finite φ-orbits. Let $\{g_0, g_1, \ldots, g_{s-1}\}$ be such an orbit, where $\varphi(g_i) = g_{i+1}$, $g_s = g_0$, and c_i $(= C_{g_i})$ are all nonzero. By (3) we see that

$$c_0 = \lambda c_1 = \ldots = \lambda^{s-1} c_{s-1} = \lambda^s c_0 .$$

This proves that λ is a root of unity with order no greater than the cardinality s of the orbit of g_0. Q.E.D.

We denote the centre of a C^*-algebra A by $Z(A)$.

LEMMA 2. *Suppose θ is a character of a discrete group G. If $\theta(G)$ is an infinite set, then*

$$Z(C^*_r(G) \times_\theta \mathbf{Z}) = C^*_r(\ker\theta) \cap Z(C^*_r(G)),$$

where $C^*_r(\ker\theta)$ is identified with a C^*-subalgebra of $C^*_r(G)$ in a natural way.

Proof. Assume x is in the centre of $C^*_r(G) \times_\theta \mathbf{Z}$. Let E be the conditional expectation from $C^*_r(G) \times_\theta \mathbf{Z}$ onto $C^*_r(G)$, and let $E_k(x) = E(x W^{*k})$, $k \in \mathbf{Z}$, where W is the unitary implementing α_θ. Then all $E_k(x)$ are in the fixed point algebra of the $*$-automorphism α_θ, which is just the C^*-algebra $C^*_r(\ker\theta)$ by [7]. From the bimodule property of E, we have, for any $g \in G$,

$$U_g E_k(x) U_g^* = E(x U_g W^{*k} U_g^*) = \theta(g)^k E_k(x).$$

Thus if $E_k(x) \neq 0$, it would be an eigenvector of the inner $*$-automorphism $\operatorname{Ad} U_g$ of $C^*_r(G)$ with respect to the eigenvalue $\theta(g)^k$. However, $\operatorname{Ad} U_g = \alpha_\varphi$ with $\varphi = \operatorname{Ad} g \in \operatorname{Aut}(G)$. Now let $h \in G$ be in the support of $E_k(x)$ and let g above run over all G. The union of all orbits of h under $\operatorname{Ad} g$ is just the conjugacy class of h in G. From the proof of Lemma 1, we see that all coefficients of $E_k(x)$ which are supported on this conjugacy class have the same absolute value, and hence this conjugacy class is of finite cardinality, say s. Since $\theta(G)$ is infinite by hypothesis, we can find, for fixed $k \neq 0$, some $g \in G$ such that $\theta(g)^{kl} \neq 1$ for $1 \leqslant l \leqslant s$. From Lemma 1, the order of the eigenvalue $\theta(g)^k$ is no greater than the cardinality of the $\operatorname{Ad} g$-orbit of h, the latter is no greater than s. Thus there is some l, $1 \leqslant l \leqslant s$, satisfying $\theta(g)^{kl} = 1$, contradicting the choice of g. This forces $E_k(x) = 0$ if $k \neq 0$. Hence $x = E_0(x) \in C_r^*(G)$. This completes the proof of one direction. The reverse is obvious since α_θ acts trivially on $C_r^*(\ker \theta)$. Q.E.D.

Return to $C^*(H) \times_\theta \mathbf{Z}$. Since $c = aba^{-1}b^{-1}$ is in the centre of H and since $\theta(c) = 1$, the C^*-algebra generated by U_c, $C^*(\{U_c\})$, is contained in the centre of $C^*(H) \times_\theta \mathbf{Z}$.

COROLLARY 3. *The centre of $C^*(H) \times_\theta \mathbf{Z}$ is $C^*(\{U_c\})$ ($\simeq C(\mathbf{T})$) if $\theta(H)$ is an infinite set, and it is $C^*(\{U_c, W^q\})$ ($\simeq C(\mathbf{T}^2)$) if $\theta(H)$ is finite of order q, where W is the unitary implementing α_θ.*

Proof. Note that $Z(C^*(H)) = C^*(\{U_c\}) \subseteq C^*(\ker \theta)$. If $\theta(H)$ is infinite, we have, from Lemma 2, that

$$Z(C^*(H) \times_\theta \mathbf{Z}) = C^*(\ker \theta) \cap Z(C^*(H)) = C^*(\{U_c\}).$$

If $\theta(H)$ is finite of order q, we argue as in the proof of Lemma 2 and use the fact that if $g \in H$ is not in the centre, then g is of infinite conjugacy class. This will show $Z(C^*(H) \times_\theta \mathbf{Z}) \subseteq C^*(\{U_c, W^q\})$. The reverse is obvious. Q.E.D.

THEOREM 4. *$C^*(H) \times_{\theta_1} \mathbf{Z} \simeq C^*(H) \times_{\theta_2} \mathbf{Z}$ if and only if $\theta_1 = \theta_2 \circ \varphi$ for some $\varphi \in \operatorname{Aut}(H)$.*

Proof. Let $A_j = C^*(H) \times_{\theta_j} \mathbf{Z}$ and let $f: A_1 \to A_2$ be a $*$-isomorphism. By (2), the trace invariant of A_j is $T(A_j) = \theta_j(H)$. Hence $\theta_1(H) = \theta_2(H)$. Let $\dot{\theta}_j$ be the character of H_{ab} induced from θ_j. We have $\dot{\theta}_1(H_{\mathrm{ab}}) = \dot{\theta}_2(H_{\mathrm{ab}})$. If the free rank of $\dot{\theta}_j(H_{\mathrm{ab}})$ is zero or two, we can find some $\psi \in \mathrm{GL}(2, \mathbf{Z})$ such that $\dot{\theta}_1 = \dot{\theta}_2 \circ \psi$ by [7]. Let φ be a lifting of ψ to H. We obtain $\theta_1 = \theta_2 \circ \varphi$. Thus in the following we can assume that $\dot{\theta}_j(H_{\mathrm{ab}})$ has free rank one. Let $\lambda \in \mathbf{T}$ be a generator of the free part of $\dot{\theta}_j(H_{\mathrm{ab}})$. By [7; 4.2], we can find $\psi_j \in \mathrm{GL}(2, \mathbf{Z})$ such that $\dot{\theta}_j \circ \psi_j$ is in its standard form, that is,

$$\dot{\theta}_j \circ \psi_j([a]) = \lambda \quad \text{and} \quad \dot{\theta}_j \circ \psi_j([b]) = \mathrm{e}^{2\pi i p_j/q}, \quad j = 1, 2,$$

with $(p_j, q) = 1$ and $0 \leqslant p_j \leqslant [q/2]$. Here $[a]$, $[b]$ denote the image of a, b in H_{ab}. Since θ_j and $\theta_j \circ \psi_j$ give rise to $*$-isomorphic crossed product C^*-algebras, we can assume that θ_j is already in its standard form for notational ease. Now $\theta_j(H)$ is infinite, hence the centre of A_j is $C^*(\{U_c\}) \simeq C(\mathbf{T})$ by Corollary 3. Choose $\xi_1 \in \mathbf{T}$ such that the free rank of the subgroup of $\mathbf{T}$ generated by ξ_1 and $\theta_1(H)$

is two. Let M_1 be a maximal ideal of $C^*(\{U_c\})$ such that the quotient map $C^*(\{U_c\}) \to C^*(\{U_c\})/M_1$ takes U_c to ξ_1. Let I_1 be the closed two-sided ideal of A_1 generated by M_1. Then A_1/I_1 is a noncommutative torus of dimension three. In fact, A_1/I_1 is the C^*-algebra generated by three unitaries $[U_a]$, $[U_b]$ and $[W]$ with the relations

$$[W][U_a][W]^* = \lambda[U_a],$$

$$[W][U_b][W]^* = e^{2\pi i p_1/q}[U_b],$$

and

$$[U_a][U_b] = \xi_1[U_b][U_a],$$

the last equation coming from $U_aU_b = U_cU_bU_a$ in A_1. Let A_{ρ_1} be the noncommutative torus of dimension three with structure coefficients $\{\lambda, e^{2\pi i p_1/q}, \xi_1\}$. Then there is a $*$-homomorphism from A_{ρ_1} onto A_1/I_1. Since the structure coefficients given above determine a nondegenerate character ρ_1 of $\mathbf{Z}^3 \wedge \mathbf{Z}^3$, A_{ρ_1} is simple [1], and hence $A_{\rho_1} \simeq A_1/I_1$ in a canonical way. Applying the Pimsner-Voiculescu six term exact sequence [4], we obtain (cf. [1]) that

$$\mathrm{K}_0(A_1/I_1) = \mathbf{Z}[1] \oplus \mathbf{Z}[P_a] \oplus \mathbf{Z}[P_b] \oplus \mathbf{Z}[P_{\xi_1}],$$

where P_a, P_b, and P_{ξ_1} are Rieffel projections in A_{ρ_1} such that if τ is a tracial state of A_{ρ_1}, then

$$\exp\circ\tau(P_a) = \lambda, \quad \exp\circ\tau(P_b) = e^{2\pi i p_1/q}, \quad \exp\circ\tau(P_{\xi_1}) = \xi_1\,. \tag{4}$$

Now we apply the twist to A_1/I_1. By our choice of ξ_1, we have

$$\{x \in \mathrm{K}_0(A_1/I_1) : \tau_*(x) \in \mathbf{Q}\} = \mathbf{Z}[1] \oplus \mathbf{Z}[P_b].$$

Hence

$$t(A_1/I_1) = \mathrm{dist}(\tau_*([P_b]), \mathbf{Z}) = p_1/q.$$

Since the $*$-isomorphism $f: A_1 \to A_2$ takes centre to centre, it takes M_1 to a maximal ideal M_2 of the centre $Z(A_2)$. Assume U_c goes to $\xi_2 \in \mathbf{T}$ under $Z(A_2) \to Z(A_2)/M_2$. Similarly, we get a quotient of A_2, A_2/I_2. Clearly $A_1/I_1 \xrightarrow{\sim} A_2/I_2$. Hence they have the same trace invariant $T(A_1/I_1) = T(A_2/I_2)$. From (4) we see that $T(A_j/I_j)$ is just the subgroup of $\mathbf{T}$ generated by $\theta_j(H)$ and ξ_j. Thus $\theta_2(H)$ and ξ_2 also generate a group of free rank two. A similar computation then gives $t(A_2/I_2) = p_2/q$. From $t(A_1/I_1) = t(A_2/I_2)$, we obtain $p_1 = p_2$. This implies that $\theta_1 = \theta_2$ after standardization. Therefore $A_1 \simeq A_2$ implies $\theta_1 = \theta_2\circ\varphi$ for some $\varphi \in \mathrm{Aut}(H)$. The converse is straightforward. Q.E.D.

§ 3

In this section we classify $C^*(H, \omega)$, the twisted group C^*-algebra of the discrete Heisenberg group H, up to $*$-isomorphism for all 2-cocycles ω. We show that the $*$-isomorphism class of $C^*(H, \omega)$ is determined by a character θ_ω of H up to some equivalence, where θ_ω is defined by

$$\theta_\omega(g) = \frac{\omega(c, g)}{\omega(g, c)}, \quad g \in H.$$

This defines a character because c is in the centre of H. Now θ_ω induces a character $\dot{\theta}_\omega$ on $H_{\mathrm{ab}} \simeq \mathbf{Z}^2$. In [7; § 4] a twist, denoted $t(\cdot)$, is defined for characters of $\mathbf{Z}^n$. This is a rational number in the interval [0, 0.5] which is invariant under automorphisms of $\mathbf{Z}^n$. Here we define the twist of θ_ω, $t(\theta_\omega)$, by setting $t(\theta_\omega) = t(\dot{\theta}_\omega)$. Since automorphisms of H factor to H_{ab}, we see that $t(\theta_\omega)$ is also invariant under automorphisms of H.

The following theorem is essentially due to Packer [3].

THEOREM 5. *Suppose ω_1 and ω_2 are two 2-cocycles of H. The following are equivalent:*

(i) $C^*(H, \omega_1) \simeq C^*(H, \omega_2)$;

(ii) *These two C^*-algebras have the same trace invariant and the same twist;*

(iii) $\theta_{\omega_1}(H) = \theta_{\omega_2}(H)$ *and* $t(\theta_{\omega_1}) = t(\theta_{\omega_2})$;

(iv) $\theta_{\omega_1} = \theta_{\omega_2} \circ \varphi$ *for some* $\varphi \in \mathrm{Aut}(H)$;

(v) ω_1 *is cohomological to* $\omega_2 \circ \varphi$ *for some* $\varphi \in \mathrm{Aut}(H)$.

Proof. We first recall some general facts about the K-theory of $C^*(H, \omega)$, which are contained in [3]. $C^*(H, \omega)$ is the C^*-algebra generated by three unitaries $\{U_a, U_b, U_c\}$ with the relation

$$U_g U_h = \omega(g, h) U_{gh}, \quad g, h \in H.$$

From this, a simple computation gives

$$U_a U_c = \theta_\omega(a) U_c U_a,$$

$$U_b U_c = \theta_\omega(b) U_c U_b,$$

and

$$U_a U_b = \xi U_c U_b U_a,$$

where $\xi \in \mathbf{T}$ is a scalar. Note that the C^*-subalgebra generated by U_a and U_c, $C^*(\{U_a, U_c\})$, is a rotation algebra. Thus $C^*(H, \omega)$ is isomorphic to $C^*(\{U_a, U_c\}) \times_\gamma \mathbf{Z}$, where γ is the composition of an action dual to the rotation with an action arising from $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbf{Z})$. The Pimsner-Voiculescu six term exact sequence enables us quickly to get

$$K_0(C^*(H, \omega)) = \mathbf{Z}[1] \oplus \mathbf{Z}[P_a] \oplus \mathbf{Z}[P_b], \tag{5}$$

where P_a and P_b are Rieffel projections sitting in the rotation algebras $C^*(\{U_a, U_c\})$ and $C^*(\{U_b, U_c\})$ respectively, such that for any tracial state τ of $C^*(H, \omega)$,

$$\exp \circ \tau(P_a) = \theta_\omega(a), \quad \exp \circ \tau(P_b) = \theta_\omega(b). \tag{6}$$

Therefore, all tracial states agree on $K_0(C^*(H, \omega))$.

Now we proceed to prove the theorem.

(i) $\Rightarrow$ (ii). From above knowledge of $K_0(C^*(H, \omega))$, we see that the twist is defined for $C^*(H, \omega)$.

(ii) $\Rightarrow$ (iii). From (5) and (6), we obtain $T(C^*(H, \omega)) = \theta_\omega(H)$. It remains to show that $t(C^*(H, \omega)) = t(\theta_\omega)$. If $\theta_\omega(H)$ has free rank two or zero, it is easy to see that $t(C^*(H, \omega)) = 0 = t(\theta_\omega)$. When $\theta_\omega(H)$ has free rank one, we can find $\varphi \in \mathrm{Aut}(H)$, as in the proof of Theorem 4, such that

$$\theta_\omega \circ \varphi(a) = \lambda, \quad \theta_\omega \circ \varphi(b) = e^{2\pi i p/q},$$

with $\lambda \in \mathbf{T}$ of infinite order, $(p, q) = 1$ and $0 \leqslant p \leqslant [q/2]$. It follows that $t(\theta_\omega) = t(\theta_\omega \circ \varphi) = p/q$. Note that if $\det \varphi = 1$, then $\varphi(c) = c$ and hence $\theta_\omega \circ \varphi = \theta_{\omega \circ \varphi}$; if $\det \varphi = -1$, let $\psi = \varphi \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and we have $\theta_\omega \circ \psi = \theta_{\omega \circ \psi}$. However,

$$\theta_{\omega \circ \psi}(a) = \theta_\omega \circ \varphi(a) = \lambda,$$

and

$$\theta_{\omega \circ \psi}(b) = \theta_\omega \circ \varphi(b^{-1}) = e^{2\pi i (q-p)/q}.$$

As in the proof of Theorem 4, we can compute the twist of $C^*(H, \omega \circ \varphi)$ and $C^*(H, \omega \circ \psi)$. In both cases, it is p/q. Since $C^*(H, \omega)$ is isomorphic to both $C^*(H, \omega \circ \varphi)$ and $C^*(H, \omega \circ \psi)$, we obtain $t(C^*(H, \omega)) = p/q = t(\theta_\omega)$.

(iii) $\Rightarrow$ (iv). This follows from [7; § 4]. The results there on $\mathbf{Z}^2$ can be lifted to H.

(iv) $\Rightarrow$ (v). If $\det \varphi = 1$, $\theta_{\omega_2} \circ \varphi = \theta_{\omega_2 \circ \varphi}$. If $\det \varphi = -1$, then $\varphi(c) = c^{-1}$, and hence $\theta_{\omega_2} \circ \varphi = \overline{\theta}_{\omega_2 \circ \varphi}$ because of $\omega_2(g^{-1}, h^{-1}) = \overline{\omega_2(h, g)}$. Let $\psi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\overline{\theta}_{\omega_2 \circ \varphi} = \theta_{\omega_2 \circ \varphi} \circ \psi = \theta_{\omega_2 \circ \varphi \circ \psi}$. Hence $\theta_{\omega_1} = \theta_{\omega_2} \circ \varphi$ implies $\theta_{\omega_1} = \theta_{\omega_2 \circ \varphi}$ or

$\theta_{\omega_1} = \theta_{\omega_2 \circ \varphi \circ \psi}$. Since the map $\omega \to \theta_\omega$ only depends on the cohomology class of ω, we have a group homomorphism $H^2(H; T) \to \hat{H}$, where $\hat{H}$ is the abelian group of all characters of H under pointwise operation. The above map is, in fact, a group isomorphism. This is a pure algebraical result, which is almost implicitly contained in [3]. From this isomorphism, we get $[\omega_1] = [\omega_2 \circ \varphi]$ or $[\omega_1] = [\omega_2 \circ \varphi \circ \psi]$.

(v) $\Rightarrow$ (i). Well-known. Q.E.D.

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REFERENCES

1. Elliott, G. A., On the K-theory of the C^*-algebra generated by a projective representation of a torsion-free abelian group, in *Operator algebras and group representations*, Vol. I, Pitman, 1984, pp. 157–184.
2. Høegh-Krohn, R.; Skjelbred, T., Classification of C^*-algebras admitting ergodic actions of the two-dimensional torus, *J. Reine Angew. Math.*, **328**(1981), 1–8.
3. Packer, J. A., C^*-algebras generated by projective representations of the discrete Heisenberg group, *J. Operator Theory*, **18**(1987), 41–66.
4. Pimsner, M.; Voiculescu, D., Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras, *J. Operator Theory*, **4**(1980), 93–118.
5. Rieffel, M., C^*-algebras associated with irrational rotations, *Pacific J. Math.*, **93**(1981), 415–429.
6. Yin, H. S., Classification of C^*-crossed products associated with characters on free groups, *C. R. Math. Rep. Acad. Sci. Canada*, **9**(1987), 1–4.
7. Yin, H. S., Classification of crossed product C^*-algebras associated with characters on free groups, *Trans. Amer. Math. Soc.*, to appear.

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