J. OPERATOR THEORY
19(1988), 283–305



# COMPACT GROUP ACTIONS ON $C^*$-ALGEBRAS

SERGIO DOPLICHER and JOHN E. ROBERTS

## INTRODUCTION

If one is given a family of Hilbert spaces of finite dimension together with linear mappings between them satisfying suitable axioms, the classical Tannaka-Krein duality theorems for a compact group will allow one to recognize this structure as the representation theory of a compact group $G$. The Hilbert spaces can then be endowed with continuous unitary representations of $G$ in such a way that the given linear mappings correspond precisely to the intertwining operators between these representations.

The problem of proving the existence of a compact group of internal gauge symmetries in quantum field theory pinpoints the inadequacy of these classical theorems. In fact, the $C^*$-algebra of local observables defines through its superselection structure an abstract structure akin to the representation category of a compact group but where the objects are not explicitly associated with finite dimensional Hilbert spaces. Instead, the objects of the category form a semigroup of endomorphisms of a $C^*$-algebra $\mathscr{A}$ and the arrows of the category are the intertwining operators between these endomorphisms, i.e. we have a full subcategory of the monoidal $C^*$-category $\mathrm{End}\,\mathscr{A}$ of endomorphisms of $\mathscr{A}$.

Hence it is natural to take as the input for the abstract duality theory of compact groups monoidal $C^*$-categories fulfilling suitable axioms related to permutation symmetry and the existence of conjugates. It will be shown elsewhere that such a category can be embedded in the category of endomorphisms of a $C^*$-algebra.

This step puts the problem of formulating an abstract duality theory for compact groups into the framework of $C^*$-algebras and reduces it to the problem met in quantum field theory. We shall show elsewhere [1] how a $C^*$-algebra $\mathscr{A}$ with centre $\mathbf{C}I$ and a subcategory of $\mathrm{End}\,\mathscr{A}$ fulfilling natural axioms lead to a compact group $G$ given as a group of automorphisms of a $C^*$-algebra $\mathscr{B}$ containing $\mathscr{A}$ as its fixed-point algebra under $G$. To the objects $\rho$ in the given category, a semigroup of endomorphisms of $\mathscr{A}$, now correspond explicit Hilbert

[1] [6]; cf. the announcements in [3], [4].

spaces $H_\rho$ in $\mathscr{B}$ which are stable under $G$ and induce on $\mathscr{A}$ the endomorphisms $\rho$. In other words, a concrete representation category for $G$ is realized within the $C^*$-algebra $\mathscr{B}$.

In fact, as we shall see in § 7, any concrete representation theory may be realized in this way within a suitable $C^*$-algebra, a $C^*$-tensor product of Cuntz algebras. Thus the $C^*$-systems $(\mathscr{B}, G)$ as above provide a model for a concrete representation category whereas the fixed-point algebras $\mathscr{A}$ together with the endomorphisms induced by the Hilbert spaces provide a model for the abstract representation category.

After these motivating remarks we turn to the specific contents of this paper. It treats certain aspects of the spectral analysis of the actions $\alpha$ of compact groups $G$ on infinite $C^*$-algebras with unit $\mathscr{B}$. The main technical tool in analyzing the $C^*$-systems $(\mathscr{B}, \alpha)$ will be to look at $\alpha$-stable Hilbert spaces $H$ within $\mathscr{B}$. These Hilbert $G$-modules $H$ defined in Section 2 combine spectral information, the equivalence class of the action of $G$ on $H$, with the algebraic property of being a Hilbert space.

The interest centres here on certain classes of actions typical of $C^*$-systems which arise as the cross product by an action of the dual of a compact group on the fixed-point algebra. Thus attention is directed to the property of having full Hilbert spectrum, i.e. each equivalence class of irreducible continuous unitary representations of $G$ is realized on some Hilbert $G$-module $H$. Another property studied here and characteristic of a class of cross products is for the relative commutant of the fixed-point algebra to be trivial.

After discussing the Hilbert spectrum of an action (Section 1) and the categories associated with Hilbert spaces in a $C^*$-algebra (Section 2), we prove variants of the Tannaka-Krein theorems in our framework. When a representation theory is realized within a $C^*$-algebra $\mathscr{B}$, such theorems give conditions for a compact group $G$ to agree with the stabilizer in $\mathrm{Aut}\,\mathscr{B}$ of the fixed-point algebra (Theorem 3.3 and Corollary 6.5).

In Section 4 we study a Galois correspondence between subalgebras of $\mathscr{B}$ and normal subgroups of $G$.

In Section 5 we characterize $C^*$-systems with $\mathscr{B}^{\alpha\prime} \cap \mathscr{B} = \mathscr{B}' \cap \mathscr{B}$; this condition plays an important role in the construction, to be given elsewhere, of minimal cross products with $\mathscr{B}^{\alpha\prime} \cap \mathscr{B} = \mathbf{C}I$ [3], [5], [6].

In Section 6 we study $C^*$-systems $(\mathscr{B}, \alpha)$ where sufficiently many representations of determinant one are realized on Hilbert spaces within $\mathscr{B}$. This condition replaces the stability of the spectrum under conjugation.

Section 7 gives examples of $C^*$-systems constructed from representation theories which illustrate the setting of this paper. It is shown (Theorem 7.3) that under natural conditions the $C^*$-algebras of fixed-points are simple.

A further paper will complement this analysis of $C^*$-systems with a synthesis of $C^*$-systems as crossed products by the action of the duals of compact groups [6].

## 1. THE SPECTRAL CATEGORY

A *$C^*$-system* is a pair $(\mathscr{B}, \alpha)$ consisting of a $C^*$-algebra $\mathscr{B}$ and a continuous homomorphism $\alpha : G \to \mathrm{Aut}\mathscr{B}$ of a topological group $G$ into the group $\mathrm{Aut}\mathscr{B}$ of automorphisms of $\mathscr{B}$ equipped with the strong topology, i.e. the topology of pointwise norm convergence. We will suppose that the $C^*$-algebra $\mathscr{B}$ has a unit $I$ and $G$ will usually be a compact group. We are, of course, interested here in aspects of spectral analysis where the $C^*$-nature of $\mathscr{B}$ plays a special role. Hence we may as well assume that the action $\alpha$ is faithful and that $G$ is an automorphism group. In this case, we denote the $C^*$-system by $(\mathscr{B}, G)$ rather than $(\mathscr{B}, \alpha)$.

A first step towards spectral analysis is to consider a multiplet from $\mathscr{B}$ transforming according to some continuous, finite dimensional, unitary matrix representation $\xi$ or, as we shall say, a *$\xi$-tensor* with values in $\mathscr{B}$. A $\xi$-tensor $B$ is a multiplet $B_1, B_2, \ldots, B_d$ of elements of $\mathscr{B}$, where $d$ is the dimension of $\xi$, such that

$$\alpha_g(B_i) = \sum_{j=1}^{d} B_j \xi_{ji}(g).$$

As $G$ is compact, the entries of irreducible tensors with values in $\mathscr{B}$ form a total set in $\mathscr{B}$ (cf. remarks following Proposition 2.2 of [2]).

The $\xi$-tensors with values in $\mathscr{B}$ form a linear space $\mathscr{B}_\xi$ which can also be looked on as follows: let $H_\xi$ denote the underlying Hilbert space of $\xi$ and let $\iota$ denote the trivial representation of $G$ on $\mathbf{C} = H_\iota$. Then if $(H_\xi, H_\iota)$ denotes the set of linear mappings from $H_\xi$ to $H_\iota$, $(H_\xi, H_\iota) \otimes \mathscr{B}$ may be regarded as a space of matrices with entries from $\mathscr{B}$ and carries a natural action of $G$

$$g(L \otimes B) = L\xi(g)^* \otimes \alpha_g(B).$$

The set of fixed points under this action is precisely $\mathscr{B}_\xi$. This way of looking at $\mathscr{B}_\xi$ is independent of the choice of basis in $\xi$ and makes sense for an arbitrary finite-dimensional, continuous unitary representation. Hence in what follows $\xi$ is not restricted to being a matrix representation. Note, further, that $\mathscr{B}_\iota$ is just the fixed-point algebra. The spaces $\mathscr{B}_\xi$, with $\xi$ irreducible, are one possible generalization of spectral subspaces to the action of non-Abelian groups (cf. [8; § 2]).

The space $\mathscr{B}_\xi$ of $\xi$-tensors is best looked at as a space of arrows in a $C^*$--category, the spectral category $\mathrm{Sp}(\mathscr{B}, \alpha)$ [8] of the $C^*$-system $(\mathscr{B}, \alpha)$. Its objects are written $\xi \otimes \alpha$[1], where $\xi$ is a finite-dimensional, continuous unitary representation of $G$. The set $(\xi \otimes \alpha, \eta \otimes \alpha)$ of arrows in $\mathrm{Sp}(\mathscr{B}, \alpha)$ from $\xi \otimes \alpha$ to $\eta \otimes \alpha$

[1] This notation will be retained even where the $C^*$-system is denoted $(\mathscr{B}, G)$.

in $\mathrm{Sp}(\mathscr{B}, \alpha)$ is the fixed-point set of $(H_\xi, H_\eta) \otimes \mathscr{B}$ under the natural action,

$$g(L \otimes B) = \eta(g)L\xi(g)^* \otimes \alpha_g(B),$$

of $G$. The composition law for arrows and the definitions of adjoint and norm are the obvious ones and serve to make $\mathrm{Sp}(\mathscr{B}, \alpha)$ into a $C^*$-category. In terms of the spectral category we have

$$\mathscr{B}_\xi = (\xi \otimes \alpha,\ \iota \otimes \alpha); \quad \mathscr{B}_\iota = \mathscr{B}^\alpha = (\iota \otimes \alpha,\ \iota \otimes \alpha).$$

The spectral category leads naturally to a number of spectral invariants (cf. [8; § 2]). The natural definition of *spectrum* in this framework is

$$\mathrm{Sp}(\alpha) = \{\xi : \mathscr{B}_\xi = (\xi \otimes \alpha,\ \iota \otimes \alpha) \neq 0\}.$$

The $C^*$-system will be said to have *full spectrum* if $\mathrm{Sp}(\alpha)$ contains each irreducible, continuous unitary representation. These definitions do not depend on whether $\mathscr{B}$ is a $C^*$-algebra so they do not tell much about the $C^*$-system $(\mathscr{B}, \alpha)$.

We define the *Hilbert spectrum* of $(\mathscr{B}, \alpha)$ to be

$$\mathrm{HSp}(\alpha) = \{\xi : \xi \otimes \alpha \leqslant \iota \otimes \alpha\},$$

e. $\xi \in \mathrm{HSp}(\alpha)$ if $\mathscr{B}_\xi = (\xi \otimes \alpha,\ \iota \otimes \alpha)$ contains an isometry. The $C^*$-system will be said to have *full Hilbert spectrum* if $\mathrm{HSp}(\alpha)$ contains each irreducible, continuous unitary representation.

If $\mathscr{B}_\xi$ contains an isometry $\underline{\psi}$ and $\underline{B} \in \mathscr{B}_\xi$ then we have $\underline{B} = (\underline{B}\underline{\psi}^*)\underline{\psi}$ so that every element $\underline{B}$ of $\mathscr{B}_\xi$ is of the form $A\underline{\psi}$, where $A \in \mathscr{B}^\alpha = \mathscr{B}_\iota$. Expressing $A$ as a linear combination of unitaries, we also see that every $\underline{B} \in \mathscr{B}_\xi$ is a linear combination of at most four isometries from $\mathscr{B}_\xi$.

In terms of the entries of $\underline{\psi}$, the condition $\underline{\psi}^*\underline{\psi} = 1_{\xi\otimes\alpha}$ reads

$$\psi_i^*\psi_j = \delta_{ij}I, \quad i,j = 1, 2, \ldots, d \tag{1.1}$$

where $d$ is the dimension of $\xi$. In other words, the entries form a multiplet of isometries with pairwise orthogonal ranges. The linear span of entries $\psi_1, \psi_2, \ldots, \psi_d$ is now an $\alpha$-invariant Hilbert space $H$ in $\mathscr{B}$ carrying a representation of class $\xi$. If $\underline{\psi} \in \mathscr{B}_\xi$ is actually unitary, the condition $\underline{\psi}\underline{\psi}^* = 1_{\iota\otimes\alpha}$ holds too and reads in terms of the entries

$$\sum_{i=1}^{d} \psi_i\psi_i^* = I \tag{1.2}$$

and we have a Hilbert space with support $I$. Hilbert spaces in $C^*$-systems form the subject of the next section.

## 2. HILBERT SPACES IN $C^*$-SYSTEMS

By a Hilbert space $H$ in a $C^*$-algebra with unit $\mathscr{B}$ we understand a closed subspace of $\mathscr{B}$ with $\psi^*\psi \in \mathbf{C}I$ for $\psi \in H$. The scalar product on the Hilbert space $H$ is then given by

$$(\psi, \psi')I = \psi^*\psi'. \tag{2.1}$$

Thus if $\psi_1, \psi_2, \ldots$ is an orthonormal basis in $H$ then (1.1) is satisfied. In this paper, we shall solely be concerned with finite-dimensional Hilbert spaces. If we pick an orthonormal basis $\psi_1, \psi_2, \ldots, \psi_d$ for $H$ then the projection

$$1_H := \sum_{i=1}^{d} \psi_i\psi_i^* \tag{2.2}$$

is independent of the choice of orthonormal basis and is called the *support* [1)] of $H$. We have

$$1_H\psi = \psi, \quad \psi \in H \tag{2.3}$$

and an element $\psi'$ of $\mathscr{B}$ lies in $H$ if and only if $1_H\psi' = \psi'$ and $\psi^*\psi' \in \mathbf{C}I$ for each $\psi \in H$.

Given finite-dimensional Hilbert spaces $H$ and $H'$ in $\mathscr{B}$ the set $(H, H')$ of linear mappings from $H$ to $H'$ can and will be realized as the set of elements $T$ of $\mathscr{B}$ satisfying $TH \subset H'$, $T = T1_H = 1_{H'}T$. In particular $1_H$, the support of $H$, is just the identity mapping on $H$ as the notation suggests. We denote by $\mathscr{H}(\mathscr{B})$ the $C^*$-category of finite-dimensional Hilbert spaces in $\mathscr{B}$ realized in this way (cf. [11; Example 3.3]).

The $C^*$-algebra generated by a $d$-dimensional Hilbert space, $d > 1$, with support $I$ in a $C^*$-algebra is a universal simple $C^*$-algebra $\mathcal{O}_d$, the Cuntz algebra [1]. We write $\mathcal{O}_H$ in place of $\mathcal{O}_d$, if it is necessary to stress the generating Hilbert space $H$. If the generating finite-dimensional Hilbert space $H$ does not have support $I$, the $C^*$-algebra $\mathcal{O}_H$ is no longer simple but we have instead an exact sequence [1], [7]

$$0 \to \mathscr{K} \to \mathcal{O}_H \to \mathcal{O}_d \to 0.$$

A 1-dimensional Hilbert space with support $I$ in a $C^*$-algebra $\mathscr{B}$ is just the scalar multiples of a unitary and therefore has canonically associated with it an inner automorphism of $\mathscr{B}$. A finite-dimensional Hilbert space $H$ in $\mathscr{B}$ generates an endomorphism $\sigma_H$ through

$$\sigma_H(B) = \sum_i \psi_i B\psi_i^*, \quad B \in \mathscr{B} \tag{2.4}$$

---

1) The Hilbert spaces in von Neumann algebras first met in Quantum Field Theory [2] and generally introduced in [11, § 2] have support $I$ by definition.

where the $\psi_i$ run through an orthonormal basis in $H$. Note that $\sigma_H(I)$ is just the support of $H$. We also have

$$\psi B = \sigma_H(B)\psi, \quad \psi \in H, \ B \in \mathscr{B}. \tag{2.5}$$

The endomorphisms of the form $\sigma_H$ may be regarded as the inner endomorphisms of $\mathscr{B}$. If $T \in (H, H')$ then

$$T\sigma_H(B) = \sigma_{H'}(B)T, \quad B \in \mathscr{B}. \tag{2.6}$$

There is a natural definition of the tensor product of two Hilbert spaces $H_1$, $H_2$ in $\mathscr{B}$: $H_1 \otimes H_2$ is the closed linear span of elements of the form $\psi_1\psi_2$ with $\psi_1 \in H_1$ and $\psi_2 \in H_2$. When $H_1$ and $H_2$ are finite dimensional so is $H_1 \otimes H_2$ and we have

$$\sigma_{H_1 \otimes H_2} = \sigma_{H_1}\sigma_{H_2}. \tag{2.7}$$

If we define the tensor product of linear mappings $T_1 \in (H_1, H_1')$, $T_2 \in (H_2, H_2')$ by

$$T_1 \otimes T_2 = T_1\sigma_1(T_2) = \sigma_1'(T_2)T_1 \tag{2.8}$$

then $\mathscr{H}(\mathscr{B})$ becomes a monoidal $C^*$-category. Since we are dealing with a category of Hilbert spaces, there is also a natural symmetry associated with this monoidal structure: the operator which permutes the order of factors in the tensor product is denoted $\theta(H, H')$ and is given by

$$\theta(H, H') = \sum_{i,j} \psi_i'\psi_j\psi_i'^*\psi_j^*, \tag{2.9}$$

where the sum is taken over orthonormal bases $\psi_j$, $j = 1, 2, \ldots, d$ and $\psi_i'$, $i = 1, 2, \ldots, d'$ of $H$ and $H'$ respectively.

Let $\mathscr{S}(\mathscr{B})$ denote the $C^*$-category whose objects are the inner endomorphisms of $\mathscr{B}$, i.e. the endomorphisms induced by the objects of $\mathscr{H}(\mathscr{B})$, and whose arrows are the intertwiners between these endomorphisms, i.e. $T \in (\sigma, \sigma')$ if $T = T\sigma(I) = \sigma'(I)T$ and

$$T\sigma(B) = \sigma'(B)T, \quad B \in \mathscr{B}. \tag{2.10}$$

$\mathscr{S}(\mathscr{B})$ is a monoidal $C^*$-category if we again use (2.8) to define the monoidal structure (cf. [11; Example 3.7]). If $\sigma = \sigma_H$ and $\sigma' = \sigma_{H'}$ then we define

$$\theta(\sigma, \sigma') = \theta(H, H') \tag{2.11}$$

and it is easy to check that $\theta(\sigma, \sigma')$ does not depend on the choice [1] of $H$ and $H'$. Furthermore, the corresponding results for $\theta(H, H')$ show that

(2.12) $$\theta(\sigma, \sigma')\,\theta(\sigma', \sigma) = 1_{\sigma'\sigma},$$

(2.13) $$\theta(\sigma, \sigma'\sigma'') = 1_{\sigma'} \otimes \theta(\sigma, \sigma'')\,\theta(\sigma, \sigma') \otimes 1_{\sigma''},$$

(2.14) $$\theta(\iota, \sigma) = \theta(\sigma, \iota) = 1_\sigma,$$

where $1_\sigma = \sigma(I)$ denotes the unit of $(\sigma, \sigma)$ and $\iota$ denotes the identity automorphism. A simple computation shows that if $T_i \in (\sigma_i, \sigma_i')$, $i = 1, 2$ then

(2.15) $$\theta(\sigma_1', \sigma_2')\,T_1 \otimes T_2 = T_2 \otimes T_1\theta(\sigma_1, \sigma_2).$$

These last four equations may be summed up as

2.1. LEMMA. *$(\mathscr{H}(\mathscr{B}), \theta)$ and $(\mathscr{S}(\mathscr{B}), \theta)$ are symmetric monoidal $C^*$-categories.*

By a *Hilbert G-module H* in a $C^*$-system $(\mathscr{B}, \alpha)$ we understand an $\alpha$-invariant Hilbert space $H$ in $\mathscr{B}$ endowed with the continuous unitary representation $\alpha_H$ of the underlying group $G$ obtained by restricting the action $\alpha$ to $H$. The $C^*$-algebra $\mathscr{O}_H$ generated by the Hilbert space $H$ then carries a natural action so that the Hilbert $G$-module $H$ generates a $C^*$-system which will, abusing notation, be denoted by $(\mathscr{O}_H, \alpha_H)$.

The endomorphism $\sigma_H$ of $\mathscr{B}$ generated by $H$ restricts to an endomorphism $\rho_H$ of the fixed-point algebra $\mathscr{B}^\alpha$. $\rho_H$ will, in general, not be an inner endomorphism of $\mathscr{B}^\alpha$.

Let $H$ and $H'$ be Hilbert $G$-modules in $(\mathscr{B}, \alpha)$. We write $T \in (H, H')$ if $T$ is a $G$-module homomorphism, i.e. if $T$ is a map of the underlying Hilbert spaces and intertwines $\alpha_H$ and $\alpha_{H'}$. This condition can be written

(2.16) $$\alpha_g(T\psi) = T\alpha_g(\psi), \quad \psi \in H.$$

Since $T = T1_H$, (2.16) reduces simply to $T \in \mathscr{B}^\alpha$.

We let $\mathscr{U}(\mathscr{B}, \alpha)$ denote the category whose objects are the finite-dimensional Hilbert $G$-modules in $(\mathscr{B}, \alpha)$ and whose arrows are the $G$-module homomorphisms. Since the tensor product of Hilbert $G$-modules is again a Hilbert $G$-module and the operators $\theta(H, H')$ permuting Hilbert spaces are $G$-module homomorphisms, we have

2.2. LEMMA. *$(\mathscr{U}(\mathscr{B}, \alpha), \theta)$ is a symmetric monoidal $C^*$-category.*

---

[1] If $\mathscr{B} \cap \mathscr{B}' = CI$ then $\sigma_H$ determines $H$ uniquely, in fact $H = (\iota, \sigma_H)$.

The Hilbert spectrum HSp($\alpha$) of an action was defined in Section 1 and we see that $\xi \in \mathrm{HSp}(\alpha)$ if and only if there is a Hilbert $G$-module $H$ in $(\mathscr{B}, \alpha)$ such that $\alpha_H$ is equivalent to $\xi$.

A notion of cross product of a von Neumann algebra by a group dual was introduced in [11] [1]. In this approach, a $W^*$-system over a compact group is the crossed product of the fixed-point algebra by the action of a group dual if and only if $\xi \otimes \alpha \sim \imath \otimes \alpha$ for each irreducible $\xi$. This notion of crossed product is adequate only if the fixed-point algebra is a properly infinite von Neumann algebra. There are $C^*$-systems arising as the cross product of the fixed-point algebra by the action of a group dual where only the weaker condition $\xi \otimes \alpha \leqslant \imath \otimes \alpha$ for each irreducible $\xi$ is satisfied, i.e. the system has full Hilbert spectrum.

2.3. LEMMA. *Let $(\mathscr{B}, \alpha)$ be a $C^*$-system with full Hilbert spectrum. Then the finite sums of elements from the objects of $\mathscr{U}(\mathscr{B}, \alpha)$ form a dense $*$-subalgebra $\mathscr{B}_0$ in $\mathscr{B}$.*

*Proof.* Since the objects of $\mathscr{U}(\mathscr{B}, \alpha)$ are closed under tensor products, $\mathscr{B}_0$ is a subalgebra. Every element of $\mathscr{B}_0$ is a finite sum of entries from irreducible tensors. Conversely if $B_1, B_2, \dots, B_d$ is an irreducible tensor under $G$, let $\psi_1, \psi_2, \dots, \psi_d$ be an orthonormal basis of a Hilbert $G$-module in $\mathscr{B}$ transforming in the same way. Then $X = \sum_i B_i \psi_i^* \in \mathscr{B}^\alpha$ and $B_i = X\psi_i$. Writing $X$ as a linear combination of unitaries we see that $B_i \in \mathscr{B}_0$. Hence $\mathscr{B}_0$ is a dense $*$-subalgebra since the sums of entries of irreducible tensors are closed under $*$ and are dense.

REMARK. $\mathscr{B}_0$ is just the dense $*$-subalgebra of $G$-finite elements in $\mathscr{B}$. An element $B$ of $\mathscr{B}$ is $G$-finite if it is contained in a finite-dimensional $G$-invariant subspace.

The next result relates $\mathscr{U}(\mathscr{B}, \alpha)$ to a monoidal $C^*$-category of endomorphisms; it should be compared with [2; Theorem 3.6]. Let $\mathscr{S}(\mathscr{B}, \alpha)$ denote the category whose objects are the endomorphisms of $\mathscr{B}^\alpha$ induced by the objects of $\mathscr{U}(\mathscr{B}, \alpha)$ and whose arrows are the intertwiners between these endomorphisms, i.e. $T \in (\rho, \rho')$ if $T \in \mathscr{B}^\alpha$, $T = T1_\rho = 1_{\rho'}T$ and

$$T\rho(A) = \rho'(A)T, \quad A \in \mathscr{B}^\alpha, \tag{2.17}$$

where $1_\rho = \rho(I) \in (\rho, \rho)$. Now the map $H \to \rho_H$ extends to a functor $F$ from $\mathscr{U}(\mathscr{B}, \alpha)$ to $\mathscr{S}(\mathscr{B}, \alpha)$ in fact we have

2.4. LEMMA. *Let $(\mathscr{B}, \alpha)$ be a $C^*$-system over a group G then there is a faithful functor F from $\mathscr{U}(\mathscr{B}, \alpha)$ to $\mathscr{S}(\mathscr{B}, \alpha)$. If the fixed-point algebra has trivial*

---

[1] For other approaches cf. [9].

*relative commutant,*

$$\mathscr{B}^{\alpha\prime} \cap \mathscr{B} = \mathbf{C}I$$

*then $F$ is an isomorphism of categories.*

*Proof.* We define for an object $H$ of $\mathscr{U}(\mathscr{B}, \alpha)$, $F(H) = \rho_H$. If $T: H \to H'$ in $\mathscr{U}(\mathscr{B}, \alpha)$ then $T\rho_H(I) = T = \rho_{H'}(I)T$ and $T \in \mathscr{B}^\alpha$. Furthermore

$$T\rho_H(A)\psi = T\psi A = \rho_{H'}(A)T\psi, \quad \psi \in H, \ A \in \mathscr{B}^\alpha,$$

and we conclude that $T \in (\rho_H, \rho_{H'})$ in $\mathscr{S}(\mathscr{B}, \alpha)$ so that we get a faithful functor defining $F(T) = T \in (\rho_H, \rho_{H'})$. If $\psi \in \mathscr{B}$ with $\psi A = \rho_H(A)\psi$, $A \in \mathscr{B}^\alpha$ and $\psi_j$, $j = 1, 2, \ldots, d$ is an orthonormal basis of $H$, then $\psi_j^* \psi \in \mathscr{B}^{\alpha\prime} \cap \mathscr{B}$ so that when this relative commutant is trivial $\psi_j^*\psi = \lambda_j I$ and $\psi = \rho_H(I)\psi = \sum_j \psi_j \psi_j^* \psi = \sum_j \psi_j \lambda_j \in H$. Hence

$$H = \{\psi \in \mathscr{B} : \psi A = \rho_H(A)\psi, \ A \in \mathscr{B}^\alpha\}. \tag{2.18}$$

So $\rho_H$ determines $H$ and $F$ is an isomorphism on objects. Now let $T \in (\rho_H, \rho_{H'})$ and let $\psi_i'$, $i = 1, 2, \ldots, d'$ be an orthonormal basis of $H'$. Then $\psi_i'^* T \psi_j \in \mathscr{B}^{\alpha\prime} \cap \mathscr{B}$ so $\psi_i'^* T \psi_j = \lambda_{ij} I$. Thus $T = \rho_{H'}(I) T \rho_H(I) = \sum_{i,j} \psi_i' \lambda_{ij} \psi_j^*$ and $T$ is a linear map from $H$ to $H'$. Furthermore $T\alpha_g(\psi) = \alpha_g(T)\alpha_g(\psi) = \alpha_g(T\psi)$ so that $T$ is an arrow in $\mathscr{U}(\mathscr{B}, \alpha)$ and $F$ is an isomorphism.

REMARK. In the same spirit, if $\mathscr{B}$ is a $C^*$-algebra with unit and $\mathscr{B}' \cap \mathscr{B} = \mathbf{C}I$ then the categories $\mathscr{H}(\mathscr{B})$ and $\mathscr{S}(\mathscr{B})$ are isomorphic.

## 3. THE STABILIZER

Given a $C^*$-system $(\mathscr{B}, \alpha)$ over a group $G$, there are many other $C^*$-systems intrinsically associated with it. We may, for example, replace $G$ by its image $\alpha_G$ in $\operatorname{Aut}\mathscr{B}$. As far as the action goes, this is a trivial change. It is of more interest to replace $G$ by the stabilizer $\hat{G}$, say, of $\mathscr{B}^\alpha$ in $\operatorname{Aut}\mathscr{B}$. We have a canonical homomorphism $G \to \hat{G}$ with image $\alpha_G$. The following result is a simple consequence of Lemma 2.4.

3.1. LEMMA. *Let $(\mathscr{B}, \alpha)$ be a $C^*$-system and suppose the fixed-point algebra has trivial relative commutant,*

$$(\mathscr{B}^\alpha)' \cap \mathscr{B} = \mathbf{C}I.$$

*Let $\hat{G}$ denote the stabilizer of $\mathscr{B}^\alpha$ in* $\operatorname{Aut}\mathscr{B}$*; then $\mathscr{U}(\mathscr{B}, \alpha) = \mathscr{U}(\mathscr{B}, \hat{G})$.*

*Proof.* Let $H$ be an object of $\mathcal{U}(\mathcal{B}, \alpha)$, $\psi, \psi' \in H$, $A \in \mathcal{B}^\alpha$ and $\beta \in \hat{G}$ then

$$\psi^*\beta(\psi')A = \psi^*\beta(\psi'A) = \psi^*\beta(\rho_H(A')\psi') = \psi^*\rho_H(A)\beta(\psi') = A\psi^*\beta(\psi').$$

Thus $\psi^*\beta(\psi') \in (\mathcal{B}^\alpha)' \cap \mathcal{B} = \mathbf{C}I$. Let $\psi_i$, $i = 1, 2, \ldots, d$ be an orthonormal basis of $H$, and $E = \sum_i \psi_i\psi_i^*$ be the support of $H$. Set $\lambda_i I = \psi_i^*\beta(\psi)$ then since $E \in \mathcal{B}^\alpha$ we have

$$\beta(\psi) = \beta(E\psi) = E\beta(\psi) = \sum_i \psi_i\psi_i^*\beta(\psi) = \sum_i \psi_i\lambda_i.$$

Hence $\beta(\psi) \in H$ and $\mathcal{U}(\mathcal{B}, \alpha)$ and $\mathcal{U}(\mathcal{B}, \hat{G})$ have the same objects. Thus $\mathcal{S}(\mathcal{B}, \alpha)$ and $\mathcal{S}(\mathcal{B}, \hat{G})$ coincide. Hence by Lemma 2.4, $\mathcal{U}(\mathcal{B}, \alpha)$ and $\mathcal{U}(\mathcal{B}, \hat{G})$ coincide.

The following result gives a criterion for the stabilizer $\hat{G}$ to be a compact group.

3.2. LEMMA. *Let $\mathcal{A}$ be a $C^*$-subalgebra with unit of a $C^*$-algebra with unit $\mathcal{B}$. Let $G$ be the stabilizer of $\mathcal{A}$ in* Aut$\mathcal{B}$. *If $\mathcal{B}$ is generated by its finite-dimensional $G$-invariant Hilbert spaces then $G$ is compact in the strong topology.*

REMARK. If $U$ is a unitary of $\mathcal{A}$ then $\mathbf{C}U$ is a 1-dimensional $G$-invariant Hilbert space in $\mathcal{B}$ so the $C^*$-subalgebra of $\mathcal{B}$ generated by its finite-dimensional, $G$-invariant Hilbert spaces automatically contains $\mathcal{A}$.

*Proof.* Let $\Gamma$ denote the set of finite-dimensional $G$-invariant Hilbert spaces in $\mathcal{B}$. If $H \in \Gamma$ we have a continuous unitary representation $u_H$, say, of $G$ on $H$. The map $u_\Gamma : g \in G \to u_\Gamma(g)$, $u_\Gamma(g) := \prod_{H\in\Gamma} u_H(g)$, is an isomorphism of $G$ with a subgroup $u_\Gamma(G)$ of the compact group $\mathcal{U}(\Gamma) := \prod_{H\in\Gamma} \mathcal{U}(H)$, where $\mathcal{U}(H)$ denotes the unitary group of $H$. Suppose $u_\Gamma(g_i) \to u$ in $\mathcal{U}(\Gamma)$, then $u_\Gamma(g_i^{-1}) \to u^{-1}$ and, since the elements of $\Gamma$ generate $\mathcal{B}$ as a $C^*$-algebra, $g_i$ and $g_i^{-1}$ converge strongly. The limit $g$ is thus an automorphism stabilizing $\mathcal{A}$ and $u_\Gamma(g) = u$. Hence $G$ is homeomorphic to the closed subgroup $u_\Gamma(G)$ of the compact Hausdorff group $\mathcal{U}(\Gamma)$. Thus $G$ is compact in the strong topology.

As pointed out at the beginning of this section, we get $C^*$-systems intrinsically associated with a $C^*$-system $(\mathcal{B}, \alpha)$ by replacing the underlying group $G$ by $\alpha_G$ or the stabilizer $\hat{G}$ of $\mathcal{B}^\alpha$ in Aut$\mathcal{B}$. We can, however, get new $C^*$-systems by replacing $\mathcal{B}$ by some intrinsically defined $\alpha$-invariant $C^*$-subalgebras. In particular, if $\Gamma$ denotes some intrinsic set of $G$-invariant Hilbert spaces in $\mathcal{B}$, i.e. Hilbert $G$-modules in $(\mathcal{B}, \alpha)$, and $\mathcal{B}_\Gamma$ the $C^*$-algebra generated by the elements of $\Gamma$ then $\mathcal{B}_\Gamma$ is invariant under the action and we have an associated $C^*$-system $(\mathcal{B}_\Gamma, \alpha_\Gamma)$ over $G$. If $\mathcal{B}_\Gamma = \mathcal{B}$

then we say that $(\mathscr{B}, \alpha)$ has *sufficient* Hilbert $G$-modules of class $\Gamma$. For example, in the proof of Lemma 3.2 we used $\Gamma$ to denote the set of all finite-dimensional $G$-invariant Hilbert spaces in $\mathscr{B}$.

3.3. THEOREM. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$. Let $\hat{G}$ denote the stabilizer of $\mathscr{B}^G$ in* $\mathrm{Aut}\,\mathscr{B}$. *Suppose*

a) *$(\mathscr{B}, G)$ has sufficient finite-dimensional Hilbert $G$-modules,*

b) *$\mathscr{B}^{G'} \cap \mathscr{B} = \mathbf{C}I$,*

c) *$(\mathscr{B}$, G) has full Hilbert spectrum;*

*then* $G = \hat{G}$.

*Proof.* $\hat{G}$ is compact by Lemma 3.2, $\mathscr{U}(\mathscr{B}, G) = \mathscr{U}(\mathscr{B}, \hat{G})$ by Lemma 3.1. If we knew that $(\mathscr{B}, \hat{G})$ had full Hilbert spectrum, $G$ would be irreducibly represented in each irreducible representation of $\hat{G}$ so that $G = \hat{G}$ (cf. [7; Corollary 3.3]). Hence it remains to show that $(\mathscr{B}, \hat{G})$ has full Hilbert spectrum. Let $\Sigma$ denote the set of equivalence classes of irreducible continuous unitary representations of $\hat{G}$ occuring as objects of $\mathscr{U}(\mathscr{B}, \hat{G})$. $\Sigma$ is closed under decomposition of tensor products of elements into irreducibles. Since $(\mathscr{B}, G)$ has full Hilbert spectrum and $\mathscr{U}(\mathscr{B}, G) = \mathscr{U}(\mathscr{B}, \hat{G})$, $\Sigma$ is closed under conjugates. For, if $H$, $\bar{H}$ are irreducible Hilbert $G$-modules whose associated representations are conjugate to each other, they are also irreducible $\hat{G}$-modules and $\{0\} \neq (\mathbf{C}, H\bar{H})_G = (\mathbf{C}, H\bar{H})_{\hat{G}}$. Hence the linear space spanned by functions $f \in \mathscr{C}(\hat{G})$ of the form

$$f(g)I = \psi^* g(\psi'), \quad \psi, \psi' \in H$$

where $H$ is an object of $\mathscr{U}(\mathscr{B}, \hat{G})$ is a $*$-subalgebra of $\mathscr{C}(\hat{G})$. Since $(\mathscr{B}, \hat{G})$ has sufficient finite-dimensional Hilbert $\hat{G}$-modules this $*$-subalgebra separates the points of $\hat{G}$. It is therefore dense in $\mathscr{C}(\hat{G})$ and a fortori in $L^2(\hat{G})$. Hence the orthogonality relations for $\hat{G}$ allow one to conclude that $(\mathscr{B}, \hat{G})$ has full Hilbert spectrum completing the proof.

## 4. A GALOIS CORRESPONDENCE

Given a $C^*$-system $(\mathscr{B}, G)$, we construct a Galois correspondence between certain subalgebras of $\mathscr{B}$ and subgroups of $G$. Given any closed *normal* subgroup $K$ of $G$, define

$$\mathscr{B}^K = \{B \in \mathscr{B} : k(B) = B,\ k \in K\}$$

then $\mathscr{B}^K$ is a $G$-invariant $C^*$-subalgebra of $\mathscr{B}$. If $K_1 \subset K_2$ then $\mathscr{B}^{K_2} \subset \mathscr{B}^{K_1}$ and, in particular, $\mathscr{B}^G \subset \mathscr{B}^K \subset \mathscr{B}$. Conversely, if $\mathscr{A}$ is a $G$-invaiiant $C^*$-algebra with $\mathscr{B}^G \subset \mathscr{A} \subset \mathscr{B}$ define

$$G(\mathscr{A}) = \{g \in G : g(A) = A,\ A \in \mathscr{A}\}$$

then $G(\mathscr{A})$ is a closed normal subgroup of $G$ and $G/G(\mathscr{A})$ is the automorphism group associated with the action of $G$ on $\mathscr{A}$. If $\mathscr{A}_1 \subset \mathscr{A}_2$, then $G(\mathscr{A}_2) \subset G(\mathscr{A}_1)$. Obviously $\mathscr{A} \subset \mathscr{B}^{G(\mathscr{A})}$ and $K \subset G(\mathscr{B}^K)$ so that we have two Galois closure operations $\mathscr{A} \to \mathscr{B}^{G(\mathscr{A})}$ and $K \to G(\mathscr{B}^K)$. In particular, $G(\mathscr{B}^{G(\mathscr{A})}) = G(\mathscr{A})$ and $\mathscr{B}^{G(\mathscr{B}^K)} = \mathscr{B}^K$. If $K = G(\mathscr{B}^K)$, we say $K$ is Galois closed and if $\mathscr{A} = \mathscr{B}^{G(\mathscr{A})}$, we say $\mathscr{A}$ is Galois closed. $\{e\}, G, \mathscr{B}$ and $\mathscr{B}^G$ are trivially Galois closed.

4.1. LEMMA. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$. Let $\mathscr{A}$ be a $G$-invariant $C^*$-subalgebra with $\mathscr{B}^G \subset \mathscr{A} \subset \mathscr{B}$. Thus $Q = G/G(\mathscr{A})$ is an automorphism group of $\mathscr{A}$ and $\mathscr{A}^Q = \mathscr{B}^G$. Suppose $(\mathscr{A}, Q)$ has full Hilbert spectrum; then $\mathscr{A}$ is Galois closed.*

*Proof.* $Q$ is also an automorphism group of $\mathscr{B}^{G(\mathscr{A})}$ and $(\mathscr{B}^{G(\mathscr{A})})^Q = (\mathscr{B}^{G(\mathscr{A})})^{G/G(\mathscr{A})} = \mathscr{B}^G$.

To show that $\mathscr{A} = \mathscr{B}^{G(\mathscr{A})}$, is suffices to show that the entries of an irreducible tensor $\underline{B}$ in $\mathscr{B}^{G(\mathscr{A})}$ lie in $\mathscr{A}$. Since $(\mathscr{A}, Q)$ has full Hilbert spectrum, there is an irreducible tensor $\underline{\psi}$ of the same type in $\mathscr{A}$ which is an isometry. Setting $A = \sum_i B_i\psi_i^*$ then $A \in (\mathscr{B}^{G(\mathscr{A})})^Q = \mathscr{B}^G \subset \mathscr{A}$ and $B_i = A\psi_i$, so $B_i \in \mathscr{A}$ and $\mathscr{A}$ is Galois closed as required.

We now have the following result

4.2. THEOREM. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$. Suppose $(\mathscr{B}, G)$ has full Hilbert spectrum; then a $G$-invariant $C^*$-algebra $\mathscr{A}$ with $\mathscr{B}^G \subset \mathscr{A} \subset \mathscr{B}$ is Galois closed if and only if $(\mathscr{A}, G/G(\mathscr{A}))$ has full Hilbert spectrum.*

*Proof.* The condition is sufficient by Lemma 4.1. To see that it is necessary, let $\mathscr{A}$ be Galois closed then $\mathscr{A} = \mathscr{B}^K$, where $K = G(\mathscr{A})$ is a normal subgroup of $G$. Since $\mathscr{B}^K$ is $G$-invariant, it is generated by its irreducible tensors under $G$. The entries of a $\xi$-tensor in $\mathscr{B}$ lie in $\mathscr{B}^K$ if and only if $K \subset \operatorname{Ker} \xi$. Since $(\mathscr{B}, G)$ has full Hilbert spectrum then so has $(\mathscr{B}^K, G/K)$ and the condition is necessary.

## 5. THE RELATIVE COMMUTANT

We have made frequent use of the hypothesis $\mathscr{B}^{G'} \cap \mathscr{B} = CI$. In this section we examine the structure of the relative commutant.

5.1. LEMMA. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$ and suppose $(\mathscr{B}, G)$ has full Hilbert spectrum; then $\mathscr{B}^{G'} \cap \mathscr{B}$ is generated as a closed linear space by sets of the form $(\rho_H, \iota)H$ where $H$ is a Hilbert $G$-module in $(\mathscr{B}, G)$.*

*Proof.* $(\rho_H, \iota)H \subset \mathscr{B}^{G'} \cap \mathscr{B}$ trivially. Since the action of $G$ induces an action on $\mathscr{B}^{G'} \cap \mathscr{B}$, $\mathscr{B}^{G'} \cap \mathscr{B}$ is generated linearly by its irreducible tensors under $G$. Let $T_1, T_2, \ldots, T_d$ be such a tensor from $\mathscr{B}^{G'} \cap \mathscr{B}$, then since $(\mathscr{B}, G)$ has full Hilbert spectrum there is an orthonormal basis $\psi_1, \psi_2, \ldots, \psi_d$ of a Hilbert $G$-module $H$ transforming in the same way. Let $X = \sum_{i=1}^{d} T_i\psi_i^* \in \mathscr{B}^G$, then we have $T_i = X\psi_i$ and $X1_H = X$. It suffices to prove that $X \in (\rho_H, \iota)$. Since $T_iA = AT_i$ for $A \in \mathscr{B}^G$, we have

$$(X\rho_H(A) - AX)\psi_i = 0, \quad A \in \mathscr{B}^G, \ i = 1, 2, \ldots, d.$$

Thus $(X\rho_H(A) - AX)1_H = 0$. Since $1_H \in (\rho_H, \rho_H)$ and $X1_H = X$, we conclude that $X\rho_H(A) = AX$ for $A \in \mathscr{B}^G$, i.e. $X \in (\rho_H, \iota)$ as required.

The next result is of interest because it links minimality of the relative commutant of the fixed-point algebras to properties of permutation symmetry.

5.2. THEOREM. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$. If $(\mathscr{B}, G)$ has full Hilbert spectrum then the following are equivalent:*

a) $\mathscr{B}^{G'} \cap \mathscr{B} = \mathscr{B}' \cap \mathscr{B}$,

b) $(\rho_H, \rho_{H'}) \subset (\sigma_H, \sigma_{H'})$ *for any pair $H$, $H'$ of Hilbert $G$-modules in $\mathscr{B}$,*

c) $(\mathscr{S}(\mathscr{B}, G), \theta)$ *is a symmetric monoidal $C^*$-category.*

*Proof.* a) $\Rightarrow$ b) Let $X \in (\rho_H, \rho_{H'})$ then picking bases $\psi_i$ and $\psi_j'$ in $H$ and $H'$ respectively we have $X = \sum_{i,j} \psi_j' C_{ij} \psi_i^*$ where $C_{ij} = \psi_j'^* X \psi_i \in \mathscr{B}' \cap \mathscr{B}$ but then $X \in (\sigma_H, \sigma_{H'})$.

b) $\Rightarrow$ a) By Lemma 5.1, it suffices to show $(\rho_H, \iota)H \subset \mathscr{B}' \cap \mathscr{B}$ but $(\rho_H, \iota)H \subset (\sigma_H, \iota)H \subset \mathscr{B}' \cap \mathscr{B}$.

a) $\Rightarrow$ c) $\theta(H, H') \in \mathscr{B}^G$ and we show that $\theta$ depends only on $\rho_H$ and $\rho_{H'}$. We already know, cf. (2.11), that $\theta$ depends only on $\sigma_H$ and $\sigma_{H'}$. Let $\sigma$ and $\sigma'$ be two inner endomorphisms restricting to $\rho$ on $\mathscr{B}^G$. In particular $\sigma(I) = \sigma'(I)$. If $\psi_i \in (\iota, \sigma)$ with $\sum_i \psi_i\psi_i^* = \sigma(I)$ and $\psi_j' \in (\iota, \sigma')$ with $\sum_j \psi_j'\psi_j'^* = \sigma'(I)$ then $\psi_i^*\psi_j' \in \mathscr{B}^{G'} \cap \mathscr{B} = \mathscr{B}' \cap \mathscr{B}$ so

$$\sigma(B) = \sum_{i,j} \psi_i\psi_i^*\sigma(B)\psi_j'\psi_j'^* = \sum_{i,j} \psi_i B\psi_i^*\psi_j'\psi_j'^* = \sum_{i,j} \psi_i\psi_i^*\psi_j' B\psi_j'^* = \sigma'(B).$$

The fact that $\theta(\rho, \rho')$ acts as a symmetry for $\mathscr{S}(\mathscr{B}, G)$ now follows from b) and Lemma 2.1.

c) $\Rightarrow$ a) By Lemmas 2.3 and 5.2, it suffices to show that

$$(X\psi)\psi' = \psi'(X\psi) \quad \text{for } X \in (\rho_H, \iota),\ \psi \in H,\ \psi' \in H'.$$

Now $\psi\psi' = \theta(H',H)\psi'\psi$ so by c)

$$X\psi\psi' = X\theta(H', H)\psi'\psi = \rho_{H'}(X)\psi'\psi = \psi' X\psi$$

completing the proof.

$C^*$-systems $(\mathscr{B}, \alpha)$ with $\mathscr{B}^{\alpha'} \cap \mathscr{B} = \mathscr{B}' \cap \mathscr{B}$ were investigated in [5] and shown to be obtained canonically by inducing up from a $C^*$-system over a closed subgroup with the same fixed-point algebra but which now has trivial relative commutant.

The following variant of Theorem 5.2 will be used in a subsequent paper; we provide a proof which avoids the hypothesis that $(\mathscr{B}, G)$ has full Hilbert spectrum.

5.3. LEMMA. *Let $G$ be a compact group of automorphisms of $\mathscr{B}$, a $C^*$-algebra with unit, let $H$ be a $G$-stable Hilbert space in $\mathscr{B}$ and set $\rho = \rho_H$, $\sigma = \sigma_H$. Suppose $\mathscr{B}^G$ and $H$ together generate $\mathscr{B}$; then the following conditions are equivalent:*

a) $\mathscr{B}^{G'} \cap \mathscr{B} = \mathscr{B}' \cap \mathscr{B}$;

b) $(\rho^r, \rho^s) \subset (\sigma^r, \sigma^s)$, $r, s \in \mathbf{N}_0$;

c) *If* $T \in (\rho^r, \rho^s)$ *then* $\theta(s, 1)T = \rho(T)\theta(r, 1)$, $r, s \in \mathbf{N}_0$, *where* $(n, 1)$ *denotes the* $n+1$-*cycle* $(1, 2 \ldots n+1)$;

d) $\mathscr{B}^{G'} \cap \mathscr{B}$ *is pointwise* $\sigma$-*invariant.*

*Proof.* a) $\Rightarrow$ b) is proved as in Theorem 5.2. b) $\Rightarrow$ c) by virtue of Lemma 2.1. d) $\Rightarrow$ a) is proved by taking $T \in \mathscr{B}^{G'} \cap \mathscr{B}$ and $\psi \in H$ then $\psi T = \sigma(T)\psi = T\psi$ but $\mathscr{B}^G$ and $H$ generate $\mathscr{B}$ so $T \in \mathscr{B}' \cap \mathscr{B}$. It remains to show that c) $\Rightarrow$ d). Since $\mathscr{B}^{G'} \cap \mathscr{B}$ is $G$-stable it suffices to consider an irreducible tensor $T_1, T_2, \ldots, T_d$, say, in $\mathscr{B}^{G'} \cap \mathscr{B}$. Since $\mathscr{B}^G$ and $H$ together generate $\mathscr{B}$, $G$ is faithfully represented on $H$ and hence a subgroup of $\mathscr{U}(H)$. We can therefore find $p, n \in \mathbf{N}_0$ such that $S^{p^*}\tilde{\psi}_i$, $i = 1, 2, \ldots, d$ transform in the same way as $T_i$ where $S$ is defined by (6.2) and $\tilde{\psi}_i$ is an orthonormal set in $H^{\otimes n}$. Hence $X := \sum_i S^p T_i \tilde{\psi}_i^*$ is in $\mathscr{B}^G$ and $T_i = S^{p^*} X \tilde{\psi}_i$ by (6.4). Since $X \in (\rho^n, \rho^{pd})$, $\tilde{\psi}_i \in (\iota, \sigma^n)$ and $S^{p^*} \in (\sigma^{pd}, \iota)$ we have by c) $\sigma(T_i) = \sigma(S^{p^*})\rho(X)\sigma(\tilde{\psi}_i) = S^{p^*}X\tilde{\psi}_i = T_i$ so that c) $\Rightarrow$ d) completing the proof.

## 6. SPECIAL HILBERT $G$-MODULES

A finite-dimensional Hilbert $G$-module $H$ in a $C^*$-system $(\mathscr{B}, \alpha)$ is said to be *special* if $\alpha_H$, the induced action on $H$, has determinant 1. This notion is related to the structure of $(\mathscr{U}(\mathscr{B}, \alpha), \theta)$ as a symmetric monoidal $C^*$-category.

A symmetry $\varepsilon$ on a symmetric monoidal $C^*$-category $\mathcal{T}$ determines for each object $\rho$ of $\mathcal{T}$ canonical unitary representations $\varepsilon_\rho^{(n)}$ of the permutation group $\mathbf{P}_n$ on $n$ symbols in $(\rho^n, \rho^n)$. In the case at hand, we have simple explicit formulae: given an object $H$ of $\mathcal{U}(\mathcal{B}, \alpha)$, we have for $p \in \mathbf{P}_n$

$$\theta_H^{(n)}(p) = \sum_{i_1, i_2, \ldots i_n} \psi_{i_1}\psi_{i_2} \ldots \psi_{i_n}\psi_{i_{p(n)}}^* \ldots \psi_{i_{p(1)}}^* \tag{6.1}$$

where $\psi_1, \psi_2, \ldots, \psi_d$ is an orthonormal basis of $H$.

The question of whether $H$ is special or not depends on the behaviour under $\alpha$ of

$$S := \frac{1}{\sqrt{d!}} \sum_{p \in \mathbf{P}_d} \operatorname{sign}(p)\psi_{p(1)}\psi_{p(2)} \ldots \psi_{p(d)}. \tag{6.2}$$

6.1. LEMMA.

$$\alpha_g(S) = \det \alpha_H(g) S \tag{6.3}$$

$$S^*S = I \tag{6.4}$$

$$SS^* = \frac{1}{d!} \sum_{p \in \mathbf{P}_d} \operatorname{sign}(p)\theta_H^{(d)}(p) \tag{6.5}$$

*so that the range of the isometry $S$ is the totally antisymmetric projection of $H^{\otimes d}$. Furthermore, we have*

$$S^*\sigma_H(S) = (-1)^{d-1} \frac{1}{d} 1_H. \tag{6.6}$$

*If we set*

$$\hat{\psi}_i = \frac{1}{\sqrt{(d-1)!}} \sum_{p \in \mathbf{P}_d(i)} \operatorname{sign}(p)\psi_{p(2)} \ldots \psi_{p(d)} \tag{6.7}$$

*where $\mathbf{P}_d(i)$ denotes the subset of $\mathbf{P}_d$ of permutations $p$ with $p(1) = i$, then*

$$\psi_i^* = (-1)^{d-1}\sqrt{d}\, S^*\hat{\psi}_i. \tag{6.8}$$

The proof is a matter of simple computations and will be omitted here. Note that $H$ is special if and only if $S \in \mathcal{B}^\alpha$ and that the $\hat{\psi}_i$ then transform under $G$ conjugately to the $\psi_i$.

6.2. LEMMA. *Let $\mathcal{B}$ be a $C^*$-algebra with unit, $G$ a compact automorphism group of $\mathcal{B}$ and $\Gamma$ a family of Hilbert $G$-modules stable under products and $G$-submo-*

*dules. Suppose $\mathscr{B}$ is generated as a $C^*$-algebra by $\mathscr{B}^G$ and the special elements of $\Gamma$. Then any equivalence class of irreducibles of $G$ is realized on some element of $\Gamma$.*

*Proof.* Let $\Sigma$ denote the set of equivalence classes of irreducible, continuous, unitary representations of $G$ occurring as Hilbert $G$-submodules of some special element of $\Gamma$. $\Sigma$ is closed under decomposition of tensor products of elements into irreducibles. It is also closed under conjugates since, when $H$ is special, the $\hat{\psi}_i$ of Lemma 6.1 transform conjugately to the $\psi_i$. Hence the linear space spanned by functions $f \in \mathscr{C}(G)$ of the form

$$f(g)I = \psi^* g(\psi'), \quad \psi, \psi' \in H,$$

where $H$ is a special element of $\Gamma$ is a $*$-subalgebra of $\mathscr{C}(G)$. Since $\mathscr{B}$ is generated as a $C^*$-algebra by $\mathscr{B}^G$ and the special elements of $\Gamma$, this $*$-subalgebra separates the points of the compact automorphism group $G$. It is therefore dense in $\mathscr{C}(G)$ and a fortiori in $L^2(G)$. Hence the orthogonality relations for $G$ allow one to conclude that $\Sigma$ contains all equivalence classes of irreducibles completing the proof.

As an immediate consequence we have

6.3. COROLLARY. *Let $G$ be a compact automorphism group of $\mathscr{B}$ and suppose $(\mathscr{B}, G)$ has sufficient special Hilbert $G$-modules; then $(\mathscr{B}, G)$ has full Hilbert spectrum.*

Corollary 6.3 and Lemma 4.1 have the following simple corollary.

6.4. COROLLARY. *Suppose $G$ is a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$ and suppose a $C^*$-subalgebra $\mathscr{A} \supset \mathscr{B}^G$ is generated by its special Hilbert $G$-modules; then $\mathscr{A}$ is $G$-invariant and Galois closed in $(\mathscr{B}, G)$.*

*Proof.* $\mathscr{A}$ is $G$-invariant since it is generated by its special Hilbert $G$-modules. $Q := G/G(\mathscr{A})$ acts as an automorphism group of $\mathscr{A}$ and $\mathscr{A}$ has a fortiori sufficient special Hilbert $Q$-modules. Hence, by Corollary 6.3, $(\mathscr{A}, Q)$ has full Hilbert spectrum. Thus $\mathscr{A}$ is Galois closed in $(\mathscr{B}, G)$ by Lemma 4.1.

The following result is an immediate corollary of Corollary 6.3 and Theorem 3.3.

6.5. COROLLARY. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$. Let $\hat{G}$ denote the stabilizer of $\mathscr{B}^G$ in* Aut $\mathscr{B}$. *Suppose*

a) *$(\mathscr{B}, G)$ has sufficient special Hilbert $G$-modules,*

b) *$\mathscr{B}^{G'} \cap \mathscr{B} = \mathbf{C}I$;*

*then $\hat{G} = G$.*

Corollary 6.5 has in its turn the following corollary.

6.6. COROLLARY. *Let $G$ be a compact automorphism group of a $C^*$-algebra with unit $\mathscr{B}$. Let $\mathscr{B}_1$, $\mathscr{B}^G \subset \mathscr{B}_1 \subset \mathscr{B}$ be a $C^*$-algebra generated by its special Hilbert $G$-modules and let $G_1$ denote the stabilizer of $\mathscr{B}^G$ in* $\mathrm{Aut}\mathscr{B}_1$. *Suppose $\mathscr{B}^{G'} \cap \mathscr{B} = \mathbf{C}I$; then $G_1 = G/G(\mathscr{B}_1)$.*

*Proof.* $\mathscr{B}_1$ is a $G$-invariant $C^*$-subalgebra of $\mathscr{B}$, $Q = G/G(\mathscr{B}_1)$ is an automorphism group of $\mathscr{B}_1$ and $\mathscr{B}_1^Q = \mathscr{B}^G$ (cf. Lemma 4.1). Trivially $\mathscr{B}_1^{Q'} \cap \mathscr{B}_1 = \mathbf{C}I$ so the result follows on applying Corollary 6.5 to the $C^*$-system $(\mathscr{B}_1, Q)$.

For the remainder of the section we change the perspective and pick a fixed $C^*$-algebra with unit $\mathscr{A}$ and trivial centre and study a class of extensions of $\mathscr{A}$. Let $\mathscr{B}$ be a $C^*$-algebra with $\mathscr{A} \subset \mathscr{B}$ and let

$$\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}) = \{g \in \mathrm{Aut}\mathscr{B} : g(A) = A,\ A \in \mathscr{A}\} \tag{6.9}$$

be the stabilizer of $\mathscr{A}$ in $\mathrm{Aut}\mathscr{B}$. The class of extensions we study are characterized by

a) $\mathscr{A}' \cap \mathscr{B} = \mathbf{C}I$

b) $\mathscr{A}$ is the fixed-point algebra of $\mathscr{B}$ under the action of $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B})$

c) $\mathscr{B}$ has sufficient special Hilbert $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B})$-modules.

Note that, by Lemma 3.2, $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B})$ is a compact automorphism group.

6.7. THEOREM. *Consider a family of $C^*$-algebras each satisfying* a), b) *and* c) *above which is partially ordered and directed upwards under inclusion. Then the $C^*$-algebra generated by the family also satisfies* a), b) *and* c).

*Proof.* Let $\mathscr{B}_1 \subset \mathscr{B}_2$ be two $C^*$-algebras in the given family, we first show that

$$\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2)|\mathscr{B}_1 = \mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_1). \tag{6.10}$$

This will follow from Corollary 6.6 if we show that $\mathscr{B}_1$ is generated by its special Hilbert $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2)$-modules. Let $H \subset \mathscr{B}_1$ be an $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_1)$-Hilbert module then $\psi A = \rho_H(A)\psi$ for each $\psi \in H$. Now $g(\psi)A = \rho_H(A)g(\psi)$ for $g \in \mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2)$ and since $\mathscr{A}' \cap \mathscr{B}_2 = \mathbf{C}I$, $H$ is $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2)$-stable. If $H$ is even special for $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_1)$ then the totally antisymmetric vector $S$ given by (6.2) is in $\mathscr{A}$ and hence invariant under $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2)$. Thus $H$ is special for $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2)$. But such Hilbert spaces generate $\mathscr{B}_1$ so (6.10) follows from Corollary 6.6. The surjective homomorphism $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_2) \to \mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_1)$ is a continuous homomorphism of compact groups and we have a projective system. The corresponding projective limit $\tilde{G}$ is a compact group [10; § 43].

Let $\mathscr{B}$ be the $C^*$-algebra generated by the given family. We can define a continuous action of $\tilde{G}$ on $\mathscr{B}$ as the inductive limit of the actions of $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_i)$ on the algebras $\mathscr{B}_i$ of the family. Therefore for each $C^*$-algebra $\mathscr{B}_i$ in the family, the

$\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_i)$-stable Hilbert spaces are actually $\tilde{G}$-stable Hilbert spaces in $\mathscr{B}$. By varying $\mathscr{B}_i$ and applying Lemma 6.2 to this collection of Hilbert spaces we see that all equivalence classes of irreducibles of $\tilde{G}$ are realized there. Furthermore $\mathscr{B}$ is generated by its special Hilbert $\tilde{G}$-modules.

Next note that $\mathscr{B}^{\tilde{G}} = \mathscr{A}$ because averaging over $\tilde{G}$ provides a conditional expectation $m$ of $\mathscr{B}$ onto $\mathscr{B}^{\tilde{G}}$ which agrees on each $C^*$-algebra $\mathscr{B}_i$ with the conditional expectation derived from $\mathrm{Aut}_{\mathscr{A}}(\mathscr{B}_i)$. Since the union of the $\mathscr{B}_i$ is dense in $\mathscr{B}$ we have $m(\mathscr{B}) = \mathscr{A}$ as claimed.

As $\mathscr{A}' \cap \mathscr{B}$ is $\tilde{G}$-stable, it is generated by its irreducible $\tilde{G}$-tensors. Let $T_1, T_2, \ldots, T_d$ be such a tensor then, as we have seen above, there is a basis $\psi_1, \psi_2, \ldots, \psi_d$ of a Hilbert space in some $C^*$-algebra $\mathscr{B}_1$ of the given family transforming in the same way under $\tilde{G}$. Then $A = \sum_i T_i\psi_i^* \in \mathscr{A}$ and $T_i = A\psi_i \in \mathscr{B}_1$. Since $\mathscr{A}' \cap \mathscr{B}_1 = \mathbf{C}I$ we conclude that $\mathscr{B}$ satisfies a). Corollary 6.5 shows that $\tilde{G} = \mathrm{Aut}_{\mathscr{A}}(\mathscr{B})$ so b) and c) are also satisfied and we have completed the proof of the theorem.

## 7. A CLASS OF EXAMPLES

In this section we discuss how a representation theory of a compact group $G$ can be embedded in a $C^*$-system of the type considered in the previous sections and give conditions for the associated fixed-point $C^*$-algebra to be simple.

Let $\mathscr{R}$ be a category whose objects are a set $|\mathscr{R}|$ of continuous unitary representations of the compact group $G$ on Hilbert spaces each of finite dimension greater than one and whose arrows are the corresponding intertwining operators. If $u \in |\mathscr{R}|$ let $\mathcal{O}_{H_u}$ be the Cuntz algebra generated by the representation space $H_u$ of $u$. We get a $C^*$-system $(\mathcal{O}_{H_u}, \alpha_u)$ over $G$ by letting $\alpha_{u(g)}$ denote the automorphism of $\mathcal{O}_{H_u}$ defined by

$$\alpha_{u(g)}(\psi) = u(g)\psi, \quad \psi \in H_u.$$

Let $(\mathscr{B}, \alpha)$ be the $C^*$-system got by taking the infinite tensor product of the simple nuclear $C^*$-algebras $\mathcal{O}_{H_u}$ together with the tensor product of the actions $\alpha_u$

$$\mathscr{B} = \bigotimes_{u \in |\mathscr{R}|} \mathcal{O}_{H_u} \tag{7.1}$$

$$\alpha_g = \bigotimes_{u \in |\mathscr{R}|} \alpha_{u(g)}, \quad g \in G \tag{7.2}$$

$\alpha$ is obviously a strongly continuous action on $\mathscr{B}$ and $\mathscr{B}$ is generated by its $\alpha$-stable

Hilbert spaces with support $I$. The image of $H_u$ under the canonical map $\mathcal{O}_{H_u} \to \mathcal{B}$ will again be denoted $H_u$. $\mathcal{R}$ is now naturally embedded as a full subcategory of the category $\mathcal{U}(\mathcal{B}, \alpha)$ defined in Section 2. By [13; page 117, Corollary] and [12; Corollary 1.23.9] $\mathcal{B}$ is a simple $C^*$-algebra.

7.1. LEMMA. *Let* $(\mathcal{B}, \alpha)$ *be the* $C^*$*-system described above; then*

$$\mathcal{B}^{\alpha'} \cap \mathcal{B} = \mathbf{C}I.$$

*Proof.* Let $T$ denote the compact product of circle groups

$$T = \prod_{u \in |\mathcal{R}|} \mathbf{T}. \tag{7.3}$$

$\mathcal{B}$ carries a natural strongly continuous action $\beta$ of $T$

$$\beta_t(\psi) = t(u)\psi, \quad t \in T, \ \psi \in H_u, \tag{7.4}$$

where $t(u) \in \mathbf{T} \subset \mathbf{C}$ is the $u$th coordinate of $T$.

Since $\alpha$ and $\beta$ commute elementwise $\mathcal{C} = \mathcal{B}^{\alpha'} \cap \mathcal{B}$ is globally stable under $\beta$. By Fourier analysis, $\mathcal{C}$ is generated (as a Banach space) by elements $C \in \mathcal{C}$ such that $\beta_t(C) = \langle z, t\rangle C$ for all $t \in T$ and some $z \in \hat{T}$, i.e. $z : |\mathcal{R}| \to \mathbf{Z}$ has finite support and $\langle z, t\rangle = \prod_{u \in |\mathcal{R}|} t(u)^{z(u)}$. Such elements are precisely the range of the map

$$m_z = \int \overline{\langle z, t\rangle}\, \beta_t d\mu(t) \tag{7.5}$$

where $\mu$ is the normalized Haar measure on $T$. The range of $m_z$ is clearly given by

$$m_z(\mathcal{B}) = \bigotimes_{u \in |\mathcal{R}|} \mathcal{O}_{H_u}^{z(u)}$$

where $\mathcal{O}_H^k$ denotes the elements of $\mathcal{O}_H$ of grade $k$ (cf. [7; § 3]). Now if $C \in \mathcal{C}$, $C$ commutes in particular with the images in $\mathcal{B}$ of all the permutation operators $\theta_u(p) \in \mathcal{O}_{\mathcal{U}(H_u)} \subset \mathcal{O}_{H_u}$, $u \in |\mathcal{R}|$, $p \in \mathbf{P}_n$, $n \in \mathbf{N}$ (cf. Equation (6.1)). Reasoning as in [7; Lemma 3.2] we see that if $C \in m_z(\mathcal{B}) \cap \mathcal{C}$ either $C = 0$ or $z = 0$. But if $z = 0$, $\sigma_{H_u}(C) = \lim_{r \to \infty} \theta_u(r, 1) C \theta_u(r, 1)^* = C$ so $C$ commutes with all the Hilbert spaces $H_u \subset \mathcal{B}$, i.e. it belongs to the centre of the simple $C^*$-algebra $\mathcal{B}$ so we have $C \in \mathbf{C}I$ completing the proof.

Combining Lemmas 7.1 and 2.4 we conclude

7.2. COROLLARY. $\mathcal{R}$ *is embedded as a full subcategory of* $\mathcal{S}(\mathcal{B}, \alpha)$.

We conclude this section by showing that, under a certain condition on $\mathscr{R}$, $\mathscr{B}^\alpha$ is simple.

We will say that $\mathscr{R}$ is *specially directed* if for each finite subset $u_1, u_2, \dots u_n \in |\mathscr{R}|$ there is a $u \in |\mathscr{R}|$ with determinant one *dominating* each $u_i$, i.e. $u_i$ is quasi-contained in $\bigoplus_n u^{\otimes n}$.

A subset $\sigma \subset |\mathscr{R}|$ is said to be *special* if it is finite and contains a representation with determinant one dominating all of the representations in $\sigma$.

If $\mathscr{R}$ is specially directed, the special subsets of $|\mathscr{R}|$ partially ordered under inclusion form a directed subset with union $|\mathscr{R}|$.

7.3. THEOREM. *If $\mathscr{R}$ is specially directed then $\mathscr{B}^\alpha$ is a simple $C^*$-algebra.*

*Proof.* When $\mathscr{R}$ is specially directed, $\mathscr{B}$ is the $C^*$-completion of the union of the partially ordered directed set of subalgebras $\mathscr{B}_\sigma = \bigotimes_{u \in \sigma} \mathscr{O}_{H_u}$ when $\sigma$ varies over the special subsets of $|\mathscr{R}|$. Now $\mathscr{B}^\alpha$ is the range of the conditional expectation

$$m = \int_G \alpha_g d\mu(g)$$

on $\mathscr{B}$ and each $\mathscr{B}_\sigma$ is $\alpha$-stable, hence $\mathscr{B}^\alpha$ is the inductive limit of the $\mathscr{B}_\sigma^\alpha$, $\sigma \subset |\mathscr{R}|$ special. It will therefore ' uffice to show that each $\mathscr{B}_\sigma^\alpha$ is simple. We now fix attention on a special subset $(u_1, u_2, \dots, u_n, u)$ where $u$ is the dominating representation of determinant one. For brevity we will now denote $\mathscr{B}_\sigma$ by $\mathscr{B}$, $\mathscr{B}_\sigma^\alpha$ by $\mathscr{A}$ and $H_u$ by $H$. For each $H_{u_i}$ there are partial isometries $W_i^j \in (u_i, u^{\otimes m_j}) \subset \mathscr{A}$ with $\sum_j W_i^{j*} W_i^j = 1_{u_i}$. Hence $H_{u_i}$ is included in $\sum_j W_i^{j*} H^{m_j}$ and $\mathscr{B}$ is generated by $\mathscr{A}$ and $H$. The proof now follows closely some arguments of [7]. Let $\mathscr{D} = \mathscr{O}'_H \cap \mathscr{B}$ we have $\mathscr{D} = \mathscr{O}_{H_{u_1}} \otimes \otimes \dots \otimes \mathscr{O}_{H_{u_n}}$ and $\mathscr{B} = \mathscr{D} \otimes \mathscr{O}_H$.

Let $\gamma$ denote the action of $\mathbf{T}$ on $\mathscr{B}$ defined by

$$\gamma_z(D \otimes \psi) = zD \otimes \psi, \quad D \in \mathscr{D}, \ \psi \in H. \tag{7.6}$$

Let $\{\mathscr{B}^k : k \in \mathbf{Z}\}$ denote the associated $\mathbf{Z}$-grading of $\mathscr{B}$, i.e. $X \in \mathscr{B}^k$ if $\gamma_z(X) = z^k X$, $z \in \mathbf{T}$. As $\gamma$ and $\alpha$ commute $\mathscr{A}$ will be $\mathbf{Z}$-graded too.

Let ${}^\circ\mathscr{B}$ denote the algebraic part of $\mathscr{B}$ relative to the factor $\mathscr{O}_H$ in the tensor product, i.e.

$${}^\circ\mathscr{B} = \mathscr{D} \odot {}^\circ\mathscr{O}_H,$$

where $\odot$ denotes the algebraic tensor product and ${}^\circ\mathscr{O}_H$ the algebraic part of $\mathscr{O}_H$ (cf. [7; § 1]). Let ${}^\circ\mathscr{A} = ({}^\circ\mathscr{B})^\alpha$ denote the algebraic part of $\mathscr{A}$.

The proof of the theorem follows from the following two lemmas.

7.4. LEMMA. *If $\pi$ is a non-zero Hilbert space representation of $\mathscr{A}$ then* $\ker\pi \cap {}^\circ\mathscr{A} = \{0\}$.

7.5. LEMMA. *There is a unique $C^*$-norm on ${}^\circ\mathscr{A}$.*

To prove the theorem granted these two lemmas, let $\pi$ be any non-zero representation of $\mathscr{A}$ then $\|\pi(A)\|$, $A \in {}^\circ\mathscr{A}$ defines a $C^*$-norm on ${}^\circ\mathscr{A}$ so $\|\pi(A)\| = \|A\|$, $A \in {}^\circ\mathscr{A}$. ${}^\circ\mathscr{A}$ is dense in $\mathscr{A}$ so $\pi$ is isometric on $\mathscr{A}$ and $\ker\pi = \{0\}$. Thus $\mathscr{A}$ is simple.

*Proof of Lemma 7.4.* First note that ${}^\circ\mathscr{B}$ is $\mathbf{Z}$-graded and is the algebraic direct sum of ${}^\circ\mathscr{B}^k$, $k \in \mathbf{Z}$, where

$$ {}^\circ\mathscr{B}^k = \mathscr{D} \odot {}^\circ\mathcal{O}_H^k . $$

So given $X \in {}^\circ\mathscr{B}^k$ we have

$$ X = \sum_i D_i \otimes X_i, \quad X_i \in (H^{r_i}, H^{r_i+k}). $$

Thus for $r \in \mathbf{N}$ large enough we have

$$ X\sigma^r(A) = \sigma^{r+k}(A)X, \quad A \in I \otimes \mathcal{O}_H \tag{7.7} $$

where $\sigma$ is the endomorphism generated by $H$. Since $\det u(g) = 1$, $g \in G$, we have $I \otimes \mathcal{O}_{SU(H)} \subset \mathscr{A}$ and identifying $\mathcal{O}_{SU(H)}$ with its image in $\mathscr{A}$, we have isometries $S_m$, $m = 1, 2, \ldots$ as in [7; Corollary 2.3]. By Equation (7.7) we have in particular that for every $X \in {}^\circ\mathscr{A}^k$ there is an $r \in \mathbf{N}$ such that

$$ X\sigma^r(S_m) = \sigma^{r+k}(S_m)X, \quad m = 1, 2, \ldots . \tag{7,8} $$

The arguments in [7] following Corollary 2.3 and in Lemma 2.4 now show that every Hilbert space representation of ${}^\circ\mathscr{A}$ has a kernel which is a graded ideal and is hence $\sigma$-stable by the argument of [7; Lemma 2.1].

Now let $\hat{\pi}$ be the representation of $\mathscr{B}$ obtained by inducing up from the representation $\pi$ of $\mathscr{A}$ using the conditional expectation $m: \mathscr{B} \to \mathscr{A}$. Since $u$ is special we see, taking the adjoint of Equation (6.8), that $\mathscr{H}_{\hat{\pi}}$ is the closed linear span of the subspaces $\hat{\pi}(H^m)^*\mathscr{H}_\pi$. If $A \in {}^\circ\mathscr{A}$ is such that $\pi(A) = 0$ then by the above arguments $\pi(\sigma^m(A)) = 0$ and for each $X \in \hat{\pi}(H^m)^*$ and $\Phi \in \mathscr{H}_\pi$ we have

$$ \hat{\pi}(A)X\Phi = X\pi(\sigma^m(A))\Phi = 0. $$

Thus $\hat{\pi}(A) = 0$. Now $\mathscr{B}$ being the (minimal) tensor product of simple $C^*$-algebras is itself simple [13; page 117, Corollary]. Thus $A = 0$ and $\ker\pi \mid {}^\circ\mathscr{A} = \{0\}$ completing the proof.

*Proof of Lemma* 7.5. First note that $^\circ\mathscr{B}^\circ$ is the ascending union of the $C^*$-algebras $\mathscr{D} \otimes (H^m, H^m)$, each stable under $\alpha$. Denoting the fixed points by $\mathscr{E}_m$ we see that $^\circ\mathscr{A}^\circ$ is the ascending union of the $C^*$-algebras $\mathscr{E}_m$ and hence has a unique $C^*$-norm. The same use of Equation (7.8) as in [7], cf. Equations (2.14) and (2.15) and Lemma 2.10, shows that the projection of $^\circ\mathscr{A}$ onto $^\circ\mathscr{A}^\circ$ is continuous for any $C^*$-norm on $^\circ\mathscr{A}$. Now $^\circ\mathscr{A}^\circ$ has a unique $C^*$-norm so [7; Lemma 2.11] applied to $^\circ\mathscr{A}$ and the action $\gamma$ of $\mathbf{T}$ on $^\circ\mathscr{A}$ shows that $^\circ\mathscr{A}$ has a unique $C^*$-norm too.

*Acknowledgements.* Our collaboration was facilitated by visits on various occasions of S.D. to the University of Hamburg and of J.E.R. to the Department of Mathematics of the University of Rome. We would like to thank Professor Rudolf Haag and the Università di Roma "La Sapienza" for hospitality and support. S.D. also gratefully acknowledges the hospitality extended to him by Professor Ludwig Streit at ZIF, University of Bielefeld in an early stage of this work.

*The first author was supported by the Ministero della Pubblica Istruzione and CNR—GNAFA.*

## REFERENCES

1. Cuntz, J., Simple $C^*$-algebras generated by isometries, *Comm. Math. Phys.*, **57**(1977), 173—185.
2. Doplicher, S.; Roberts, J. E., Fields, statistics and non-Abelian gauge groups, *Comm. Math. Phys.*, **28**(1972), 331—348.
3. Doplicher, S.; Roberts, J. E., Compact Lie groups associated with endomorphisms of $C^*$--algebras, *Bull. Amer. Math. Soc.*, **11**(1984), 333—338.
4. Doplicher, S.; Roberts, J. E., $C^*$-algebras and duality for compact groups: why there is a compact group of internal symmetries, in *Particle physics*, Proceedings of the International Conference on Mathematical Physics, Marseille 1986, ed. R. Seneor and M. Mebkhout, World Scientific, 1986.
5. Doplicher, S.; Roberts, J. E., A remark on compact automorphism groups of $C^*$-algebras, *J. Funct. Anal.*, **66**(1986), 67—72.
6. Doplicher, S.; Roberts, J. E., Endomorphisms of $C^*$-algebras, cross products and duality for compact groups, preprint.
7. Doplicher, S.; Roberts, J. E., Duals of compact Lie groups realized in the Cuntz algebras and their actions on $C^*$-algebras, *J. Funct. Anal.*, **74**(1987), 96—120.
8. Ghez, P.; Lima, R.; Roberts, J. E., The spectral category and the Connes invariant $\Gamma$, *J. Operator Theory*, **14**(1985), 129—146.
9. Nakagami, Y.; Takesaki, M., *Duality for crossed products of von Neumann algebras*, Lecture Notes in Mathematics, **271**, Springer Verlag, 1979.
10. Pontrjagin, L., *Topological groups*, Princeton University Press, 1958.
11. Roberts, J. E., Cross products of von Neumann algebras by group duals, *Symposia Mathematica*, **20**(1976), 335—363.

12. SAKAI, S., *$C^*$-algebras and $W^*$-algebras*, Springer Verlag, Berlin—Heidelberg—New York, 1971.

13. TAKESAKI, M., On the cross-norm of the direct product of $C^*$-algebras, *Tôhoku Math. J.*, 16(1964). 111—122.

*SERGIO DOPLICHER*
*Dipartimento di Matematica,*
*Università di Roma "La Sapienza",*
*00185 Roma,*
*Italy.*

*JOHN E. ROBERTS*
*Fachbereich Physik,*
*Universität Osnabrück,*
*D-4500 Osnabrück,*
*Federal Republic of Germany.*

Received May 4, 1987.