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# WEIGHTED COMPOSITION OPERATOR ON $C(X, E)$

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## INTRODUCTION

Let $X$ be a Hausdorff topological space and $E$ a real or complex Banach space. The space of continuous $E$ valued functions on $X$ will be denoted by $C(X, E)$. This space is a Banach space when endowed with the usual norm

$$\|f\| = \sup\{\|f(x)\| : x \text{ belongs to } X\}.$$

In what follows we will study a class of operators on $C(X, E)$ known as weighted composition operators. This class of operators has been the subject of several papers in recent years, see for example [1], [2], [6], and [10]. In our setting these operators take the following form:

$$Tf(x) = W(x)f(\varphi(x)),$$

where $\varphi$ is a selfmap of $X$ and for each $x$, $W(x)$ is a bounded linear operator on $E$. It is well known that for many Banach spaces $E$, the surjective isometries of $C(X, E)$ are of this form [3]. In addition, it is known [4] that the extreme points of the unit sphere in $L(C(X), C(Y))$ are weighted composition operators. In the first section of the paper we show that these operators arise in another very natural way. We say that an operator $T$ on $C(X, E)$ has the disjoint support property if $\|f(x)\|\,\|g(x)\| = 0$ for every $x$ in $X$ implies that $\|Tf(x)\|\,\|Tg(x)\| = 0$ for every $x$ in $X$. We show in Section 1 that every operator on $C(X, E)$ that satisfies the disjoint support property is a weighted composition operator. This result should be compared with Sourour's [9] result for such operators in the Bochner $L^p$ spaces.

Weighted composition operators have been studied by Kamowitz [6] in the $C(X)$ setting. In the second section of the paper we extend his results in two ways. We extend his results to the setting of vector valued functions and we also allow for a weakening of his hypothesis on the selfmap $\varphi$. In particular, we do not require that the map $\varphi$ be continuous everywhere. This is stated as a hypothesis in both the papers of Kamowitz [6] and Uhlig [10]. The following example illustrates why this continuity hypothesis is not necessary.

EXAMPLES. Let $E = \mathbf{C}$, the complex numbers and $X = [0, 1]$. Let

$$W(x) = \begin{cases} 0 & \text{for } x \text{ in } [0, 1/2] \\ (x - 1/2)^2 & \text{for } x \text{ in } [1/2, 1] \end{cases}$$

$$\varphi(x) = \begin{cases} x & \text{for } x \text{ in } [0, 1/4] \\ 1 & \text{for } x \text{ in } [1/4, 1/2] \\ x/2 & \text{for } x \text{ in } [1/2, 1]. \end{cases}$$

With this choice of $W$ and $\varphi$ the operator $T$ acts as follows:

$$Tf(x) = \begin{cases} 0 & \text{for } x \text{ in } [0, 1/2] \\ (x - 1/2)^2 f(x/2) & \text{for } x \text{ in } [1/2, 1]. \end{cases}$$

Clearly, $Tf \in C[0, 1]$ for $f \in C[0, 1]$ and $\|Tf\| \leqslant (1/4)\|f\|$. Therefore, $T$ is a bounded operator on $C[0, 1]$ and $\varphi$ is not continuous on all of $[0, 1]$.

A more extreme example is obtained by taking $X = [0, 1]$, $E = \mathbf{C}$ and letting $W(x) = 0$ for every $x$ and $\varphi(x) = 1$ on the rationals and 0 on the irrationals. The resulting operator $T$ is continuous and even compact but $\varphi$ is discontinuous at every point.

In the examples above, we can replace the selfmap $\varphi$ by a continuous function. In the first example the $\varphi$ can be replaced by $x/2$ in $[0, 1]$ and in the second example the $\varphi$ can be replaced by the constant function "1" on $[0, 1]$. However we now give an example where the $\varphi$ can not be replaced by a continuous function. Let $T: C([0, 1], \mathbf{R}) \to C([0, 1], \mathbf{R})$ be defined as follows:

$$Tf(x) = xf(\varphi(x)) \quad \text{where } \varphi(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that there does not exist a continuous selfmap $\psi$ of $[0, 1]$ and a continuous function $w(x)$ on $[0, 1]$ such that $Tf(x) = w(x)f(\psi(x))$ for all $f$'s in $C([0, 1], \mathbf{R})$. For if such $w(x)$ and $\psi$ existed, choosing $f = 1$ we get $w(x) = x$ for all $x$ in $[0, 1]$. Now choosing $f(x) = x$ we get $Tf(x) = x\sin(1/x)$ for $x \neq 0$. Therefore $\psi$ must be equal to $\sin(1/x)$ on $[0, 1]$ and there is no continuous function with this property.

## 1. OPERATORS ON $C(X, E)$ WITH THE DISJOINT SUPPORT PROPERTY

As noted in the introduction, in this section we will give a characterization of those linear operators on $C(X, E)$ which have the disjoint support property. This property has been studied mainly in the lattice setting, see for example the papers by Feldman and Porter [5] as well as the paper by Arendt and Hart [2]. In the case that $E$ is a Banach lattice $C(X, E)$ is also a Banach lattice. However, we do not assume any lattice structure for $E$. We do rely on the characterization of such operators in the scalar case. This result is easily obtained using duality arguments. In particular, if $T$ is a bounded linear operator on $C(X, \mathbf{R})$ with the disjoint support property, then $Tf = hf(\varphi)$ where $h = T(1)$ and $\varphi$ is a map of $X$ which is continuous on $\{x \mid h(x) \neq 0\}$. See [1] for a proof of this result.

THEOREM 1. *A bounded linear operator $T$ on $C(X, E)$ has the disjoint support property iff there is a selfmap $\varphi$ of $X$ and strongly continuous operator valued function $W(x)$ defined on $X$ with values in $B(E)$ such that $\varphi$ is continuous on the set $X \setminus N$, where $N = \{x \mid W(x) = 0\}$ and*

$$Tf(x) = W(x)f(\varphi(x))$$

*for every $f$ in $C(X, E)$.*

*Proof.* It is clear that any bounded linear operator on $C(X, E)$ with this description has the disjoint support property.

Now suppose that $T$ is a bounded linear operator on $C(X, E)$ and that $T$ has the disjoint support property. For $f$ in $C(X, \mathbf{R})$ and $v$ in $E$, let $f_v$ be defined by $f_v(x) = f(x)v$ for every $x$ in $X$. Clearly $f_v$ belongs to $C(X, E)$. In addition $1_v$ denotes the function defined by $1_v(x) = v$ for every $x$ in $X$.

Given a fixed $v$ in $E$ and $v^*$ in $E^*$ we define a map $T'$on $C(X, \mathbf{R})$ as follows:

$$T'f(x) = ((Tf_v)(x), v^*) \quad \text{for } x \text{ in } X.$$

$T'$ is a bounded linear operator on $C(X, \mathbf{R})$ and furthermore $T'$ inherits the disjoint support property from $T$. From [1] we know that this fact implies the existence of a selfmap $\varphi_{v,v^*}$ and a scalar valued function $w_{v,v^*}$ such that for each $f$ in $C(X, \mathbf{R})$

$$T'f(x) = w_{v,v^*}(x)f(\varphi_{v,v^*}(x)).$$

If we let $f = 1$ (the constant function) then we see that for each $x$, $w_{v,v^*}(x) = ((T1_v)(x), v^*)$. Thus, $w_{v,v^*}$ is weak* continuous for fixed $x$ and $v$. Let $v^*$ be fixed and define for fixed $x$ in $X$ and $v$ in $E$

$$W(x)v = (T1_v)(x).$$

Then for each $x$, $W(x)$ is a linear operator and furthermore $\|W(x)\| \leqslant \|T\|$. Moreover, the map $x \to W(x)$ is continuous in the strong operator topology.

Observe that for each $f$ in $C(X, \mathbf{R})$, $v$ in $E$, $v^*$ in $E^*$,

$$((Tf_v)(x),\ v^*) = w_{v,v^*}(x)f(\varphi_{v,v^*}(x)) =$$

$$= (W(x)v,\ v^*)f(\varphi_{v,v^*}(x)) = (W(x)f_v(\varphi_{v,v^*}(x)),\ v^*).$$

We must show that $\varphi_{v,v^*}$ does not depend on $v$ and $v^*$. To that end, we first note that the scalar function $((Tf_v)(x), v^*)$ is jointly continuous in $x$, $v$, and $v^*$. This is apparent from the following inequality:

$$|(Tf_v(x),\ v^*) - (Tf_w(y),\ w^*)| \leqslant$$

$$\leqslant |(Tf_v(x),\ v^*) - (Tf_v(y),\ v^*)| + |(Tf_v(y),\ v^*) - (Tf_w(y),\ v^*)| +$$

$$+ |(Tf_w(y),\ v^*) - (Tf_w(y),\ w^*)|.$$

We also claim that $\varphi_{v,v}$ is continuous in a neighborhood of any $x_0$ for which $W(x_0)v \neq 0$. Choose $v^*$ such that $(W(x_0)v, v^*) \neq 0$. By continuity of the map $(y, w, w^*) \to (T1_w(y), w^*)$ there exists a neighborhood $U$ of $(x_0, v, v^*)$ such that $(T1_w(y), w^*)$ does not vanish for $(y, w, w^*)$ in $U$. It follows from our formula for $T'$ that for every $f$ in $C(X, \mathbf{R})$ and $(y, w, w^*)$ in $U$ we have

$$f(\varphi_{w,w^*}(y)) = (Tf_w(y),\ w^*)/(W(y)w,\ w^*).$$

As a consequence of this last equality a net argument shows that $\varphi_{w,w^*}$ must be continuous on $U$.

Let $x_0$ in $X$ and $v$ in $E$ be given. We claim that if there exists $v^*$ and $w^*$ in $E^*$ such that $(W(x_0)v, v^*) \neq 0$ and $(W(x_0)v, w^*) \neq 0$ then $\varphi_{v,v^*}(x_0) = \varphi_{v,w^*}(x_0)$. To see this let $x_1 = \varphi_{v,v^*}(x_0)$ and $x_2 = \varphi_{v,w^*}(x_0)$ and suppose that $x_1 \neq x_2$. Choose open sets $U_1$ and $U_2$ in $X$ which contain $x_1$ and $x_2$ respectively and also have disjoint closures. Let $f_1$ and $f_2$ be continuous functions on $X$ such that

$$f_1 = \begin{cases} 0 & \text{on } X \setminus U_1 \\ 1 & \text{at } x_1 \end{cases} \quad f_2 = \begin{cases} 0 & \text{on } X \setminus U_2 \\ 1 & \text{at } x_2. \end{cases}$$

Since $f_1 v$ and $f_2 v$ are disjoint, $Tf_1 v$ and $Tf_2 v$ must also be disjoint. However,

$$(Tf_1(x_0)v,\ v^*) = (W(x_0)v,\ v^*)f_1(\varphi_{v,v^*}(x_0)) = (W(x_0)v,\ v^*) \neq 0,$$

and

$$(Tf_2(x_0)v,\ w^*) = (W(x_0)v,\ w^*)f_2(\varphi_{v,w^*}(x_0)) = (W(x_0)v,\ w^*) \neq 0,$$

and therefore we have a contradiction. Hence, if $(W(x_0)v, v^*)$ is not zero, then for every $w^*$ in $H = \{w^* \mid (W(x_0)v, w^*) \neq 0\}$ we have $\varphi_{v,v^*}(x_0) = \varphi_{v,w^*}(x_0)$. Since $H$ is weak* dense in $E^*$ and $\varphi_{v,v^*}(x)$ is weak* continuous, it follows that $\varphi_{v,v^*}(x)$ is independent of $v^*$ on the set $\{x \mid W(x) \neq 0\}$. Let $\varphi_v(x)$ denote the value of $\varphi_{v,v^*}(x)$ for a given $x$ and $v$. The argument to show that $\varphi_v(x)$ is independent of $v$ on the set $\{x \mid W(x) \neq 0\}$ is essentially the same as the previous one and so we omit it. In addition, $\varphi$ can be taken to be the identity on the set $N$.

We have now shown that there exists a strongly continuous $B(E)$ valued function $W(x)$ and a selfmap $\varphi$ which is continuous on the set $\{x \mid W(x) \neq 0\}$. Furthermore,

$$((Tf_v)(x), v^*) = (W(x)f_v(\varphi(x)), v^*)$$

for every $f$ in $C(X, \mathbf{R})$, $v$ in $E$, $v^*$ in $E^*$ and $x$ in $X \setminus N$. This fact implies that the following formula is valid:

$$Tf_v(x) = \begin{cases} 0 & \text{for } x \text{ in } N \\ W(x)f_v(\varphi(x)) & \text{for } x \text{ in } X \setminus N. \end{cases}$$

Since the closed linear span of the set $\{f_v \mid f \text{ in } C(X, \mathbf{R}) \text{ and } v \text{ in } E\}$ is dense in $C(X, E)$ [8, p. 237], it follows that the formula is valid for all $f$ in $C(X, E)$.

Remarks. It should be noted that the map $x \to W(x)$ is only required to be continuous in the strong operator topology. This is made clear in the next example. In Section 2 we will show that if $T$ is compact, then the map $x \to W(x)$ is necessarily continuous in the uniform operator topology.

Example. Let $E$ be a separable infinite dimensional Hilbert space and let $X = \{0, 1, 1/2, 1/3, \ldots,\}$ with the usual subspace topology. Let $\{e_i\}$ denote an orthonormal basis for the Hilbert space $E$. For $x$ in $X$ define the linear transformation $W(x)$ as follows:

$$W(0) = 0 \quad \text{and} \quad W(1/k)v = (v, e_k)e_1.$$

We take $\varphi(x) = x$ for each $x$ in $X$. For $f$ in $C(X, E)$, the action of the resulting weighted composition operator is given by:

$$Tf(x) = \begin{cases} 0 & \text{when } x = 0 \\ W(1/k)f(1/k) = (f(1/k), e_k)e_1 & \text{for } x = 1/k. \end{cases}$$

It can be seen that $x \to W(x)$ is strong operator continuous on $X$. However, $\|W(1/n)\| = 1$ for all $n$ and $W(0) = 0$, therefore $x \to W(x)$ is not continuous in the uniform operator topology.

## 2. COMPACT WEIGHTED COMPOSITION OPERATORS ON $C(X, E)$

Given a weighted composition operator $T$ we denote by $N$ the set of $x$ in $X$ for which $W(x)$ is the zero operator on $E$. We use $|G|$ to denote the cardinality of a set $G$. A sequence $\{f_n\}$ is said to be $\varepsilon$ uniform Cauchy on $Q \subseteq X$ if given $\varepsilon > 0$ there exists $N_\varepsilon$ such that

$$\|f_n(x) - f_m(x)\| < \varepsilon$$

for every $m, n > N_\varepsilon$ and $x$ in $\Omega$. It is well known that a sequence $\{f_n\}$ converges in $C(X, E)$ if and only if it is $\varepsilon$ uniform Cauchy on $X$ for every $\varepsilon > 0$.

The next lemma gives an equivalence between two conditions which are associated with the compactness of the weighted composition operator $Tf = W(\cdot)f(\varphi)$.

LEMMA. *The following are equivalent:*

(i) *If $F$ is a compact subset of $X \setminus N$, then $|\varphi(F)| < \infty$.*

(ii) *If $F$ is a connected component of $X \setminus N$, there exists an open subset $U$ of $X$ such that $U \subseteq X \setminus N$, $F \subseteq Y$, and $|\varphi(U)| = 1$.*

*Proof.* (i) $\Rightarrow$ (ii). Let $F$ be a connected component contained in $X \setminus N$ and suppose that $x_0 \in F$. Let $G = \{x \mid x \in F \text{ and } \varphi(x) = \varphi(x_0)\}$. First we show that $F = G$, and so we deduce that $\varphi$ is constant on $F$. To show that $F = G$ it is sufficient to prove that $G$ is nonempty and clopen relative to $F$.

To show $G$ is open relative to $F$ let $x \in G$ and let $K \subseteq F$ be a compact neighborhood of $x$ relative to $F$. Since $K$ is a compact subset of $X \setminus N$, $|\varphi(K)| < < \infty$. Therefore, $\varphi(K) = \{z_0, z_1, \ldots, z_n\}$. Since $x \in K$, we may assume (without loss of generality) that $\varphi(x) = z_0$. If we set $U_x = \varphi^{-1}(z_0) \cap F$, then $U_x$ is a neighborhood of $x$ relative to $F$. Since $\varphi(U_x) = \{z_0\}$, it follows that $U_x \subseteq G$ and thus $G$ is open relative to $F$.

To prove that $G$ is closed relative to $F$ let $x \in F$ and $x \in G$. Let $(x_n)$ be a net in $G$ and $x_n \to x$. Since $\varphi$ is continuous at $x$, $\varphi(x_n) \to \varphi(x)$. Since $\varphi(x_n) = z_0$ for every $n$, it follows that $\varphi(x) = z_0$. Thus $x \in G$ and $G$ is closed relative to $F$.

To get a neighborhood $U$ as advertised in (ii) let $\varphi(F) = \{z_0\}$. If $x \in F$ there exists an open set $U_x \subseteq X \setminus N$ such that $\varphi(U_x) = \{\varphi(x)\} = \{z_0\}$. Find such a neighborhood for each $x$ in $F$ and then set $U = \bigcup U_x$. Then $U$ is open, $F \subseteq U$ and $\varphi(U) = \bigcup \varphi(U_x) = \{z_0\}$.

(ii) $\Rightarrow$ (i). Let $K$ be a compact subset of $X \setminus N$. For each $p \in K$, there exists a component $F_p \subseteq X \setminus N$ containing $p$. There exists a set $U_p$ open in $X$ such that $U_p \supseteq F_p$ and $|\varphi(U_p)| = 1$. $K \subseteq \bigcup U_p$ and by compactness, $K$ is contained in a finite union of the $U_p$'s. It follows that $|\varphi(K)| < \infty$.

THEOREM 2. *The following conditions are necessary and sufficient for the weighted composition operator*

$$Tf(x) = W(x)f(\varphi(x))$$

*to be a compact operator on* $C(X, E)$.

(2.1) $\varphi: X \to X$ *and* $\varphi$ *is continuous on* $X \setminus N$.

(2.2) $x \to W(x)$ *is continuous in the uniform operator topology.*

(2.3) *If* $F$ *is a compact subset of* $X \setminus N$, *then* $|\varphi(F)| < \infty$.

(2.3′) *If* $F$ *is a connected component of* $X \setminus N$, *there exists an open subset* $U$ *of* $X$, *such that*

$$U \subseteq X \setminus N, \quad F \subseteq U, \quad \text{and} \quad |\varphi(U)| < \infty.$$

(2.4) *If* $\{e_n\}$ *is a sequence in* $E$, $\varepsilon > 0$, *and* $F$ *is a compact subset of* $X \setminus N$, *then there exists a subsequence* $\{e_{n(k)}\}$ *such that* $\{W(x)e_{n(k)}\}$ *is* $\varepsilon$*-uniformly Cauchy on* $F$.

(2.5) *Given a bounded sequence* $\{e_n\}$ *in* $C(X, E)$, *let* $Z = \{x \mid T(e_n)(x) = W(x)e_n = 0$ *for every* $n\}$. *If* $\varepsilon > 0$ *there exists a subsequence* $\{e_{k(n)}\}$ *and a neighborhood* $U_\varepsilon \supseteq Z$ *such that*

$$\|Te_{n(k)}(x)\| < \varepsilon \quad \text{for every } x \text{ in } U_\varepsilon.$$

REMARK. At this point we note that condition (2.4) implies that $W(x)$ is a compact operator on $E$ for each $x$ in $X$.

*Proof.* We first show that the conditions stated above are sufficient. Conditions (2.1) and (2.2) insure continuity of $T$. To prove compactness, let $N$ be the set $N = \{x \mid W(x) = 0\}$ and $\varepsilon$ be a positive number. Let $\{f_i\}$ be a norm 1 sequence in $C(X, E)$. Hypothesis (2.5) implies that there exists a subsequence $\{f_n'\}$ and a neighborhood $U_\varepsilon \supseteq N$ such that $\|Tf_n'(x)\| < \varepsilon$ for every $x \in U_\varepsilon$. If we let $K_\varepsilon = X \setminus U_\varepsilon$ then $K_\varepsilon$ is compact and by (2.3) $|\varphi(K_\varepsilon)| < \infty$. Thus there exist $x_1, x_2, \ldots, x_p$ such that

$$\varphi(K_\varepsilon) = \{x_1, x_2, \ldots, x_p\}.$$

By continuity the set $\varphi^{-1}(x_i)$ is closed and consequently the set $F_i = \varphi^{-1}(x_i) \cap K_\varepsilon$ is compact and $|\varphi(F_i)| = 1$.

We can (without loss of generality) relabel the original subsequence as $\{f_n\}$. Then, for $x \in F_1$, $\{f_j \circ \varphi(x)\}$ is a sequence of constant vectors in $E$. By (2.4) there exists a subsequence $\{f_{1k}\}$ of $\{f_n\}$ such that $\{W(x)f_{1k} \circ \varphi\}$ is $\varepsilon$ uniform Cauchy on $F_1$. The sequence $\{f_{1k} \circ \varphi\}$ is again a sequence of constant vectors in $E$. Hypothesis (2.4) yields a subsequence $\{f_{2k}\}$ of $\{f_{1k}\}$ such that $\{W(x)f_{2k} \circ \varphi\}$ is $\varepsilon$ uniform Cauchy on $F_1 \cup F_2$. Continuing this process we obtain a sequence $\{f_{pk}\}$ such that $\{W(x)f_{pk} \circ \varphi\}$

is $\varepsilon$ uniform Cauchy on $K_\varepsilon$. Furthermore, since $\|W(x)f_{pk}\circ\varphi\| < \varepsilon$ for $x$ in $U$ it follows that $\{Tf_{pk}\}$ is $\varepsilon$ uniform Cauchy on all of $X$. By completeness, and the Cantor diagonal process there exists a $g$ in $C(X, E)$ and a subsequence $\{f_{p_{n_k}}\}$ such that $\{Tf_{p_{n_k}}\}$ converges to $g$. This establishes the compactness of $T$.

To show that these conditions are necessary we first note that (2.1) follows from the continuity of the operator $T$. To see that (2.2) is true we assume that the map $x \to W(x)$ is not continuous in uniform norm at some $x_0$ in $X$. Then there is a $\delta > 0$ such that for each open set $V$ containing $x_0$, there exists an $x_V$ such that $\|W(x_V) - W(x_0)\| > \delta$. As a consequence, there is a net $\{e_V\}$ such that $\|e_V\| \leqslant 1$, and $\|W(x_V)e_V - W(x_0)e_V\| > \delta$ for all $V$ in $\Omega$, where $\Omega$ denotes the class of all neighborhoods of $x_0$. For each compact neighborhood $V$ of $x_0$ let $g_V$ be the constant function defined by $g_V(x) = e_V$. Then, $Tg_V(x) = W(x)e_V$ for each $x$ in $X$. From the compactness of $T$ we know there is a subnet of $\{g_V\}$, which we can without loss of generality also label $\{g_V\}$, and a function $h$ in $C(X, E)$ such that $\{Tg_V\}$ converges to $h$ along $\Omega$. Therefore, $\|h(x_V) - h(x_0)\| \to 0$ along $\Omega$ since $x_V \to x_0$. Furthermore, we also have that

$$\|Tg_V(x_V) - h(x_V)\| \to 0$$

and

$$\|Tg_V(x_0) - h(x_0)\| \to 0 \quad \text{along } \Omega.$$

However, since it is true that

$$\|Tg_V(x_0) - Tg_V(x_V)\| = \|W(x_V)e_V - W(x_0)e_V\| > \delta$$

we clearly have a contradiction.

To show that (2.3) is necessary, suppose that $F$ is a compact subset of $X\setminus N$ and that $x_0 \in F$. We claim that there exists a neighborhood $U$ of $x_0$ such that $|\varphi(U)| < \infty$. To see this, note that since $x_0 \in X\setminus N$, $W(x_0) \neq 0$ and so there exists $z$ in $E$ with $\|z\| = 1$ and $\|W(x_0)z\| > 0$. Since the map $x \to W(x)$ is strongly continuous there exists an $\varepsilon > 0$ and a compact neighborhood $U$ of $x_0$ such that $U \subseteq X\setminus N$ and $\|W(y)z\| > 0$ for all $y$ in $U$. If $|\varphi(U)|$ were not finite, then there would exist a sequence $\{f_n\}$ defined on $\varphi(U)$ such that $\|f_n\| = 1$ and $\{f_n\}$ contains no convergent subsequence. For $x$ in $U$ set

$$h_n(x) = f_n(\varphi(x))$$

and let $\{g_n\}$ be a (norm preserving) Tietze extension of the $h_n$'s to all of $X$. The functions $g_n(x)z$ belong to $C(X, E)$ and furthermore,

$$T(g_n z)(x) = g_n(\varphi)W(x)z = h_n(x)W(x)z \quad \text{on } U.$$

Clearly, $T(g_n z)$ contains no convergent sequence and this contradicts the compactness of $T$.

We will show that (2.4) is also necessary. To that end let $F$ be a compact subset of $X \setminus N$ and $\{e_n\}$ a sequence in $E$. Let $g$ be a function in $C(X)$ such that $g(x) = 1$ on $F$ and $0 < g(x) < 1$. Let $f_n(x) = g(x)e_n$. Since $\|f_n\| < \infty$ there exists a subsequence $f_{n(k)} = g(x)e_{n(k)}$ such that $\{Tf_{n(k)}\}$ converges in $C(X, E)$. It follows that the subsequence $Tf_{n(k)} = W(x)e_{n(k)}$ is $\varepsilon$ uniform Cauchy on $F$.

Finally, to show that (2.5) is necessary let $\{e_n\}$ be a bounded sequence. Then by the compactness of $T$ there exists a subsequence $e_{n(k)}$ such that $Te_{n(k)}$ converges to some $g$ in $C(X, E)$. Clearly $g(x) = 0$ for $x$ in $Z = \{x \mid Te_n(x) = 0\}$. Hence, if $\varepsilon > 0$, there exists an open set $U_\varepsilon \supseteq Z$ such that $\varepsilon > \|g(y) - g(x)\| = \|g(y)\|$ for every $y$ in $U_\varepsilon$, i.e., $\|Te_{n(k)}(y)\| < \varepsilon$ for every $y$ in $U_\varepsilon$. With this the proof of the theorem is complete.

REMARKS. The map $x \to W(x)$ is automatically uniformly continuous in the case that $E$ is finite dimensional because in that setting $B(E)$ is finite dimensional. As a consequence this hypothesis does not appear in [6] and [10].

## 3. THE SPECTRUM OF A COMPACT WEIGHTED COMPOSITION OPERATOR

The spectrum of a compact weighted composition operator on $C(X)$ has been studied by Kamowitz [6] and by Uhlig [10]. In this section we generalize and extend some of their results to vector setting of $C(X, E)$.

THEOREM 3. *Let $T$ be a compact weighted composition operator on $C(X, E)$. A scalar $\alpha \neq 0$ is an eigenvalue of the operator $T$ if and only if for some integer $n > 0$, $\alpha^n$ is an eigenvalue of $W(c)W(\varphi(c))W(\varphi(\varphi(c))) \dots W(\varphi_{n-1}(c))$ and $c$ is a fixed point of order $n$ of $\varphi$,* i.e. *$\varphi_n(c) = c$ and $\varphi_k(c) \neq c$ for $k < n$ ($\varphi_n(x) := \varphi(\varphi_{n-1}(x))$).*

*Proof.* We first prove that these conditions are necessary. Let $\alpha$ be an eigenvalue of $T$. Then for some $f$ in $C(X, E)$ of norm 1, $Tf = \alpha f$, and so $W(x)f(\varphi(x)) = \alpha f(x)$ for $x$ in $X$. We obtain by iteration that the following holds for every $x$ in $X$:

$$W(x)W(\varphi(x)) \dots W(\varphi_{n-1}(x))f(\varphi_n(x)) = \alpha^n f(x)$$

for every positive integer $n$. Since $\|f\| = 1$, $f(q) \neq 0$ for some $q$ in $X$. Let $G = \{q, \varphi(q), \varphi_2(q), \dots,\}$. If $G$ is finite then for some $m < n$, $\varphi_n(q) = \varphi_m(q)$. In this case, $c = \varphi_m(q)$ is a fixed point of $\varphi_{n-m}$. Furthermore, $f(\varphi_m(q)) \neq 0$ because this would imply that $f(q) = 0$. Therefore, we would have that $\alpha^{n-m}$ is an eigenvalue of the operator product $W(c)W(\varphi(c)) \dots W(\varphi_{n-m}(c))$, where $c = \varphi_m(q)$. In the case that $G$ is infinite, the fact that the map $x \to W(x)$ is uniformly continuous enables us to give an argument analogous to that in Proposition 4 of [6] to show that $f(q) = 0$. Thus, we reach a contradiction and hence the set $G$ must be finite. This completes this part of the proof.

To finish the proof, we assume that $c$ is a fixed point of order $n$ of $\varphi$ and $\alpha^n$ is a eigenvalue of the operator product $W(c)W(\varphi(c)) \ldots W(\varphi_{n-1}(c))$ with eigenvector $v$. To obtain an $f$ in $C(X, E)$ which satisfies the eigenvalue equation $Tf = \alpha f$, we proceed as follows: Let $y_1 = c, y_2 = \varphi_1(c), \ldots, y_n = \varphi_{n-1}(c)$. Define $f(y_n) = v, f(y_{n-1}) = (1/\alpha)W(y_n)v, \ldots,$ and $f(y_1) = (1/\alpha^{n-1})W(y_2)W(y_3) \ldots W(y_n)v$. Next we define

$$K = \bigcup\{\varphi^{-i}\{y_1, y_2, \ldots, y_n\} \mid 1 \leqslant i < \infty\}.$$

Then if we define,

$$f(x) = \begin{cases} (1/\alpha)W(x)f(\varphi(x)) & \text{for } x \text{ in } K \\ 0 & \text{for } x \text{ in } X \setminus K \end{cases}$$

the function $f$ is defined everywhere on $X$. That the function $f$ belongs to $C(X, E)$ follows from an argument completely analogous to that of Proposition 1 of [10]. This transference is possible because of the uniform continuity of the map $x \to W(x)$. It is clear from the construction of the function $f$ that $Tf(x) = \alpha f(x)$ for every $x$ in $X$ and the proof is complete.

We also have the following corollary.

COROLLARY. *Let $Tf(x) = W(x)f(\varphi(x))$ be a compact weighted composition operator on $C(X, E)$ and suppose that $W(x)$ is never the zero operator. Then $\varphi(X) = \{y_1, \ldots, y_n\}$ for some $n$ and the range of $T$ is $\{W(x)v_i \mid v_i \in E \text{ and } x \in \varphi^{-1}(y_i)\}$.*

Since we know what the eigenfunctions of $T$ are, we can give a precise description of the eigenspaces of $T$. We use the notation of Theorem 3. Each eigenvector $v$ of $W(c)W(\varphi(c)) \ldots W(\varphi_{n-1}(c))$ corresponding to eigenvalue $\alpha^n$ determines a unique eigenvector $f_v$ of $T$ with eigenvalue $\alpha$. Thus we have a map $\Omega : v \to f_v$, from the eigenspace of $W(c) \ldots W(\varphi_{n-1}(c))$ for eigenvalue $\alpha^n$ to the eigenspace of $T$ with eigenvalue $\alpha$. This map is linear and one-to-one. The range of this map will be called the "*canonical eigenspace of $T$ for $\alpha$*". It is easy to see that the canonical eigenspace of $T$ for $\alpha$ determined by the product $W(c)W(\varphi(c)) \ldots W(\varphi_{n-1}(c))$ is the same as the eigenspace of $W(\varphi(c)) \ldots W(\varphi_{n-1}(c))W(c)$ or any other cyclic permutation of $\{c, \varphi(c), \ldots, \varphi_{n-1}(c)\}$.

With this terminology, the following theorem gives a complete description of the eigenspaces of $T$. The proof should be clear from the arguments in the proof of the second part of the previous result.

THEOREM 4. *Let* $\alpha$ *be a nonzero eigenvalue of* $T$. *For each finite set of the form* $\Gamma = \{c, \varphi(c), \varphi_2(c), \ldots, \varphi_{n-1}(c)\}$, *such that* $\varphi_n(c) = c$, *and* $\alpha^n$ *belongs to* $\sigma(W(c)W(\varphi(c)) \ldots W(\varphi_{x-1}(c)))$, *let* $V_\Gamma^*$ *denote the corresponding canonical eigenspace of* $T$ *for* $\alpha$. *Then the eigenspace of* $T$ *corresponding to* $\alpha$ *is the direct sum of all such* $V_\Gamma^*$.

REMARK. It might be interesting to determine necessary and sufficient conditions for two weighted composition operators to commute. This is probably more tractable in the case that both operators are compact.

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