J. OPERATOR THEORY
20(1988), 3–20

MODELS FOR OPERATORS USING WEIGHTED SHIFTS

VLADIMIR MÜLLER

INTRODUCTION

It is well-known that the shift operators in a Hilbert space can be regarded as universal operators. This was first pointed out by Rota [9], see also Foiaş [5] The most important result in this direction is the following theorem, [10], [3]:

THEOREM. *Let T be a contraction in a separable Hilbert space, $T^n \to 0$ strongly. Then T is unitarily equivalent to a part of the (unweighted) backward shift of infinite multiplicity.*

Later the conditions under which an operator is similar to a part of a weighted shift were studied. Some Rota type models for operators using weighted shifts were obtained in [6] and [12].

It is natural to ask when an operator is unitarily equivalent to a part of a weighted shift. This question was first studied by Agler [1], [2] who proved a generalization of the above mentioned theorem for hypercontractions.

The aim of this paper is to study the question which operators are unitarily equivalent to a part of a given weighted shift.

The paper is divided into three sections. The first section is introductory and presents motivations and some general ideas which are used in the remaining two sections. In Section 2, the weighted shifts with non-increasing weights are studied and some generalizations of results of [6] and [12] are obtained. In Section 3 we study the weighted shifts whose weights satisfy some recurrent relation. This section is closely related to the work of Alger [1], [2] and generalize some of his results.

1.

Let H' be an infinite-dimensional, separable complex Hilbert space with an orthonormal basis $\{e_0, e_1, \ldots\}$. Denote by $B(H')$ the algebra of all bounded operators on H'. Let $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ be a bounded sequence of positive numbers. Then the back ward shift $S'_\alpha \in B(H')$ with weights α_i is defined by $S'_\alpha e_0 = 0$, $S'_\alpha e_i = \alpha_i e_{i-1}$ $(i = 1, 2, \dots)$.

Let $H = \ell^2(H')$ be the Hilbert space of all square summable sequences $x = (x_0, x_1, \dots)$ of vectors $x_i \in H'$ and let the weighted shift $S_\alpha \in B(H)$ of infinite multiplicity be defined by

$$S_\alpha(x_0, x_1, \dots) = (\alpha_1 x_1, \alpha_2 x_2, \dots).$$

Let T be a bounded operator in a separable complex Hilbert space K. We are looking for necessary and sufficient conditions on T to be unitarily equivalent to a restriction of S_α to some invariant subspace $M \subset H$, $S_\alpha M \subset M$. We say shortly that T is unitarily equivalent to a part of S_α. Equivalently, we are looking for an isometry $V: K \to H$ such that

(1) $$S_\alpha V = VT.$$

As $H = \bigoplus_{i=0}^{\infty} H'$ we may write $Vx = (A_0 x, A_1 x, \dots)$ where $A_i: K \to H'$ are bounded operators. Then condition (1) gives $\alpha_i A_i x = A_{i-1} T x$ $(i = 1, 2, \dots, x \in K)$, so $A_i = \alpha_i^{-1} A_{i-1} T$ and by induction

$$A_s = \alpha_s^{-1} \alpha_{s-1}^{-1} \dots \alpha_1^{-1} A_0 T^s .$$

If we put

(2) $$b_0 = 1,\ b_s = \alpha_s^{-2} \alpha_{s-1}^{-2} \dots \alpha_1^{-2} \quad (s \geqslant 1),$$

then $A_s = \sqrt{b_s} A_0 T^s$ $(s \geqslant 1)$ and condition (1) is equivalent to the formula

(3) $$Vx = (A_0 x, \sqrt{b_1} A_0 T x, \sqrt{b_2} A_0 T^2 x, \dots) \quad (x \in K),$$

where A_0 is an arbitrary operator $A_0: K \to H'$.

We would like to find A_0 such that the operator V defined by (3) will be an isometry. Let $x \in K$. Then

(4) $$\|Vx\|^2 = \|A_0 x\|^2 + b_1 \|A_0 T x\|^2 + b_2 \|A_0 T^2 x\|^2 + \dots .$$

If V is an isometry then

$$\begin{aligned} \|x\|^2 &= \|A_0 x\|^2 + b_1 \|A_0 T x\|^2 + b_2 \|A_0 T^2 x\|^2 + \dots \\ \|Tx\|^2 &= \qquad\qquad \|A_0 T x\|^2 + b_1 \|A_0 T^2 x\|^2 + \dots \\ \|T^2 x\|^2 &= \qquad\qquad\qquad\qquad \|A_0 T^2 x\|^2 + \dots \\ &\vdots \end{aligned}$$

hence

$$(5)\qquad \begin{pmatrix} \|x\|^2 \\ \|Tx\|^2 \\ \|T^2x\|^2 \\ \vdots \end{pmatrix} = B \begin{pmatrix} \|A_0x\|^2 \\ \|A_0Tx\|^2 \\ \|A_0T^2x\|^2 \\ \vdots \end{pmatrix}$$

where

$$B = \begin{pmatrix} 1, & b_1, & b_2, & b_3, & \dots \\ & 1, & b_1, & b_2, & \dots \\ & & 1, & b_1, & \dots \\ & & & 1, & \dots \\ & 0 & & & \ddots \end{pmatrix}.$$

Put $c_0 = 1$ and consider the following system of linear equations:

$$(6)\qquad \begin{aligned} &b_1 + c_1 = 0, \\ &b_2 + b_1c_1 + c_2 = 0, \\ &\vdots \\ &b_s + \sum_{i=1}^{s-1} b_{s-i}c_i + c_s = 0, \\ &\vdots \end{aligned}$$

Clearly, the numbers $b_1, b_2, \dots$ determine $c_1, c_2, \dots$ satisfying (6) uniquely and vice versa. The Toeplitz matrix

$$C = \begin{pmatrix} 1, & c_1, & c_2, & c_3, & \dots \\ & 1, & c_1, & c_2, & \dots \\ & & 1, & c_1, & \dots \\ & & & 1, & \dots \\ & 0 & & & \ddots \end{pmatrix}$$

is formally the inverse matrix of B (in general, B and C do not define bounded operators).

It is reasonable to expect that the operator $A_0 : K \to H'$ will satisfy

$$\begin{pmatrix} \|A_0x\|^2 \\ \|A_0Tx\|^2 \\ \|A_0T^2x\|^2 \\ \vdots \end{pmatrix} = C \begin{pmatrix} \|x\|^2 \\ \|Tx\|^2 \\ \|T^2x\|^2 \\ \vdots \end{pmatrix}.$$

In particular,

$$\|A_0x\|^2 = \|x\|^2 + \sum_{i=1}^{\infty} c_i\|T^ix\|^2.$$

So we can expect that a necessary condition on T to be unitarily equivalent to a part of S_α will be

$$\|x\|^2 + \sum_{i=1}^{\infty} c_i\|T^ix\|^2 \geqslant 0 \quad (x \in K).$$

If this condition is satisfied, the operator

$$A_0 = \left(I + \sum_{i=1}^{\infty} c_iT^{*i}T^i\right)^{1/2} \tag{8}$$

will satisfy (7).

Example. In the theory of Sz.-Nagy and Foiaş we have $\alpha_1 = \alpha_2 = \ldots = 1$, $b_1 = b_2 = \ldots = 1$. It follows from (6) that $c_1 = -1$, $c_2 = c_3 = \ldots = 0$, hence $A_0 = (I - T^*T)^{1/2}$. This operator naturally appears in the theory of Sz.-Nagy and Foiaş and is called the defect operator of T. Analogous operators are used also in [2].

The considerations in this section were not quite correct. We neglected questions of convergence in several places. Nevertheless, in reasonable cases, these considerations will work. Two such cases will be studied in the following two sections.

2.

In this section we suppose that the weights form a non-increasing sequence $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots > 0$.

Let b_i, c_i $(i = 0, 1, \ldots)$ be defined by (2) and (6). Denote $b_k^{(i)} = b_k + b_{k-1}c_1 + \ldots \ldots + b_{k-i}c_i$ $(k > i \geqslant 0)$. Then $b_k^{(0)} = b_k$, $b_k^{(k-1)} = -c_k$ and $b_k^{(i)} = b_k^{(i-1)} + b_{k-i}c_i$ $(k > i \geqslant 1)$. Further, $b_{k-1}/b_k = \alpha_k^2$.

2.1. LEMMA. a) $b_k^{(i)} \geqslant 0 \quad (k > i \geqslant 0)$

b) $-b_{i+1} \leqslant c_{i+1} \leqslant 0 \quad (i \geqslant 0)$

c) $\dfrac{b_k^{(i)}}{b_l^{(i)}} \geqslant \dfrac{b_k^{(i-1)}}{b_l^{(i-1)}} \quad (k > l > i \geqslant 1).$

Proof. We prove Lemma 2.1 by induction on i. For $i = 0$ the assertions a) and b) follow from the definitions. Let $i = 1$.

a)
$$b_k^{(1)} = b_k + b_{k-1}c_1 = b_k b_1\left(\frac{1}{b_1} - \frac{b_{k-1}}{b_k}\right) = b_k b_1(\alpha_1^2 - \alpha_k^2) \geqslant 0.$$

b)
$$c_2 = -b_2^{(1)} \leqslant 0 \qquad \text{by a)},$$

$$c_2 = -b_2 - b_1 c_1 = -b_2 + b_1^2 \geqslant -b_2 .$$

c) We have

$$\frac{b_k}{b_l} = \frac{1}{\alpha_{l+1}^2 \ldots \alpha_k^2} \geqslant \frac{1}{\alpha_l^2 \ldots \alpha_{k-1}^2} = \frac{b_{k-1}}{b_{l-1}},$$

so

$$b_l b_{k-1} \leqslant b_k b_{l-1},$$

$$b_k b_l + b_l b_{k-1} c_1 \geqslant b_k b_l + b_k b_{l-1} c_1,$$

$$b_l b_k^{(1)} \geqslant b_k b_l^{(1)},$$

$$b_k^{(1)}/b_l^{(1)} \geqslant b_k/b_l = b_k^{(0)}/b_l^{(0)}.$$

Suppose now that the assertions a), b) and c) hold for every j, $0 \leqslant j \leqslant i-1$. We prove them for i.

a)
$$b_k^{(i)} = b_k^{(i-1)} + b_{k-i}c_i = b_i^{(i-1)}\left(\frac{b_k^{(i-1)}}{b_i^{(i-1)}} - b_{k-i}\right)$$

where $b_i^{(i-1)} \geqslant 0$ by the induction assumption and

$$\frac{b_k^{(i-1)}}{b_i^{(i-1)}} \geqslant \frac{b_k^{(0)}}{b_i^{(0)}} = \frac{b_k}{b_i} = \frac{1}{\alpha_{i+1}^2 \ldots \alpha_k^2} \geqslant \frac{1}{\alpha_1^2 \ldots \alpha_{k-i}^2} = b_{k-i} .$$

b)
$$c_{i+1} = -b_{i+1}^{(i)} \leqslant 0 \qquad \text{by a)},$$

$$c_{i+1} = -b_{i+1} - b_i c_1 - \ldots - b_1 c_i \geqslant -b_{i+1} .$$

c) We have

$$\frac{b_k^{(i-1)}}{b_l^{(i-1)}} \geqslant \frac{b_k^{(0)}}{b_l^{(0)}} = \frac{b_k}{b_l} = \frac{1}{\alpha_{l+1}^2 \dots \alpha_k^2} \geqslant \frac{1}{\alpha_{l-i+1}^2 \dots \alpha_{k-i}^2} = \frac{b_{k-i}}{b_{l-i}},$$

$$b_{k-i} b_l^{(i-1)} \leqslant b_{l-i} b_k^{(i-1)},$$

$$b_k^{(i-1)} b_l^{(i-1)} + c_i b_{k-i} b_l^{(i-1)} \geqslant b_k^{(i-1)} b_l^{(i-1)} + c_i b_{l-i} b_k^{(i-1)},$$

$$b_l^{(i-1)} b_k^i \geqslant b_k^{(i-1)} b_l^i,$$

$$b_k^{(i)} / b_l^{(i)} \geqslant b_k^{(i-1)} / b_l^{(i-1)}.$$

2.2. THEOREM. *Let* $\alpha_1 \geqslant \alpha_2 \geqslant \dots > 0$, $b_i = \alpha_i^{-2} \dots \alpha_1^{-2}$ $(i \geqslant 1)$. *Let* $T \in B(K)$ *satisfy* $\sum_{s=1}^{\infty} \|T^s\|^2 b_s \leqslant 1$. *Then* T *is unitarily equivalent to a part of* S_α.

Proof. Let $c_0, c_1, c_2, \dots$ be defined by (6). By the previous lemma, $0 \geqslant c_i \geqslant -b_i$ $(i = 1, 2, \dots)$. Let $x \in K$, $\|x\| = 1$. Then

$$\|x\|^2 + \sum_{i=1}^{\infty} c_i \|T^i x\|^2 \geqslant 1 - \sum_{i=1}^{\infty} \|T^i\|^2 b_i \geqslant 0.$$

Put $D = (I + \sum_{i=1}^{\infty} c_i T^{*i} T^i)^{1/2}$ and define the operator $V : K \to \bigoplus_{i=0}^{\infty} H'$ by $Vx = \bigoplus_{s=0}^{\infty} \sqrt{b_s} D T^s x$ $(x \in K)$. Then $S_\alpha V = VT$ (see (3)).

It is sufficient to prove that V is an isometry. We have $\|Dx\|^2 = \sum_{j=0}^{\infty} c_j \|T^j x\|^2$ and

$$\|Vx\|^2 = \sum_{i=0}^{\infty} b_i \|DT^i x\|^2 = \sum_{i=0}^{\infty} b_i \sum_{j=0}^{\infty} c_j \|T^{i+j} x\|^2 = \sum_{k=0}^{\infty} \|T^k x\|^2 \sum_{i+j=k} b_i c_j = \|x\|^2$$

as $\sum_{i+j=k} b_i c_j = 0$ for $k \geqslant 1$. The re-arrangement of the series is correct as

$$\sum_{i=0}^{\infty} |b_i| \sum_{j=0}^{\infty} |c_j| \|T^{i+j} x\|^2 \leqslant \sum_{i,j=0}^{\infty} b_i b_j \|T^i\|^2 \|T^j\|^2 \|x\|^2 =$$

$$= \left(\sum_{i=0}^{\infty} b_i \|T^i\|^2 \right)^2 \|x\|^2 \leqslant 4 \|x\|^2$$

and the series converges absolutely.

The following corollary improves the result of Foiaş, Pearcy [6], (see also Yadav, Bansal [12]) that every quasinilpotent operator is similar to a part of some quasinilpotent weighted shift.

2.3. COROLLARY. *Let $T \in B(K)$. Then there exist weights $\alpha_1, \alpha_2, \ldots$ such that T is unitarily equivalent to a part of the weighted shift S_α and the operators T and S_α have equal spectral radii.*

Proof. Put $\alpha_s = 1/s + \sup_{k \geqslant s}(2k\|T^k\|)^{1/k}$ $(s = 1, 2, \ldots)$. Then $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ $\ldots > 0$ and

$$|S_\alpha|_\sigma = \lim_{s\to\infty} \|S_\alpha^s\|^{1/s} = \lim_{s\to\infty} (\alpha_1 \ldots \alpha_s)^{1/s} = \lim_{s\to\infty} \alpha_s = |T|_\sigma .$$

Further

$$\sum_{s=1}^{\infty} \|T^s\|^2 b_s = \sum_{s=1}^{\infty} \frac{\|T^s\|^2}{\alpha_1^2 \ldots \alpha_s^2} \leqslant \sum_{s=1}^{\infty} \frac{\|T^s\|^2}{\alpha_s^{2s}} \leqslant$$

$$\leqslant \sum_{s=1}^{\infty} \frac{\|T^s\|^2}{[(2s\|T^s\|)^{1/s}]^{2s}} = \sum_{s=1}^{\infty} \frac{1}{4s^2} = \frac{\pi^2}{24} < 1.$$

The assertion of Corollary 2.3 follows now from Theorem 2.2.

Note that if T is quasinilpotent then $\lim_{s\to\infty} \alpha_s = 0$ and the weighted shift S'_α is compact (compare with [6]).

3.

In this section we suppose that the sequence $\{b_s\}_{s=0}^{\infty}$ of positive numbers satisfies a recurrent relation of order n

(9) $$b_s = -c_1 b_{s-1} - c_2 b_{s-2} - \ldots - c_n b_{s-n} \quad (s = 1, 2, \ldots)$$

where we put formally $b_s = 0$ for $s < 0$ and $b_0 = 1$.

By this we always mean that $\{b_s\}_{s=0}^{\infty}$ satisfies no recurrent relation of order $n' < n$.

In other words only finite number of c_i's defined by (6) are non-zero and obviously all of them are real numbers.

Let $p(z) = \sum_{i=0}^{n} c_i z^{n-i}$ be the characteristic polynomial of the recurrent sequence $\{b_s\}_{s=0}^{\infty}$. Recall that we always put $c_0 = 1$.

3.1. DEFINITION. Let $p(z) = \sum_{i=0}^{n} c_i z^{n-i}$ be a polynomial and let K be a separable complex Hilbert space. We say that an operator $T \in B(K)$ belongs to the class $C(p)$ if

$$\sum_{i=0}^{n} c_i \|T^i x\|^2 \geqslant 0 \qquad \text{for every } x \in K.$$

3.2. EXAMPLE. $T \in B(K)$ is a contraction if and only if $T \in C(z-1)$. In [2], n-hypercontractions were defined. In the present notation $T \in B(K)$ is an n-hypercontraction iff $T \in \bigcap_{k=1}^{n} C((z-1)^k)$.

3.3. PROPOSITION. *Let $p(z) = (z-\lambda_1)\dots(z-\lambda_n)$ be a polynomial, $\lambda_1, \dots, \lambda_n$ its (complex) roots and let $\mu > 0$. Then $T \in C(p)$ if and only if $\mu T \in C(q)$ where*

$$q(z) = \left(z - \frac{\lambda_1}{\mu^2}\right)\dots\left(z - \frac{\lambda_n}{\mu^2}\right).$$

Proof. The following conditions are equivalent:

$T \in C(p)$,

$\sum_{i=0}^{n} c_i \|T^i x\|^2 \geq 0 \quad (x \in K)$, where $p(z) = \sum_{i=0}^{n} c_i z^{n-i}$,

$\sum_{i=0}^{n} \frac{c_i}{\mu^{2i}} \|(\mu T)^i x\|^2 \geq 0 \quad (x \in K)$,

$\mu T \in C(q)$, where $q(z) = \sum_{i=0}^{n} \frac{c_i}{\mu^{2i}} z^{n-i} = \left(z - \frac{\lambda_1}{\mu^2}\right)\dots\left(z - \frac{\lambda_n}{\mu^2}\right)$.

Proposition 3.3 shows that we may consider without loss of generality only the case of a polynomial $p(z)$ with roots $\lambda_1, \dots, \lambda_n$ satisfying $\max_{1\leq i\leq n} |\lambda_i| = 1$. As $\lim_{\substack{t\to\infty \\ t \text{ real}}} p(t) = \infty$ and p has no roots between 1 and ∞, we get $p(1) \geq 0$. The following proposition proves that the condition $b_i > 0$ $(i = 1, 2, \dots)$ implies $p(1) = 0$.

Let λ be a root of the polynomial p. We denote by $m(\lambda)$ its multiplicity.

3.4. PROPOSITION. *Let $\{b_s\}_{s=0}^{\infty}$ be a sequence of positive numbers satisfying a recurrent relation (9) of order n. Let $p(z) = \sum_{i=0}^{n} c_i z^{n-i}$ be the characteristic polynomial of the recurrent sequence $\{b_s\}_{s=0}^{\infty}$. Suppose $\max\{|\lambda|, p(\lambda) = 0\} = 1$. Then $p(1) = 0$ and $m(1) = \max\{m(\lambda), |\lambda| = 1$ and $p(\lambda) = 0\}$.*

Proof. Suppose on the contrary that $k = \max\{m(\lambda), |\lambda| = 1 \text{ and } p(\lambda) = 0\} > m(1)$. It is well-known that $b_s = \sum_{\{\lambda;\, p(\lambda)=0\}} \sum_{j=0}^{m(\lambda)-1} a_{\lambda,j} s^j \lambda^s$ for some (complex) numbers $a_{\lambda,j}$. Put $b'_s = \frac{b_s}{s^{k-1}}$. Then $b'_s = B_s + R_s$ where $B_s = \sum_{\substack{|\lambda|=1 \\ m(\lambda)=k}} a_{\lambda,k-1}\lambda^s$ and $\lim_{s\to\infty} R_s = 0$.

Further,

$$\left|\sum_{s=0}^{N}\lambda_s\right| = \left|\frac{1-\lambda^{N+1}}{1-\lambda}\right| \leqslant \frac{2}{|1-\lambda|}$$

for every λ, $|\lambda| = 1$, hence $\lim_{N\to\infty} N^{-1}\sum_{s=0}^{N} B_s = 0$. Let $\{\lambda, p(\lambda) = 0, |\lambda| = 1$ and $m(\lambda) = k\} = \{\lambda_1, \ldots, \lambda_{t'}\}$ and let $\lambda_t = e^{2\pi i\varphi_t}$ where $0 \leqslant \varphi_t < 1$ $(t = 1, \ldots, t')$. Let $\sigma_1, \ldots, \sigma_{t''}$ $(t'' \geqslant 0)$ be some irrational numbers such that $1, \sigma_1, \ldots, \sigma_{t''}$ are linearly independent over the rationals and

$$\varphi_t = \sum_{j=1}^{t''}\frac{r_{t,j}}{d}\sigma_j + \frac{r_{t,0}}{d} \quad (t = 1, \ldots, t')$$

for some integers $r_{t,j}, r_{t,0}, d$, $d \geqslant 1$.

By [7, p. 48] $\lim_{N\to\infty} N^{-1}\mathrm{card}\{s \leqslant N, -\varepsilon \leqslant s\sigma_t \leqslant \varepsilon \pmod 1),\ t = 1, \ldots, t''\} = (2\varepsilon)^{t''}$ for every $\varepsilon > 0$. Then

$$\liminf_{N\to\infty} N^{-1}\mathrm{card}\{s \leqslant N,\ -\varepsilon < s\varphi_t < \varepsilon \ (\mathrm{mod}\ 1),\ t = 1, \ldots, t'\} \geqslant$$

$$\geqslant \liminf_{N\to\infty} N^{-1}\mathrm{card}\{s' \leqslant N/d,\ -\varepsilon < s'd\varphi_t < \varepsilon \ (\mathrm{mod}\ 1),\ t = 1, \ldots, t'\} \geqslant$$

$$\geqslant d^{-1}\liminf_{N/d\to\infty}(N/d)^{-1}\mathrm{card}\{s' \leqslant N/d, -\varepsilon/S < s'\sigma_j < \varepsilon/S \ (\mathrm{mod}\ 1),$$

$$j = 1, \ldots, t''\} = d^{-1}\left(\frac{2\varepsilon}{S}\right)^{t''} > 0,$$

where $S = \max_{1\leqslant t\leqslant t'}\sum_{j=0}^{t''}|r_{t,j}|$. Let $s_0 \geqslant 0$ and $\varepsilon > 0$. Then

$$\liminf_{N\to\infty} N^{-1}\mathrm{card}\{s \leqslant N, |B_s - B_{s_0}| < \varepsilon\} \geqslant \frac{1}{d}\left(\frac{2\varepsilon}{S\sum_{t=1}^{t'}|a_{\lambda_t,k-1}|}\right)^{t''} > 0.$$

Denote by $\mathbf{R}_+$ the set of all non-negative real numbers and suppose $h = \mathrm{dist}(B_{s_0}, \mathbf{R}_+) > 0$ for some s_0. Then $\mathrm{dist}(B_s, \mathbf{R}_+) > h/2$ for infinitely many s so $b'_s \notin \mathbf{R}_+$ and $b_s \notin \mathbf{R}_+$ for some s sufficiently large, a contradiction. Thus $B_s \geqslant 0$ for all s. Suppose $B_{s_0} > 0$ for some s_0. Then

$$\liminf_{N\to\infty} N^{-1}\mathrm{card}\{s \leqslant N,\ B_s \geqslant B_{s_0}/2\} = h > 0,$$

hence

$$\lim_{N\to\infty} N^{-1} \sum_{s=0}^{N} b'_s = \lim_{N\to\infty} N^{-1} \sum_{s=0}^{N} B_s \geqslant hB_{s_0}/2 > 0,$$

a contradiction.

We have proved that $B_s = 0$ for all s, hence $\{b_s\}_{s=0}^{\infty}$ satisfies a recurrent relation of order $n' < n$, a contradiction.

As the root $\lambda = 1$ plays an important role, we are going to study this root in more detail.

Suppose now $p(z) = (z-1)^k q(z)$ where $n \geqslant k \geqslant 1$, $\deg q = n - k$. Let an operator $T \in B(K)$ belong to $C(p)$, $x \in K$, $\|x\| = 1$. For $r = 0, 1, \ldots k$, $s = 0, 1, \ldots$ define

$$d_{r,s} = \sum_{i=0}^{n-k+r} c_i^{(r)} \|T^{s+i}x\|^2 \tag{10}$$

where $c_i^{(r)}$'s are the coefficients of the polynomial

$$(z-1)^r q(z) = \sum_{i=0}^{n-k+r} c_i^{(r)} z^{n-k+r-i}.$$

It is easy to check that $c_i^{(r)} = c_i^{(r-1)} - c_{i-1}^{(r-1)}$ $(i = 1, \ldots, n-k+r-1)$ and

$$d_{r,s} = d_{r-1,s} - d_{r-1,s+1} \quad (r = 1, \ldots, k,\ s = 0, 1, \ldots). \tag{11}$$

3.5. Theorem. *Let $p(z) = (z-1)^k q(z)$, $k \geqslant 2$, $\deg q = n-k$. Let $T \in B(K)$ be a power bounded operator* (i.e. $\sup_{s=0,1,\ldots} \|T^s\| < \infty$), *$T \in C(p)$. Then $T \in C((z-1)^r q(z))$ for every r, $1 \leqslant r \leqslant k$.*

Proof. Fix $x \in K$, $\|x\| = 1$ and let $d_{r,s}$ be the numers defined by (10). As $T \in C(p)$, we have $d_{k,s} \geqslant 0$ $(s = 0, 1, \ldots)$, i.e. $\{d_{k-1,s}\}_{s=0}^{\infty}$ is a non-increasing sequence. Further

$$\left| \sum_{s=0}^{N} d_{k-1,s} \right| = \left| \sum_{s=0}^{N} (d_{k-2,s} - d_{k-2,s+1}) \right| =$$

$$= |d_{k-2,0} - d_{k-2,N+1}| \leqslant 2\left(\sum_{i=0}^{n-k} |c_i^{(k-2)}| \right) \sup_{s=0,1,\ldots} \|T^s\|$$

which is a constant independent on N. Therefore $d_{k-1,s} \geqslant 0$ $(s = 0, 1, \ldots)$. As $x \in K$ was arbitrary we conclude that $T \in C((z-1)^{k-1}q(z))$. Using this argument repeatedly we obtain $T \in C((z-1)^r q(z))$ for every r, $1 \leqslant r \leqslant k$.

Theorem 3.5 gives the following characterization of n-hypercontractions. This characterization was proved in [2] for n-hypercontractions T satisfying $T^s \to 0$ strongly. Note that then T is power bounded by the Banach-Steinhaus theorem.

3.6. COROLLARY. *The following conditions are equivalent:*

a) *T is an n-hypercontraction*, i.e. $T \in \bigcap_{k=1}^{n} C((z-1)^k)$.

b) *$T \in C((z-1)^n)$ and T is power bounded.*

c) *$T \in C((z-1)^n)$ and $\|T\| \leqslant 1$.*

3.7. LEMMA. *Let $p(z) = (z-1)^k q(z)$, $k \geqslant 1$, $\deg q = n - k$. Let $T \in B(K)$ be a power bounded operator, $T \in C(p)$ and $x \in K$, $\|x\| = 1$. Let $d_{r,s}$ ($r = 0, 1, \ldots, k$, $s = 0, 1, \ldots$) be the numbers defined by* (10). *Then for every r, $0 \leqslant r \leqslant k-1$*

$$1) \quad \sum_{s=0}^{N} \binom{s}{r} d_{k,s} = d_{k-r-1,r} - \sum_{i=0}^{r} \binom{N+i-r}{i} d_{k-r+i-1,N+1}$$

for every $N \geqslant r$ $\left(\text{we put } \binom{s}{r} = 0 \text{ for } s < r\right)$.

2) $\lim\limits_{N\to\infty} d_{k-r-1,N}$ *exists and*

$$\sum_{s=0}^{\infty} \binom{s}{r} d_{k,s} = d_{k-r-1,r} - \lim_{N\to\infty} d_{k-r-1,N}.$$

If $k \geqslant 2$ then $\lim\limits_{s\to\infty} \binom{s}{r} d_{k-1,s} = 0$.

Proof. We prove statements 1) and 2) by induction on r. Let $r = 0$. Then

$$1) \quad \sum_{s=0}^{N} d_{k,s} = \sum_{s=0}^{N} (d_{k-1,s} - d_{k-1,s+1}) = d_{k-1,0} - d_{k-1,N+1}.$$

2) As $T \in C(p)$ the numbers $d_{k,s}$ are non-negative. Further

$$\sum_{s=0}^{N} d_{k,s} = d_{k-1,0} - d_{k-1,N+1} \leqslant$$

$$\leqslant 2\left(\sum_{i=0}^{n-1} |c_i^{(k-1)}|\right) \sup\{\|T^s\|, \ s = 0, 1, \ldots\}$$

where $c_i^{(k-1)}$ are coefficients of the polynomial $(z-1)^{k-1}q(z)$, see (10). Therefore $\sum_{s=0}^{\infty} d_{k,s} < \infty$ which means that $\lim\limits_{N\to\infty} d_{k-1,N+1}$ exists and $\sum_{s=0}^{\infty} d_{k,s} = d_{k-1,0} - \lim\limits_{N\to\infty} d_{k-1,N}$.

Further $\lim_{s\to\infty} d_{k,s} = 0$. Suppose $k \geqslant 2$ and $r = 0$. Then $T \in C((z-1)^{k-1}q(z))$ by Theorem 3.5 and thus also $\lim_{s\to\infty} d_{k-1,s} = 0$.

Let $r \geqslant 1$. Then $k \geqslant 2$ and $T \in C((z-1)^{k-1}q(z))$ by Theorem 3.5.

1) The induction assumption gives

$$\sum_{s=0}^{N} \binom{s}{r-1} d_{k-1,s} = d_{k-r-1,r-1} - \sum_{i=0}^{r-1} \binom{N+i-r+1}{i} d_{k-r+i-1,N+1}.$$

Then

$$\sum_{s=0}^{N} \binom{s}{r} d_{k,s} = \sum_{s=0}^{N} \binom{s}{r} (d_{k-1,s} - d_{k-1,s+1}) =$$

$$= \sum_{s=0}^{N-1} \left(\binom{s+1}{r} - \binom{s}{r} \right) d_{k-1,s+1} - \binom{N}{r} d_{k-1,N+1} =$$

1 2)

$$= \sum_{s=0}^{N-1} \binom{s}{r-1} d_{k-1,s+1} - \binom{N}{r} d_{k-1,N+1} =$$

$$= d_{k-r-1,r} - \sum_{i=0}^{r} \binom{N+i-r}{i} d_{k-r+i-1,N+1}.$$

2a) First we prove $\lim_{s\to\infty} \binom{s}{r} d_{k-1,s} = 0$ or, equivalently, $\lim_{s\to\infty} s^r d_{k-1,s} = 0$.

As $T \in C(p)$ and $T \in C((z-1)^{k-1}q(z))$, we have $d_{k-1,s} \geqslant 0$ and $d_{k,s} = d_{k-1,s} - d_{k-1,s+1} \geqslant 0$ $(s = 0, 1, \dots)$. Therefore $\{d_{k-1,s}\}_{s=0}^{\infty}$ is a non-increasing sequence of non-negative numbers. By the induction assumption $\sum_{s=0}^{\infty} \binom{s}{r-1} d_{k-1,s} < \infty$. As $\binom{s}{r-1} \geqslant \frac{s^{r-1}}{2(r-1)!}$ for all s large enough we have $\sum_{s=0}^{\infty} s^{r-1} d_{k-1,s} < \infty$.

The rest follows from the following lemma:

3.8. LEMMA. *Let $\{a_s\}_{s=0}^{\infty}$ be a non-increasing sequence of non-negative numbers, $r \geqslant 1$. Let $\sum_{s=0}^{\infty} s^{r-1} a_s < \infty$. Then $\lim_{s\to\infty} s^r a_s = 0$.*

Proof of Lemma 3.8. We have

$$\sum_{s=0}^{N} s^{r-1} \geqslant \int_0^N t^{r-1}\,\mathrm{d}t = \frac{N^r}{r}.$$

Let $\varepsilon > 0$ and let N be an integer such that $\sum_{s=N+1}^{\infty} s^{r-1}a_s < \varepsilon/2r$. We prove that $s^r a_s < \varepsilon$ for all $s > 2N$. Suppose on the contrary that there exists $N' > 2N$ such that $N'^r a_{N'} \geqslant \varepsilon$, i.e. $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_{N'} \geqslant \varepsilon/N'^r$. Then

$$\sum_{s=N+1}^{\infty} s^{r-1}a_s \geqslant \sum_{s=N+1}^{N'} s^{r-1}\frac{\varepsilon}{N'^r} \geqslant \frac{\varepsilon}{2N'^r}\sum_{s=0}^{N'} s^{r-1} \geqslant \frac{\varepsilon}{2r},$$

a contradiction.

Proof of Theorem 3.7 (continuation). b) By (12),

$$\sum_{s=0}^{N}\binom{s}{r} d_{k,s} = \sum_{s=0}^{N-1}\binom{s}{r-1} d_{k-1,s+1} - \binom{N}{r} d_{k-1,N+1}$$

where $\sum_{s=0}^{\infty}\binom{s}{r-1} d_{k-1,s+1} < \infty$ by the induction assumption and $\lim_{N\to\infty}\binom{N}{r} d_{k-1,N+1} = 0$. Therefore $\sum_{s=0}^{N}\binom{s}{r} d_{k,s}$ is bounded by a constant independent on N. As $d_{k,s} \geqslant 0$ $(s = 0, 1, \ldots)$ we conclude $\sum_{s=0}^{\infty}\binom{s}{r} d_{k-1,N+1} < \infty$. Using (12) and the induction assumption we get

$$\sum_{s=0}^{\infty}\binom{s}{r} d_{k,s} = \sum_{s=0}^{\infty}\binom{s}{r-1} d_{k-1,s+1} = d_{k-r-1,r} - \lim_{N\to\infty} d_{k-r-1,N}.$$

3.9. THEOREM. *Let $\{b_s\}_{s=0}^{\infty}$ be a sequence of positive numbers satisfying a recurrent relation* (9) *of order n, let $p(z)$ be the corresponding characteristic polynomial. Suppose* $\max\{|\lambda|, p(\lambda) = 0\} = 1$ *and* $p(\lambda) = 0$, $|\lambda| = 1 \neq \lambda$ *implies that λ is a simple root of p. Suppose further that the weighted shift S_α with weights α_i defined by* (2) *is a bounded operator*, i.e. $\inf_{s=0,1,\ldots} b_s/b_{s+1} > 0$. *Let $T \in B(K)$, $\|T\| \leqslant 1$ belong to class $C(p)$. Then $\|Vx\|^2 = \|x\|^2 - \lim_{r\to\infty}\|T^r x\|^2$ for every $x \in K$, where V is the operator defined by* (3) *and* (8).

Proof. Recall that $Vx = (A_0 x, \sqrt{b_1}A_0Tx, \sqrt{b_2}A_0T^2x, \ldots)$ where $A_0 = := \left(\sum_{i=0}^{n} c_i T^{*i}T^i\right)^{1/2}$. Then

$$\|Vx\|^2 = \sum_{s=0}^{\infty} b_s\|A_0T^sx\|^2 = \sum_{s=0}^{\infty} b_s d_{k,s} = \lim_{N\to\infty} S_N$$

where $S_N = \sum_{s=0}^{N} b_s d_{k,s}$, k is the multiplicity of the root $\lambda = 1$ of p and $d_{k,s} = \sum_{i=0}^{n} c_i \|T^{s+i}x\|^2$ (see (10)). Here $k \geq 1$, i.e. $p(1) = 0$ by Proposition 3.4. Using (9) we have

$$S_N = \sum_{s=0}^{N} b_s \sum_{i=0}^{n} c_i \|T^{s+i}x\|^2 = \tag{13}$$

$$= \|x\|^2 + \sum_{r=N+1}^{N+n} \|T^r x\|^2 \sum_{\substack{s+i=r \\ 0\leq i\leq n \\ 0\leq s\leq N}} b_s c_i .$$

As $\{b_s\}_{s=0}^{\infty}$ is a recurrent sequence there exist numbers $a_{\lambda,j}$ $(p(\lambda) = 0,\ 0 \leq j \leq m(\lambda) - 1)$ such that

$$b_s = \sum_{\{\lambda; p(\lambda)=0\}} \sum_{j=0}^{m(\lambda)-1} a_{\lambda,j} s^j \lambda^s \quad (s = 0, 1, \ldots).$$

As s^j can be expressed as a linear combination of $\binom{s}{j}, \binom{s}{j-1}, \ldots, \binom{s}{1}, \binom{s}{0}$ we can find numbers $a'_{1,j}$ $(j = 0, 1, \ldots, k - 1)$ such that

$$b_s = B_s + R_s \tag{14}$$

where

$$B_s = \sum_{j=1}^{k-1} a'_{1,j} \binom{s}{j}$$

and

$$R_s = \sum_{|\lambda|<1} \sum_{j=0}^{m(\lambda)-1} a_{\lambda,j} s^j \lambda^s + \sum_{|\lambda|=1\neq\lambda} a_{\lambda,0}\lambda^s + a'_{1,0}$$

(if $k = 1$ then we have $B_s = 0$ and all considerations are much simpler). Clearly, R_s is a bounded sequence.

Using (13) and Theorem 3.7 we get

$$S_N = \sum_{s=0}^{N} \left(R_s d_{k,s} + \sum_{j=1}^{k-1} a'_{1,j} \binom{s}{j} d_{k,s} \right) =$$

$$= \sum_{s=0}^{N} R_s d_{k,s} + \sum_{j=1}^{k-1} a'_{1,j} \left(d_{k-j-1,j} - \sum_{i=0}^{j} \binom{N+i-j}{i} d_{k-j+i-1,N+1} \right) =$$

$$= S'_N + S''_N + S'''_N$$

where

$$S'_N = \sum_{s=0}^{N} R_s d_{k,s},$$

$$S''_N = \sum_{j=1}^{k-1} a'_{1,j}(d_{k-\ -1,j} - d_{k-j-1,N+1})$$

and

$$(15) \qquad S'''_N = -\sum_{j=1}^{k-1} a'_{1,j} \sum_{i=1}^{j} \binom{N+i-j}{i} d_{k-j+i-1,N+1}.$$

By Lemma 3.7, $\lim_{N\to\infty} S'''_N = 0$ so $\|Vx\|^2 = \lim_{N\to\infty} (S'_N + S''_N)$. Substitute from (10) and express S_N, S'_N, S''_N and S'''_N as linear combinations of $\|T^r x\|^2$:

$$S_N = \sum_{r=0}^{N+n} \gamma_{r,N}\|T^r x\|^2, \quad S'_N = \sum_{r=0}^{N+n} \gamma'_{r,N}\|T^r x\|^2,$$

$$S''_N = \sum_{r=0}^{N+n} \gamma''_{r,N}\|T^r x\|^2 \quad \text{and} \quad S'''_N = \sum_{r=0}^{N+n} \gamma'''_{r,N}\|T^r x\|^2.$$

Then (13) and (15) gives $\gamma_{0,N} = 1$, $\gamma_{1,N} = \gamma_{2,N} = \ldots = \gamma_{N,N} = 0$, $\gamma'''_{0,N} = \gamma'''_{1,N} = \ldots = \gamma'''_{N,N} = 0$, therefore $\gamma'_{0,N} + \gamma''_{0,N} = 1$, $\gamma'_{1,N} + \gamma''_{1,N} = \ldots = \gamma'_{N,N} + \gamma''_{N,N} = 0$.

Let $c_i^{(r)}$ ($i = 0, \ldots, n-k+r$, $r = 1, \ldots, k$) be the coefficients of the polynomial $(z-1)^r q(z)$ (see (10)). Then $\sum_{i=0}^{n-k+r} c_i^{(r)} = 0$ ($r \geqslant 1$), so the sum of coefficients of $d_{r,s}$ is equal to 0 for $r \geqslant 1$. Thus $\sum_{s=0}^{N+n} \gamma_{s,N} = \sum_{s=0}^{N+n} \gamma'''_{s,N} = 0$, hence $\sum_{s=N+1}^{N+n} (\gamma'_{s,N} + \gamma''_{s,N}) = -1$ for every N.

Further

$$\sum_{s=0}^{N+n} |\gamma''_{s,N}| \leqslant \left(\sum_{j=1}^{k-1} |a'_{1,j}|\right) \cdot 2 \cdot \max_{1\leqslant j\leqslant k-1} \sum_{i=0}^{n-j-1} |c_i^{(k-i-1)}|$$

which is a constant independent on N.

Analogously to (13) $\gamma'_r = \sum_{\substack{s+i=r \\ 0\leqslant i\leqslant n \\ \leqslant s\leqslant N}} R_s c_i$ ($r = N+1, \ldots, N+n$),

thus

$$\sum_{r=N+1}^{N+n} |\gamma'_{r,N}| \leqslant n^2 \max_{i=0,\ldots,n} |c_i| \cdot \max_{s=0,1,\ldots} |R_s|.$$

Therefore

$$\|Vx\|^2 = \lim_{N\to\infty} (S_N' + S_N''),$$

$$S_N' + S_N'' = \|x\|^2 + \sum_{r=N-1}^{N+n} (\gamma_{r,N}' + \gamma_{r,N}'')\|T^r x\|^2,$$

$\sum_{r=N-1}^{N+n} (\gamma_{r,N}' + \gamma_{r,N}'') = -1$ for all N and $\sum_{r=N-1}^{N+n} |\gamma_{r,N}' + \gamma_{r,N}''|$ is bounded by a constant independent on N.

As $\lim_{r\to\infty} \|T^r x\|^2$ exists we conclude that

$$\|Vx\|^2 = \|x\|^2 - \lim_{r\to\infty} \|T^r x\|^2.$$

If $T^r \to 0$ strongly, the operator V will be an isometry. In general, the well-known functional model of Sz.-Nagy and Foiaş [11] gives that any contraction $T \in B(K)$ is unitarily equivalent to a part of $S \oplus U$, where $S \in B(H)$ is a backward unweighted shift and $U \in B(H_2)$ is a unitary operator. In other words there exist operators $W_1: K \to H$, $W_2: K \to H_2$ such that $SW_1 = W_1 T$, $UW_2 = W_2 T$ and $\|W_1 x\|^2 + \|W_2 x\|^2 = \|x\|^2$ for every $x \in K$. Also (see [11], [4] or [8]) $\|W_1 x\|^2 = \|x\|^2 - \lim_{r\to\infty} \|T^r x\|^2$, i.e. $\|W_2 x\|^2 = \lim_{r\to\infty} \|T^r x\|^2$. So we may take the operator W_2 from the model of Sz.-Nagy and Foiaş to complete the operator V from the previous theorem to an isometry. This gives the following theorem:

3.10. THEOREM. *Let $\{b_s\}_{s=0}^\infty$ be a sequence of positive numbers satisfying a recurrent relation* (9) *of order n, let $p(z)$ be its characteristic polynomial. Suppose* $\max\{|\lambda|, p(\lambda) = 0\} = 1$ *and $p(\lambda) = 0$, $|\lambda| = 1 \neq \lambda$ implies that λ is a simple root of p. Suppose further that the corresponding weighted shift S_α is bounded. Let $T \in B(K)$ be a contraction. Then*

1) *$T \in C(p)$ if and only if T is unitarily equivalent to a part of $S_\alpha \oplus U$ where U is a unitary operator.*

2) *If $T^r \to 0$ strongly then $T \in C(p)$ if and only if T is unitarily equivalent to a part of S_α.*

Proof. It remains to prove that $S_\alpha \oplus U \in C(p)$ for every unitary operator $U \in B(H_2)$. Let $p(z) = \sum_{i=0}^{n} c_i z^{n-i}$. By Proposition 3.4, $\sum_{i=0}^{n} c_i = p(1) = 0$, therefore

$$\sum_{i=0}^{n} c_i \|U^i x\|^2 = \left(\sum_{i=0}^{n} c_i\right) \|x\|^2 = 0 \quad \text{for every } x \in H_2.$$

Let $x = (x_0, x_1, \dots) \in H$. Then

$$\sum_{i=0}^{n} c_i \| S_\alpha^i x\|^2 = \sum_{i=0}^{n} c_i \|(\alpha_i \alpha_{i-1} \dots \alpha_1 x_i, \alpha_{i+1} \dots \alpha_2 x_{i+1}, \dots)\|^2 =$$

$$= \sum_{i=0}^{n} c_i \sum_{j=1}^{\infty} \|x_j\|^2 \cdot \frac{b_{j-i}}{b_j} = \sum_{j=0}^{\infty} \frac{\|x_j\|^2}{b_j} \sum_{i=0}^{j} c_i b_{j-i} = \|x_0\|^2 \geqslant 0$$

by (9).

3.11. REMARK. If $m(1) \geqslant 2$ or if p has no roots λ, $|\lambda| = 1 \neq \lambda$ then S_α is always bounded, i.e. $\inf\limits_{s=0,1,\dots} b_s b_{s+1}^{-1} > 0$.

Consider now the general case of polynomial $p(z) = \sum\limits_{i=0}^{n} c_i z^{n-i} = (z - \lambda_1) \dots \dots (z - \lambda_n)$ with $\max\{|\lambda_i|, i = 1, \dots, n\} = M$. Clearly $M \neq 0$ as $p(z) \neq z^n$ ($c_1 = -b_1 \neq 0$). The previous theorem applied to the polynomial $\tilde{p}(z) = (z - \lambda_1/M) \dots \dots (z - \lambda_n/M)$ gives:

3.12. THEOREM. *Let $\{b_s\}_{s=1}^{\infty}$ be a sequence of positive numbers satisfying a recurrent relation* (9) *of order n such that* $\inf\limits_{s=0,1,\dots} b_s b_{s+1}^{-1} > 0$. *Let $p(z)$ be the corresponding characteristic polynomial. Suppose $p(\lambda) = 0$ and $|\lambda| = M \neq \lambda$ implies that λ is a simple root of p, where $M = \max\{|\lambda|, p(\lambda) = 0\}$. Let $T \in B(K)$, $\|T\| \leqslant M^{-1/2}$. Then*

1) *$T \in C(p)$ if and only if T is unitarily equivalent to a part of $S_\alpha \oplus M^{-1/2}U$ where U is a unitary operator.*

2) *If $M^{r/2}T^r \to 0$ strongly then $T \in C(p)$ if and only if T is unitarily equivalent to a part of S_α.*

Proof. Put $\tilde{T} = \sqrt{M}T$, $\tilde{p}(z) = (z - \lambda_1/M) \dots (z - \lambda_n/M)$, $\tilde{c}_i = M^{-i}c_i$ ($i = 0, 1, \dots$), $\tilde{b}_s = M^{-s}b_s$ and $\tilde{\alpha}_s = \sqrt{M}\alpha_s$, i.e. $S_{\tilde{\alpha}} = \sqrt{M}S_\alpha$. Use Proposition 3.3 and Theorem 3.10.

3.13. EXAMPLE. A. If $p(z) = z - 1$ then Theorem 3.12 gives the model of Sz.-Nagy and Foiaş for contractions.

B. Let $p(z) = (z - 1)^n$. Then we get the theorem of Agler [2]:

1) $T \in B(K)$ is an n-hypercontraction iff T is unitarily equivalent to a part of $S_\alpha \oplus U$.

2) T is an n-hypercontraction and $T^r \to 0$ strongly iff T is unitarily equivalent to a part of S_α.

Here (9) and (2) gives $b_s = \binom{s+n-1}{n-1}$ and $\alpha_s = \sqrt{\dfrac{b_{s-1}}{b_s}}$.

C. As an example of general situation let us consider the Fibonacci sequence $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 3$, $b_4 = 5$, $b_5 = 8, \ldots, b_{n+2} = b_{n+1} + b_n$. The characteristic polynomial is

$$p(z) = z^2 - z - 1 = \left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right),$$

$M = \dfrac{1+\sqrt{5}}{2}$. Then we have:

THEOREM. *Let $T \in B(K)$, $\|T\| \leqslant M^{-1/2}$. Then T satisfies $\|x\|^2 \geqslant \|Tx\|^2 + \|T^2x\|^2$ for every $x \in K$ if and only if T is unitarily equivalent to a part of $S_\alpha \oplus M^{-1/2}U$. If $M^{r/2}T^r \to 0$ strongly then T is unitarily equivalent to a part of S_α.*

REFERENCES

1. AGLER, J., The Arveson extension theorem and coanalytic models, *Integral Equations Operator Theory*, **5**(1982), 608—631.
2. AGLER, J., Hypercontractions and subnormality, *J. Operator Theory*, **13**(1985), 203—217.
3. DE BRANGES, L.; ROVNYAK, J., *Perturbation theory and its applications in quantum mechanics*, Calvin H. Wilcox ed., Wiley, New York, 1966.
4. DURSZT, E., Contractions as restricted shifts, *Acta Sci. Math. (Szeged)*, **48**(1985), 129—134.
5. FOIAŞ, C., A remark on the universal model for contractions of G. C. Rota, *Comm. Acad. R. P. Române*, **13**(1963), 349—352.
6. FOIAŞ, C.; PEARCY, C., A model for quasinilpotent operators, *Michigan Math. J.*, **21**(1974), 399—404.
7. KUIPERS, L.; NIEDERREITER, H., *Uniform distribution of sequences*, Wiley-Interscience New York—London—Sydney—Toronto, 1974.
8. PTÁK, V.; VRBOVÁ, P., An abstract model for compressions, *Časopis Pěst. Mat.*, **113** (1988), 252—266
9. ROTA, G. C., On models for linear operators, *Comm. Pure Appl. Math.*, **13**(1960), 469—472.
10. SZ.-NAGY, B.; FOIAŞ, C., Sur les contractions de l'espace de Hilbert. VIII, *Acta Sci. Math. (Szeged)*, **25**(1964), 38—71.
11. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators in Hilbert space*, Akadémiai Kiadó, North Holland, Budapest—Amsterdam—London, 1970.
12. YADAV, B. S.; BANSAL, R., On Rota's models for linear operators, *Rocky Mountain J. Math.*, **13**(1983), 553—556.

VLADIMIR MÜLLER
Mathematical Institute,
Czechoslovak Academy of Sciences,
Žitná 25, 115 67 Praha 1,
Czechoslovakia.

Received October 22, 1986; revised March 31, 1988.