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ESTIMATING THE DISTANCE BETWEEN UNITARY ORBITS

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In this paper, we examine how well an operator is determined by its restrictions to finite dimensional subspaces. The finite dimensional pieces are taken only up to unitary equivalence, so it is not, a priori, apparent that the operator can be recovered up to unitary equivalence from these pieces. Indeed, as we will be concerned with approximation of pieces of one operator by pieces of another, it is more natural to consider approximate unitary equivalence. As a consequence of our methods, it follows that an operator is determined up to approximate unitary equivalence by its finite dimensional restrictions. Thus we are able to recover some results of Hadwin [9] describing the norm closure and strong* operator topology closure of the unitary orbit of T.

The main use we had in mind is a method of estimating the distance between the unitary orbits of two operators based on finite dimensional information. We quantitatively measure how well finite dimensional pieces of one operator, say B, can be approximated by finite dimensional pieces of another operator A. It is shown (Theorem 4.1) to be a good measure of how close B is to being a summand of A. This can be used to estimate the distance between the unitary orbits of A and B. As this estimate is based on finite dimensional information, it is perhaps surprising that definitive results are obtained when $C^*(A)$ (or $C^*(B)$) contains no compact operator (Section 2). In general there is no operator theoretic "Schroeder-Bernstein" theorem to turn "*A is almost a summand of B and B is almost a summand of A*" into "*A is almost unitarily equivalent to B*". An example is given (Section 5) to show that a finite dimensional summand of an operator can provide an obstruction to good approximation which cannot be detected by comparing finite dimensional pieces.

I would like to take this opportunity to thank the referee. His thoughtful comments indicated a simpler proof of Theorem 2.1 which led to better constants than were obtained in the original draft. He also asked a good question which led to the inclusion of Section 4 of this paper.

The description of the closure of the unitary orbit of an operator in terms of approximate unitary equivalence is due to Voiculescu [14]. Our methods were motivated in part by Arveson's approach [1] to Voiculescu's Theorem. The distance between unitary orbits of special classes of operators have been computed only in a few cases (see [2, 3, 5, 6, 10].)

All operators in this paper are continuous linear operators on a separable Hilbert space $\mathscr{H}$. The set of all operators is denoted by $\mathscr{B}(\mathscr{H})$, and $\mathscr{K}$ denotes the ideal of compact operators. The set of unitary operators on $\mathscr{B}(\mathscr{H})$ is denoted by $\mathscr{U}$. The unitary orbit of an operator A is

$$\mathscr{U}(A) = \{UAU^* : U \in \mathscr{U}\}.$$

Two operators A and B are approximately unitarily equivalent ($A \underset{a}{\sim} B$) if there is a sequence of unitary operators U_n such that

$$B = \lim_{n\to\infty} U_n A U_n^*.$$

It is readily verified that $\underset{a}{\sim}$ is an equivalence relation, and that $\{B : B \underset{a}{\sim} A\}$ is the norm closure of $\mathscr{U}(A)$, denoted $\overline{\mathscr{U}}(A)$. The distance between unitary orbits is $d(\mathscr{U}(A), \mathscr{U}(B)) = \inf_{U\in\mathscr{U}} \|A - UBU^*\|$. Let $\mathscr{F}$ denote the set of finite dimensional subspaces of $\mathscr{H}$. If M is a subspace, let P_M denote the orthogonal projection onto M.

1. A MEASURE OF APPROXIMATION AND PRELIMINARY ESTIMATES

Consider the following estimate of how well finite dimensional pieces of an operator B can be approximated by finite dimensional pieces of A:

$$\delta_A(B) = \sup_{M\in\mathscr{F}} \inf_{U\in\mathscr{U}} \max\{\|(AU - UB)P_M\|, \|(A^*U - UB^*)P_M\|\}.$$

Also, consider an "essential" version in which UM must be orthogonal to an (arbitrary) finite dimensional subspace:

$$\delta_A^e(B) = \sup_{M,N\in\mathscr{F}} \inf_{\{U\in\mathscr{U}: UM\perp N\}} \max\{\|(AU - UB)P_M\|, \|(A^*U - UB^*)P_M\|\}.$$

As well, for A and B in $\mathscr{B}(\mathscr{H})$, define:

$$\rho(A, B) = \max\{\delta_A(B), \delta_B(A)\}$$

$$\rho^e(A, B) = \max\{\delta_A^e(B), \delta_B^e(A)\}.$$

The following propositions are very easy. The proofs are omitted, but they are a useful exercise to gain a working understanding of these definitions.

PROPOSITION 1.1. *For all A, B and C in $\mathscr{B}(\mathscr{H})$, $d(\mathscr{U}(A), \mathscr{U}(B)) \geqslant \rho(A, B)$, and $d(\mathscr{U}(A), \mathscr{U}(B \oplus C)) \geqslant \delta_A(B)$.*

PROPOSITION 1.2. *If $A \underset{a}{\sim} A'$, then $\rho(A, A') = 0$. If B is any operator, $\delta_A(B) = \delta_{A'}(B)$, $\delta_A^e(B) = \delta_{A'}^e(B)$, $\delta_B(A) = \delta_B(A')$ and $\delta_B^e(A) = \delta_B^e(A')$.*

PROPOSITION 1.3. *For A, B and C in $\mathscr{B}(\mathscr{H})$, $\delta_A(B) + \delta_B(C) \geqslant \delta_A(C)$ and $\delta_A^e(B) + \delta_B^e(C) \geqslant \delta_A^e(C)$. Hence ρ and ρ^e satisfy the triangle inequality.*

Let $A^{(\infty)}$ denote the direct sum of countably many copies of A acting on a Hilbert space $\mathscr{H}^{(\infty)}$ which is the ℓ^2 direct sum of countably many copies of $\mathscr{H}$, $\mathscr{H}^{(\infty)} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \dots$.

PROPOSITION 1.4. *For A and B in $\mathscr{B}(\mathscr{H})$,*

$$\delta_{A^{(\infty)}}(B) = \delta^e_{A^{(\infty)}}(B) = \delta_{A^{(\infty)}}(B^{(\infty)}) = \delta^e_{A^{(\infty)}}(B^{(\infty)}).$$

Proof. Clearly, $\delta_{A^{(\infty)}}(B) \leqslant \delta^e_{A^{(\infty)}}(B^{(\infty)})$ are respectively the smallest and largest of these quantities. Let $\delta = \delta_{A^{(\infty)}}(B)$. Let M and N be finite dimensional subspaces of $\mathscr{H}^{(\infty)}$, and let $\varepsilon > 0$ be given. It is possible to choose an integer k and a subspace $M_0 \in \mathscr{F}$ so that $M_0(k) = M_0 \oplus \dots \oplus M_0 \oplus 0^{(\infty)}$ almost contains M in the sense that $\|P^{\perp}_{M_0(k)} P_M\| < \varepsilon$, and so that $\mathscr{H}^{(k)} = \mathscr{H} \oplus \dots \oplus \mathscr{H} \oplus 0^{(\infty)}$ almost contains N in a similar fashion. Let U be a unitary so that

$$\max\{\|(A^{(\infty)}U - UB)P_{M_0}\|, \|(A^{*(\infty)}U - UB^*)P_{M_0}\|\} < \delta + \varepsilon.$$

Decompose $\mathscr{H}^{(\infty)}$ as a direct sum of $k + 1$ copies of $\mathscr{H}^{(\infty)}$, say $\mathscr{K}_0 \oplus \mathscr{K}_1 \oplus \dots \oplus \mathscr{K}_k$, so that $\mathscr{K}_0$ contains $\mathscr{H}^{(k)}$ and $A^{(\infty)}|\mathscr{K}_i$ is unitarily equivalent to $A^{(\infty)}$ for each $0 \leqslant i \leqslant n$. Define a unitary W whose restriction to $\mathscr{H}_i$ maps into $\mathscr{K}_i$, $1 \leqslant i \leqslant k$, and corresponds via the unitary equivalence of $\mathscr{H}_i$ with $\mathscr{H}^{(\infty)}$ to the unitary U. Then

$$\max\{\|(A^{(\infty)}W - WB^{(\infty)})P_{M_0^{(k)}}\|, \|(A^{*(\infty)}W - WB^{*(\infty)})P_{M_0^{(k)}}\|\} < \delta + \varepsilon.$$

It follows, by letting ε tend to zero, that $\delta^e_{A^{(\infty)}}(B^{(\infty)}) = \delta_{A^{(\infty)}}(B)$. ▨

PROPOSITION 1.5. *For A and B in $\mathscr{B}(\mathscr{H})$,*

$$\delta_A^e(B) = \lim_{n \to \infty} \delta_A(B^{(n)}) = \delta_A(B^{(\infty)}).$$

Proof. Clearly, $\delta_A(B^{(n)}) \leqslant \delta_A(B^{(\infty)})$ for every $n \geqslant 1$. On other the hand, as in the proof of Proposition 1.4, every finite dimensional subspace of $\mathscr{H}^{(\infty)}$ can be approximated by a subspace of $\mathscr{H}^{(n)}$ for n sufficiently large. So a minor modification of the previous argument shows that $\delta_A(B^{(\infty)}) \leqslant \sup_n \delta_A(B^{(n)}) = \lim_{n\to\infty} \delta_A(B^{(n)})$.

Let M be a finite dimensional subspace of $\mathscr{H}$, and let $\delta = \delta_A^e(B)$. If n is a positive integer, one can recursively find unitaries $U_1, \ldots, U_n$ so that

i) $\max\{\|(AU_j - U_jB)P_M\|, \|(A^*U_j - U_jB^*)P_M\|\} < \delta + 1/n$;

ii) $U_jM \perp \operatorname{span}\{XU_iM, 1 \leqslant i < j, X \in \mathscr{X}\}$ where

$$\mathscr{X} = \{YZ : Y = I,\ A,\ A^2,\ A^*,\ A^{*2},\ AA^* \text{ or } A^*A,\ Z = I,\ B \text{ or } B^*\}.$$

Let $M_1 = \operatorname{span}\{M, BM, B^*M\}$, and define a unitary W from $\mathscr{H}^{(n)}$ onto $\mathscr{H}$ such that $W|0 \oplus \ldots \oplus M_1 \oplus 0 \oplus \ldots \oplus 0$ agrees with $U_i|M_1$. The subspaces $N_i = \operatorname{span}\{U_iM_1, AU_iM_1, A^*U_iM_1\}$ are pairwise orthogonal by (ii). Thus if $x = \sum_{i=1}^{n}{}^{\oplus} x_i$ belongs to $M^{(n)}$,

$$\|(AW - WB^{(n)})x\| = \left\| \sum_{i=1}^{n} (AU_i - U_iB)\, x_i \right\|.$$

Since $(AU_i - U_iB)x_i$ belongs to N_i, this is an orthogonal sum. Hence $\|(AW - WB^{(n)})P_M\| < \delta + 1/n$. Similarly, $\|(A^*W - WB^{*(n)})P_M\| \leqslant \delta + 1/n$. So,

$$\delta_A(B)^{(\infty)} = \lim \delta_A(B^{(n)}) \leqslant \delta = \delta_A^e(B).$$

On the other hand, $\delta_A^e(B) \leqslant \delta_A^e(B^{(\infty)}) = \delta_A(B^{(\infty)})$ by Proposition 1.4. ▨

The next propositions are easy, and again the proofs are omitted.

PROPOSITION 1.6. *Let A and B_n, $n \geqslant 1$ belongs to $\mathscr{B}(\mathscr{H})$. Then*

$$\delta_A^e\left(\sum_{n=1}^{\infty}{}^{\oplus} B_n\right) = \sup \delta_A^e(B_n).$$

COROLLARY 1.7. *For A and B in $\mathscr{B}(\mathscr{H})$, $\delta_A^e(B^{(\infty)}) = \delta_A^e(B) = \delta_A(B^{(\infty)})$.*

PROPOSITION 1.8. *For A_i, B_i in $\mathscr{B}(\mathscr{H})$, $1 \leqslant i < k \leqslant \infty$, let $A = \sum_{i<k}{}^{\oplus} A_i$ and $B = \sum_{i<k}{}^{\oplus} B_i$. Then $\delta_A(B) \leqslant \max\{\delta_{A_i}(B_i)\}$.*

COROLLARY 1.9. *Suppose A and B belong to $\mathscr{B}(\mathscr{H})$ and $\delta_A(A \oplus B) = 0$. Then $\delta_A^e(B) = 0$.*

Proof. By Proposition 1.5, it suffices to show that $\delta_A(B^{(n)}) = 0$ for every $n \geqslant 1$. Proceed by induction on n. Clearly, $\delta_A(B) \leqslant \delta_A(A \oplus B) = 0$. Suppose $\delta_A(B^{(n-1)}) = 0$. By Propositions 1.3 and 1.8

$$\delta_A(B^{(n)}) \leqslant \delta_A(A \oplus B) + \delta_{A\oplus B}(B^{(n-1)} \oplus B) \leqslant 0 + \delta_A(B^{(n-1)}) + \delta_B(B) = 0. \quad ▨$$

REMARK. At this point, it seems reasonable to address the question: does $\delta_A(B)$ really measure finite dimensional information? It would appear, in fact, that although $B|M$ is finite rank for M in $\mathscr{F}$, its range is arbitrary and hence it is not a finite object. Perhaps it would be more reasonable to measure compressions $P_M B|M$ versus $P_{UM}A|UM$ as U runs over all unitaries. It turns out that compressions are adequate provided that we consider, simultaneously, compressions of B^*B and BB^* as well. Let $\varphi_i(x, y)$ be the (noncommuting) polynomials x, xy, and yx for $i = 1, 2, 3$, and consider the measure

$$\delta'_A(B) = \sup_{M\in\mathscr{F}} \inf_{U\in\mathscr{U}} \max_{1\leqslant i\leqslant 3} \| P_{UM}\varphi_i(A, A^*)P_{UM}U - UP_M\varphi_i(B, B^*)P_M\|.$$

The comparison with $\delta_A(B)$ is based on the following computation. For M in $\mathscr{F}$, let $\tilde{M} = \operatorname{span}\{M, BM, B^*M\}$. Fix U in $\mathscr{U}$.

$$\|(AU - UB)P_M\|^2 = \|P_M(U^*A^*AU - B^*B + B^*(B-U^*AU)+(B^*-U^*AU)B)P_M\| \leqslant$$

$$\leqslant \|P_{UM}A^*AP_{UM}U - UP_MB^*BP_M\| + 2\|P_MB^*P_{\tilde{M}}(U^*AU - B)P_{\tilde{M}}P_M\| \leqslant$$

$$\leqslant \|P_{U\tilde{M}}A^*AP_{U\tilde{M}}U - UP_{\tilde{M}}B^*BP_{\tilde{M}}\| + 2\|B\|\,\|P_{U\tilde{M}}AP_{U\tilde{M}}U - UP_{\tilde{M}}BP_{\tilde{M}}\|.$$

A similar estimate holds for $\|(A^*U - UB^*)P_M\|$. Thus

$$\delta_A(B)^2 \leqslant (1 + 2\|B\|)\delta'_A(B).$$

On the other hand, it is easy to show that

$$\delta'_A(B) \leqslant \max\{\|A\| + \|B\|,\ 1\}\delta_A(B).$$

So these two measures are closely related. However, the measure $\delta'_A(B)$ is not homogeneous, and is not as convenient to our purposes. The main reasons for introducing it here is to demonstrate that $\delta_A(B)$ is indeed a measure of finite dimensional phenomena.

2. APPLICATIONS OF VOICULESCU'S THEOREM

In this section, we obtain better estimates by employing the following Weyl--von Neumann type theorem due to Voiculescu [14].

VOICULESCU'S THEOREM. *Let T be a bounded operator on $\mathcal{H}$, and let ρ be a separable $*$-representation of $C^*(T)$ which annihilates $C^*(T) \cap \mathcal{K}$. Then $T \underset{a}{\sim} T \oplus \rho(T)$.*

The following result is an immediate and useful consequence.

COROLLARY. *Let T be a bounded operator on $\mathcal{H}$. Then $T \underset{a}{\sim} T^{(\infty)}$ if and only if $C^*(T)$ contains no non-zero compact operator.*

Operators of "approximate infinite multiplicity" (i.e. $T \underset{a}{\sim} T^{(\infty)}$) are easier to handle for purposes of approximation since there is less rigidity, or more "room" in which to manoeuvre. For this reason, the measure $\delta_A^e(\cdot)$ is more tractable than the more important $\delta_A(\cdot)$.

THEOREM 2.1. *Let A and B belong to $\mathcal{B}(\mathcal{H})$. Then there exists a T in $\mathcal{B}(\mathcal{H})$ so that*

$$d(\mathcal{U}(A), \mathcal{U}(T \oplus B^{(\infty)})) \leqslant \delta_A^e(B).$$

Hence

$$d(\mathcal{U}(A), \mathcal{U}(A \oplus B)) \leqslant 2\delta_A^e(B).$$

Proof. The idea is to perturb A to a nearby operator T which contains pairwise orthogonal pieces that are unitarily equivalent to arbitarily large pieces of B. This will allow us to define a $*$-representation of $C^*(T)$ taking T onto B which annihilates the compacts. Then Voiculescu's Theorem may be applied. We take a certain amount of care in the construction to ensure a perturbation of norm close to $\delta_A^e(B)$.

Let P_k be a fixed sequence of finite rank projections increasing to the identity. Fix $\varepsilon > \delta_A^e(B)$. Then there is a unitary operator U_1 such that

$$\|(AU_1 - U_1B)P_1\| < \varepsilon \quad \text{and} \quad \|(A^*U_1 - U_1B^*)P_1\| < \varepsilon.$$

Let $Q_1 = U_1P_1U_1^*$ and let R_1 be the projection onto the span of the ranges of $\{Q_1, AQ_1, A^*Q_1, U_1BP_1, U_1B^*P_1\}$. Then

$$\|R_1(A - U_1BU_1^*)Q_1\| = \|(AU_1 - U_1B)P_1\| < \varepsilon$$

and

$$\|Q_1(A - U_1BU_1^*)R_1\| = \|(A^*U_1 - U_1B^*)P_1\| < \varepsilon.$$

Choose an operator $K_1 = R_1K_1R_1$ so that

$$R_1K_1Q_1 = R_1(A - U_1BU_1^*)Q_1 \quad \text{and} \quad Q_1K_1R_1 = Q_1(A - U_1BU_1^*)R_1,$$

such that $\|K_1\| < \varepsilon$. This is possible by [7], Theorem 1.2. Thus

$$(A - K_1)Q_1 = R_1(A - K_1)Q_1 = R_1(U_1BU_1^*)Q_1 = (U_1BU_1^*)Q_1$$

and similarly,

$$Q_1(A - K_1) = Q_1(U_1BU_1^*).$$

Now let N_1 be the span of the ranges of $\{R_1, AR_1, A^*R_1\}$.

By induction, we will construct sequences of unitaries U_k, pairwise orthogonal finite rank projections R_k, and operators $K_k = R_kK_kR_k$ such that

i) $Q_k = U_kP_kU_k^*$ and the range of R_k equals the range of $\{Q_k, AQ_k, A^*Q_k, U_kBP_k, U_kB^*P_k\}$,

ii) $(A - K_k)Q_k = (U_kBU_k^*)Q_k$,

ii′) $Q_k(A - K_k) = Q_k(U_kBU_k^*)$,

iii) $\|K_k\| < \varepsilon$.

We also define finite dimensional subspaces N_k equal to the span of $\{N_{k-1}, R_k\mathscr{H}, AR_k\mathscr{H}, A^*R_k\mathscr{H}\}$ and M_k equal to the span of $\{P_k\mathscr{H}, BP_k\mathscr{H}, B^*P_k\mathscr{H}\}$.

Assume U_{k-1}, R_{k-1}, K_{k-1} and N_{k-1} are defined. Define N_k and M_k as above. Since $\varepsilon > \delta_A^e(B)$, choose a unitary U_{k+1} such that U_kM_k is orthogonal to N_k such that

$$\|(AU_k - U_kB)P_{M_k}\| < \varepsilon$$

and

$$\|(A^*U_k - U_kB^*)P_{M_k}\| < \varepsilon.$$

Define Q_k and R_k by (i). First we verify that R_k is orthogonal to each R_j for $j < k$. The ranges of Q_k, U_kBP_k and $U_kB^*P_k$ are contained in U_kM_k which is orthogonal to N_k and hence to R_j for all $j < k$. In particular, the range of Q_k is orthogonal to the range of AR_j and A^*R_j for $j < k$; whence AQ_k and A^*Q_k have ranges orthogonal to each R_j. Consequently, R_k is orthogonal to R_j for $j < k$.

As in the first paragraph, one has

$$R_k(A - U_kBU_k^*)Q_k = (AU_k - U_kB)P_kU_k^*$$

and

$$Q_k(A - U_k B U_k^*) R_k = U_k[(A^* U_k - U_k B^*) P_k]^*.$$

Thus by [7], there is an operator $K_k = R_k K_k R_k$ with $\|K_k\| < \varepsilon$ such that

$$(A - K_k) Q_k = R_k (A - K_k) Q_k = R_k U_k B U_k^* Q_k = U_k B U_k^* Q_k$$

and

$$Q_k (A - K_k) = Q_k (A - K_k) R_k = Q_k (U_k B U_k^*) R_k = Q_k U_k B U_k^*.$$

Thus ii), ii') and iii) have been verified.

Let $K = \sum_{k \geq 1} K_k$. Then $\|K\| \leq \varepsilon$. Define $T = A - K$. By ii) and ii') it follows that

$$(U_k^* T U_k - B) P_k = 0 = P_k (U_k^* T U_k - B)$$

for all $k \geq 1$. Consequently, $U_k^* T U_k$ converges to B in the strong-* topology. Define a representation ρ of $C^*(T)$ by

$$\rho(X) = \text{s}^*\text{-}\lim_{k \to \infty} U_k^* X U_k.$$

This limit exists for every polynomial in T and T^*, and thus for all X in $C^*(T)$. Moreover, $\rho(T) = B$. Since $U_k P_k$ are pairwise orthogonal, the sequence U_k converges to zero in the weak operator topology. Thus ρ annihilates every compact operator.

By Voiculescu's Theorem applied to $\rho^{(\infty)}$, one has $T \underset{a}{\sim} T \oplus B^{(\infty)}$. Thus

$$d(\mathscr{U}(A), \mathscr{U}(T \oplus B^{(\infty)})) \leq \|A - T\| + d(\mathscr{U}(T), \mathscr{U}(T \oplus B^{(\infty)})) \leq \varepsilon.$$

Thus $d(\mathscr{U}(A \oplus B), \mathscr{U}(T \oplus B^{(\infty)})) \leq \varepsilon$ also, whence $d(\mathscr{U}(A), \mathscr{U}(A \oplus B)) \leq 2\varepsilon$. Let ε decrease to $\delta_A^e(B)$ to finish the proof. ▨

COROLLARY 2.2. *Let A and B belong to $\mathscr{B}(\mathscr{H})$. Then*

$$\delta_A^e(B) \leq d(\mathscr{U}(A), \mathscr{U}(A \oplus B^{(\infty)})) \leq 2\delta_A^e(B) = 2\delta_A(B^{(\infty)}).$$

Proof. This is an immediate consequence of Theorem 2.1, Corollary 1.7 and Proposition 1.1. ▨

COROLLARY 2.3. *Suppose A is an operator such that $C^*(A)$ contains no non-zero compact operator. Then for every operator B in $\mathscr{B}(\mathscr{H})$,*

$$\delta_A(B) \leq d(\mathscr{U}(A), \mathscr{U}(A \oplus B)) \leq 2\delta_A(B)$$

and

$$\delta_B(A) \leqslant d(\mathcal{U}(B), \mathcal{U}(A \oplus B)) \leqslant 2\delta_B(A).$$

Proof. By the corollary to Voiculescu's Theorem, $A \sim A^{(\infty)}$. Hence by Proposition 1.4, $\delta_A(B) = \delta_A^e(B)$; and by Proposition 1.5, $\delta_B(A) = \delta_B^e(A)$. The non trivial half of the two inequalities now follows from Theorem 2.1, and the trivial part is given in Proposition 1.1. ▨

The following result is an immediate corollary of Corollary 2.3. We call it a theorem to emphasize its importance.

THEOREM 2.4. *Suppose that A is an operator such that $C^*(A)$ contains no non-zero compact operator. Then for every operator B in $\mathcal{B}(\mathcal{H})$,*

$$\rho(A, B) \leqslant d(\mathcal{U}(A), \mathcal{U}(B)) \leqslant 4\rho(A, B).$$

Proof. Indeed, the upper bound obtained is $2\delta_A(B) + 2\delta_B(A)$ which is at most $4\rho(A, B)$. ▨

Two more corollaries follow easily.

COROLLARY 2.5. *Let A and B belong to $\mathcal{B}(\mathcal{H})$. Then $\rho^e(A, B) = 0$ if and only if $A \underset{a}{\sim} A^{(\infty)} \underset{a}{\sim} B^{(\infty)} \underset{a}{\sim} B$.*

In particular, $\delta_A^e(A) = 0$ if and only if $A \underset{a}{\sim} A^{(\infty)}$.

Proof. We assume $\rho^e(A, B) = 0$ and prove the non-trivial assertion. By Theorem 2.1, $A \underset{a}{\sim} A \oplus B \underset{a}{\sim} A \oplus B^{(\infty)}$ and $B \underset{a}{\sim} B \oplus A \underset{a}{\sim} B \oplus A^{(\infty)}$. So $A \underset{a}{\sim} B$ and

$$A \underset{a}{\sim} A \oplus B^{(\infty)} \underset{a}{\sim} A \oplus A^{(\infty)} \simeq A^{(\infty)} \underset{a}{\sim} B^{(\infty)}.$$ ▨

COROLLARY 2.6. *Let A and B belong to $\mathcal{B}(\mathcal{H})$. Then*

$$d(\mathcal{U}(A), \mathcal{U}(B)) \leqslant 4\rho^e(A, B).$$

Proof. By Theorem 2.1,

$$d(\mathcal{U}(A), \mathcal{U}(B)) \leqslant d(\mathcal{U}(A), \mathcal{U}(A \oplus B)) + d(\mathcal{U}(A \oplus B), \mathcal{U}(B)) \leqslant$$

$$\leqslant 2\delta_A^e(B) + 2\delta_B^e(A) \leqslant 4\rho^e(A, B).$$ ▨

3. CONSEQUENCES OF $\delta_A(B) = 0$

The aim of this section is to show that $\delta_A(B) = 0$ implies that B is an approximate direct summand of A. As a corollary, it follows that $\rho(A, B) = 0$ if and only if $A \underset{a}{\sim} B$. Thus ρ is a metric on the unitary equivalence classes. In Section 5 we will see that ρ is not equivalent to d. However, the "one sided version" is.

LEMMA 3.1. *Let B belong to $\mathscr{B}(\mathscr{H})$. Let k be a positive integer, and suppose that D is a direct summand of $B^{(k)}$. Then B has a direct summand D_0 unitarily equivalent to a summand of D.*

Proof. Let P be the projection of $\mathscr{H}^{(k)}$ onto the domain of D, and write P as a $k \times k$ matrix $[P_{ij}]$. We can assume, without loss of generality, that $P_{11} \neq 0$. Let X be the operator from $\mathscr{H}$ into $\mathscr{H}^{(k)}$ given by $k \times 1$ matrix $[P_{i1}]_{1 \leqslant i \leqslant k}$. Since P commutes with $B^{(k)}$, one obtains $B^{(k)}X = XB$ and $B^{*(k)}X = XB^*$. The range of X is contained in the range of P, so by restricting, we obtain $DX = XB$ and $D^*X = XB^*$. Let $X = UR$ be the polar decomposition of X. Then

$$R^2B = X^*XB = X^*DX = (D^*X)^*X = BX^*X = BR^2.$$

Thus, B commutes with R, and hence with the projection U^*U onto the range of R (which is non-zero since $X \neq 0$). So

$$(DU - UB)R = DX - XB = 0$$

and

$$(D^*U - UB^*)R = D^*X - XB^* = 0.$$

Thus if W is the restriction of U to U^*U and D_0 is the restriction of B to $U^*U\mathscr{H}$, then W is an isometry and $DW = WD_0$ and $D^*W = WD_0^*$. So WD_0W^* a summand of D unitarily equivalent to D_0. ▨

THEOREM 3.2. *Suppose A and B are operators on $\mathscr{H}$, and $\delta_A(B) = 0$. Then there are operators A_0, B_0 and C such that $A \cong A_0 \oplus C$, $B \cong B_0 \oplus C$, and $A \underset{a}{\sim} A \oplus B_0^{(\infty)}$.*

Proof. Let $\{(A_\lambda, B_\lambda),\ \lambda \in \Lambda\}$ be a maximal family of pairs of unitarily equivalent direct summands of A and B respectively so that A_λ (resp. B_λ) are supported on pairwise orthogonal subspaces. Let $C = \sum^\oplus A_\lambda \cong \sum^\oplus B_\lambda$. Then $A \cong C \oplus A_0$ and $B \cong C \oplus B_0$, and no summand of B_0 is unitarily equivalent to a summand of A_0. It suffices to show that $\delta_A^e(B_0) = 0$. Let $\{B_\gamma,\ \gamma \in \Gamma\}$ be a maximal family of pairwise orthogonal summands of B_0 such that $\delta_A^e(B_\gamma) = 0$. Let $B_1 = \sum^\oplus B_\gamma$. By Proposition 1.6, $\delta_A^e(B_1) = 0$. Let $B_0 = B_1 \oplus B_2$. Thus no summand D of B_2 satisfies either $\delta_A^e(D) = 0$ or is unitarily equivalent to a summand of A_0. Let $\mathscr{H}_2$ be the domain of B_2.

Now $\delta_A(B_2) = 0$. So choose a sequence P_n of finite rank projections in $\mathscr{B}(\mathscr{H}_2)$ increasing to the identity. One can find unitary operators U_n such that

$$\max\{\|(AU_n - U_nB_2)P_n\|,\ \|(A^*U_n - U_nB_2^*)P_n\|\} < 1/n.$$

Let X be a weak operator topology cluster point of $\{U_n\}$, say $U_{n_\alpha} \xrightarrow{w} X$. Let $X = UR$ be the polar decomposition of X. One has $AX = XB_2$ and $A^*X = XB_2^*$. Proceeding as in Lemma 3.1,

$$R^2B_2 = X^*XB_2 = X^*AX = (A^*X)^*X = B_2X^*X = B_2R^2.$$

Hence B_2 and B_2^* commute with R. In particular, the closure of the range of R reduces B_2, so B_2 commutes with $Q = U^*U$. Moreover,

$$(AU - UB_2)R = AX - XB_2 = 0$$

and

$$(A^*U - UB_2^*)R = A^*X - XB_2^* = 0.$$

Thus, if we decompose $B_2 = B_3 \oplus B_4$ on $Q\mathscr{H}_2 \oplus Q^\perp\mathscr{H}_2$, then B_3 is unitarily equivalent to a direct summand of A.

Suppose that $U^*U \neq I$, so that B_4 is a proper summand of B_2. Let $W_n = U_n|Q^\perp\mathscr{H}$. Then for every finite dimensional space M,

$$\max\{\|(AW_n - W_nB_4)P_M\|,\ \|(A^*W_n - W_nB_4^*)P_M\|\}$$

tends to zero. Moreover, $W_{n_\alpha} \xrightarrow{w} 0$. Let N be any finite dimensional subspace of $\mathscr{H}$. Then $\lim_\alpha \|P_NW_{n_\alpha}P_n\| = 0$. Hence one may choose n so that $\|P_NW_nP_M\|$ is arbitrarily small. Thus a small perturbation of W_n yields a unitary W_n' so that $W_n'M$ is orthogonal to N. It follows that $\delta_A^e(B_4) = 0$, contrary to the construction of B_1 and B_2. Hence U is unitary, and B_2 is unitarily equivalent to a direct summand of A.

Let $A \cong A_0 \oplus C$ with respect to the decomposition $\mathscr{H}_0 \oplus \mathscr{H}_1$. With respect to this decomposition, U has a matrix $[U_0U_1]$. Suppose that $P_{\mathscr{H}_0}U = U_0 \neq 0$. Then since $P_{\mathscr{H}_0}$ commutes with A, one obtains $A_0U_0 = U_0B_2$ and $A_0^*U_0 = U_0B_2^*$. Using the polar decomposition of U_0, one obtains (as in Lemma 3.1) a summand of B_2 unitarily equivalent to a summand of A_0. This is contrary to the construction of C, hence the range of U is contained in $\mathscr{H}_1$. So B_2 is a summand of C.

It will be shown by induction that $B_2^{(k)}$ is a summand of C for every $k \geqslant 1$. This has been verified for $k = 1$. Suppose that $B_2^{(k)}$ is a summand of C. Then $B_2^{(k+1)}$

is a summand of $B \cong C \oplus B_1 \oplus B_2$, and hence $\delta_A(B_2^{(k+1)}) = 0$. Repeat the argument given above for $B_2^{(k+1)}$ in place of B_2. There are three possibilities:

(i) $B_2^{(k+1)}$ has a summand D such that $\delta_A^e(D) = 0$,

(ii) $B_2^{(k+1)}$ has a summand D unitarily equivalent to a summand of A_0, or

(iii) $B_2^{(k+1)}$ is a summand of C.

Now in cases (i) and (ii), one uses Lemma 3.1 to obtain a non-trivial summand D_0 of B_2 unitarily equivalent to a summand of D. In case (i), one obtains $\delta_A^e(D_0) = 0$ contrary to the construction of B_1. In case (ii), D_0 is unitarily equivalent to a summand of A_0 contrary to the construction of C. Thus only case (iii) is possible, and $B_2^{(k)}$ is a summand of C for all $k \geq 1$.

Clearly, $\delta_A(B_2^{(k)}) = 0$ for all $k \geq 1$. Hence by Proposition 1.5, $\delta_A^e(B_2) = 0$. From the definition of B_1, one finds that B_2 is vacuous. Thus A and B have the desired decompositions. ▨

COROLLARY 3.3. *If $\rho(A, B) = 0$, then $A \underset{a}{\sim} B$. Hence ρ is a metric on $\mathscr{B}(\mathscr{H})/\underset{a}{\sim}$.*

Proof. By Theorem 3.2 and its proof, A and B can be decomposed as $A \cong A_0 \oplus C$ and $B \cong B_0 \oplus C$ so that $\delta_A^e(B_0) = 0 = \delta_B^e(A_0)$. By Theorem 2.1 one obtains

$$A \underset{a}{\sim} A_0^{(\infty)} \oplus B_0^{(\infty)} \oplus C \underset{a}{\sim} B.$$

That ρ is a metric on the collection of closed unitary orbits now follows from Proposition 1.3. ▨

Before considering $\rho(A, B)$ in general, we mention a few applications of this corollary. In [8], Hadwin defines an operator valued spectrum $\Sigma(T) = \{A \in \mathscr{B}(\mathscr{H}') : T = \lim_{n\to\infty} T_n$ and $T_n \cong A \oplus B_n\}$. The space $\mathscr{H}'$ may be finite or infinite dimensional. The following theorem due to Hadwin can be deduced from our results.

THEOREM 3.4. *Let T belong to $\mathscr{B}(\mathscr{H})$.*

(i) *$A \in \Sigma(T)$ if and only if $T \underset{a}{\sim} T' \cong A \oplus B$.*

(ii) $\overline{\mathscr{U}(T)} = \{A \in \mathscr{B}(\mathscr{H}) : \Sigma(A) = \Sigma(T)\}$.

(iii) *The *-strong operator closure* $\overline{\mathscr{U}(T)}^{*s} = \Sigma(T) \cap \mathscr{B}(\mathscr{H}) = \{A \in \mathscr{B}(\mathscr{H}) : \delta_T(A) = 0\}$.

Proof. It is clear from the definition of $\Sigma(T)$ that A belongs to $\Sigma(T)$ if and only if $\delta_T(A) = 0$. So (i) follows from Theorem 3.2. Hence $\Sigma(A) = \Sigma(T)$ implies $\rho(A, T) = 0$. By Corollary 3.3, $A \underset{a}{\sim} T$. The converse follows from Proposition 1.2. This proves (ii). If A belongs to $\Sigma(T) \cap \mathscr{B}(\mathscr{H})$, one has $T \sim T' \simeq A \oplus B$ by (i). Let P_n be a sequence of finite rank projections increasing to the identity of $\mathscr{H}_A$. Let U_n be unitary taking $\mathscr{H}_A$ onto $\mathscr{H}$ which identifies P_n with $P_n \oplus 0$ in $\mathscr{H}' = \mathscr{H}_A \oplus \mathscr{H}_B$. Then $W_n^* T' W_n P_n = AP_n$ and $P_n W_n^* T' W_n = P_n A$. Thus $W_n^* T' W_n$

converges to A in the *-strong topology. Consequently, A belongs to $\overline{\mathcal{U}(T)}^{*s}$. Conversely, if A is the *-strong limit of $W_n^*TW_n$ with W_n unitary for $n \geqslant 1$, then for every finite dimensional space M,

$$\max\{\|(TW_n - W_nA)P_M\|, \ \|(T^*W_n - W_nA^*)P_M\|\}$$

tends to zero. Thus $\delta_T(A) = 0$ and hence A belongs to $\Sigma(T) \cap \mathcal{B}(\mathcal{H})$. ▨

EXAMPLE 3.5. Let A be an invertible, bilateral weighted shift. That is, one has an orthonormal basis $\{e_n, \ n \in \mathbf{Z}\}$ such that $Ae_n = w_ne_{n+1}$ and $\inf|w_n| > 0$. The operators $A^*A, \ldots, A^{*n}A^n$ are commuting diagonal operators with joint spectrum

$$\Sigma_n(A) = \{(|w_k|^2, |w_kw_{k+1}|^2, \ldots, |w_kw_{k+1} \ldots w_{k+n-1}|^2) : k \in \mathbf{Z}\}^-.$$

There is a natural correspondence between this subset of $\mathbf{R}^n$ and

$$\tilde{\Sigma}_n(A) = \{(|w_k|, |w_{k+1}|, \ldots, |w_{k+n-1}|) : k \in \mathbf{Z}\}^-.$$

It is clear that $\Sigma_n(A)$ (or $\tilde{\Sigma}_n(A)$) are unitary invariants of A. The following theorem is a consequence of O'Donovan [12].

THEOREM. *Let A and B be invretible, bilateral weighted shifts. Then $A \underset{a}{\sim} B$ if and only if $\tilde{\Sigma}_n(A) = \tilde{\Sigma}_n(B)$ for all $n \geqslant 1$.*

Proof. The only if direction has already been noted. Conversely, if $\tilde{\Sigma}_{n+1}(A) = \tilde{\Sigma}_{n+1}(B)$, it is immediate that the restriction of B to $M = \operatorname{span}\{e_{k+1}, \ldots, e_{k+n}\}$ can be approximated to arbitrary accuracy by a piece of A. Namely, take the weights $|v_k|, \ldots, |v_{k+n}|$ of B and some $\varepsilon > 0$ and find an integer l so that the weights $|w_l|, \ldots, |w_{l+n}|$ approximate these weights within ε. The unitary given by $Ue_j = e_{j+l-k}$ satisfies

$$\max\{\|(AU - UB)P_M\|, \ \|(A^*U - UB^*)P_M\|\} < \varepsilon.$$

The projections P_M of this type almost dominate every finite rank projection. Hence $\delta_A(B) = 0 = \delta_B(A)$. By Corollary 3.3, $A \underset{a}{\sim} B$. ▨

Unfortunately, it is not easy to estimate $d(\mathcal{U}(A), \mathcal{U}(B))$ from these weights when A is not approximately unitarily equivalent to B. In his doctoral thesis [11], L. Marcoux has obtained interesting upper and lower bounds for this distance.

The case of unilateral shifts is somewhat different, but can be proven along the same lines. Again, this theorem is a consequence of O'Donovan's results.

THEOREM. *Let A and B be unilateral weighted shifts which are bounded below. Then $A \underset{a}{\sim} B$ if and only if $A \cong B$ if and only if their weights $\{w_k\}$ and $\{v_k\}$ satisfy $|w_k| = |v_k|$ for all $k \geqslant 1$.*

Again, computing $d(\mathscr{U}(A), \mathscr{U}(B))$ from the weights is an interesting and non-trivial task. Refer to Marcoux [11] for some partial results.

4. $\delta_A(B)$ AS A MEASURE OF B AS A SUMMAND OF A

In Proposition 1.1, it is shown that for all operators A, B and C, one obtains the trivial estimate $d(\mathscr{U}(A), \mathscr{U}(B \oplus C)) \geqslant \delta_A(B)$. The purpose of this section is to prove the converse. That is, for each A and B, there is an operator C so that $d(\mathscr{U}(A), \mathscr{U}(B \oplus C)) = \delta_A(B)$. By applying this to A and B in turn, one would hope to generalize Theorem 2.4 to all operators. In spite of the positive evidence of Corollary 3.3 and Theorem 4.1 below, this turns out to be false. The example will be given in the next section.

First, we describe a representation of $\mathscr{B}(\mathscr{H})$ closely related to a representation used by Calkin [4]. Let $\mathscr{U}$ be a non-principal ultrafilter on N. Take $\ell^\infty(\mathscr{H})$ to be the space of all bounded sequences $x = (x_n)$ of vectors in $\mathscr{H}$. Define a sesquilinear form

$$\langle x, y\rangle = \lim_{\mathscr{U}} (x_n, y_n).$$

Standard arguments show that $\mathscr{N} = \{x : \langle x, x\rangle = 0\}$ is a subspace and the completion $\mathscr{H}_{\mathscr{U}}$ of $\ell^\infty(\mathscr{H})/\mathscr{N}$ is a Hilbert space. Let $\dot{x} = (\dot{x}_n)$ denote a typical element of the quotient. Define a representation σ of $\mathscr{B}(\mathscr{H})$ on $\mathscr{H}_{\mathscr{U}}$ by

$$\sigma(T)(\dot{x}_n) = (T\dot{x}_n)$$

and extend by continuity.

Define a unitary map W of $\mathscr{H}$ into $\mathscr{H}_{\mathscr{U}}$ by sending a vector x to the constant sequence $x_n = x$. Clearly, the range $\mathscr{H}_0$ of W is invariant for σ. Thus $\mathscr{H}_{\mathscr{U}}$ splits as $\mathscr{H}_{\mathscr{U}} \simeq \mathscr{H}_0 \oplus \mathscr{H}^0_{\mathscr{U}}$ and σ splits as $\sigma \cong \sigma_0 \oplus \tau$ where $\sigma_0 = \operatorname{Ad} W$ is equivalent to the identity representation. We claim that τ factors through the Calkin algebra. To see this, let K be a compact operator and let $\dot{y} = (\dot{y}_n)$ be a unit vector in $\mathscr{H}^0_{\mathscr{U}}$. For $\varepsilon > 0$, choose a finite rank projection P so that $\|KP^\perp\| < \varepsilon$. Let e_i, $1 \leqslant i \leqslant p = \operatorname{rank} P$ be an orthonormal basis for $\operatorname{Ran} P$, and let $x_i = We_i$. Then

$$0 = \langle y, x_i\rangle = \lim_{\mathscr{U}}(y_n, e_i) \quad 1 \leqslant i \leqslant p.$$

Thus, one can choose a subset S in $\mathscr{U}$ so that $\|Py_n\| < \varepsilon$ for all n in S. Hence

$$\|Ky_n\| \leqslant \|K\|\,\|Py_n\| + \|KP^\perp\|\,\|y_n\| < (\|K\| + 1)\varepsilon \quad \text{for } n \in S.$$

Consequently, $\|\tau(K)y\| \leqslant (\|K\| + 1)\varepsilon$. This is independent of ε, whence $\tau(K) = 0$.

Now, we describe how certain bounded operators in $\mathscr{B}(\mathscr{H}, \mathscr{H}_{\mathscr{U}})$ may be obtained. To each bounded sequence (A_n), define

$$Ax = (A_n^{\bullet} x).$$

This is easily seen to be linear with $\|A\| = \lim_{\mathscr{U}} \|A_n\| \leqslant \sup \|A_n\|$. If T is a bounded operator on $\mathscr{H}$, then AT is given by the sequence $(A_n T)$; and $\sigma(T)A$ is given by the sequence (TA_n).

The main result of this section can now be stated. Notice that Theorem 2.1 is an immediate corollary.

4.1. THEOREM. *Given A and B in $\mathscr{B}(\mathscr{H})$, there is an operator C such that $d(\mathscr{U}(A), \mathscr{U}(B \oplus C)) = \delta_A(B)$.*

Proof. Let P_n be a sequence of finite rank projections increasing to the identity. From the definition of $\delta_A(B)$, one can obtain unitary operators U_n such that

$$\|(AU_n - U_n B)P_n\| < \delta_A(B) + 1/n$$

and

$$\|(A^* U_n - U_n B^*)P_n\| < \delta_A(B) + 1/n.$$

Fix a nonprincipal ultrafilter $\mathscr{U}$ as above, and define an operator U in $\mathscr{B}(\mathscr{H}, \mathscr{H}_{\mathscr{U}})$ by the sequence (U_n). Then $\sigma(A)U - UB$ is given by the sequence $(AU_n - U_n B)$. But for every $x = P_k x$, one has $\|(AU_n - U_n B)x\| \leqslant \delta_A(B) + 1/n$ for $n \geqslant k$ and hence $\|(\sigma(A)U - UB)x\| \leqslant \delta_A(B)$. But these vectors are dense, so $\|\sigma(A)U - UB\| \leqslant \leqslant \delta_A(B)$. Similarly, $\|\sigma(A^*)U - UB^*\| \leqslant \delta_A(B)$. Next, notice that U is an isometry. For x in $\mathscr{H}$,

$$\|Ux\|^2 = \lim_{\mathscr{U}} (U_n x, U_n x) = \lim_{\mathscr{U}} \|x\|^2 = \|x\|^2.$$

Thus we have identified an "approximate summand" of $\sigma(A)$ close to unitarily equivalent to B.

Let $\mathscr{H}_1$ be the smallest reducing subspace of $\mathscr{H}_{\mathscr{U}}$ for $\sigma(A)$ containing $\mathscr{H}_0$ and the range of U. Since $\mathscr{H}_0$, $\operatorname{Ran} U$, and $C^*(\sigma(A))$ are all separable, the space $\mathscr{H}_1$ is separable. Now $\mathscr{H}_1$ decomposes as $\mathscr{H}_1 = \mathscr{H}_0 \oplus \mathscr{H}_1^0$. Define a representation σ_1 of $C^*(A)$ on $\mathscr{H}_1$ by restricting σ to $C^*(A)$, and restricting this to $\mathscr{H}_1$. This is a separable representation, and it decomposes as $\sigma_1 = \sigma_0 \oplus \tau_1$ where $\sigma_0 = \operatorname{Ad} W$ is equivalent to the identity representation, and τ_1 factors through the Calkin algebra. By Voiculescu's Theorem, $A \underset{a}{\sim} \sigma_1(A)$. Translating the results of the previous paragraph, we obtain

$$\|\sigma_1(A)U - UB\| \leqslant \delta_A(B) \quad \text{and} \quad \|\sigma_1(A^*)U - UB^*\| \leqslant \delta_A(B).$$

Now split $\mathscr{H}_1$ as $U\mathscr{H} \oplus (U\mathscr{H})^{\perp}$. Note that $\sigma_1(A)$ decomposes as

$$\sigma_1(A) = \begin{bmatrix} UBU^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & A_{22} \end{bmatrix}$$

and $\|[X_{11}\ X_{12}]\| \leqslant \delta_A(B)$ and $\left\|\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}\right\| \leqslant \delta_A(B)$. By [7], there is an operator X_{22} so that $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ has $\|X\| \leqslant \delta_A(B)$. Let $C = A_{22} - X_{22}$. It is now apparent that

$$\|\sigma_1(A) - (UBU^* \oplus C)\| = \|X\| \leqslant \delta_A(B).$$

Hence

$$d(\mathscr{U}(A), \mathscr{U}(B \oplus C)) \leqslant d(\mathscr{U}(A), \mathscr{U}(\sigma_1(A))) + d(\mathscr{U}(\sigma_1(A)), \mathscr{U}(B \oplus C)) \leqslant$$

$$\leqslant 0 + \delta_A(B) = \delta_A(B).$$

The reverse inequality is trivial, and follows from Proposition 1.1. ∎

5. FAILURE OF $\rho(A, B)$ AS A MEASURE OF $d(\mathscr{U}(A), \mathscr{U}(B))$

From Theorem 4.1, we obtain, for each A and B, a pair of operators A and D and unitary operators U and V so that

$$\|AU - U(B \oplus D)\| \leqslant \rho(A, B) \quad \text{and} \quad \|BV - V(A \oplus C)\| \leqslant \rho(A, B).$$

That is, A and B are almost summands of each other. An operator analogue of the Schroeder-Bernstein Theorem would yield a good estimate of $d(\mathscr{U}(A), \mathscr{U}(B))$ in terms of $\rho(A, B)$. The following example shows that this is not possible.

Let $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ be the 2×2 Jordan block. Let $N > 1$ be a positive integer. Define $F_n = \sum_{k=1}^{N-1}{}^{\oplus} \frac{k}{N-1} J$ and $G_N = 0_1 \oplus F_n$ in $M_{2N-2}(\mathbf{C})$ and $M_{2N-1}(\mathbf{C})$ respectively. Let $A_N = F_N \oplus J^{(\infty)}$ and $B_N = G_N \oplus J^{(\infty)}$. Since A_N is a direct summand of B_N, $\delta_{B_N}(A_N) = 0$. Now B_N acts on the space $\mathbf{C} \oplus \mathbf{C}^2 \oplus \ldots \oplus \mathbf{C}^2 \oplus (\mathbf{C}^2)^{(\infty)}$, and A_N acts on $\mathbf{C}^2 \oplus \ldots \oplus \mathbf{C}^2 \oplus (\mathbf{C}^2)^{(\infty)}$ where in each case there are $N - 1$ single copies of $\mathbf{C}^2$. Let e be the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathbf{C}^2$, and consider the isometry W given by

$$W\left(\lambda, x_1, x_{N-1}, \sum_{k=1}^{\infty}{}^{\oplus} y_k\right) = \left(\lambda e, x_1, \ldots, x_{N-2}, x_{N-1} \oplus \sum_{k=2}^{\infty}{}^{\oplus} y_{k-1}\right).$$

Then

$$(A_N W - WB_N)\left(\lambda, x_1, \ldots, x_{N-1}, \sum_{k=1}^{\infty}{}^{\oplus} y_k\right) =$$

$$= \left(0, \frac{2}{N} Jx_1, \ldots, \frac{N-1}{N} Jx_{N-2}, Jx_N \oplus \sum_{k=2}^{\infty}{}^{\oplus} Jy_{k-1}\right) -$$

$$- W\left(0, \frac{1}{N} Jx_1, \ldots, \frac{N-1}{N} Jx_{N-1}, \sum_{k=1}^{\infty}{}^{\oplus} Jy_k\right) =$$

$$= \left(0, \frac{2}{N} Jx_1, \ldots, \frac{N-1}{N} Jx_{N-2}, Jx_N \oplus \sum_{k=1}^{\infty}{}^{\oplus} Jy_{k-1}\right) -$$

$$- \left(0, \frac{1}{N} Jx_1, \ldots, \frac{N-2}{N} Jx_{N-2}, \frac{N-1}{N} Jx_{N-1} \oplus \sum_{k=2}^{\infty}{}^{\oplus} Jy_k\right) =$$

$$= \left(0, \frac{1}{N} Jx_1, \ldots, \frac{1}{N} Jx_{N-2}, \frac{1}{N} Jx_N \oplus 0\right).$$

Hence $\|A_N W - WB_N\| = 1/N$. A similar computation yields $\|A_N^* W - WB_N^*\| = 1/N$. Thus $\rho(A_N, B_N) \leqslant 1/N$. It will be shown that $d(\mathcal{U}(A_N), \mathcal{U}(B_N)) \geqslant 1/\sqrt{5}$.

A simple and well known lemma is required.

LEMMA. *Let P be a finite rank projection such that $\|[P, J^{(\infty)}]\| < 1/\sqrt{5}$. Then* rank$(P)$ *is even.*

Proof. Now $J^{(\infty)}$ is unitarily equivalent to $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$. With respect to this same decomposition, P has the matrix $[P_{ij}]$. Compute

$$[P, J^{(\infty)}] = \begin{bmatrix} -P_{21} & P_{11} - P_{22} \\ 0 & P_{21} \end{bmatrix}.$$

Therefore, $\|P_{21}\| = \|P_{12}\| < 1\sqrt{5}$ and $\|P_{11} - P_{22}\| < 1/\sqrt{5}$. Since $P = P^2$,

$$\|P_{ii} - P_{ii}^2\| = \|P_{21}\|^2 < 1/5.$$

Thus, the spectrum $\sigma(P_{ii})$ belongs to $[0, \beta) \cup (1 - \beta, 1]$ where $\beta = (5 - \sqrt{5})/10$ is the smaller root of $x - x^2 = 1/5$. From the Hermitian functional calculus, one obtains projections E_{ii} such that $\|P_{ii} - E_{ii}\| < \beta$. Now

$$\|E_{11} - E_{22}\| \leqslant \|E_{11} - P_{11}\| + \|P_{11} - P_{22}\| + \|P_{22} - E_{22}\| <$$

$$< 2\beta + 1/5 = 1 - 1/\sqrt{5} + 1/5 < 1.$$

So $\operatorname{rank} E_{11} = \operatorname{rank} E_{22}$. Also

$$\|P - E_{11} \oplus E_{22}\| \leqslant \max_i \|P_{ii} - E_{ii}\| + \max\{\|P_{12}\|, \|P_{21}\|\} <$$

$$< \beta + 1/\sqrt{5} = 1 - \beta < 1.$$

Hence $\operatorname{rank}(P) = \operatorname{rank}(E_{11} \oplus E_{22}) = 2 \operatorname{rank}(E_{11})$ is even. ∎

Now return to the example. Suppose that U were a unitary satisfying $\|A_N - UB_NU^*\| = \delta < 1/\sqrt{5}$. The projections $P_k = I_{2N-1} \oplus (I_2^{(k)} \oplus 0^{(\infty)})$ commute with $B_N = G_N \oplus J^{(\infty)}$, and increase strongly to the identity. Let $\varepsilon = (1/4)(1/\sqrt{5} - \delta)$. Choose k so large that

$$\|(UP_k^{\perp} U^*)(I_{2N-2} \oplus 0^{(\infty)})\| < \varepsilon.$$

This says that UP_kU^* almost dominates the projection $E = I_{2N-2} \oplus 0^{(\infty)}$. It is possible to find a projection $Q \geqslant E$ so that $\|Q - UP_kU^*\| < 2\varepsilon$. So $\operatorname{rank} Q =$ $= \operatorname{rank} P = 2N - 1 + 2k$ is odd. Moreover,

$$\|[Q, A_N]\| \leqslant \|[UP_kU^*, A_N]\| + \|[UP_kU^* - Q, A_N]\| \leqslant$$

$$\leqslant \|[P_k, U^*A_NU - B_N]\| + 2\|UP_kU^* - Q\| < \|U^*A_NU - B_N\| + 4\varepsilon = 1/\sqrt{5}.$$

So the projection $Q - E$ is of odd rank, and

$$\|[Q - E, J^{(\infty)}]\| = \|[Q, A_N]\| < 1/\sqrt{5}$$

contrary to the lemma. This establishes that $d(\mathscr{U}(A_N), \mathscr{U}(B_N)) \geqslant 1/\sqrt{5}$. ∎

The curious aspect of this example is that it is the finite rank pieces of A_N and B_N that cause the trouble — even though $\rho(A, B)$ is expressly designed to examine and compare the finite pieces of A and B. To put it in another light, this example shows that there is no good analogue of Corollary 1.9 for close orbits. That is, if $\delta_A(A \oplus B) = \varepsilon$, then the best one can prove is $\delta_A(B^{(n)}) \leqslant n\varepsilon$. An easier example of this phenomena is obtained by taking $A_N = \operatorname{diag}\left\{\frac{1}{N}, \frac{2}{N}, \ldots\right.$ $\left.\ldots, \frac{N-1}{N}\right\} \oplus 0^{(\infty)}$ and $B = I_1$ acting on $\mathbf{C}$. Then $\delta_{A_N}(A_N \oplus B) = 1/N$ and $\delta_{A_N}(B^{(k)}) = \min\left\{\frac{k}{N}, 1\right\}$.

Also, notice that $\delta_{A\oplus F}(B \oplus F) \leqslant \delta_A(B)$ and may be much less even when F acts on a finite dimensional space. For example, take $A = 0$ and B to be a rank one projection. Then $F = \operatorname{diag}\left\{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$ yields $\delta_{A\oplus F}(B \oplus F) = 1/N = d(\mathscr{U}(A \oplus F), \mathscr{U}(B \oplus F))$.

QUESTION 5.1. Do ρ and d (the distance metric) yield the same topology on $\mathscr{B}(\mathscr{H})/\underset{a}{\sim}$?

QUESTION 5.2. Find comparable upper and lower bounds for the distance between the unitary orbits of two binormal operators.

In the case of binormal operators, $\overline{\mathscr{U}(T)}$ is determined by $\Sigma^{(2)}(T) = \{\mathscr{U}(\rho(T))\, ;\, \rho\colon C^*(T) \to M_2$ is a $*$-representation$\}$ and its multiplicity function. For example, see Ernest [8]. In particular, it follows from Pearcy-Salinas [13] that if A belongs to the essential spectrum $\Sigma_e^{(2)}(T)$, then $T \underset{a}{\sim} T \oplus A^{(\infty)}$. The theory is analogous to the approximate unitary equivalence of normal operators. In the case of normal operators, estimation of the distance between unitary orbits can be computed but it is not trivial. See Bhatia-Davis-MacIntosh [3], Azoff-Davis [2], and Davidson [5, 6].

Binormal operators, and in general n-normal operators, are members of the larger class of operators which are limits of operators unitarily equivalent to $X^{(\infty)} \oplus F$ where F acts on a finite dimensional space. In computing the distance between the unitary orbits of $A = X^{(\infty)} \oplus F$ and $B = Y^{(\infty)} \oplus G$, one finds using the results of Sections 1 and 2 that $\rho(A, B)$ is a good measure of $d(\mathscr{U}(X^{(\infty)}), \mathscr{U}(Y^{(\infty)}))$ as well as a measure of how well F embeds as a summand of B, and G embeds as a summand of A. So one obtains Theorem 4.3 in this context much more easily with better constants. The problem remains to find some way of measuring an obstruction of the type described in this section.

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