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OUTER AUTOMORPHISM SUBGROUPS OF A COMPACT ABELIAN ERGODIC ACTION

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For a compact abelian ergodic automorphism group G of a simple separable C^*-algebra, the only invariant state is tracial. However, for any (dense) subgroup H of G which is a continuous image of a separable locally compact abelian group and consists of outer automorphisms (except the identity automorphism) there is an *almost invariant* pure state; in other words, there is an irreducible representation in which the automorphism group H is implemented by a unitary group. This is shown by computing the invariant Γ_1 of H, which is introduced in [3]. If furthermore $H \cong \mathbf{R}$, a small perturbation of H by an inner derivation implemented by an analytic element gives an invariant pure state.

1.

Let A be a C^*-algebra, G a locally compact abelian group, and α a continuous action of G on A. In [3] we have defined $\Gamma_1(\alpha)$ to be the set of $p \in \Gamma \equiv \hat{G}$ satisfying the following condition: For any non-zero $x \in A$, any compact neighbourhood U of p and any $\varepsilon > 0$, there is an $a \in A^{\alpha}(U)$ such that $\|a\| = 1$ and

$$(*) \qquad \|xax^*\| \geqslant (1-\varepsilon)\|x\|^2.$$

Subsequently in [4] we have defined $\Gamma_2(\alpha)$ to be the set of $p \in \Gamma$ satisfying the same condition with

$$\|x(a+a^*)x^*\| \geqslant 2(1-\varepsilon)\|x\|^2$$

in place of $(*)$. (Clearly $\Gamma_2(\alpha) \subset \Gamma_1(\alpha)$ and it is likely that Γ_2 coincides with Γ_1.)

If A is separable and prime, G is separable, and $\Gamma_1(\alpha) = \Gamma$, then it follows that there is an α-covariant faithful irreducible representation. (See [3], [4] for other related results.)

If A is simple and unital, and (A, G, α) is asymptotically abelian in the sense that there is an automorphism σ of A such that

$$\sigma \circ \alpha_t = \alpha_t \circ \sigma, \quad t \in G$$

and

$$\|[x, \sigma^n(y)]\| \to 0 \quad \text{as } n \to \infty,\ x, y \in A,$$

it easily follows that $\Gamma_1(\alpha) = \Gamma_2(\alpha)$ is equal to the Connes spectrum $\Gamma(\alpha)$. In this section we shall compute Γ_1 (and Γ_2) for certain C^*-dynamical systems which are not asymptotically abelian.

THEOREM 1.1. *Let G be a compact abelian group and H a locally compact abelian group. Let φ be an isomorphism of $\hat{G}$ into G and ψ a continuous isomorphism of H into G such that* $\mathrm{Range}\,\varphi$ *is dense in G and* $\mathrm{Range}\,{}^t\varphi \cap \mathrm{Range}\,\psi = \{1\}$. *Then it follows that for any open neighbourhood U of $1 \in G$ and any non-empty open subset V of $\hat{H}$, there exists an $a \in \hat{G}$ such that $\varphi(a) \in U$ and ${}^t\psi(a) \in V$.*

Proof. Note that ${}^t\psi$ is the homomorphism of $\hat{G}$ into $\hat{H}$ defined by the equality:

$$\langle \psi(t), a \rangle = \langle t, {}^t\psi(a) \rangle, \quad t \in H,\ a \in \hat{G}.$$

It follows that $\mathrm{Range}\,{}^t\psi$ is dense in $\hat{H}$. Because, otherwise, there is an $s \in H \setminus \{1\}$ such that $\langle s, {}^t\psi(a) \rangle = 1$ for all $a \in \hat{G}$, which implies that $\psi(s) = 1$. Since ψ is an injection, one obtains that $s = 1$, which is a contradiction.

Let L be the closure of

$$\{(\varphi(a), {}^t\psi(a)) \in G \times \hat{H} \mid a \in \hat{G}\}.$$

Note that L is a closed subgroup of $G \times \hat{H}$. Define

$$S = \{p \in \hat{H} \mid (1, p) \in L\}.$$

Then S is a closed subgroup of $\hat{H}$ and it suffices to show that $S = \hat{H}$. Let $H_1 = H \cap S^\perp$, i.e.,

$$H_1 = \{t \in H \mid \langle t, p \rangle = 1,\ p \in S\}.$$

Then H_1 is a closed subgroup of H. We replace H by H_1, ψ by $\psi_1 = \psi \mid H_1$, noting that the conditions for (G, H, φ, ψ) are satisfied for $(G, H_1, \varphi, \psi_1)$. Since ${}^t\psi_1$ is the composition of ${}^t\psi$ followed by the quotient of $\hat{H}$ onto $\hat{H}_1 \cong \hat{H}/S$, the closure L_1 of

$$\{(\varphi(a), {}^t\psi_1(a)) \in G \times \hat{H}_1 \mid a \in \hat{G}\}$$

satisfies that

$$S_1 \equiv \{p \in \hat{H}_1 \mid (1, p) \in L_1\} = \{1\}.$$

Since Range φ is dense in G, there is a continuous homomorphism ν of G into $\hat{H}_1$ such that

$$L_1 = \{(t, \nu(t)) \mid t \in G\}.$$

In particular, it follows that

$$\nu \circ \varphi(a) = {}^t\psi_1(a), \quad a \in \hat{G}.$$

Then for $s \in H_1$,

$$\langle s, \nu \circ \varphi(a) \rangle = \langle \psi_1(s), a \rangle,$$

which implies that ${}^t\varphi \circ {}^t\nu(s) = \psi(s)$, $s \in H_1$. By the assumption that Range ${}^t\varphi \cap$ $\cap$ Range $\psi = \{1\}$, it follows that $\psi_1(s) = 1$, $s \in H_1$, i.e., H_1 is trivial. This implies that $S = \hat{H}$. Q.E.D.

Let G be a compact abelian group and let α be an ergodic action of G on a simple C^*-algebra A such that α is faithful. For each $p \in \hat{G}$ let u_p be a unitary of A such that $\alpha_t(u_p) = \langle t, p \rangle u_p$, $t \in G$. Since Ad u_p maps each eigenspace of A under α into itself, there is a $\varphi(p) \in G$ such that $\alpha_{\varphi(p)} = \mathrm{Ad}\, u_p$ (cf. [1]). The map φ of $\hat{G}$ into G is a homomorphism. Since A is simple, in fact φ is an isomorphism. Note that $u_p u_q u_p^* = \langle \varphi(p), q \rangle u_q$, $p, q \in \hat{G}$. This implies that

$$u_q u_p^* u_q^* = \langle \varphi(p), q \rangle u_p^*,$$

or

$$u_q u_p u_q^* = \langle \varphi(p^{-1}), q \rangle u_p, \quad p, q \in G.$$

Hence one obtains that

$$\langle \varphi(q), p \rangle = \langle \varphi(p^{-1}), q \rangle, \quad \text{i.e.,} \quad {}^t\varphi(p) = \varphi(p^{-1}).$$

In particular, since ${}^t\varphi$ is injective, Range φ is dense.

COROLLARY 1.2. *Let (A, G, α) be as above. Let H be a locally compact abelian group and let ψ be a continuous isomorphism of H into G such that* Range $\varphi \cap$ Range $\psi = \{1\}$ *(or equivalently, $\alpha_{\psi(t)}$ is outer for any $t \in H \setminus \{1\}$). Denote by γ the action of H on A defined by $\gamma_t = \alpha_{\varphi(t)}$. Then $\Gamma_1(\gamma) = \Gamma_2(\gamma) = \hat{H}$.*

Proof. Let $p \in \hat{H}$ and let $\{U_n\}$ (resp. $\{V_n\}$) be a decreasing basis for the open neighbourhoods of $1 \in G$ (resp. $p \in \hat{H}$). Then by the previous theorem, there is an $a_n \in \hat{G}$ such that $\varphi(a_n) \in U_n$, ${}^t\psi(a_n) \in V_n$ for each n. Let u_n be a unitary in the eigenspace of a_n with respect to the action α. Then, since $\varphi(a_n) \to 1$ in G, $\{u_n\}$ is a central net in A and so for any $x \in A$,

$$\|x(u_n + u_n^*)x^*\| = \|xx^*(u_n + u_n^*) + x[u_n + u_n^*, x^*]\| \to 2\|xx^*\|$$

where $\|u_n + u_n^*\| = \|u_n^2 + 1\| = 2$ as $\mathrm{Sp}(u_n) = \mathbf{T}$. Since $\mathrm{Sp}_\gamma(u_n) = \{{}^t\psi(a_n)\} \subset V_n$, this implies that $\Gamma_2(\gamma) \ni p$. Q.E.D.

REMARK 1.3. In the above corollary, if Range ψ is dense in G, then (A, H, γ) is not asymptotically abelian. Because if σ is an automorphism of A which commutes with γ_t, $t \in H$, and so with α_t, $t \in G$, then it easily follows that there exists an $s \in G$ such that $\sigma = \alpha_s$. Since G is compact, α_s cannot satisfy the property that $[\alpha_s^n(x), y] \to 0$ as $n \to \infty$.

2.

One has the following version of Weyl's theorem with an irreducibly acting C^*-algebra in place of the compact operators.

THEOREM 2.1. *Let A be a C^*-algebra acting irreducibly on a separable Hilbert space $\mathscr{H}$, and let H be a self-adjoint operator on $\mathscr{H}$. Then for any $\varepsilon > 0$ there is a self-adjoint element h of A such that $\|h\| \leqslant \varepsilon$ and $H - h$ is diagonal (or has a pure point spectrum).*

This is shown in the same way as Weyl's theorem is in ([2], X.2) if we use Kadison's transitivity theorem ([6], 1.21.16) for what is trivial in the case of compact operators.

Instead of the above theorem we shall show:

THEOREM 2.2. *Let A be a C^*-algebra acting irreducibly on a separable Hilbert space $\mathscr{H}$, and let H be a self-adjoint operator on $\mathscr{H}$. Suppose that $\mathrm{Ad}\, e^{itH} \mid A$ defines a strongly continuous one parameter automorphism group α of A with infinitesimal generator δ. Then for any $t > 0$, there exists a self-adjoint element h of A such that $H - h$ is diagonal, h is analytic for δ, and*

$$\sum_{k=1}^{\infty} \frac{1}{k!} \|\delta^k(h)\| t^k < 1.$$

Proof. Denote by E the spectral measure of H. Let $\{\eta_n\}$ be a dense sequence in $\mathscr{H}$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers.

Let $\xi_1 = \eta_1$ and let F_1 be a finite family of mutually disjoint translates of $[-\varepsilon_1/6, \varepsilon_1/6)$ such that $\xi_1(I) \equiv E(I)\xi_1 \neq 0$ for $I \in F_1$, and

$$\|\xi_1\|^2 - \sum_{I \in F_1} \|\xi_1(I)\|^2 < 1.$$

Denote by $m(I)$ the middle point of $I \in F_1$ and let

$$K_1 = \sum_{I \in F_1} (H - m(I))\, E(I).$$

Then K_1 is self-adjoint and $\|K_1\| \leqslant \varepsilon_1/6$. Denote by e_1 the projection onto the subspace spanned by $\xi_1(I)$, $I \in F_1$. By the transitivity theorem there exists an $h_1 \in A$ such that $h_1 = h_1^*$, $\|h_1\| < 2\|K_1\| \leqslant \varepsilon_1/3$, and $h_1 e_1 = K_1 e_1$. Thus,

$$(*) \qquad (H - h_1)\xi_1(I) = m(I)\xi_1(I), \quad I \in F_1.$$

That is, e_1 is a finite-dimensional projection commuting with $H - h_1$, and satisfies that $\|(1 - e_1)\xi_1\|^2 < 1$.

Denote by $\bar{\alpha}$ the one parameter automorphism group of $B(\mathscr{H})$ defined by $\bar{\alpha}_t = \mathrm{Ad}\, e^{itH}$, and by $\mathrm{Sp}_{\bar{\alpha}}(x)$ the $\bar{\alpha}$-spectrum of $x \in B(\mathscr{H})$ (cf. [5]). Since $\mathrm{Sp}_{\bar{\alpha}}(e_1) \subset$ $\subset [-\varepsilon_1/3, \varepsilon_1/3]$, and $\mathrm{Sp}_{\bar{\alpha}}(K_1) = \{0\}$, we may replace h_1 by

$$\varepsilon_1 \int \alpha_t(h_1) f(\varepsilon_1 t)\, \mathrm{d}t$$

where f is a real-valued integrable function on $\mathbf{R}$ such that $\hat{f}(p) = 0$ if $|p| \geqslant 1$, and $\hat{f}(p) = 1$ if $|p| \leqslant 2/3$. Thus we may suppose that $h_1 = h_1^*$ satisfies:

$$\|h_1\| \leqslant C\varepsilon_1, \quad \mathrm{Sp}_\alpha(h_1) \subset [-\varepsilon_1, \varepsilon_1]$$

in addition to $(*)$, where $C = \|f\|_1/3$.

We apply this procedure for $H_1 = H - h_1$, $\xi_2 = (1 - e_1)\eta_2$, and ε_2 in place of H, ξ_1, and ε_1. Thus, let F_2 be a finite family of mutually disjoint translates of $[-\varepsilon_2/6, \varepsilon_2/6)$ such that $\xi_2(I) \equiv E_1(I)\xi_2 \neq 0$ for $I \in F_2$, and

$$\|\xi_2\|^2 - \sum_{I \in F_1} \|\xi_2(I)\|^2 < 1$$

(where E_1 is the spectral measure for H_1). Let

$$K_2 = \sum_{I \in F_2} (H_1 - m(I))E_1(I).$$

Then K_2 is self-adjoint and $\|K_2\| \leqslant \varepsilon_2/6$. Denote by e_2 the projection onto the subspace spanned by $\xi_2(I)$, $I \in F_2$. Then, since $e_1\xi_2(I) = 0$ for $I \in F_2$ and $e_2\xi_2 =$ $= \sum_{I \in F_2} \xi_2(I)$, it follows that $e_1 e_2 = 0$ and $\|(1 - e_2 - e_2)\eta_2\| < 1$. One finds a self-adjoint $h_2 \in A$ such that $\|h_2\| < 2\|K_2\| \leqslant \varepsilon_2/3$ and $h_2(e_1 + e_2) = K_2(e_1 + e_2) = K_2 e_2$. Thus $h_2 e_1 = 0$ and

$$h_2\xi_2(I) = (H - h_1 - m(I))\xi_2(I), \quad I \in F_2.$$

Define a one-parameter automorphism group $\alpha^{(1)}$ of A by

$$\alpha_t^{(1)}(x) = e^{itH_1} x e^{-itH_1}, \quad x \in A, \ t \in \mathbf{R},$$

which is an inner perturbation of $\alpha^{(0)} = \alpha$. As above we may then assume that

$$\|h_2\| \leqslant C\varepsilon_2, \quad \mathrm{Sp}_{\alpha^{(1)}}(h_2) \subset [-\varepsilon_2, \varepsilon_2],$$

in addition to the property that $e_1 + e_2$ commutes with $H - h_1 - h_2$ and $h_2 e_1 = 0$.

We repeat this procedure. Thus we obtain a sequence $\{e_n\}$ of finite-dimensional projections and a sequence $\{h_n\}$ of self-adjoint elements of A such that $\|h_n\| \leqslant C\varepsilon_n$, $\|(1 - e_1 - e_2 - \ldots - e_n)\eta_n\| < 1$, $e_m e_n = 0$ for $m \neq n$, $h_m e_n = 0$ for $m > n$, $\mathrm{Sp}_{\alpha^{(n-1)}}(h_n) \subset [-\varepsilon_n, \varepsilon_n]$, and $e_1 + \ldots + e_n$ commutes with $H_n = H - h_1 - h_2 - \ldots - h_n$, where $\alpha_t^{(n)} = \mathrm{Ad}\, e^{itH_n} \,|\, A$, $t \in \mathbf{R}$.

Let $\varepsilon > 0$ and let $\varepsilon_n = \varepsilon 2^{-n}$. Then $h = \sum_{n=1}^{\infty} h_n$ converges in norm and is a self-adjoint element of A with $\|h\| \leqslant C\varepsilon$. It follows that $H - h$ commutes with $E_n \equiv e_1 + e_2 + \ldots + e_n$ for $n = 1, 2, \ldots$. Since E_n is finite-dimensional, $H - h$ is diagonal on $E = \lim E_n$.

Since $\|(1 - E_n)\eta_n\| < 1$, it follows that $\|(1 - E)\eta_n\| < 1$ for the dense sequence $\{\eta_n\}$. Thus $E = 1$.

For each $n = 1, 2, \ldots$, define δ_n to be the infinitesimal generator of $\alpha^{(n)}$, i.e., $\delta_n = \delta - i[h_1 + \ldots + h_n, \cdot]$ with $D(\delta_n) = D(\delta)$, and let $\delta_0 = \delta$. There exists a constant $C_1 > 0$ such that for any $a \in A$ with $\mathrm{Sp}_{\alpha^{(n)}}(a) \subset [-p, p]$,

$$\|\delta_n(a)\| \leqslant C_1 p \|a\|.$$

By replacing C and C_1 by $\max(C, C_1)$, we use the same symbol C for C_1 in the above inequality and for $\|h_n\| \leqslant C\varepsilon_n$.

Now we shall show that h satisfies the additional properties.

LEMMA 2.3. *For any $n = 1, 2, \ldots$, any $a \in A$ with $\mathrm{Sp}_{\alpha^{(n-1)}}(a) \subset [-\varepsilon_n, \varepsilon_n]$, and any $k = 1, 2, \ldots$ it follows that*

$$(*) \qquad \|\delta^k(a)\| \leqslant k!(2C\varepsilon)^k \|a\|.$$

Proof. We prove this by induction on k. If $k = 1$, then

$$(**) \qquad \delta(a) = \delta_{n-1}(a) + i[l_n, a]$$

where $l_n = h_1 + h_2 + \ldots + h_{n-1}$, and hence

$$\|\delta(a)\| \leqslant \|\delta_{n-1}(a)\| + 2\|a\|\,\|l_n\|.$$

Since $\|\delta_{n-1}(a)\| \leqslant C\varepsilon_n \|a_n\| = C\varepsilon 2^{-n}\|a\|$ and $\|l_n\| \leqslant \sum_{i=1}^{n-1} \|h_i\| \leqslant C\varepsilon(1-2^{-n})$, one obtains that

$$\|\delta(a)\| \leqslant C\varepsilon 2^{-n}\|a\| + 2C\varepsilon(1-2^{-n})\|a\| \leqslant 2C\varepsilon\|a\|.$$

Suppose that (*) is true for $k = 1, 2, \ldots, m$. Then it follows that for $k = 1, 2, \ldots, m$,

$$\|\delta^k(l_n)\| \leqslant \sum_{i=1}^{n-1} \|\delta^k(h_i)\| \leqslant \sum_{i=1}^{n-1} k!(2C\varepsilon)^k\|h_i\| < k!(2C\varepsilon)^k C\varepsilon(1-2^{-n})$$

where we have used that $\|h_i\| \leqslant C\varepsilon 2^{-i}$. From (**) it follows that

$$\delta^{m+1}(a) = \delta^m(\delta_{n-1}(a)) + \mathrm{i}\sum_{k=0}^{m} \frac{m!}{k!(m-k)!}[\delta^k(l_n), \delta^{m-k}(a)].$$

Then as $\mathrm{Sp}_{\alpha^{(n-1)}}(\delta_{n-1}(a)) \subset [-\varepsilon_n, \varepsilon_n]$, it follows that

$$\|\delta^{m+1}(a)\| \leqslant m!(2C\varepsilon)^m\|\delta_{n-1}(a)\| + 2m!\sum_{k=0}^{m}(2C\varepsilon)^k C\varepsilon(1-2^{-n})(2C\varepsilon)^{m-k}\|a\| \leqslant$$

$$\leqslant m!(2C\varepsilon)^m C\varepsilon 2^{-n}\|a\| + 2(m+1)!(2C\varepsilon)^m C\varepsilon(1-2^{-n})\|a\| \leqslant$$

$$\leqslant (m+1)!(2C\varepsilon)^{m+1}\|a\|.$$

This concludes the proof of the lemma.

By the lemma one has that

$$\|\delta^k(h_n)\| \leqslant k!(2C\varepsilon)^{k+1}2^{-n-1}.$$

Hence for any k,

$$\sum_{n=1}^{m} \delta^k(h_n) = \delta^k\left(\sum_{n=1}^{m} h_n\right)$$

converges in norm as $m \to \infty$, i.e., $h \in D(\delta^k)$ and

$$\|\delta^k(h)\| \leqslant k!(2C\varepsilon)^{k+1}.$$

Since $\varepsilon > 0$ is arbitrary and C is independent of ε, this implies that h satisfies the additional properties.

REMARK 2.4. When A is the compact operators, Theorem 2.2 follows easily from Weyl's theorem, by using the fact that the spectral projections of H are multipliers of A. In fact we can even require h to have an arbitrarily small α-spectrum around $0 \in \hat{\mathbf{R}}$ and arbitrarily small norm.

Let A_θ be the irrational rotation algebra generated by unitaries u, v satisfying $uv = e^{2\pi i\theta}vu$ with $\theta \in [0, 1] \cap \mathbf{Q}^c$. Denote by α the action of $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ on A_θ defined by $\alpha_{(s,t)}(u) = e^{2\pi is}u$, $\alpha_{(s,t)}(v) = e^{2\pi it}v$. Let $\mu \in \mathbf{R}$ be such that θ and μ are linearly independent over $\mathbf{Q}$, and let $\gamma_t = \alpha_{(t,\mu t)}$, $t \in \mathbf{R}$. Then, since α is ergodic and γ_t is outer for $t \neq 0$, by Corollary 1.2 there is a γ-covariant irreducible representation of A_θ. Since the closure of $\gamma_{\mathbf{R}}$ is $\alpha_{\mathbf{T}^2}$, there are no γ-invariant pure states. But Theorem 2.2 shows that, by perturbing γ by an inner derivation with arbitrarily small norm, there are infinitely many invariant states (all of which may be equivalent).

COROLLARY 2.5. *Let $(A_\theta, \mathbf{R}, \gamma)$ be as above. There exists a type* II$_1$ *orbit in the pure states of this system.*

Proof. Let $\mu_1 \in \mathbf{R}$ be such that $\mu_1 \neq \mu$ and θ and μ_1 are linearlv independent over $\mathbf{Q}$ and define a one-parameter automorphism group $\gamma^{(1)}$ for μ_1 in the same way as γ is defined for μ. Let f be a pure state of A_θ such that the GNS representation π_θ is $\gamma^{(1)}$-covariant. Then

$$\rho_f = \int_{\mathbf{R}}^{\oplus} \pi_f \circ \gamma_t \, dt$$

is $\gamma \circ \gamma^{(1)}$-covariant, where $\gamma \circ \gamma^{(1)}$ is the action of $\mathbf{R}^2$ defined by $(\gamma \circ \gamma^{(1)})_{(s,t)} = \gamma_s \circ \gamma_t^{(1)}$, $(s, t) \in \mathbf{R}^2$. Since $\mathbf{R}^2 \ni (s, t) \to (s + t, \mu s + \mu_1 t) \in \mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ is a quotient map, it is easy to conclude that ρ_f is α-covariant. Thus, ρ_f is quasi-equivalent to the GNS representation associated with the unique α-invariant tracial state. Q.E.D.

COROLLARY 2.6. *Let A be a simple, unital, separable C^*-algebra and let α be a strongly continuous one parameter automorphism group of A. Suppose that α satisfies that*

$$(*) \qquad \|[\alpha_t(x), y]\| \to 0 \quad \text{as } t \to \infty, \text{ for } x, y \in A.$$

The above property is not stable under arbitrarily small inner perturbations. More precisely, for any $t > 0$ there is a self-adjoint element h of A such that the one-parameter automorphism group $\alpha^{(h)}$ obtained by perturbing α by the inner derivation i$[h, \cdot]$ *does not satisfy $(*)$, h is α-analytic, and*

$$\sum_{k=1}^{\infty} \frac{\|\delta^k(h)\| t^k}{k!} < 1.$$

Proof. Denoting by δ the infinitesimal generator of α, α^h has $\delta - i[h, \cdot]$ as its infinitesimal generator.

It is known ([6], 3.1) that if $(A, \mathbf{R}, \alpha)$ satisfies $(*)$, then it is $\mathbf{R}$-abelian, or the set of α-invariant states forms a Choquet simplex. Since $(A, \mathbf{R}, \alpha)$ has a covariant irreducible representation [4], Theorem 2.2 implies that for some $h \in A$ satisfying the additional properties, $(A, \mathbf{R}, \alpha^{(h)})$ is not $\mathbf{R}$-abelian.

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