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ON BEURLING TYPE INVARIANT SUBSPACES OF $L^2(\mathbf{T}^2)$ AND THEIR EQUIVALENCE

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In [2], Beurling characterized the invariant subspaces of the Hardy space $H^2(\mathbf{T})$ ($\mathbf{T}$ the circle). In [3], Beurling theorem is extended to obtain the invariant subspaces of $L^2(\mathbf{T})$. The most general technique indicated is the Wold-type decomposition. In [11] and [6] an extension of the Wold-type decomposition is given for two doubly commuting isometries. Our purpose in this note is to use this decomposition to characterize invariant subspaces of $L^2(\mathbf{T}^2)$ on which V_i $(i = 1,2)$, the multiplication operator by t_i $(i = 1, 2)$, are doubly commuting where $\mathbf{t} = (t_1, t_2) \in \mathbf{T}^2$.

In [7], the invariant subspaces M of $H^2(\mathbf{T}^2)$ of the Beurling form were characterized as the subspaces on which multiplication operators V_i $(i = 1, 2)$ are doubly commuting. Motivated from this we call these subspaces of $L^2(\mathbf{T}^2)$ of Beurling type. It is observed in [7] that any invariant subspace N of $H^2(\mathbf{T}^2)$ equivalent to M in the sense of [1] is again of Beurling form. We also examine the analogous results for the subspaces of $L^2(\mathbf{T}^2)$. We characterize in terms of φ the subspace $N = \varphi M$ of Beurling type equivalent to invariant subspace M and satisfying $N \subseteq M$. We relate these to recent work in [5], [7], [9]. The basic techniques used are found in [11], [8], [6].

Let $\mathbf{Z}$ be the set of integers. We denote by $m, n \ldots$ etc the elements of $\mathbf{Z}$. Let U be the open unit disc and $\mathbf{T}$ the boundary of U in the complex plane $\mathbf{C}$. Let $\mathbf{Z}^2$, $\mathbf{C}^2$, U^2 and $\mathbf{T}^2$ be the respective cartesian products and σ_2 the normalized Lebesgue measure on $\mathbf{T}^2$. The characteristic function of a set E is denoted by 1_E for $E \subseteq \mathbf{T}^2$. For $p > 0$, we denote by $L^p(\mathbf{T}^2, \sigma_2)$ (L^p, for short) the Lebesgue space of equivalence classes of p-integrable functionns. We shall be interested in the following subspaces of L^2.

(a) $H^2 = \overline{\operatorname{span}}\{t_1^m,\ t_2^n,\ m, n \geqslant 0\}$

(1) (b) $\mathscr{H}_1 = \overline{\operatorname{span}}\{t_1^m,\ t_2^n,\ m \geqslant 0,\ n \in \mathbf{Z}\}$

(c) $\mathscr{H}_2 = \overline{\operatorname{span}}\{t_1^m,\ t_2^n,\ m \in \mathbf{Z},\ n \geqslant 0\}$.

We say that M is an invariant subspace of L^2 if for each $f \in M$, $t_i f \in M$ for $i = 1, 2$ with $\mathbf{t} = (t_1, t_2) \in \mathbf{T}$. As in [7], we denote by V_i multiplication by t_i.

We note that if M is an invariant subspace, then $V_1 | M$ commutes with $V_2 | M$. We now get the following generalization of the Beurling Theorem in [7].

2. Theorem. *Let M be an invariant subspace of L^2. Then $V_i | M$ doubly commute on M (i.e. $V_1 | M$ commutes with $V_2^* | M$) iff*

$$M = qH^2 + 1_{E_2} q_1 \mathscr{H}_1 + 1_{E_1} q_2 \mathscr{H}_2 + 1_E L^2$$

where q, q_1, q_2 are unimodular functions, $1_E \in L^2$ and $1_{E_i} \in \operatorname{span}\{t_i^k, k \in \mathbf{Z}\}$.

Remark. The subspaces of the above type will be called of Beurling type.

Proof. We note that if the above decomposition is written in the form

$$M = M_0 \oplus M_1 \oplus M_2 \oplus M_\infty$$

then

a) $V_i(M_i) \subsetneq M_i$ and $V_i | M_i$ is a shift for $i = 1, 2$;

b) $V_i(M_0) \subsetneq M_0$ and $V_i | M_0$ is a shift for $i = 1, 2$;

c) $V_i(M_\infty) = M_\infty$, $(i = 1, 2)$ and $V_2(M_1) = M_1$, $V_2(M_2) = M_2$ giving that $V_1 | M_2$, $V_1 | M_\infty$, $V_2 | M_1$, and $V_2 | M_\infty$ are unitary.

Thus V_1, V_2 are doubly commuting on $M_1 \oplus M_2 \oplus M_\infty$. We note that

$$M_0 = \sum_{m, n \geq 0}^{\oplus} V_1^m V_2^n(L)$$

with L being the subspace generated by q. Now using the implication (i) $\Rightarrow$ (ii) of Theorem 1 of [11] we get the sufficiency.

To prove the converse, we use Theorem 4.2 (c) $\Rightarrow$ (b) of [6] to get the Wold type decomposition of M in the form

$$M = \sum_{m, n \geq 0}^{\oplus} V_1^m V_2^n (R_1^\perp \cap R_2^\perp) \oplus \sum_{m \geq 0}^{\oplus} V_1^m \Big(\bigcap_{n \geq 0} V_2^n (R_1^\perp)\Big) \oplus$$

$$\oplus \sum_{n \geq 0}^{\oplus} V_2^n \Big(\bigcap_{m \geq 0} V_1^m (R_2^\perp)\Big) \oplus \bigcap_{m, n \geq 0} V_1^m V_2^n (M).$$

Here $R_i = V_i(M)$ and $R_i^\perp = M \ominus V_i M$ for $(i = 1, 2)$.

Using arguments as in the proof of Theorem 2 in [7], we get $R_1^\perp \cap R_2^\perp$ is one dimensional and is generated by a unimodular function q giving

$$M_0 = \sum_{m, n \geq 0}^{\oplus} V_1^m V_2^n (R_1^\perp \cap R_2^\perp) = qH^2.$$

On $M_1 = \sum_{m\geqslant 0}^{\oplus} V_1^m(\bigcap_n V_2^n(R_1^\perp))$. V_1 is a shift and V_2 is unitary. Using the result of Merill and Lal ([8], p. 472—73) we get $M_1 = 1_{E_2}q_1\mathscr{H}_1$ as the second part in [8] is zero using the fact that V_1 is a shift. By symmetry we get

$$M_2 = \sum_{n\geqslant 0}^{\oplus} V_2^n(\bigcap_m V_1^m(R_1^\perp)) = 1_{E_1}q_2\mathscr{H}_2.$$

To get the form of $M_\infty = \bigcap_{m,n} V_1^m V_2^n(M)$, we use the following analogue of Wiener Theorem. An invariant subspace M of L^2 is called doubly invariant if $V_i(M) = M$ $(i = 1, 2)$.

3. Lemma. *Every doubly invariant subspace of L^2 is of the form $1_E L^2$ for some measurable set E of $\mathbf{T}^2$.*

Proof. Note $1_E L^2$ is clearly a doubly invariant subspace of L^2. To prove the converse we follow ideas in [3]. Let 1 be the indicator of $\mathbf{T}^2$ and g be the projection of 1 onto M. Then $(1-g)\perp M$ giving $\int_{\mathbf{T}^2} t_1^{n_1}t_2^{n_2}g\overline{(1-g)}\,d\sigma_2 = 0$ for all $(n_1, n_2)\in \mathbf{Z}^2$. Hence $g = |g|^2$ a.e. giving g is non-negative and $g = g^2$ a.e. Hence g takes only values 1 or 0. Let

$$E = \{\mathbf{t}\in\mathbf{T}^2 : g(\mathbf{t}) = 1\}.$$

Then $g = 1_E$.

Since $g\in M$ and M is doubly invariant, we get $gL^2 \subseteq M$. Suppose $h\in M$ and $h\perp gL^2$ then $\int \bar{h}gt_1^{n_1}t_2^{n_2}\,d\sigma_2 = 0$ for $(n_1, n_2)\in\mathbf{Z}^2$. Hence $\bar{h}g = 0$ a.e. But $(1-g)\perp t_1^{n_1}t_2^{n_2}h$ for all $(n_1, n_2)\in\mathbf{Z}^2$ giving $\bar{h}(1-g) = 0$, i.e. $\bar{h} = 0$; or equivalently $M = gL^2 = 1_E L^2$.

4. Corollary. *Let $M\neq\{0\}$ be an invariant subspace of L^2. Then $V_1\,|\,M$, $V_2\,|\,M$ are doubly commuting shifts on M iff $M = qH^2$ where q is unimodular.*

5. Corollary. ([7], Theorem 2). *Let $M\neq\{0\}$ be an invariant subspace of H^2. Then $V_i\,|\,M$ $(i = 1, 2)$ are doubly commuting on M iff $M = qH^2$ and q is inner.*

Proof. Since $M\subseteq H^2$ we get $V_i\,|\,M$ are shifts giving $M = qH^2$ (q unimodular). But $qH^2\subseteq H^2$ giving q inner. The converse is obvious.

We now relate our results to some recent results of Nakazi [9]. Let V_1, V_2 be two commuting isometries on a Hilbert space H. Denote by $R_i^\perp = H\ominus V_i(H)$.

Then we have the following proposition. We note that the sufficiency part is already contained in the proof of Theorem 1 of [11] ((ii) $\Rightarrow$ (iii)).

6. PROPOSITION. *$V_2(R_1^\perp) \subseteq R_1^\perp$ iff V_1 and V_2 are doubly commuting on H. In either case, $V_1(R_2^\perp) \subseteq R_2^\perp$.*

Proof. By Halmos decomposition [4], for V_1 on H

$$H = \sum_{m\geq 0}^{\oplus} V_1^m(R_1^\perp) \oplus \bigcap_m V_1^m(H).$$

Let $x \in \bigcap_m V_1^m(H)$; then $x = V_1^m y_m$ for each m giving $V_2 x = V_1^m V_2 y_m = V_1^m z_m$ for each m, i.e. $V_2 x \in \bigcap_m V_1^m(H)$. Now use Halmos decomposition for $V_2 | R_1^\perp$ on $R_1^\perp$ to get

$$R_1^\perp = \sum_{n\geq 0}^{\oplus} V_2^n(R_1^\perp \cap R_2^\perp) \oplus \bigcap_{n=0} V_2^n(R_1^\perp).$$

Thus

$$H = \sum_{m,n\geq 0}^{\oplus} V_1^m V_2^n(R_1^\perp \cap R_2^\perp) \oplus \sum_{m\geq 0}^{\oplus} V_1^m(\bigcap_n V_2^n(R_1^\perp)) \oplus$$

$$\oplus \bigcap_{m\geq 0} V_1^m(H) = H_1 \oplus H_2 \oplus H_3 \qquad \text{(say)}.$$

On H_2, H_3, $V_2 | H_2$, $V_1 | H_3$ are unitary, $V_i(H_k) \subseteq H_k$ for $i = 1, 2$, $k = 1, 2, 3$; thus $V_i | H_2$, $V_i | H_3$ ($i = 1, 2$) doubly commute. Now $V_i | H_1$ doubly commutes with $V_2 | H_1$ by ([11], Theorem 1 i) $\Rightarrow$ ii). The converse is simple as proved in ([6], Theorem 4.1).

REMARK. In view of the above theorem and Theorem 4.1 of [6], we get V_1 reduces $R_2^\perp$ and V_2 reduces $R_1^\perp$ as soon as $V_2(R_1^\perp) \subseteq R_1^\perp$ *or* $V_1(R_2^\perp) \subseteq R_2^\perp$. Using Proposition 6 and Theorem 2, we get

7. COROLLARY. ([9], Theorem 5). *Let M be an invariant subspace of L^2 and $M \ominus V_2 M = S \neq \{0\}$.*

(i) *$V_1(S) = S$ iff $M = 1_{E_1} q_2 \mathscr{H}_2 + 1_{E_2} L^2$ where $1_{E_1} \in \overline{\text{span}}\{t_1^k,\ k \in \mathbf{Z}\}$,*

(ii) *$V_1(S) \subsetneq S$ iff $M = qH^2$ for some unimodular function q.*

One can also derive Theorem 6 of [9] similarly. We now turn our attention to equivalence of subspaces introduced in [1]. We say that an invariant subspace M of L^2 is unitarily equivalent to an invariant subspace N if there is a unitary operator U on M onto N such that $UV_k | MU^{-1} = V_k | N$ ($k = 1, 2$). In view of Theorem 2 and the definition above we get

8. COROLLARY. *A subspace equivalent to a subspace of Beurling type is of Beurling type.*

Now we examine subspaces M and N of Beurling type with only one part of decomposition of the Theorem 2 present and $N \subseteq M$.

9. COROLLARY. *Let M be a subspace of Beurling type of the form $M = qH^2$. If N is an invariant subspace and $N \subseteq M$ then $N = \varphi M$ with $\varphi \in H^2$ if and only if N is equivalent to M.*

Proof. In view of Theorem 1 from [1], we have $N = \varphi M$, φ unimodular. But $N = \varphi q H^2 \subseteq qH^2$ implies $\varphi H^2 \subseteq H^2$ giving $\varphi \in H^2$. The converse is obvious.

REMARK. In view of Theorem 2 of [7] and the example of Rudin [10], we note that unless $N \subseteq M$, the necessity is false.

10. COROLLARY. *Let M be a subspace of Beurling type of the form $M = 1_{E_2} q_1 \mathscr{H}_1$ with q_1, 1_{E_2} as in Theorem 2. Then N is equivalent to M with $N \subseteq M$ iff $\varphi = \psi$ on the set E_2 where $\psi \in \mathscr{H}_1$.*

Proof. Again the sufficiency is obvious. To prove the converse we know that $N = \varphi 1_{E_2} q_1 \mathscr{H}_1 \subseteq 1_{E_2} q_1 \mathscr{H}_1$ gives $1_{E_2}\varphi \in \mathscr{H}_1$. Choose $\psi = 1_{E_2}\varphi$ to get the result.

Note that a similar result holds for the case $1_{E_1} q_2 \mathscr{H}_2$. Clearly for the space of the form $M = 1_E L^2$, φM is of the form $1_E \varphi L^2$. And $1_E \varphi L^2 \subseteq 1_E L^2$ so in this case no condition is needed on φ.

In [5], conditions are given for φ to belong to H^2; we now rederive these results from Corollary 10 in case φ is unimodular and $n = 2$. Given a subspace M of H^2, define the subspace

$$(M)_k = \overline{\operatorname{span}}\{L_k^\infty M\} \quad \text{in } L^2$$

following [5]. Here L_k^∞ is the weak*-closure of the algebra generated by $\{1, t_i, \bar{t}_i, i = 1, 2\} \setminus \{t_k, \bar{t}_k\}$. Then $V_1 | (M)_1$, $V_2 | (M)_1$ are doubly commuting with $V_1 | (M)_1$ a shift and $V_2 | (M)_1$ unitary. Thus we get $(M)_1 = 1_{E_1} q \mathscr{H}_1$. In addition $(M)_1$ contains $M \neq \{0\}$ and it contains a function in H^2 giving $(M)_1 = q_1 \mathscr{H}_2$. Similarly $(M)_2 = q_2 \mathscr{H}_1$. Using Corollary 10 and the remark following it, we get the following as $\varphi \in \mathscr{H}_1 \cap \mathscr{H}_2$ implies $\varphi \in H^2$.

11. COROLLARY. *Let M, N be invariant subspaces of H^2 with N equivalent to M. Then $N = \varphi M$ and $\varphi \in H^2$ iff $(N)_k \subseteq (M)_k$ for $k = 1, 2$.*

The idea of the above proof is due to Izuchi [5]. However, we derive it as a consequence of our structure of Beurling type spaces.

In [1], two sufficient conditions were given for the validity of the sufficiency condition in Corollary 11 and they are shown not to be necessary by Izuchi [5] for $n \geqslant 3$. We show that full range condition on M ([1]) is equivalent to $(M)_2 = \mathscr{H}_1$ and $(M)_1 = \mathscr{H}_2$ in case $n = 2$. Consider $(M)_2 = \overline{\text{span}}\{fh,\ h \in M,\ f \in L_1^\infty\}$. Clearly $(M)_2 = \overline{\left\{\sum_{m,n\geqslant 0}^{\text{finite}} a_{mn} t_1^n \bar{t}_1^{m} h_{m,n},\ h_{m,n} \in M\right\}}$ where M is an invariant subspace of $H^2(\mathrm{T}^2)$ and $\overline{}$ denotes closure in $L^2(\mathrm{T}^2)$.

LEMMA. $(M)_2 = \overline{\text{span}}\{\bigcup_{n\geqslant 0} \bar{t}_1^n M\}$.

Proof. Note that $(M)_2 \supseteq \overline{\text{span}}\{\bigcup_{n\geqslant 0} \bar{t}_1^n M\}$. Let $f \in (M)_2$ orthogonal to $\bigcup_{n\geqslant 0} \bar{t}_1^n M$. Then $t_1^n f \perp M$. Given $\varepsilon > 0$, choose $g_\varepsilon = \sum_{m=0}^{k} \sum_{n=0}^{l} a_{m,n} t_1^n \bar{t}_1^m h_{m,n}$ such that $\|g_\varepsilon - f\|_{L_2} < \varepsilon/2$ with l, k finite (possibly depending on ε). Since V_1^j is an isometry we get $\|V_1^j g_\varepsilon - V_1^j f\| < \varepsilon$ for all j. Choose now j_ε such that for $j \geqslant j_\varepsilon$, $V_1^j g_\varepsilon \in M$.

Consider

$$\|V_1^j f\|_{L^2} \leqslant \|V_1^j f - V_1^j g_\varepsilon\|_{L^2} + \|V_1^j g_\varepsilon\|_{L^2} \leqslant \frac{\varepsilon}{2} + \|V_1^j g_\varepsilon\|.$$

But

$$\|V_1^j g_\varepsilon\|_{L^2} = \sup_{\substack{\|h\|\leqslant 1\\ h\in M}} |\langle V^j g_\varepsilon, h\rangle|$$

which does not exceed

$$\sup_{\substack{\|h\|\leqslant 1\\ h\in M}} \{|\langle V_1^j g_\varepsilon - V_1^j f, h\rangle| + |\langle V_1^j f, h\rangle|\}$$

which is $\leqslant \varepsilon/2$. Thus for any $\varepsilon > 0$, and $j \geqslant j_\varepsilon$, $\|f\|_{L_2} = \|V_1^j f\|_{L_2} \leqslant \varepsilon$ giving $f = 0$. Using the above lemma and Lemma 2 of [5], we get

13. THEOREM. *$(M)_2 = \mathscr{H}_1$ and $(M)_1 = \mathscr{H}_2$ iff M is full range.*

14. REMARK. The example given in [5] of an invariant subspace M of $H^2(\mathrm{T}^3)$ which does not contain a function independent of z_3 can be modified easily to show that Condition (ii) of Lemma 1 in [5] is not necessary even for $n = 2$.

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