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SHIFTS ON THE HYPERFINITE FACTOR OF TYPE II_1

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0. INTRODUCTION

Following Powers, we call a shift on the hyperfinite II_1-factor R a unit-preserving $*$-endomorphism σ of R such that $\bigcap_{k=1}^{\infty} \sigma^k(R) = \mathbf{C}$. We introduce a class of shifts, which we call group shifts, constructed by realizing R as a twisted group von Neumann algebra on a discrete abelian group. We obtain an intrinsic characterization of such shifts, and, for those satisfying $\sigma(R)' \cap R = \mathbf{C}$, a classification up to conjugacy. We then examine in detail the special case of the group $\bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ with the canonical shift, thereby unifying and generalizing results of R. Powers, G. Price and M. Choda.

In Section 1 we give the details of the construction of a group shift given a discrete abelian group G, a shift s on G, and an s-invariant 2-cocycle ω on G. In Section 2 we give first an intrinsic characterization of group shifts (Proposition 2.1); secondly, when $\sigma(R)' \cap R = \mathbf{C}$, we determine the normalizer of σ (Proposition 2.2); finally we classify such σ (up to conjugacy) in terms of G, s, ω (Proposition 2.5).

In Sections 3 and 4 we study n-shifts: group shifts with $G = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ and s the canonical shift on G. For $n = 2$ these are the binary shifts of [5] and [6]; for general n these include the n-unitary shifts of [1]. In Section 3 we obtain a necessary and sufficient condition for $W^*(G, \omega)$, the twisted group von Neumann algebra determined by the group G and cocycle ω, to be a factor (Proposition 3.1). This result was proved for $n = 2$ in [6]. And in [7], a preprint which we received during the final preparations of this paper, G. Price proves a result equivalent to our Proposition 3.1 by somewhat different methods. In Section 4 we obtain an intrinsic characterization for n-shifts (Proposition 4.1) and we find an explicit conjugacy invariant when n is square-free (Proposition 4.4). In Section 5 we conclude with some simple examples of shifts with integer index which are not group shifts, and of group shifts of finite index n which are not n-shifts.

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1. GROUP SHIFTS

In this section we construct certain shifts of the hyperfinite II_1-factor R by realizing R as a twisted group von Neumann algebra. First let us recall some definitions.

DEFINITION 1.1. ([5]). A *shift* σ of a unital C^*-algebra A is a $*$-endomorphism of A such that $\sigma(1) = 1$ and $\bigcap_{k=1}^{\infty} \sigma^k(A) = \mathbf{C}$.

DEFINITION 1.2. A *shift* s of a group G is an one-to-one endomorphism of G such that $\bigcap_{k=1}^{\infty} s^k(G) = \{e\}$.

In the following let G be a discrete abelian group, ω a normalized 2-cocycle of G with values in the unit circle $\mathbf{T}$ and s a shift of G. Assume s is compatible with ω, that is $\omega(s(g), s(h)) = \omega(g, h)$, for $g, h \in G$. The (reduced) twisted group C^*-algebra of G, $C^*(G, \omega)$, is the C^*-algebra generated by the left regular projective representation of G, $g \to U_g$, associated with ω on $\ell^2(G)$. These unitaries U_g satisfy the relation

$$U_g U_h = \omega(g, h) U_{gh}, \quad g, h \in G. \tag{1.1}$$

The weak closure of $C^*(G, \omega)$ is the (reduced) twisted group von Neumann algebra of G, $W^*(G, \omega)$. The shift s of G induces a $*$-endomorphism σ of $W^*(G, \omega)$, as well as of $C^*(G, \omega)$, by $\sigma(U_g) = U_{s(g)}$, $g \in G$. If H is a subgroup of G, $W^*(H, \omega|H)$ can be identified in a natural way with the von Neumann subalgebra of $W^*(G, \omega)$ generated by $\{U_g : g \in H\}$.

PROPOSITION 1.1. *σ is a shift of $W^*(G, \omega)$ as well as of $C^*(G, \omega)$.*

Proof. It is obvious that $\sigma^k(W^*(G, \omega)) = W^*(s^k(G), \omega)$. If $\{H_i\}$, $i \in I$, is a family of subgroups of G, then $\bigcap_i W^*(H_i, \omega) = W^*(\bigcap_i H_i, \omega)$ which follows from $\bigcap_i \ell^2(H_i) = \ell^2(\bigcap_i H_i)$. Hence $\bigcap_{k=0}^{\infty} \sigma^k(W^*(G, \omega)) = W^*\left(\bigcap_{k=0}^{\infty} s^k(G), \omega\right) = \mathbf{C}$. Q.E.D.

The 2-cocycle ω of G gives rise to a character ρ of the second exterior product $G \wedge G$ via

$$\rho(g \wedge h) = \omega(g, h)\overline{\omega(h, g)}, \quad g, h \in G. \tag{1.2}$$

PROPOSITION 1.2. *If H is a subgroup of G, then*

$$W^*(H, \omega)' \cap W^*(G, \omega) = W^*(D_H, \omega),$$

where D_H is the subgroup $\{g \in G : \rho(g \wedge H) = 1\}$ of G.

Proof. From (1.1), $U_g U_h = \rho(g \wedge h) U_h U_g$. Hence $W^*(D_H, \omega)$ is the relative commutant of $W^*(H, \omega)$. For the reverse inclusion, let $T \in W^*(H, \omega)' \cap W^*(G, \omega)$ and let $\{f_g : g \in G\}$ be the canonical orthonormal basis of $\ell^2(G)$, so that $U_g(f_h) = \omega(g, h) f_{gh}$. Assume $Tf_e = \sum_{g \in G} c_g f_g$, where e is the identity of G, $c_g \in \mathbf{C}$, $\sum_{g \in G} |c_g|^2 < \infty$. For any $h \in H$, we have

$$(U_h T) f_e = \sum_g c_g \omega(h, g) f_{hg},$$

$$(TU_h) f_e = T(R_{h^{-1}} f_e) = R_{h^{-1}}(Tf_e) = \sum_g c_g \omega(g, h) f_{gh},$$

where $k \to R_k$ is the right regular ω-representation of G: $R_k(f_g) = \omega(g, k^{-1}) f_{gk^{-1}}$ with R_k commuting with the U_g. Since $U_h T = TU_h$, we get $\sum_g c_g \omega(h, g) f_{hg} = \sum_g c_g \omega(g, h) f_{gh}$. Therefore $c_g \omega(g, h) = c_g \omega(h, g)$, $g \in G$, $h \in H$. If $c_g \neq 0$, then $\rho(g \wedge h) = 1$. This shows that T is supported on D_H. It follows that $T \in W^*(D_H, \omega)$. Q.E.D.

COROLLARY 1.3. (i) *$W^*(G, \omega)$ is a factor if and only if $\rho(g \wedge G) = 1$ implies $g = e$.*

(ii) *$(\sigma^k(W^*(G, \omega)))' \cap W^*(G, \omega) = \mathbf{C}$ if and only if $\rho(g \wedge s^k(G)) = 1$ implies $g = e$.*

When ω (resp. ρ) satisfies the condition of Corollary 1.3(i), we say it is *nondegenerate.*

PROPOSITION 1.4. *Suppose that G is a countable discrete abelian group, that ω is a nondegenerate 2-cocycle of G and that s is a shift of G compatible with ω. Let σ be the shift of $W^*(G, \omega)$ induced by s. Then*

(i) *$W^*(G, \omega) = R$, the hyperfinite II_1-factor;*

(ii) *The Jones index $[R: \sigma(R)] = [G: s(G)]$;*

(iii) *$\sigma(R)' \cap R = \mathbf{C}$ provided $[G: s(G)]$ is a prime number.*

Proof. (i) The nondegeneracy of ω implies that $W^*(G, \omega)$ is a (finite) factor and that s is one-to-one on G. Then $\bigcap_k s^k(G) = \{e\}$ implies that G is an infinite group. Hence $W^*(G, \omega)$ is a II_1-factor. The dual group of G, $\hat{G}$, acts on $W^*(G, \omega)$ via $\theta(U_g) = \theta(g) U_g$, $\theta \in \hat{G}$, $g \in G$. It is a standard result that this action is ergodic. Thus by [4; 5.15], $W^*(G, \omega) = R$, the unique hyperfinite II_1-factor.

(ii) This follows from [2; 2.32]. In fact, we can replace the crossed products there by twisted crossed products and the proof still works.

(iii) By Corollary 1.3 (ii), it is enough to show that $\rho(g \wedge s(G)) = 1$ implies $g = e$. Assume the contrary. Then the subgroup $E = \{g \in G : \rho(g \wedge s(G)) = 1\}$ is nontrivial. We have $E \cap s(G) = \{e\}$. For, if $s(g) \in E$, then $\rho(g \wedge G) = \rho(s(g) \wedge \wedge s(G)) = 1$ by the compatibility. Since ω is nondegenerate, we get $g = e$ and so $s(g) = e$. Therefore the restriction of the quotient map $\pi: G \to G/s(G)$ to E is one-to-one. Since $G/s(G)$ has order $[G: s(G)]$ a prime number, it follows that E is a cyclic group of order the same prime, and that $G = E \oplus s(G)$. Let g be a generator of E: then $\rho(g \wedge G) = \rho(g \wedge E)\, \rho(g \wedge s(G)) = \rho(g \wedge E) = 1$. The nondegeneracy of ω implies that $g = e$. A contradiction. Q.E.D.

The shift σ of R constructed in Proposition 1.4 will be called a group shift and denoted by $\sigma(G, s, \omega)$ when there is need to indicate the data G, s, ω. Note that for a group shift $\sigma(G, s, \omega)$, ω is always nondegenerate by assumption.

We record the following result for future reference.

PROPOSITION 1.5. *The following are equivalent:*

(i) *$C^*(G, \omega)$ is simple;*

(ii) *$C^*(G, \omega)$ has trivial centre;*

(iii) *$C^*(G, \omega)$ has unique tracial state;*

(iv) *ω is nondegenerate;*

(v) *$W^*(G, \omega)$ is a factor.*

This proposition is essentially proved in [8]. The proof given in [5] can be viewed as an alternative proof in the case $G = \overset{\infty}{\oplus} \mathbf{Z}_2$.

Finally we remark that some results in this section are also true for non-abelian groups.

2. CHARACTERIZATION AND CLASSIFICATION OF GROUP SHIFTS

In this section we first determine when a given shift σ of R is conjugate to a group shift $\sigma(G, s, \omega)$ as constructed in Section 1. Secondly, for those group shifts σ of R satisfying $\sigma(R)' \cap R = \mathbf{C}$, we calculate the normalizer of σ and its conjugacy class in terms of (G, s, ω).

PROPOSITION 2.1. *A shift σ of R is conjugate to a group shift $\sigma(G, s, \omega)$ if and only if there exists a set S of unitaries of R such that*

(i) *$\{S, \sigma(S), \sigma^2(S), \ldots\}'' = R$ and*

(ii) *$uvu^*v^* \in \mathbf{C}$ for all u, v in $\{S, \sigma(S), \sigma^2(S), \ldots\}$.*

Denote by $G_\sigma(S)$ the group of unitaries generated by $\{S, \sigma(S), \sigma^2(S), \ldots\}$, and let $\pi: G_\sigma(S) \to G_\sigma(S)/G_\sigma(S) \cap \mathbf{C}$ be the quotient map. Then under conditions

(i) *and* (ii) σ *is conjugate to* $\sigma(G, s, \omega)$, *where:*

$$G = G_\sigma(S)/G_\sigma(S) \cap \mathbf{C},$$

$$s(\pi(u)) = \pi(\sigma(u)),$$

and ω *is a suitable nondegenerate 2-cocycle.*

Proof. If σ is a group shift $\sigma(G, s, \omega)$, we can take $S = \{U_g : g \in G\}$. For the converse, assume S satisfying (i) and (ii) is given. By (ii), G is an abelian group. The map s on G is naturally induced by σ and is a shift of G since σ is a shift of R. It is easy to see that s is one-to-one. Define a cross-section δ of π as follows: let $\delta(0) = 1$. For all $g \in G\setminus s(G)$, let δ satisfy $\delta(g^{-1}) = \delta(g)^{-1}$. For any $g \in s^k(G)\setminus s^{k+1}(G)$, there is a unique element $g' \in G\setminus s(G)$ such that $s^k(g') = g$. Then define $\delta(g) = \sigma^k(\delta(g'))$. Write $\delta(g) = V_g$, $g \in G$. Our choice of δ ensures $\sigma(V_g) = V_{s(g)}$. Since $\pi(V_g V_h) = gh = \pi(V_{gh})$, we have $V_g V_h = \omega(g, h)V_{gh}$ for some $\omega(g, h) \in T$. It is routine to check that ω is a normalized 2-cocycle of G. Applying σ to the both sides of the equation $\omega(g, h)V_{gh} = V_g V_h$, we obtain

$$\omega(g, h)V_{s(gh)} = V_{s(g)}V_{s(h)} = \omega(s(g), s(h))V_{s(gh)}.$$

This shows that s is compatible with ω. If $g \in G$ is such that $\rho(g \wedge G) = 1$, then V_g commutes with all V_h, $h \in G$. It follows from (i) that V_g is a scalar. So $g = 0$. This proves that ω is nondegenerate. By the universal property of twisted group C^*-algebras, there exists a $*$-homomorphism $\beta: C^*(G, \omega) \to R$ such that $\beta(U_g) = V_g$, $g \in G$. Since ω is nondegenerate, it follows from Proposition 1.5 that β is a $*$-isomorphism onto the C^*-subalgebra of R generated by $\{V_g : g \in G\}$, and β extends to a $*$-isomorphism of $W^*(G, \omega)$ onto R by the uniqueness of tracial state. Finally let $\tilde{\sigma}$ be the group shift associated to (G, s, ω). We check $\sigma \circ \beta = \beta \circ \tilde{\sigma}$. So σ is conjugate to $\tilde{\sigma}$. Q.E.D.

DEFINITION 2.1. ([5]). The *normalizer*, $N(\sigma)$, of a shift σ of R is the group of unitaries w of R such that $w\sigma^k(R)w^* = \sigma^k(R)$, $k = 1, 2, \ldots$.

PROPOSITION 2.2. *Suppose* $\sigma = \sigma(G, s, \omega)$ *is a group of* R. *Then* $N(\sigma) = \{\lambda U_g : \lambda \in \mathbf{T}, g \in G\}$ *if and only if* $\sigma(R)' \cap R = \mathbf{C}$.

Proof. First assume $\sigma(R)' \cap R = \mathbf{C}$. Since $U_g\sigma^k(U_h)U_g^* = U_g U_{s^k(h)} U_g^* = \rho(g \wedge s^k(h))U_{s^k(h)} \in \sigma^k(R)$, and since $\{U_h : h \in G\}'' = R$, we see that $U_g \in N(\sigma)$. Now let $K = \{\theta \in \hat{G} : \theta(s(G)) = 1\}$. The group K acts on $R = W^*(G, \omega)$ via $\theta(U_g) = \theta(g)U_g$, $\theta \in K$, $g \in G$. The fixed point subalgebra of K, R^K, is just

$$\bigcap_{\theta\in K} W^*(\ker\theta, \omega) = W^*(\bigcap_{\theta\in K} \ker\theta, \omega) = W^*(s(G), \omega) = \sigma(R)$$

(cf. [4], [9]). Assume $W \in N(\sigma)$. Since the linear span of $\{U_g : g \in G\}$ is weakly dense in R, we can find some $g \in G$ such that $\tau(U_g^* W) \neq 0$, where τ is the unique normal normalized trace on R. Write $U_g^* W = W_1$. Since $W_1 \in N(\sigma)$, there is a $*$-automorphism γ of R such that $W_1 \sigma(a) W_1^* = \sigma(\gamma(a))$, $a \in R$. Then $\theta(W_1)\sigma(a)\theta(W_1)^* = \sigma(\gamma(a))$, $\theta \in K$. It follows that $W_1^* \theta(W_1) \in \sigma(R)' \cap R = \mathbf{C}$. Hence $\theta(W_1) = \lambda W_1$ for some $\lambda \in \mathbf{T}$. Taking the trace, which is K-invariant, we get $0 \neq \tau(W_1) = \tau(\theta(W_1)) = \lambda\tau(W_1)$. This forces $\lambda = 1$, and so $\theta(W_1) = W_1$. Since $\theta \in K$ is arbitrary, we obtain $W_1 \in R^K = \sigma(R)$. Let $W_1 = \sigma(W_2)$. It is easy to see that $W_2 \in N(\sigma)$. The uniqueness of trace implies $\tau(W_2) = \tau(\sigma(W_2)) = \tau(W_1) \neq 0$. Repeating the above argument with W_2, we get $W_2 \in \sigma(R)$. Thus $W_1 \in \sigma^2(R)$. By induction, we obtain $W_1 \in \bigcap_k \sigma^k(R) = \mathbf{C}$. Hence $W = \lambda U_g$ for some $\lambda \in \mathbf{T}$. This completes the proof that $N(\sigma) = \{\lambda U_g : \lambda \in \mathbf{T},\ g \in G\}$.

For the converse, if $\sigma(R)' \cap R \neq \mathbf{C}$, then any unitary in $\sigma(R)' \cap R$ is in $N(\sigma)$. By Proposition 1.2, $\sigma(R)' \cap R$ is the von Neumann algebra of the subgroup $\{g \in G : \rho(g \wedge s(G)) = 1\}$. If this group is nontrivial, $\sigma(R)' \cap R$ certainly contains unitaries which are not of the form λU_g. Q.E.D.

COROLLARY 2.3. *Suppose σ is a shift of R. Then the following are equivalent:*
(i) *$N(\sigma)'' = R$ and $N(\sigma)/\mathbf{T}$ is abelian;*
(ii) *σ is a group shift with $\sigma(R)' \cap R = \mathbf{C}$.*

Proof. (i) $\Rightarrow$ (ii). Taking $S = N(\sigma)$ in Proposition 2.1, we see that σ is some group shift $\sigma(G, s, \omega)$. Since $\sigma(N(\sigma)) \subset N(\sigma)$, the group $G = N(\sigma)/\mathbf{T}$. Hence $N(\sigma) = \{\lambda U_g : \lambda \in \mathbf{T}, g \in G\}$. By Proposition 2.2, we get $\sigma(R)' \cap R = \mathbf{C}$.

(ii) $\Rightarrow$ (i). By Proposition 2.2 again, $N(\sigma) = \{\lambda U_g : \lambda \in \mathbf{T},\ g \in G\}$. Hence (i) holds. Q.E.D.

COROLLARY 2.4. *If any element in $N(\sigma)$ has square a scalar multiple of the identity and if $N(\sigma)'' = R$, then σ is a group shift with $\sigma(R)' \cap R = \mathbf{C}$.*

Proof. From the hypothesis, any element in $N(\sigma)/\mathbf{T}$ has order two. This implies, as is well-known and elementary in group theory, that $N(\sigma)/\mathbf{T}$ is abelian. Then Corollary 2.3 applies.

REMARK. By this corollary, the shifts considered in [6; §4] are in fact group shifts.

For group shifts $\sigma = \sigma(G, s, \omega)$ with $\sigma(R)' \cap R = \mathbf{C}$, Proposition 2.2 shows that the normalizer is the central extension of G. This enables us to obtain a complete classification of these shifts up to conjugacy.

PROPOSITION 2.5. *Suppose $\sigma_i = \sigma(G_i, s_i, \omega_i)$, $i = 1, 2$, are group shifts of R with $\sigma_i(R)' \cap R = \mathbf{C}$. Then σ_1 and σ_2 are conjugate if and only if there exist a group somorphism $\gamma: G_1 \to G_2$ and a map $\lambda: G_1 \to \mathbf{T}$ such that*
(i) $s_2 \circ \gamma = \gamma \circ s_1$;

(ii) $\omega_1(g, h) = \frac{\lambda(g)\lambda(h)}{\lambda(gh)} \omega_2(\gamma(g), \gamma(h)), g, h \in G_1$;

(iii) $\lambda(s_1(g)) = \lambda(g), g \in G_1$.

Proof. Assume σ_1 and σ_2 are conjugate. Then there is a $*$-automorphism ψ of R such that $\sigma_2 \circ \psi = \psi \circ \sigma_1$. Thus ψ restricts to a group isomorphism of the normalizers: $\psi: N(\sigma_1) \to N(\sigma_2)$. By Proposition 2.2, $N(\sigma_1) = \{\lambda U_g : \lambda \in \mathbf{T}, g \in G_1\}$ and $N(\sigma_2) = \{\lambda V_g : \lambda \in \mathbf{T}, g \in G_2\}$. Then ψ induces a group isomorphism $\gamma : G_1 \to G_2$ since $G_i = N(\sigma_i)/\mathbf{T}$. From $\sigma_2 \circ \psi = \psi \circ \sigma_1$, we get $s_2 \circ \gamma = \gamma \circ s_1$. It is obvious that $\psi(U_g) = \lambda(g) V_{\gamma(g)}$, $g \in G_1$, for some $\lambda(g) \in \mathbf{T}$. Applying ψ to the equation $U_g U_h = \omega_1(g, h) U_{gh}$, we obtain

$$\lambda(g) V_{\gamma(g)} \cdot \lambda(h) V_{\gamma(h)} = \omega_1(g, h) \lambda_{gh} V_{\gamma(gh)}.$$

Since $V_{\gamma(g)} V_{\gamma(h)} = \omega_2(\gamma(g), \gamma(h)) V_{\gamma(gh)}$, we get

$$\omega_1(g, h) = \frac{\lambda(g)\lambda(h)}{\lambda(gh)} \omega_2(\gamma(g), \gamma(h)).$$

Applying σ_2 to $\psi(U_g) = \lambda(g) V_{\gamma(g)}$, we obtain

$$\sigma_2 \circ \psi(U_g) = \lambda(g) \sigma_2(V_{\gamma(g)}) = \lambda(g) V_{s_2 \circ \gamma(g)} = \lambda(g) V_{\gamma \circ s_1(g)}.$$

However, $\sigma_2 \circ \psi(U_g) = \psi \circ \sigma_1(U_g) = \psi(U_{s_1(g)}) = \lambda(s_1(g)) V_{\gamma \circ s_1(g)}$. Therefore $\lambda(s_1(g)) = \lambda(g)$. This proves the necessity. For the sufficiency, assume γ and λ satisfying (i)—(iii) are given. Then $\psi: C^*(G_1, \omega_1) \to C^*(G_2, \omega_2)$, $\psi(U_g) = \lambda(g) V_{\gamma(g)}$, is a $*$-isomorphism and extends to a $*$-isomorphism of $W^*(G_1, \omega_1)$ onto $W^*(G_2, \omega_2)$ by the uniqueness of the trace (Proposition 1.5). It is easy to check that $\sigma_2 \circ \psi = \psi \circ \sigma_1$, that is, σ_1 and σ_2 are conjugate shifts. Q.E.D.

REMARK. The conditions (i)—(iii) of Proposition 2.5 are sufficient for any two group shifts $\sigma_i = \sigma(G_i, s_i, \omega_i)$ to be conjugate, without the hypothesis that $\sigma_i(R)' \cap R = \mathbf{C}$. Moreover, if we replace the map λ by $\theta \circ \lambda$ for any $\theta \in \hat{G}$, the condition (ii) remains unchanged, but the condition (iii) now becomes $\theta(s_1(g))\lambda(s_1(g)) = \theta(g)\lambda(g)$, $g \in G_1$. For certain groups, we can always find some θ to make this equation hold. Therefore, the two conditions (i) $s_2 \circ \gamma = \gamma \circ s_1$ and (ii) $[\omega_1] = [\omega_2 \circ \gamma]$ in $H^2(G_1; \mathbf{T})$ will be sufficient for σ_1 and σ_2 to be conjugate. A direct consequence of this observation is that we can use the characters ρ of $G \wedge G$ to replace the cocycles ω. More precisely, let ω_1 and ω_2 be (nondegenerate) cocycles of G with $[\omega_1] = [\omega_2]$. Let s be a shift of G satisfying $\omega_i = \omega_i \circ s$, $i = 1, 2$. Then s induces two shifts σ_1 and σ_2 on $W^*(G, \omega_1)$ and $W^*(G, \omega_2)$ respectively. There is no a priori reason that σ_1 and σ_2 should be conjugate. However, in the circumstances mentioned above, we know that σ_1 and σ_2 are actually conjugate. Thus we only need to specify the character ρ of $G \wedge G$ defined in (1.2), since those characters and cohomology classes of cocycles are in one-to-one correspondence (cf. [4]).

PROPOSITION 2.6. *Suppose* $G = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$, ω_1 *and* ω_2 *are 2-cocycles of* G *with* $\omega_1(g, h) = \frac{\lambda(g)\lambda(h)}{\lambda(gh)} \omega_2(g, h)$, $g, h \in G$, *for some map* $\lambda: G \to \mathbf{T}$. *Let* s *be the shift* $s(e_i) = e_{i+1}$, $i \geqslant 0$, *where* e_i *is a generator of* $\mathbf{Z}_n^{(i)}$. *Suppose* $\omega_i \circ s = \omega_i$, $i = 1, 2$. *Then the shifts* σ_1 *and* σ_2 *induced by* s *on* $W^*(G, \omega_1)$ *and* $W^*(G, \omega_2)$ *respectively are conjugate.*

Proof. The hypotheses implies that the map $\psi(g) = \frac{\lambda(g)}{\lambda(s(g))}$, $g \in G$, is a character of G. We define a character θ of G by $\theta(e_0) = 1$, $\theta(e_i) = \psi(e_{i-1})\theta(e_{i-1})$, $i \geqslant 1$. This guarantees $\theta(s(g))\lambda(s(g)) = \theta(g)\lambda(g)$, $g \in G$. With $G_1 = G_2 = G$, $s_1 = s_2 = s$, $\gamma = \mathrm{id}_G$ and λ being $\theta \circ \lambda$ in Propositon 2.5, all three conditions are fulfilled. Hence σ_1 and σ_2 are conjugate. Q.E.D.

3. FACTOR CONDITION

Let $n \geqslant 2$ be an integer, let $G = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$, let $s(e_i) = e_{i+1}$, where e_i is a generator of $\mathbf{Z}_n^{(i)}$, and let ω be an s-compatible 2-cocycle. Let σ be the shift of $W^*(G, \omega)$ induced by s. In this section we determine all those ω which are nondegenerate, equivalently, which make $W^*(G, \omega)$ a factor . By the remark after Theorem 2.5 and Proposition 2.6, this is equivalent to determining all nondegenerate s-compatible characters ρ of $G \wedge G$. Here the s-compatibility means $\rho(g \wedge h) = \rho(s(g) \wedge s(h))$ for all g, h in G. In the case when $n = 2$, this problem was solved previously by G. Price [6]. However, our approach is different, and we feel, much simpler.

Let $\gamma = e^{2\pi i/n}$ and let $\rho(e_0 \wedge e_j) = \gamma^{a(j)}$, where $a(j) \in \mathbf{Z}_n$. By defining

$$a(0) = 0, \quad a(-j) = -a(j), \tag{3.1}$$

we obtain a sequence $\{a(j)\}, j \in \mathbf{Z}$, of elements of $\mathbf{Z}_n$ satisfying

$$\rho(e_j \wedge e_k) = \gamma^{a(k-j)}, \quad j, k = 0, 1, 2, \ldots . \tag{3.2}$$

Conversely, each doubly infinite sequence $\{a(j)\} \subset \mathbf{Z}_n$ satisfying (3.1) determines an s-compatible character ρ by (3.2). We call $\{a(j)\}$ the defining sequence of ρ, as well as of ω.

PROPOSITION 3.1. *The following are equivalent:*

(i) $W^*(G, \omega)$ *is a factor;*

(ii) $(\sigma(W^*(G, \omega)))' \cap W^*(G, \omega) = \mathbf{C}$;

(iii) *For all primes p dividing n, the defining sequence* $\{a(j)\}$ *of* ω *fails to be periodic* mod p.

Proof. By Corollary 1.3, condition (i) is equivalent to $\rho(g \wedge G) = 1$ implying $g = 0$, and condition (ii) is equivalent to $\rho(g \wedge s(G)) = 1$ implying $g = 0$. Then the following lemmas will complete the proof.

LEMMA 3.2. *Suppose* $g = \sum_{j=0}^{\infty} g_j e_j$, *where* $g_j \in \mathbf{Z}_n$ *and* $g_j = 0$ *for all but finitely many j. Then* $\rho(g \wedge s^m(G)) = 1$ *if and only if*

$$\sum_{j=0}^{\infty} g_j a(k-j) = 0 \quad \text{for } k = m, m+1, m+2, \ldots \tag{3.3}$$

Proof. $s^m(G)$ is generated by $\{e_k : k \geqslant m\}$. Now $\rho(g \wedge s^m(G)) = 1$ if and ony if $\rho(g \wedge e_k) = 0$ for $k \geqslant m$, the latter being (3.3). Q.E.D.

LEMMA 3.3. *Suppose that there exists a prime p dividing n and such that* $\{a(j)\}_{j \in \mathbf{Z}}$ *is periodic* modulo p. *Then there exists* $g \in G, g \neq 0$ *and* $\rho(g \wedge G) = 1$.

Proof. Assume t is a positive integer such that $a(j) = a(j+t)$ (mod$p$) for all $j \in \mathbf{Z}$. Then put $g = \frac{n}{p} \cdot (e_0 - e_t)$. Q.E.D.

LEMMA 3.4. *Suppose that there exists* $g \in G$ *with* $g \neq 0$ *and* $\rho(g \wedge s(G)) = 1$. *Then there exists a prime p dividing n such that* $\{a(j)\}_{j \in \mathbf{Z}}$ *is periodic* modulo p.

Proof. Assume first that n is a prime. We show $\{a(j)\}_{j \in \mathbf{Z}}$ is periodic. Let $g = \sum_{j=0}^{\infty} g_j e_j \neq 0$ be such that $\rho(g \wedge s(G)) = 1$, so that by Lemma 3.2:

$$\sum_{j=0}^{\infty} g_j a(k-j) = 0, \quad \text{for } k = 1, 2, \ldots \tag{3.4}$$

Let j_1 be the smallest and j_2 the largest j's for which $g_j \neq 0$. Then we can solve (3.4) to obtain

$$a(k-j_1) = \varphi(a(k-j_1-1),\ a(k-j_1-2), \ldots, a(k-j_2))$$

and

$$a(k-j_2) = \psi((a(k-j_2+1),\ a(k-j_2+2), \ldots, a(k-j_1))$$

for $k = 1, 2, 3, \ldots$, where φ and ψ are fixed linear functions.

Let $r = j_2 - j_1$ and assume first that $r > 0$. Then we have

$$a(k) = \varphi(a(k-1), a(k-2), \ldots, a(k-r)) \quad \text{for all } k \geqslant 1 - j_1, \tag{3.5}$$

and

$$a(k) = \psi(a(k+1), a(k+2), \ldots, a(k+r)) \quad \text{for all } k \geqslant 1 - j_2. \tag{3.6}$$

Since there are only finitely many distinct values for an r-tuple from $\mathbf{Z}_n$, (3.5) implies that $a(k)$ is ultimately periodic as $k \to \infty$, that is, there exist positive integers t and N such that $a(k+t) = a(k)$ for all $k \geqslant N$. Then (3.6) implies that $a(k+t) = a(k)$ for all $k \geqslant 1 - j_2$. Since $a(-j) = -a(j)$ for all $j \in \mathbf{Z}$, we deduce from (3.5) that

$$a(k) = \varphi(a(k+1), a(k+2), \ldots, a(k+r)) \quad \text{for all } k \leqslant j_1 - 1. \tag{3.7}$$

Since $(1 - j_2) - 1 \leqslant j_1 - 1$ always, (3.6) and (3.7) shows that $a(k+t) = a(k)$ for all $k \in \mathbf{Z}$.

Suppose now $r = 0$. Then (3.4) becomes $g_{j_1} a(k - j_1) = 0$ for $k = 1, 2, \ldots$, or $a(k) = 0$ for $k \leqslant 1 - j_1$. Since $j_1 \geqslant 0$ and $a(-k) = -a(k)$, we obtain $a(k) = 0$ for all $k \in \mathbf{Z}$.

Now consider the general case where n has the prime decomposition $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$. Under the hypothesis $\rho(g \wedge s(G)) = 1$, we still have (3.4). Since $g \neq 0$, one of the primes $p_i = p$ must be such that $g_{j_1} \neq 0 \pmod{p_i^{\alpha_i}}$. Write $g_j = p^k h_j$ where k is the largest integer such that p^k divides all g_j. It follows that $k < \alpha_i$ and that not all h_j are $0 \bmod p$. Then we obtain from (3.4) that

$$\sum h_j a(k-j) = 0 \pmod{p} \quad \text{for } k = 1, 2, \ldots .$$

As before we now find that $\{a(k)\}$ is periodic $\bmod p$. Q.E.D.

4. n-SHIFTS

For each integer $n \geqslant 2$, let $G_n = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ and let s_n be the shift defined by $s_n(e_i) = e_{i+1}$ as in Section 3, where e_i is a generator of $\mathbf{Z}_n^{(i)}$.

DEFINITION 4.1. A shift σ of the hyperfinite II_1-factor R is called an *n-shift* if σ is conjugate to a group shift $\sigma(G_n, s_n, \omega)$.

In this section we first give a characterization of n-shifts, which shows that our results about group shifts generalize the results for the binary shifts of Powers [5] and Price [6] and the n-unitary shifts of Choda [1]. Then we discuss the classification problem for n-shifts.

PROPOSITION 4.1. *A shift σ of the hyperfinite II_1-factor R is conjugate to an n-shift if and only if there exists a unitary u in R (which is called a σ-generator) such that the following hold:*

(i) $u^n = 1$ *and* $u^k \notin \mathbf{C}$ *for* $1 \leqslant k \leqslant n-1$;

(ii) $\{u, \sigma(u), \sigma^2(u), \ldots\}'' = R$;

(iii) u *and* $\sigma^i(u)$ *commute up to a scalar for* $i = 1, 2, \ldots$.

When these conditions hold, the conjugacy is given by $u \to U_{e_0}$.

Proof. If $\sigma = \sigma(G_n, s_n, \omega)$, we can take $u = U_{e_0}$. Then $\sigma^i(u) = U_{e_i}$ and conditions (i)—(iii) are easily verified. Now assume σ is a shift of R with a unitary $u \in R$ so that the conditions (i)—(iii) are satisfied. Taking $S = \{u\}$ in Proposition 2.1, we see that σ is conjugate to a group shift $\sigma(G, s, \omega)$. By that proposition, G is the quotient group of the group generated by $\{u, \sigma(u), \sigma^2(u), \ldots\}$ modulo scalars. Denote the image of $\sigma^i(u)$ in G by f_i. Then G is the abelian group generated by $\{f_i : i \geqslant 0\}$ and s is defined by $s(f_i) = f_{i+1}$ (Proposition 2.1). Note that $kf_i = 0$ if and only if $k = 0 \mod n$.

We proceed to show that $\{f_i : i \geqslant 0\}$ is $\mathbf{Z}_n$-linearly independent, which proves $G = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$. Assume there exists a relation $\sum_{i=0}^{N} c_i f_i = 0$ with $c_N \neq 0 \pmod n$ in $\mathbf{Z}_n$. Consider all such relations where N is minimal. Among them choose one so that c_N is minimal. An Euclidean algorithm argument then shows that c_N must divide n. Let $n = dc_N$. Then in $d \sum_{i=0}^{N} c_i f_i = 0$, since $dc_N = 0$, all coefficients must be zero: $dc_i = 0 \pmod n$, $0 \leqslant i \leqslant N-1$. It follows that c_N divides c_i for all i. Applying s^j to $\sum_{i=0}^{N} c_i f_i = 0$, we obtain $\sum_{i=0}^{N} \frac{c_i}{c_N}(c_N f_{i+j}) = 0$ for all $j = 0, 1, 2, \ldots$. Now let K be the subgroup of G generated by $\{c_N f_0, c_N f_1, \ldots, c_N f_{N-1}\}$. The above equation shows that $s(K) \subset K$. Since s is one-to-one (see the proof of Proposition 2.1) and K is finite, we get $s(K) = K$. Then $\bigcap_{k=0}^{\infty} s^k(G) \supset K \neq \{0\}$ contradicting the fact that s is a shift of G. Q.E.D.

REMARK 1. The proof of the proposition shows that if G is a group possessing a one-to-one shift s and if $g \in G$ is an element of order n, then the subgroup of G generated by $\{g, s(g), s^2(g), \ldots\}$ is isomorphic to $\bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ under $s^i(g) \to e_i$. This shows that n-shifts are the basic blocks of more general group shifts.

REMARK 2. Suppose that a shift σ of R satisfies the conditions of Proposition 4.1 except that instead of (i) we assume only (i)$'$ $u^n = 1$. Then we can proceed as follows. Let m be the smallest positive integer such that $u^m \in \mathbf{C}$. Let $v = \lambda u$ where λ

is a scalar chosen so that $v^m = 1$. Then $\{\sigma, v\}$ is a pair satisfying the conditions of Proposition 4.1. Hence σ is an m-shift. If n is a prime, of course (i) and (i)′ are equivalent.

REMARK 3. Suppose that σ is an n-shift with generator u and that σ is conjugate to $\sigma(G_n, s_n, \omega)$. Then the defining sequence $\{a(j)\}_{j\in \mathbf{Z}}$ (Section 3) for ω is given by

$$u\sigma^j(u)u^*\sigma^j(u)^* = (e^{2\pi i/n})^{a(j)}.$$

COROLLARY 4.2. *If σ is an n-shift of R with generator u, then*
(i) $\sigma(R)' \cap R = \mathbf{C}$.
(ii) *The normalizer* $N(\sigma) = \{\lambda w : w$ *is a word in* $\sigma^j(u)$, $\lambda \in \mathbf{T}\}$.

Proof. (i) follows from Proposition 3.1; (ii) from Proposition 2.2. Q.E.D.

REMARK. The problem of classifying n-shifts is, of course, completely solved by Proposition 2.5. Let $G = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ and ρ a nondegenerate character of $G \wedge G$ with defining sequence $\{a(j)\}_{j\in\mathbf{Z}}$. By Proposition 2.5 and Proposition 4.1, to determine all n-shifts conjugate to the given one associated with ρ, it is sufficient to determine all elements $g \in G$ such that $\{g, s(g), s^2(g), \ldots\}$ generates G. These g's are called generators. Then the defining sequence can be computed in terms of $\{a(j)\}$ by $\rho(g \wedge s^j(g)) = e^{2\pi i b(j)}$. For example, let $n = 4$ and $g = e_0 + 2e_1$. Since $g + 2s(g) = e_0$, g is a generator. Using (3.2), we get $b(f) = 2a(j-1) + a(j) + 2a(j+1)$, $j \in \mathbf{Z}$. Thus $\{b(j)\}_{j\in\mathbf{Z}}$ defines an n-shift conjugate to the one defined by $\{a(j)\}_{j\in\mathbf{Z}}$. If $\{a(j) : j \geqslant 0\}$ is $\{0, 1, 0, 0, 0, \ldots\}$, then $\{b(j) : j \geqslant 0\}$ is $\{0, 1, 2, 0, 0, \ldots\}$.

The classification of binary shifts in [5] is achieved by showing that if u and v are two σ-generators of a binary shift σ, then $u = \pm v$. It is tempting to try to prove that, for general n, two σ-generators of an n-shift are related as $u = \lambda v^m$ for some m with $(n, m) = 1$. However, as shown in the last paragraph for $n = 4$, this is no longer true. We need some condition on n.

PROPOSITION 4.3. *Suppose that u and v are σ-generators of an n-shift σ and that n is square-free. Then $u = \lambda v^m$ for some $\lambda \in \mathbf{T}$ with $\lambda^n = 1$, and some integer m with $(m, n) = 1$.*

Proof. By Proposition 4.1, we can assume $\sigma = \sigma(G_n, s_n, \omega)$ such that u is just U_{e_0}. Since v is a generator for σ, $v \in N(\sigma)$. By Proposition 2.2, $v = \lambda U_g$ for some $\lambda \in \mathbf{T}$, $g \in G$. Since v is a generator for σ, g must be a generator for G. Hence

$$e_0 = \sum_{j=0}^{M} c_j s^j(g), \quad c_M \neq 0, \quad \text{and} \quad g = \sum_{i=0}^{N} b_i e_i, \quad b_N \neq 0. \tag{4.1}$$

Substituting and comparing the coefficients of e_{M+N}, we get $b_N c_M = 0$ in $\mathbf{Z}_n$ provided $M + N > 0$. Hence if n is a prime, we must have $M = N = 0$, so that $g = b_0 e_0$, or $v = \lambda u^{b_0}$. It is obvious that $(b_0, n) = 1$. In the general case where $n = p_1 p_2 \dots p_s$ with the p_k's distinct primes, we pass (4.1) to the quotient group $\bigoplus_{i=0}^{\infty} \mathbf{Z}_p^{(i)}$ for each $p_k = p$. The same argument as above gives $b_i = 0 \pmod p$ for $i > 0$. Hence $b_i = 0 \pmod n$ for $i > 0$ and again $g = b_0 e_0$. Q.E.D.

PROPOSITION 4.4. *Suppose that σ_1 and σ_2 are n-shifts with defining sequences $\{a(j)\}_{j\in \mathbf{Z}}$ and $\{b(j)\}_{j\in\mathbf{Z}}$ respectively, and that n is square-free. Then σ_1 and σ_2 are conjugate if and only if there exists an integer m, $(m, n) = 1$, such that $a(j) = m^2 b(j)$ for all $j \in \mathbf{Z}$.*

Proof. Assume $\psi \in \operatorname{Aut}(R)$ implementing the conjugacy. Let u and v be generators of σ_1 and σ_2 respectively. Then $\psi(u)$ is a generator of σ_2. By Proposition 4.3, $\psi(u) = \lambda v^m$ for some m, $(m, n) = 1$. Computing the defining sequences as in Remark 3 following Proposition 4.1, we get $a(j) = m^2 b(j)$. The converse is obvious: $\gamma(e_i) = me_i$ is an automorphism of $\bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ such that $\gamma \circ s = s \circ \gamma$ and that $\rho_1(e_j \wedge e_k) = \rho_2(\gamma(e_j) \wedge \gamma(e_k))$. Q.E.D.

5. SOME EXAMPLES

In this section, we first show that for each integer $n \geqslant 2$, there is a group shift of index n which is not an n-shift. Then we show, by using Jones' work on index of subfactors, that there are shifts of R which are not group shifts. We conclude with remarks on sequences of projections.

We start with the construction of some group shifts over the group $G = \bigoplus_{j=-\infty}^{+\infty} \mathbf{Z}_n^{(j)}$. The construction is a variant of that in Price [6]. Let e_j be a generator of $\mathbf{Z}_n^{(j)}$, and let $s(e_j) = e_j + e_{j+1}$. First we check that s is a shift and that $[G: s(G)] = n$.

Define a character $\theta\colon G \to \mathbf{T}$ by

$$\theta(e_j) = \begin{cases} e^{2\pi i/n}, & \text{if } j \text{ is even,} \\ e^{-2\pi i/n}, & \text{if } j \text{ is odd.} \end{cases}$$

Let $H = \ker\theta$; then $[G: H] = n$. A short calculation shows that $s(G) = H$. Hence $[G: s(G)] = n$.

LEMMA 5.1. *s is a shift of G.*

Proof. Assume $g \in \bigcap_k s^k(G)$ with $g \neq 0$. Let γ be the automorphism of G defined by $\gamma(e_j) = e_{j+1}$. Since $\gamma \circ s = s \circ \gamma$, we have $\gamma^l(g) \in \bigcap_k s^k(G)$ for all $l \in \mathbf{Z}$.

Therefore, without loss of generality, we may assume $g = \sum_{j=0}^{N} c_j e_j$ with $c_N \neq 0$. Note that

$$s^k(e_0) = e_0 + \binom{k}{1} e_1 + \binom{k}{2} e_2 + \ldots + \binom{k}{k-1} e_{k-1} + e_k .$$

Thus $g - c_N s^N(e_0)$ is a linear combination of $e_0, e_1, \ldots, e_{N-1}$. It follows from induction that $g = \sum_{j=0}^{N} b_j s^j(e_0)$ with $b_N = c_N$. Let b_k be the first nonzero b_j. Then $g - b_k s^k(e_0) \in s^{k+1}(G)$. Since $g \in s^{k+1}(G)$ by assumption, we have $b_k s^k(e_0) \in s^{k+1}(G)$. It is easy to see that s is one-to-one. Hence $b_k e_0 \in s(G) = \ker \theta$, and $1 = \theta(b_k e_0) = e^{(2\pi i/n) b_k}$. Thus $b_k = 0 \pmod{n}$. A contradiction. Q.E.D.

Next we need to define a nondegenerate character ρ on $G \wedge G$ which is compatible with s. Let $\rho(e_i \wedge e_j) = e^{(2\pi i/n) a_{i,j}}$ where $a_{i,j} \in \mathbf{Z}_n$. The compatibility condition $\rho(e_i \wedge e_j) = \rho(s(e_i) \wedge s(e_j))$ is just that

$$a_{i,j+1} + a_{i+1,j+1} + a_{i+1,j} = 0 \quad \text{for all } i, j \in \mathbf{Z}. \tag{5.1}$$

The $a_{i,j}$ satisfy also $a_{i,j} = -a_{j,i}$ and $a_{i,i} = 0$. Set $a_{0,1} = 1$ and $a_{0,k} = 0$ for $k \neq 1$. Then (5.1) and the skew symmetry determine the $a_{i,j}$ completely: Letting A_k be the $2k \times 2k$ matrix $(a_{i,j})_{i,j=-k+1,\,-k+2,\,\ldots,\,k}$, we see that

$$A_1 = \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and in general

$$A_{k+1} = \begin{pmatrix} * & \cdots\cdots & * & \pm 1 \\ \vdots & & & 0 \\ \vdots & & A_k & \vdots \\ * & & & \vdots \\ \mp 1 & 0 & \cdots\cdots & 0 \end{pmatrix}.$$

Therefore $\det A_k = \pm 1$ for all $k \geqslant 1$. Now we show that ρ is nondegenerate. For suppose $\rho(g \wedge G) = 1$. Then if $g = \sum_{j=-\infty}^{+\infty} x_j e_j$, we must have $\sum_{j=-\infty}^{\infty} a_{ij} x_j = 0$ for all i in $\mathbf{Z}$. Choosing k so large that $x_j = 0$ for $|j| \geqslant k$, we get $A_k X = 0$ where $X = (x_j)_{j=-k+1,\,-k+2,\,\ldots,\,k}$. Since $\det A_k = \pm 1$, A_k is invertible in the ring of $k \times k$ matrices over $\mathbf{Z}_n$. This forces $X = 0$, and so $g = 0$. With this nondegenerate ρ, we get a group shift $\sigma = \sigma(G, s, \omega)$ of R, of index n.

PROPOSITION 5.2. *The group shift $\sigma = \sigma(G, s, \omega)$ constructed above is a shift of index n of R but is not an n-shift.*

Proof. Assume $\sigma(G, s, \omega)$ is conjugate to an n-shift $\tilde{\sigma} = \sigma(G_n, s_n, \omega)$ in the notation of Section 4. Since $\tilde{\sigma}(R)' \cap R = \mathbf{C}$, there exists a group isomorphism $\gamma: G_n \to G$ such that $s \circ \gamma = \gamma \circ s_n$ (Proposition 2.5). Note that $e_0 \in G_n$ is a generator (Section 4), that is, $\{s_n^k(e_0) : k \geqslant 0\}$ generates G_n. Hence $\gamma(e_0)$ must be a generator of G. However, it is easy to see that there is no generator in G. Q.E.D.

REMARK. The example of Price [6; § 5] is in fact a group shift $\sigma(G, s, \omega)$, where $G = \bigoplus_{i=-\infty}^{\infty} \mathbf{Z}_2^{(i)}$, $s(e_i) = e_{i+1}$ if $i \geqslant 0$ and $s(e_i) = e_i + e_{i+1}$ if $i < 0$.

Using tensor products, we can get new shifts from old ones. Suppose $\sigma_i = \sigma(G_i, s_i, \omega_i)$, $i = 1, 2$, are group shifts. Then $\sigma_1 \otimes \sigma_2$ is a group shift $\sigma(G_1 \oplus G_2, s_1 \oplus s_2, \omega_1 \oplus \omega_2)$, where $(s_1 \oplus s_2)(g_1 \oplus g_2) = s_1(g_1) \oplus s_2(g_2)$ and $(\omega_1 \oplus \omega_2)(g_1 \oplus g_2, h_1 \oplus h_2) = \omega_1(g_1, h_1)\omega_2(g_2, h_2)$. Note that $\omega_1 \oplus \omega_2$ is nondegenerate if and only if ω_1 and ω_2 are both nondegenerate. If $\sigma_i(R)' \cap R = \mathbf{C}$, $i = 1, 2$, then $(\sigma_1 \otimes \sigma_2)(R \otimes R)' \cap (R \otimes R) = \mathbf{C}$. In particular, if $[G_i : s_i(G_i)]$ is a prime, then $\sigma_1 \otimes \sigma_2$ always satisfies this by Proposition 1.4 (iii).

PROPOSITION 5.3. *The tensor product σ of an n_1-shift σ_1 and an n_2-shift σ_2 is a shift of R of index $n_1 n_2$. σ is an $n_1 n_2$-shift if and only if $(n_1, n_2) = 1$.*

Proof. As in Section 4, let $G_n = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$ and $s_n(e_i) = e_{i+1}$. Assume $\sigma = \sigma_1 \otimes \sigma_2$ is an $n_1 n_2$-shift. By Theorem 2.5, there exists an isomorphism $\gamma: G_{n_1} \oplus G_{n_2} \to G_{n_1 n_2}$ such that $s_{n_1 n_2} \circ \gamma = \gamma \circ (s_{n_1} \oplus s_{n_2})$. Hence γ induces an isomorphism between $G_{n_1} \oplus G_{n_2}/(s_{n_1} \oplus s_{n_2})(G_{n_1} \oplus G_{n_2}) = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2}$ and $G_{n_1 n_2}/s_{n_1 n_2}(G_{n_1 n_2}) = \mathbf{Z}_{n_1 n_2}$. It follows that $(n_1, n_2) = 1$. Conversely, if $(n_1, n_2) = 1$, fix an isomorphism $\gamma: \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \to \mathbf{Z}_{n_1 n_2}$ and extend γ to $G_{n_1} \oplus G_{n_2} \to G_{n_1 n_2}$ in an obvious way so that $s_{n_1 n_2} \circ \gamma = \gamma \circ (s_{n_1} \oplus s_{n_2})$. Q.E.D.

PROPOSITION 5.4. *For each prime $n > 4$, there is a shift of R of index n which is not a group shift.*

Proof. Let σ be an n-shift over $G = \bigoplus_{i=0}^{\infty} \mathbf{Z}_n^{(i)}$. Denote U_{e_i} by U_i, and put $p_i = (1/n)(1 + U_i + U_i^2 + \ldots + U_i^{n-1})$, which is a spectral projection of Y_i. Since $U_i U_j = \lambda U_j U_i$ for some $\lambda \in \mathbf{T}$, an easy computation shows that

$$p_i p_j p_i = \frac{1}{n} p_i, \quad \text{if } \lambda \neq 1,$$

$$p_i p_j = p_j p_i, \quad \text{if } \lambda = 1,$$

$$\operatorname{tr}(w p_i) = \frac{1}{n} \cdot \operatorname{tr}(w), \quad \text{if } w \text{ is a word on } 1, p_1, \ldots, p_{i-1}.$$

Let M and N be the von Neuman subalgebra of $W^*(G, \omega)$ generated by $\{p_1, p_2, p_3, \ldots\}$ and $\{p_2, p_3, \ldots\}$ respectively. The shift σ restricts to a shift $\tilde{\sigma}$ on M such that $\tilde{\sigma}(p_i) = p_{i+1}$. So $\tilde{\sigma}(M) = N$. We choose the defining sequence of ω by $a(1) = 1$, $a(-1) = -1$, and $a(k) = 0$ otherwise. Then $\{p_1, p_2, \ldots\}$ satisfies the conditions of Jones [2; 4.1.1]. Hence M is the hyperfinite II_1-factor R and N is a subfactor of M with index $[M: N] = n$. By [2; §5], $\tilde{\sigma}(M)' \cap M \neq C$ if $n > 4$. Hence by Proposition 1.4 (iii), when $n > 4$ is a prime, $\tilde{\sigma}$ cannot be a group shift. Q.E.D.

CONCLUDING REMARK. Let S be any nonempty subset of positive integers. When $n \geqslant 3$, we can always find n-shifts so that the procedure in the proof of above proposition provides a shift given by $p_i \to p_{i+1}$ where $\{p_1, p_2, \ldots\}$ is a sequence of projections satisfying

(i) $p_i p_j p_i = \frac{1}{n} p_i$ if $|i-j| \in S$;

(ii) $p_i p_j = p_j p_i$ if $|i-j| \notin S$; and

(iii) $\mathrm{tr}(w p_i) = \frac{1}{n} \mathrm{tr}(w)$ if w is a word on $1, p_1, p_2, \ldots, p_{i-1}$.

The work of V.F.R. Jones ([2], [3]) suggests that it would be of interest to carry out further investigations of such sequences of projections, particularly when $1/n$ is replaced in certain cases by τ in the Jones index set. We wish to discuss this in future publications.

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