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ON THE CALDERON PROJECTIONS AND SPECTRAL PROJECTIONS OF THE ELLIPTIC OPERATORS

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0. INTRODUCTION AND STATEMENT OF THE RESULTS

In [13] and [14] the space Pdiff_p of pseudodifferential projections with the same principal symbol was investigated. In particular it has been shown that it has countably many path-connected components. Pseudodifferential projections often appear as the boundary conditions, hence one may expect applications of this result in the theory of the elliptic boundary value problems, especially in the calculation of the index. This is discussed in [7] where a "higher" dimensional effect also appears. The results of [7] are based on an explicit calculation of the homotopy groups of Pdiff_p and $\mathrm{Pdiff}_p^{\mathcal{G}}$, which is the subspace containing the projections satisfying a certain additional condition. One can find the calculation of the homotopy groups in [14]. In this paper we investigate two projections related to an elliptic operator on a manifold with boundary. We want to know when they belong to the same path-connected component of Pdiff_p. Along the way we obtain a formula for the index of the Atiyah-Patodi-Singer problem in terms of the projections we consider.

Let X be a smooth compact manifold with boundary Y. Let E and F be smooth complex vector bundles over X. We fix a Riemannian structure on X and Hermitian structures on E and F. We use the Riemannian structure to construct a collar neighbourhood $N = I \times Y$ of the boundary Y; let $t \in [0, 1]$ denote the normal coordinate. We identify Y with $\{0\} \times Y$ and we use the inward normal.

Let $A: C^\infty(X; E) \to C^\infty(X; F)$ be an elliptic differential operator of the first order. We assume that on the collar N, A has the form

$$(1) \qquad A(t, y) = G(y)\left(\frac{\partial}{\partial t} + B\right),$$

where $B : C^\infty(Y, E|Y) \to C^\infty(Y, E|Y)$ is a self-adjoint (elliptic) operator and where $G : E|Y \to F|Y$ is a unitary bundle isomorphism. G is the principal symbol of A evaluated on the normal vector dt.

Such operators appear in the index theorem of Atiyah, Patodi and Singer (see [1], [2]). Set

$$H(A) = \{s \mid Y \,; s \in C^{\infty}(X, E) \quad \text{and} \quad As = 0 \text{ on } X\}.$$

Let A^* denote the formal adjoint of A and define $H(A^*)$ similarly. $H(A)$ and $H(A^*)$ are called Cauchy data spaces. Denote by $P(A)$ and $P(A^*)$ the orthogonal projections of $L^2(Y, E \mid Y)$ and of $L^2(Y, F \mid Y)$ onto the closure of $H(A)$ and $H(A^*)$. Let $b(y, \zeta)$ be the principal symbol of B. Let $E_{\pm}(b)(y, \zeta)$ be the subbundle of $\pi^*(E \mid Y)$, where $\pi : SY \to Y$ is the natural projection of cotangent sphere bundle spanned by the eigenvectors of b corresponding to the positive (negative) eigenvalues. These are called the Cauchy data bundless. We have

$$E_+(b) \oplus E_-(b) = \pi^*(E).$$

Let $p_{\pm}(y, \zeta)$ be the complementary projections corresponding to this decomposition. These are orthogonal projections in the case that B is self-adjoint as it is assumed here. Following Calderon, Seeley has shown in [9], [10] that there exist pseudodifferential projections $P(A)$, $P(A^*)$ onto the closure of $H(A)$ and $H(A^*)$ with the principal symbols p_+ and p_- respectively.

If R is a projection onto the closed subspace V, then it is easy to see that

$$R_{\text{ort}} = RR^*(RR^* + (\text{Id} - R^*)(\text{Id} - R))^{-1}$$

is the orthogonal projection onto V. Thus in our case, we may assume that $P(A)$ and $P(A^*)$ are orthogonal projections. Now we describe $\Pi_+(B)$. We may consider B from the decomposition (1) as unbounded, self-adjoint operator. $\Pi_+(B)$ is the spectral projection of B for $[0, +\infty)$, i.e. onto the subspace spanned by the eigenvectors corresponding to the non-negative eigenvalues. It is well-known (see for instance [1], [3], [5]) that $\Pi_+(B)$ is a pseudodifferential operator of order 0 with the principal symbol p_+.

Now. $P(A)$ and $\Pi_+(B)$ are two elements of the space Pdiff_{p_+} and we want to know, when they belong to different path-connected components of this space. To detect such a situation we use the unbounded operator $A_{\Pi_+(B)}$

$$A_{\Pi_+(B)} = A \quad \text{with the domain}$$

(2)

$$\text{dom}\, A_{\Pi_+(B)} = \{u \in H_1(X; E) : \Pi_+(B)(u \mid Y) = 0\}$$

(see [1]; see [11], Chapter VI for the general theory of the problems of this type). $A_{\Pi_+(B)} : \mathrm{Dom}\, A_{\Pi+(B)} \to L^2(X; F)$ is a Fredholm operator. The main result of this paper is:

THEOREM 1. *Let us assume that the spaces of the interior solutions*

$$\ker_X(A) = \{u \in C^\infty(X; E) : Au = 0 \text{ and } u \,|\, Y = 0\}$$

and

$$\ker_X(A^*) = \{v \in C^\infty(X; F) : A^* v = 0 \text{ and } v \,|\, Y = 0\}$$

contain only 0*-sections, then* $\Pi_+(B)$ *and* $P(A)$ *belong to the same path-connected component of* Pdiff_{P_+} *if and only if*

$$\text{index } A_{\Pi_+(B)} = 0.$$

REMARKS. (0.1) Signature and the Euler characteristic of X are obtained as the indices of the operators of the form (2) (see [1]). Moreover let D denote the signature operator (or more generally a Dirac operator with coefficients in an auxiliary vector bundle). If $Du = 0$, then also $D^2 u = 0$. Thus u is a solution of the operator with scalar principal symbol. Operators of this type have no non-trivial solutions which vanish on a non-empty open subset of X. Let us assume that $u|Y = 0$, then $\tilde{v}$ equal u on one copy of X and 0 on the other copy is a solution of the corresponding operator on the closed double of X.

This consideration shows us that operators of Dirac type, constructed with respect to the metric which is the product on some collar of Y, satisfy the assumptions of Theorem 1.

(0.2) In case of the Dirac operator on a Spin-manifold the index can jump when we vary the metric on the collar. This leads us to a new interesting effect which will be discussed elsewhere.

Theorem 1 is a consequence of the following proposition:

PROPOSITION 2. *For any operator fulfilling assumptions of Theorem* 1

$$\text{index } A_{\Pi_+(B)} = \text{index } \Pi_+(B) P(A),$$

where we consider $\Pi_+(B)P(A)$ *as the operator acting from the range of* $P(A)$ *into the range of* $\Pi_+(B)$.

REMARK. (0.3) It is obvious that $\Pi_+(B)P(A) : \overline{H(A)} \to \mathrm{Ran}\, \Pi_+(B)$ is a Fredholm operator. It follows from the equality of the principal symbols. Detailed calculation can be found in the proof of Lemma 2.3 in [13] (see (2.8)—(2.11) in [13]).

We are able to prove Proposition 2 thanks to the following result which was first stated in [5].

THEOREM 3. *$H(A)$ and $G^{-1}(H(A^*))$ are orthogonal complementary subspaces of $C^\infty(Y; E|Y)$. $P(A)$ and $P(A^*)$ are 0-th order pseudodifferential operators. The leading symbol of $P(A)$ is $P_+(y, \zeta)$ and the leading symbol of $P(A^*)$ is $Gp_-(y, \zeta)G^{-1}$*

REMARKS. (0.4) The second part of Theorem 3 is well-known and the first is proved in a different way in [5], part II.

Here we make some restrictions on operator A (it has the form (1) in the collar and B and G do not depend on t). Due to this simplification we are able to give a simple geometric proof based on a certain result of Seeley ([9], Chapter XVII; see also [10], [11]) and a glueing construction which was given in [12].

(0.5) The proof along this line can be extended to a larger class of operators. We may assume that B is a pseudodifferential operator (depending on t) such that $b(y, \zeta)$ has no imaginary eigenvalues. This is discussed in [6].

Theorem 1 is a corollary from Proposition 2 and the following fact.

PROPOSITION 4. *Let $P, P_1 \in \mathrm{Pdiff}_{P_+}$; they belong to the same path-connected component of Pdiff_{P_+} if and only if the index of the Fredholm operator P_1P : Ran $P \to$ $\to$ Ran P_1 is equal 0.*

REMARK. (0.6) Proposition 4 is in fact Proposition 4.1 from [13]. To make the presentation complete we give in Section 2 a simple proof of the fact, which shows that $P(A)$ and $\Pi_+(B)$ can be in a different connected component of Pdiff_{P_+}.

PROPOSITION 4'. *If P and P_1 belong to the same path-connected component of Pdiff_{P_+}, then* index $P_1P = 0$.

Section 1 contains the proof of Theorem 3 which should be of independent interest. In Section 2 we prove Proposition 2 and Proposition 4' which completes the proof of Theorem 1.

1. PROOF OF THEOREM 3

We may assume that X is a submanifold of some closed manifold M. We may take for instance $M = \tilde{X}$, the double of X. Moreover, as will be explained below, we may assume that A is the restriction of the operator $\tilde{A}$ defined on the whole M. Under these assumptions we have the following result.

THEOREM 5. ([9], Chapter XVII, Theorem 2; see also [10], [11]). *Let $\tilde{A}$: $: C^\infty(M; V) \to C^\infty(M; W)$ be an elliptic first order pseudodifferential operator of the* *form* (1) *on some collar of* Y*). *Write* $X_+ = X$, $X_- = M \setminus (X \setminus Y)$, *and*

$$H_\pm(\tilde{A}) = H(\tilde{A}|X_\pm).$$

Moreover, let us assume that $\tilde{A} : C^\infty(M; V) \to C^\infty(M; W)$ *is a bijection. Then* $H_+(\tilde{A})$ *and* $H_-(\tilde{A})$ *are complementary subspaces of* $C^\infty(Y; V|Y)$ *and* $P_\pm(\tilde{A}) : C^\infty(Y; V|Y) \to C^\infty(Y; V|Y)$ *the projections onto* $H_+(\tilde{A})$ *along* $H_-(\tilde{A})$ *are pseudodifferential operators of order* 0, $p_\pm$ — *the principal symbol of* $P_\pm(\tilde{A})$ — *at any* $(y, \zeta) \in SY$ *is the projection onto* $E_\pm(b)(y, \zeta)$ *along* $E_\mp(b)(y, \zeta)$.

We shall deduce Theorem 3 from Theorem 5 using the following construction. Let $\tilde{X}$ denote the double of X, so Y has a double collar $\tilde{N} = [-1, +1] \times Y$ in $\tilde{X}$. We define vector bundles E^G and $F^{G^{-1}}$ over $\tilde{X}$ using G and G^{-1} as clutching functions

$$E^G = E \cup_G F \quad \text{and} \quad F^{G^{-1}} = F \cup_{G^{-1}} E.$$

We take E over X_+ and F over X_- and identify $E \cong F$ over Y using G. If $J : \tilde{X} \to \tilde{X}$ is the isometry which interchanges the two factors X_+ and X_-, then the pull-back $J^*(E^G)$ is equal to $F^{G^{-1}}$, and similarly $J^*(F^{G^{-1}}) = E^G$.

A smooth section of E^G is a pair (s_1, s_2) where s_1 is a smooth section of the bundle E extended to $X_+ \cup [-1, 0] \times Y$, s_2 is a smooth section of the bundle F extended to $X_- \cup [0, 1] \times Y$, and

$$s_2(t, y) = G(y) s_1(t, y) \quad \text{on } \tilde{N}. \tag{3}$$

(3) implies that the operators

$$A \cup A^* = \begin{cases} A & \text{on } X_+ \\ A^* & \text{on } X_- \end{cases} \quad \text{and} \quad A^* \cup A = \begin{cases} A^* & \text{on } X_+ \\ A & \text{on } X_- \end{cases}$$

are well-defined on $C^\infty(\tilde{X}; E^G)$ and $C^\infty(\tilde{X}; F^{G^{-1}})$.

Furthermore $(A \cup A^*)^* = A^* \cup A$ and the involution J satisfies

$$(A \cup A^*) J = J(A^* \cup A). \tag{4}$$

This glueing construction is crucial to our discussion. It has also deep application in the analytical realisation of relative cycles in K-homology (see [12]).

Now we make the following observation:

*) This requires A to be a differential operator on the collar.

LEMMA 6. *Denote by* T *the operator* $A \cup A^*: C^\infty(\tilde{X}; E^G) \to C^\infty(\tilde{X}; F^{G^{-1}})$. *We have*

$$\operatorname{index} T = 0 \quad \textit{and} \quad H_+(T) \cap H_-(T) = \{0\}.$$

Proof. The nullity of the index follows from (4), which says that T is isomorphic to the adjoint operator. We shall use Green's formula ([9], Chapter XVII, Theorem 1; see also [10] and [11]) to prove the second statement.

Let $r_\pm \in H_\pm(T)$ and choose $w^\pm \in C^\infty(X_\pm; E^G)$ so that $Tw^\pm = 0$ on $X_\pm$ and $w^\pm | Y = r_\pm$. $w^+ \in C^\infty(X; E)$ and $Aw^+ = 0$; also $w^- \in C^\infty(X; F)$ and $A^* w^- = 0$. Now we use Seeley's extension and we get $\tilde{w}^+$ which is a smooth section of E over $X_+ \cup [-1, 0] \times Y$ with support in the interior of this manifold and such that $\tilde{w}^+ | X_+ = w^+$.

A pair $v^+ = (v_1^+, v_2^+)$ given by the formula

$$v_1^+ = \tilde{w}^+ \quad \text{on } X_+ \cup [-1, 0] \times Y = X_+ \cup \tilde{N}$$

and

$$v_2^+ = G(y)\tilde{w}^+(t, y) \quad \text{on } \tilde{N}$$

is automatically a smooth section of E^G. In a similar way we get $v^- = (v_1^-, v_2^-)$ a smooth section of $F^{G^{-1}}$ such that $v_2^- | X_- = w^-$.

Using Green's formula we have

$$(v^+ | Y, v^- | Y) = (v_1^+ | Y, v_1^- | Y) = (r_+, G^{-1} r_-) =$$

$$= (Aw_+, w^-) - (w^+, A^* w^-) = 0.$$

This completes the proof.

COROLLARY 7. $\ker T \cong \ker_X(A) \oplus \ker_X(A^*) \cong \ker T^*$.

REMARK. (1.1) It has been already observed (see Remark (0.1)) that at least for the generalized Dirac operators

$$\ker_X(A) = \{u \in C^\infty(X; E) : Au = 0 \text{ and } u | Y = 0\} =$$

$$= \{u \in C^\infty(X; E) : Au = 0 \text{ and } \operatorname{supp} u \subset X \setminus Y\}.$$

To apply Theorem 5 we must change T to get an invertible operator. Let $\eta_1, \ldots, \eta_k$ denote the orthogonal base of $\ker_X(A)$ and $\psi_1, \ldots, \psi_m$ the orthogonal base of $\ker_X(A^*)$. The orthogonal projection Q onto $\ker T$ is an integral operator with (smooth) kernel

(5) $$Q(s_1, s_2) = (\Sigma(s_1, \eta_p)\eta_p, \Sigma(s_2, \psi_q)\psi_q).$$

(5) implies that there exists N' a bicollar neighbourhood of Y with the following property

(6) $$Q(s_1, s_2) = 0 \text{ iff } \operatorname{supp} s_1 \subset N' \quad \text{and} \quad \operatorname{supp} s_2 \subset N'.$$

The formula $E_x^G \ni r \to r \in F_{J(x)}^{G^{-1}}$ defines the bundle isomorphism which covers the isometry J. We denote this bundle map by $\tilde{J}$. It is obvious that the operator $\tilde{J}Q : C^\infty(\tilde{X}; E^G) \to C^\infty(\tilde{X}; F^{G^{-1}})$ has a smooth kernel and that (6) holds for $\tilde{J}Q$.

This implies that perturbed operator $T + \tilde{J}Q$ is a pseudodifferential bijection and is a differential operator when restricted to N'. Now Theorem 3 follows from Theorem 5 because we have the following equalities

(7) $$P_+(T + \tilde{J}Q) = P_+(T) = P(A) \quad \text{and} \quad P_-(T + \tilde{J}Q) = P_-(T) = P(A^*).$$

Thus $H(A)$ and $G^{-1}(H(A^*))$ are complementary subspaces and $P(A)$ is the pseudodifferential projection onto $H(A)$ along $G^{-1}(H(A^*))$. From Green's formula we know that $G^{-1}(H(A^*)) \subset H(A)^\perp$. Hence $P(A)$ must be equal to the orthogonal projection onto $H(A)$.

2. PROOF OF PROPOSITION 2 AND PROPOSITION 4'

Proposition 2 is an elementary corollary of Theorem 3. We start with the calculation of $\ker A_{\Pi_+(B)}$. Thanks to the results of [1] and [11] we know that

$$\ker A_{\Pi_+(B)} = \{u \in C^\infty(X; E) : Au = 0 \quad \text{and} \quad \Pi_+(B)(u \mid Y) = 0\}$$

and it is obvious that $u \in \ker A_{\Pi_+(B)}$ implies $P(A)(u \mid Y) = u \mid Y$. This shows us that in case $\ker_X(A)$ is trivial the condition $u \in \ker A_{\Pi_+(B)}$ is equivalent to $u \mid Y \in \ker(\Pi_+(B)P(A))$. It is shown in [1] that the cokernel of $A_{\Pi_+(B)}$ is equal to the kernel of the adjoint operator and that

$$(A_{\Pi_+(B)})^* = (A^*)_{G\Pi_-(B)G^{-1}},$$

where we put $\Pi_-(B) = \mathrm{Id} - \Pi_+(B)$ (see also [11], Chapter VI, Theorem 7). Thus (under assumption that $\ker_X(A^*) = 0$) the kernel of $(A_{\Pi_+(B)})^*$ may be identified with the kernel of the operator $G\Pi_-(B)G^{-1}P(A^*)$. Now we apply Theorem 3

$$G\Pi_-(B)G^{-1}P(A^*) = G(\mathrm{Id} - \Pi_+(B))G^{-1}G(\mathrm{Id} - P(A))G^{-1} =$$

$$= G(\mathrm{Id} - \Pi_+(B))(\mathrm{Id} - P(A))G^{-1}$$

which gives us the following lemma:

LEMMA 8. coker $A_{\Pi_+(B)}$ = coker $\Pi_+(B)P(A)$.

Proof. It follows from the standard calculations that $\ker(\mathrm{Id}-\Pi_+(B))(\mathrm{Id}-P(A))$ is equal to the kernel of the operator $P(A)\Pi_+(B) : \mathrm{Ran}\, \Pi_+(B) \to H(A)$ which is adjoint to $\Pi_+(B)P(A)$.

Now we sketch the argument which gives the proof of the Proposition 4′. It is based on the notion of the spectral flow (see [2], [5], [13]).

Let $P_0, P_1 \in \mathrm{Pdiff}_P$; for simplicity (although it is not necessary) we assume that P_0, P_1 are orthogonal.

$$\{B_t = t(2P_1 - \mathrm{Id}) + (1-t)(2P_0 - \mathrm{Id})\}_{t \in I}$$

is a family of elliptic self-adjoint operators of order 0. $\mathrm{sf}\{B_t\}$ — the spectral flow of the family $\{B_t\}$ — is the difference between the number of the eigenvalues which change sign from $-$ to $+$, when t comes from 0 to 1, and the number of the eigenvalues which change sign from $+$ to $-$ (see [2], [5], [13]). An easy computation shows that

$$\mathrm{sf}\{B_t\} = \mathrm{index}\, P_1P_0, \tag{8}$$

where it is assumed as before that $P_1P_0 : \mathrm{Ran}\, P_0 \to \mathrm{Ran}\, P_1$. Now let us assume that P_1, P_0 are in the same path-connected component of Pdiff_P. We have a continuous family $\{P_t\}_{t\in I} \subset \mathrm{Pdiff}_P$ which joins P_1 with P_0. This gives us the family of elliptic self-adjoint operators $\{C_t = 2P_t - \mathrm{Id}\}_{t\in I}$. As a result we may define the family $\{\tilde{B}_t\}$ over the circle S^1

$$\tilde{B}_t = \begin{cases} B_{2t} & \text{for } 0 \leqslant t \leqslant 1/2 \\ C_{2(1-t)} & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

This family is homotopy equivalent (through families over S^1 of self-adjoint elliptic operators with the same principal symbol) to the constant family. Now the spectral flow is a homotopy invariant of such families (see [2], [5]). We know also that $\mathrm{sf}\{C_t\} = 0$. It is immediate from (8) that

$$0 = \mathrm{sf}\{\tilde{B}_t\} = \mathrm{sf}\{B_t\} = \mathrm{index}\, P_1P_0. \tag{9}$$

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