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QUANTUM DOOB-MEYER DECOMPOSITIONS

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INTRODUCTION

Doob-Meyer decompositions were discussed in quantum probability theory in [3, 5, 6, 9, 10] for the Itô-Clifford theory. This is the fermion analogue of the stochastic calculus of classical Brownian motion: one observes that Brownian motion can be realised via boson quantum fields, and then by replacing these fields by fermion fields the Itô-Clifford theory results. Much of this latter theory is facilitated by the existence of a faithful normal tracial state which carries with it the non-commutative integration theory developed in [12, 20, 24]. Thus, one defines L^2-martingales and via the martingale representation theorem [3, 22] one shows that the (modulus) square of any such martingale is the sum of an L^1-martingale and an increasing L^1-process [3, 9, 10]. This allows one to define the random inner product (or (pointed) bracket-process) between L^2-martingales which leads to the characterisation (as in standard stochastic calculus) of quantum stochastic integrals whose integrator is itself a quantum L^2-martingale [6, 9].

A quantum stochastic calculus for non-Fock bosons and fermions was first developed in [4], where the creation and annihilation operators act independently as integrators. (The corresponding Fock-space calculus was given in [1] and [14].) These non-Fock theories inhabit certain quasi-free representation spaces of the C.C.R. and C.A.R., respectively. For these there is no tracial state and the non-commutative integration theory above is not applicable. However, these representations do possess cyclic and separating vectors and so one can apply Tomita-Takesaki modular theory to reasonable effect. In particular, this gives a correspondence between Hilbert space-valued and operator-valued processes [4, 13, 15, 16, 17, 22, 23].

The generalisation of an L^2-valued process is a Hilbert space-valued process. Thus, an L^2-martingale in these non-Fock theories is simply a vector-valued martingale. For such martingales one also has martingale representation theorems [13, 15, 16, 17, 22, 23]. The question then arises as to the meaning of the square of such a process. The answer is provided by the non-commutative (and non-tracial)

L^1-space of [19]. (See also [25, 26, 27].) Using this L^1-space, we can formulate a Doob-Meyer decomposition for the square of a vector-valued martingale and establish uniqueness. This leads to a characterisation of stochastic integrals with non-Fock martingales as "quantum times".

In Section 1, we set up the non-commutative L^1-theory for any von Neumann algebra with cyclic and separating vector and various properties are established. Conditional expectations are considered in Section 2 and in Section 3, we consider filtrations of von Neumann algebras, processes and the notion of naturalness. The results here extend those of [9] to cover the non-tracial case. (It is clear that other notions such as mean forward derivatives etc. defined for the tracial case in [9] can be defined here and will lead to analogous results.)

The Doob-Meyer decomposition (existence and uniqueness) is discussed for fermions in Section 4, and for bosons in Section 5. As pointed out there, the statements and methods of proof are identical except a few obvious changes of notation.

In Section 6, we define the random inner product between L^2-valued martingales, and L^1-valued integrals with integrators given by such a random inner product. We also define integrals where the integrator is a general centred L^2-martingale and obtain a characterisation of these as processes. A version of the Kunita-Watanabe inequality is also obtained.

In Section 7, we consider stopping times in this non-commutative context, and, using the results of Section 6, characterise stopped L^2-martingales as processes. This extends the results of [8].

1. THE NON-COMMUTATIVE L^1 SPACE

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and let Ω be a cyclic and separating unit vector for $\mathcal{M}$. Let $S = J\Delta^{1/2}$ denote, as usual, the closure of the conjugate linear operator $x\Omega \mapsto x^*\Omega$, $x \in \mathcal{M}$. Then $J\Omega = \Omega$, $J^2 = \mathbf{1}$ and $J\mathcal{M}J = \mathcal{M}'$, the commutant of $\mathcal{M}$. To each $x \in \mathcal{M}$ we associate the functional ω_x in $\mathcal{M}_*$ given by

$$\omega_x(y) = (y\Omega, Jx\Omega), \quad y \in \mathcal{M}.$$

The basic properties of this association are given in the following.

PROPOSITION 1.1. *The map $x \mapsto \omega_x$ is a linear order preserving injection of $\mathcal{M}$ into $\mathcal{M}_*$. Moreover, the image of $\mathcal{M}^+$ is norm dense in $\mathcal{M}_*^+$, the positive part of $\mathcal{M}_*$, and the image of $\mathcal{M}$ is norm dense in $\mathcal{M}_*$.*

Proof. Clearly the map $x \mapsto \omega_x$, $x \in \mathscr{M}$, is linear and, since Ω is cyclic and separating for $\mathscr{M}$, it is easy to see that it is injective. Now let $x \in \mathscr{M}^+$. Then $x = z^*z$ for some $z \in \mathscr{M}$ and we have

$$\omega_x(y) = (y\Omega, Jz^*z\Omega) = (y\Omega, Jz^*JJz\Omega) = (yJz\Omega, Jz\Omega)$$

for $y \in \mathscr{M}$. Hence $\omega_x \geqslant 0$ and so $x \mapsto \omega_x$ is order preserving.

Let $\varphi \in \mathscr{M}_*^+$. Then, since $\mathscr{M}$ has a separating vector, there is a vector ζ in $\mathscr{H}$ such that $\varphi(y) = (y\zeta, \zeta)$, $y \in \mathscr{M}$. By cyclicity, there is a sequence (x_n) in $\mathscr{M}$ such that $x_n\Omega$ converges to $J\zeta$ in $\mathscr{H}$. But then $Jx_n\Omega \to \zeta$ and

$$\varphi(y) = \lim(yJx_n\Omega, Jx_n\Omega) = \lim(y\Omega, Jx_n^*x_n\Omega) = \lim \omega_{x_n^*x_n}(y)$$

uniformly for $y \in \mathscr{M}_1$. Hence the image of $\mathscr{M}^+$ is norm dense in $\mathscr{M}_*^+$. Finally, we observe that any element of $\mathscr{M}_*$ can be written as a linear combination of elements of $\mathscr{M}_*^+$ and so we deduce that the image, $\{\omega_x : x \in \mathscr{M}\}$, of $\mathscr{M}$ is norm dense in $\mathscr{M}_*$. ▨

DEFINITION 1.2. Set $\|x\|_1 = \|\omega_x\|$ for $x \in \mathscr{M}$ and let $L^1(\mathscr{M})$ denote the completion of $\mathscr{M}$ with respect to the $\|\cdot\|_1$-norm.

It follows from Proposition 1.1 that $\|\cdot\|_1$ is indeed a norm on $\mathscr{M}$ and that the map of $\mathscr{M}$ into $\mathscr{M}_*$ given by $x \mapsto \omega_x$ extends to an isometric isomorphism of $L^1(\mathscr{M})$ onto $\mathscr{M}_*$ which we again denote by $x \mapsto \omega_x$.

If Ω defines a tracial state on $\mathscr{M}$, that is, $(xy\Omega, \Omega) = (yx\Omega, \Omega)$ for $x, y \in \mathscr{M}$, hen one has [12, 20]

$$\|x\|_1 = (|x|\Omega, \Omega) = \sup_{\|y\| \leqslant 1} |(xy\Omega, \Omega)| =$$

$$= \sup_{\|y\| \leqslant 1} |(y\Omega, x^*\Omega)| = \sup_{\|y\| \leqslant 1} |(y\Omega, Jx\Omega)|$$

for $x \in \mathscr{M}$. Thus, the $\|\cdot\|_1$-norm of Definition 1.2 is the generalisation of the tracial $\|\cdot\|_1$-norm to the non tracial case. Note that since $S = J$ when Ω is tracial we could also write $\omega_y(x) = (y\Omega, Sx\Omega)$ for Ω tracial. This, however, does not appear to yield the appropriate generalisation to the non tracial situation.

With the following operations $L^1(\mathscr{M})$ is an $\mathscr{M}$ bimodule and the action is norm contractive. The idea is simply that we regard elements of $L^1(\mathscr{M})$ as functionals and let $\mathscr{M}$ act in the usual way on such objects. So for $x \in \mathscr{M}$ and $y \in L^1(\mathscr{M})$ we define xy to be the element of $L^1(\mathscr{M})$ corresponding to the functional $x\omega_y$ where $x\omega_y(z) = \omega_y(zx)$ for $z \in \mathscr{M}$. Similarly yx is defined by $\omega_y x(z) = \omega_y(xz)$ for

$z \in \mathcal{M}$. When Ω is non tracial the actions of $\mathcal{M}$ on $L^1(\mathcal{M})$ defined above are no longer consistent with the multiplication in $\mathcal{M}$, that is, when Ω is tracial $\omega_{xy} = x\omega_y = \omega_x y$, for $x, y \in \mathcal{M}$, but, when it is not tracial $\omega_{xy} = (Jx^*J)\omega_y$ and this will not be the same as $x\omega_y$ for every $x, y \in \mathcal{M}$.

Consider now the map $x \mapsto x\Omega$, $x \in \mathcal{M}$. This is a linear injection of $\mathcal{M}$ into a dense subspace of $\mathcal{H}$. If we set $\|x\|_2 = \|x\Omega\|$, $x \in \mathcal{M}$, then $\|\cdot\|_2$ is a norm on $\mathcal{M}$. Denote by $L^2(\mathcal{M})$ the completion of $\mathcal{M}$ in $\|\cdot\|_2$. Then $L^2(\mathcal{M})$ is isometrically isomorphic to $\mathcal{H}$. Now each element $\zeta \in \mathcal{H}$ determines an element ω_ζ of $\mathcal{M}_*$ given by $y \mapsto (y\Omega, J\zeta)$, $y \in \mathcal{M}$ and we have the following.

PROPOSITION 1.3. *The map $\zeta \mapsto \omega_\zeta$ is a linear injection of $\mathcal{H}$ into a norm dense subspace of $\mathcal{M}_*$.*

Proof. It is clear that $\zeta \mapsto \omega_\zeta$ is a linear injection of $\mathcal{H}$ into $\mathcal{M}_*$. Moreover, $\{\omega_\zeta : \zeta \in \mathcal{H}\}$ contains the set $\{\omega_x : x \in \mathcal{M}\}$, since if $\zeta = x\Omega$ then $\omega_\zeta = \omega_x$. The result now follows by Proposition 1.1. ∎

Thus we have injections $\mathcal{M} \hookrightarrow L^1(\mathcal{M})$, $\mathcal{M} \hookrightarrow L^2(\mathcal{M})$ and $L^2(\mathcal{M}) \hookrightarrow L^1(\mathcal{M})$ with dense ranges. These are consistent in that the injection $\mathcal{M} \hookrightarrow L^1(\mathcal{M})$ is the composition of the injections $\mathcal{M} \hookrightarrow L^2(\mathcal{M})$ and $L^2(\mathcal{M}) \hookrightarrow L^1(\mathcal{M})$, as is easily checked.

Let $L^\infty(\mathcal{M})$ denote $\mathcal{M}$ equipped with its C^*-norm. Then, via the identifications above, we have $L^\infty(\mathcal{M}) \subseteq L^2(\mathcal{M}) \subseteq L^1(\mathcal{M})$ and we may regard the first space in each of the three inclusions as being dense in the second. It is easy to see too that $\|x\|_2 \leqslant \|x\|_\infty$ for $x \in L^\infty(\mathcal{M})$ and $\|\zeta\|_1 \leqslant \|\zeta\|_2$ when $\zeta \in L^2(\mathcal{M})$. This of course, is the usual case when $\mathcal{M}$ is abelian, or more generally, when Ω defines a tracial state on $\mathcal{M}$.

DEFINITION 1.4. For $\zeta \in \mathcal{H}$, let $|\zeta|^2 \in \mathcal{M}_*$ be given by $|\zeta|^2(x) = (xJ\zeta, J\zeta)$, $x \in \mathcal{M}$.

PROPOSITION 1.5. *For any $\zeta \in \mathcal{H}$, we have*

(i) $\||\zeta|^2\|_1 = \|\zeta\|^2$.

(ii) *If $\zeta = x\Omega$, $x \in \mathcal{M}$, then $|\zeta|^2 = \omega_{x^*x}$.*

(iii) $\{|\zeta|^2 : \zeta \in \mathcal{H}\} = \mathcal{M}_*^+$.

Proof. (i) $|\zeta|^2 \in \mathcal{M}_*^+$ and so $\||\zeta|^2\|_1 = |\zeta|^2(\mathbf{1}) = (J\zeta, J\zeta) = \|\zeta\|^2$.

(ii) This is clear.

(iii) If $\varphi \in \mathcal{M}_*^+$ then there is $\zeta \in \mathcal{M}$ such that $\varphi(x) = (xJ\zeta, J\zeta)$, $x \in \mathcal{M}$; that is $\varphi = |\zeta|^2$. ∎

More generally we can define a product between elements of $\mathcal{H}$.

DEFINITION 1.6. For $\eta, \zeta \in \mathscr{H}$, let $(\zeta\eta)$ be the element of $\mathscr{M}_*$ given by

$$(\zeta\eta)(x) = (xJ\zeta, J\eta)$$

for $x \in \mathscr{M}$.

Thus $\|\zeta\|^2 = (\zeta\zeta)$, and $(x\Omega y\Omega) = \omega_{x^*y}$ for $x, y \in \mathscr{M}$.

PROPOSITION 1.7. *The map $\mathscr{H} \times \mathscr{H} \to \mathscr{M}_*$ given by $(\zeta, \eta) \mapsto (\zeta\eta)$ is a jointly norm continuous sesquilinear mapping (linear on the right) onto $\mathscr{M}_*$.*

Proof. The map is clearly sesquilinear. Let $\zeta, \zeta', \eta, \eta' \in \mathscr{H}$. Then, for $x \in \mathscr{M}$,

$$|(\zeta'\eta')(x) - (\zeta\eta)(x)| = |(xJ\zeta', J\eta') - (xJ\zeta, J\eta)| \leqslant$$

$$\leqslant |(xJ(\zeta' - \zeta), J\eta')| + |(xJ\zeta, J(\eta' - \eta))| \leqslant$$

$$\leqslant \|x\| \, \|\zeta' - \zeta\| \, \|\eta'\| + \|x\| \, \|\zeta\| \, \|\eta' - \eta\|$$

from which joint norm continuity follows. To see that the product mapping is onto $\mathscr{M}_*$, let $\varphi \in \mathscr{M}_*$. Then φ has a polar decomposition $\varphi = u\psi$ where $u \in \mathscr{M}$ and $\psi \in \mathscr{M}_*^+$. But then there is $\xi \in \mathscr{H}$ such that $\psi(x) = (x\xi, \xi)$, $x \in \mathscr{M}$ and so $\varphi(x) = (xu\xi, \xi)$. Setting $\zeta = Ju\xi$ and $\eta = J\xi$ gives $\varphi = (\zeta\eta)$. ▨

Thus we have $L^2(\mathscr{M}) \cdot L^2(\mathscr{M}) = L^1(\mathscr{M})$ and $|L^2(\mathscr{M})|^2 = L^1(\mathscr{M})^+$ in agreement with the case when Ω is a tracial state on $\mathscr{M}$.

2. CONDITIONAL EXPECTATIONS

Let $\mathscr{N}$ be a von Neumann subalgebra of $\mathscr{M}$, and let $L^1(\mathscr{N})$ denote the completion of $\mathscr{N}$ with respect to $\|\cdot\|_1$ and $L^2(\mathscr{N})$ the completion of $\mathscr{N}$ with respect to $\|\cdot\|_2$. Then $L^1(\mathscr{N})$ is a closed subspace of $L^1(\mathscr{M}) \cong \mathscr{M}_*$ and $L^2(\mathscr{N})$ is a closed subspace of $L^2(\mathscr{M}) \cong \mathscr{H}$. Let P denote the orthogonal projection of $\mathscr{H}$ onto $\overline{\mathscr{N}\Omega}$. Bearing in mind the various identifications, P is nothing more than the orthogonal projection of $L^2(\mathscr{M})$ onto $L^2(\mathscr{N})$.

Suppose there is a conditional expectation $e : \mathscr{M} \to \mathscr{N}$ with $\omega \circ e = \omega$, where $\omega(\cdot) = (\cdot\Omega, \Omega)$. Then $Px\Omega = e(x)\Omega$ for $x \in \mathscr{M}$. Indeed, for $y \in \mathscr{N}$,

$$(Px\Omega, y\Omega) = (x\Omega, y\Omega) = (y^*x\Omega, \Omega) = (e(y^*x)\Omega, \Omega) = (e(x)\Omega, y\Omega)$$

giving $Px\Omega = e(x)\Omega$. Thus e is unique if it exists and P is the L^2 extension of $e : L^\infty(\mathscr{M}) \to L^\infty(\mathscr{N})$. We wish to extend e to $L^1(\mathscr{M})$.

DEFINITION 2.1. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ and suppose that there is a normal conditional expectation $e : \mathcal{M} \to \mathcal{N}$ with $\omega \circ e = \omega$. We define $E : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ by $E\varphi = \varphi \circ e$, $\varphi \in L^1(\mathcal{M})$.

Note that E is a well-defined contractive projection of $L^1(\mathcal{M})$ into $L^1(\mathcal{M})$. By [21], such an e, and hence E, will exist if and only if $\mathcal{N}$ is globally invariant under the modular automorphism group induced by Ω on $\mathcal{M}$. Thus if e exists as above, then J commutes with P the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{N}\Omega}$. We shall need this fact in the next result.

PROPOSITION 2.2. *With the notation above, we have*

(i) *E extends both e and P.*

(ii) *$E\varphi = \varphi$ for $\varphi \in L^1(\mathcal{N})$.*

(iii) *$E(L^1(\mathcal{M})) = L^1(\mathcal{N})$.*

Proof. (i) For any $x, y \in \mathcal{M}$,

$$E\omega_x(y) = \omega_x(e(y)) = (e(y)\Omega, Jx\Omega) =$$

$$= (Py\Omega, Jx\Omega) = (y\Omega, JPxQ) = \qquad \text{(since } JP = PJ\text{)}$$

$$= (y\Omega, Je(x)\Omega) = \omega_{e(x)}(y),$$

i.e. $E\omega_x = \omega_{e(x)}$. Furthermore, for any $\zeta \in \mathcal{H}$ and $x \in \mathcal{M}$,

$$E\omega_\zeta(x) = \omega_\zeta(e(x)) = (e(x)\Omega, J\zeta) =$$

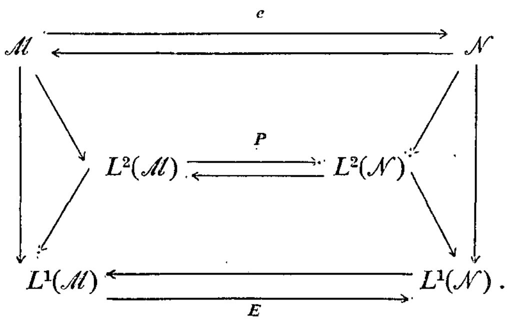
$$= (Px\Omega, J\zeta) = (x\Omega, JP\zeta) = \omega_{P\zeta}(x),$$

i.e. $E\omega_\zeta = \omega_{P\zeta}$. Thus $E \restriction L^\infty(\mathcal{M}) = e$ and $E \restriction L^2(\mathcal{M}) = P$.

(ii) For given $\varphi \in L^1(\mathcal{N})$ there is a sequence (y_n) in $\mathcal{N}$ such that $\omega_{y_n} \to \varphi$ in $\|\cdot\|_1$, and so $E\omega_{y_n} \to E\varphi$ in L^1 norm. But $E\omega_{y_n} = \omega_{e(y_n)} = \omega_{y_n}$ for all n which implies that $E\varphi = \varphi$.

(iii) Let $\varphi \in L^1(\mathcal{M})$. Then there is a sequence (x_n) in $\mathcal{M}$ such that φ is the L^1 limit of (ω_{x_n}). Of course then $E\omega_{x_n} \to E\varphi$ in L^1 too. But as above, $E\omega_{x_n} = \omega_{e(x_n)}$ with $e(x_n) \in \mathcal{N}$ for all n. It follows that $E\varphi \in L^1(\mathcal{N})$. Thus E maps $L^1(\mathcal{M})$ onto $L^1(\mathcal{N})$ by (ii). ▨

We have the following commutative diagram:



The non-horizontal arrows are linear injections with dense ranges. The horizontal arrows to the right are linear surjections and those to the left are their right inverses, the inclusion injections.

3. NATURAL PROCESSES

It is convenient to discuss the notion of a natural process in the abstract and then apply the results obtained to the particular set-up in Section 5. Natural processes were introduced into non-commutative probability theory in [9] to give uniqueness of Doob-Meyer decompositions of submartingales as in the classical theory [18]. The formalism and results of [9], cast within the tracial theory, carry over with minimal changes to the present possibly non-tracial context.

Let $\{\mathcal{M}_s : s \geqslant 0\}$ be a filtration of von Neumann subalgebras of $\mathcal{M}$; thus $\mathcal{M}_s \subseteq \mathcal{M}_t \subseteq \mathcal{M}$ for $0 \leqslant s \leqslant t$. We shall suppose that for each $t \geqslant 0$ there is an ω-invariant normal conditional expectation $e_t : \mathcal{M} \to \mathcal{M}_t$ as in Section 2. Furthermore we will suppose that the filtration is continuous; i.e.

$$\mathcal{M}_s = \bigcap_{t>s} \mathcal{M}_t \quad \text{for } s \geqslant 0$$

and

$$\mathcal{M}_t = \left(\bigcup \{\mathcal{M}_s : s < t\}\right)''$$

for each $t > 0$.

DEFINITION 3.1. An L^1 *process* is a family $\{X_t : t \geqslant 0\}$ with $X_t \in L^1(\mathcal{M}_t)$ for $t \geqslant 0$. An L^1 *martingale* is an L^1 process such that $E_s X_t = X_s$ for all $0 \leqslant s \leqslant t$.

Thus, by definition, a process means an "adapted process". In an analogous way we define L^2 ($\cong \mathcal{H}$) and L^∞($\cong \mathcal{M}$)-valued processes and martingales (using

P_s, the projection of $\mathscr{H}_s$ onto $\overline{\mathscr{M}_s\Omega}$, and e_s). Continuity of the filtration implies continuity of the conditional expectations, as the next result shows.

PROPOSITION 3.2. *The map $s \mapsto P_s$ is strongly continuous on $\mathscr{H}$ and the map $s \mapsto E_s$ is strongly continuous on $L^1(\mathscr{M})$, $s \geqslant 0$.*

Proof. The strong continuity of $s \mapsto P_s$ is proved in [7]. To establish the strong continuity of $s \mapsto E_s$ let $\varphi \in L^1(\mathscr{M})$ and $\varepsilon > 0$. We have

$$\|E_s\varphi - E_t\varphi\|_1 \leqslant \|E_s\varphi - E_s\omega_x\|_1 + \|E_s\omega_x - E_t\omega_x\|_1 + \|E_t\omega_x - E_t\varphi\|_1 \leqslant$$

$$\leqslant 2\|\varphi - \omega_x\|_1 + \|E_s\omega_x - E_t\omega_x\|_1 < 2\varepsilon + \|E_s\omega_x - E_t\omega_x\|_1,$$

for suitable $x \in \mathscr{M}$.

Bearing in mind the identifications of $L^2(\mathscr{M})$ and $L^\infty(\mathscr{M})$ with subspaces of $L^1(\mathscr{M})$ and the fact that E_s and E_t extend P_s and P_t respectively and that $\|\zeta\|_1 \leqslant \|\zeta\|_2$ for $\zeta \in L^2(\mathscr{M})$ we can rewrite the last term and estimate it by

$$\|E_s\omega_x - E_t\omega_x\|_1 = \|P_s\omega_x - P_t\omega_x\|_1 \leqslant \|P_s\omega_x - P_t\omega_x\|_2.$$

But we already know that P_s is strongly continuous in s, so the result follows.

The following definition of naturalness is the obvious rewording of that given in [9], which in turn was a reformulation of that of [18].

DEFINITION 3.3. An L^1 process $\{A_t : t \geqslant 0\}$ is *natural* if for any $t > 0$ and any sequence (σ_n) of finite subdivisions of $[0, t]$ with mesh $\sigma_n \to 0$ as $n \to \infty$ we have

$$\sum_j E_{s_j^n}(A_{s_{j+1}^n} - A_{s_j^n})(y) \to A_t(y) \tag{3.1}$$

as $n \to \infty$ for each $y \in \mathscr{M}$. Here $\sigma_n = \{s_j^n\}$.

Clearly the set of natural L^1 processes forms a linear space. We also observe that, as in [9], it follows that $A_0 = 0$ because if $y \in \mathscr{M}_0$ then the left hand side of 3.1 is just $(A_t - A_0)(y)$. Furthermore it is clear from 3.1 that if $\{Z_t : t \geqslant 0\}$ is a natural L^1 martingale then $Z_t = 0$ for all $t > 0$, and hence, by continuity, for all $t \geqslant 0$.

For any Borel measure ν on $\mathbf{R}^+$, by a process in $L^1_{loc}(\mathbf{R}^+, \nu, L^1(\mathscr{M}))$ we mean an element f of $L^1_{loc}(\mathbf{R}^+, \nu, L^1(\mathscr{M}))$ such that $f(s) \in L^1(\mathscr{M}_s)$ ν-almost everywhere. Thus if f is a process in $L^1_{loc}(\mathbf{R}^+, \nu, L^1(\mathscr{M}))$, then

$$t \mapsto \int_0^t f(s)\, d\nu(s) \equiv \int_{(0,\, t]} f\, d\nu$$

defines an L^1 process.

THEOREM 3.4. *For any process $f \in L^1_{loc}(\mathbf{R}^+, \nu, L^1(\mathcal{M}))$, the L^1 process $A_t = \int_0^t f(s)\,d\nu(s)$, $t \geqslant 0$, is natural.*

Proof. Let $t > 0$ and $\sigma_n = \{s_j^n\}$ with $0 = s_0^n < s_1^n < \ldots < s_{k(n)}^n = t$, be a sequence of subdivisions of $[0, t]$ with mesh $\sigma_n \to 0$ as $n \to \infty$. Then for any $y \in \mathcal{M}$,

$$\sum_j E_{s_j^n}(A_{s_{j+1}^n} - A_{s_j^n})(y) = \sum_j (A_{s_{j+1}^n} - A_{s_j^n})(e_{s_j^n}(y)) = \int_0^t \alpha_n(s)\,d\nu(s)$$

where $\alpha_n(s) = \sum_j f(s)(e_{s_j^n}(y))\chi_{(s_j^n, s_{j+1}^n]}(s)$.

Now by Proposition 3.2, we see that $\alpha_n(s) \to f(s)(e_s(y))$ as $n \to \infty$ for each $s \in (0, t]$. But $f(s)(e_s(y)) = E_s f(s)(y) = f(s)(y)$ ν-a.e. and so $\alpha_n(\cdot)$ converges to $f(\cdot)(y)$ ν-a.e. on $(0, t]$ as $n \to \infty$. Furthermore

$$|\alpha_n(s)| \leqslant \|y\|\,\|f(s)\|_1$$

which is ν-integrable over $(0, t]$ and so we deduce that

$$\lim_n \int_0^t \alpha_n(s)\,d\nu(s) = \int_0^t f(s)(y)\,d\nu(s) = \left(\int_0^t f\,d\nu\right)(y) = A_t(y).$$

This theorem is the direct analogue of Theorem 2.6 in [9], and will be called upon in Section 5.

4. DOOB-MEYER DECOMPOSITIONS OVER THE C.A.R.

We briefly recall the quasi-free fermion quantum stochastic calculus of [4]. We denote by $\mathfrak{A}$ the C.A.R. C^*-algebra over $L^2(\mathbf{R}^+)$ in the gauge invariant quasi-free representation determined by the state

$$\omega(b^*(f)b(g)) = \int_0^\infty f(s)\,\overline{g(s)}\,\rho(s)ds$$

where $0 < \rho < 1$ a.e. and $f, g \in L^2(\mathbf{R}^+)$. $\mathfrak{A}$ is realised concretely as operators on $\mathcal{H}$, the tensor product of two copies of the antisymmetric Fock space over $L^2(\mathbf{R}^+)$. Then $\Omega = \Omega_0 \otimes \Omega_0$, where Ω_0 is the Fock vacuum vector, is cyclic and separating for $\mathfrak{A}''$ on $\mathcal{H}$.

Let $\mathfrak{A}_t$ be the C^*-algebra generated by the operators

$$\{b(f) : f \in L^2(\mathbf{R}^+),\ \operatorname{supp} f \subseteq [0, t]\}$$

for $t \geqslant 0$, and let $\mathscr{H}_t = \overline{\mathfrak{A}_t \Omega}$. Then $\{\mathfrak{A}_t'' : t \geqslant 0\}$ is a continuous filtration of von Neumann subalgebras of $\mathfrak{A}''$, and $\{\mathscr{H}_t : t \geqslant 0\}$ is a filtration of Hilbert subspaces of $\mathscr{H}$. These filtrations determine a notion of adaptedness; i.e. the notion of process. Furthermore, each $\mathfrak{A}_t''$ is globally invariant under the automorphism group induced by ω on $\mathfrak{A}''$. Hence there exist unique normal ω-invariant conditional expectations $e_t : \mathfrak{A}'' \mapsto \mathfrak{A}_t''$, $t \geqslant 0$ as in Section 2.

Let $b_t^* = b^*(\chi_{[0,t]})$ and $b_t = b(\chi_{[0,t]})$, $t \geqslant 0$. Then $\{b_t^* : t \geqslant 0\}$ and $\{b_t : t \geqslant 0\}$ are $\mathfrak{A}$-valued martingales. If $h = \sum_i \chi_{[s_i, s_{i+1})} h_i$ is a simple $\mathfrak{A}\Omega$-valued process, the stochastic integrals $\int_0^t db_s^* h(s)$ and $\int_0^t db_s h(s)$ are defined to be $\sum_i (b^*_{s_{i+1}} - b^*_{s_i}) h_i$ and $\sum_i (b_{s_{i+1}} - b_{s_i}) h_i$ respectively. These integrals are orthogonal in $\mathscr{H}$ and satisfy the isometry relations

$$\left\| \int_0^t db_s^* h(s) \right\|^2 = \int_0^t \|h\|^2 (1 - \rho) ds$$

and

$$\left\| \int_0^t db_s h(s) \right\|^2 = \int_0^t \|h\|^2 \rho ds.$$

This allows one to define the corresponding integrals for a wider class of processes;

$$\int_0^t db^* \xi \quad \text{for } \xi \in L^2_{\text{loc}}(\mathbf{R}^+,\ (1 - \rho(s)) ds,\ \mathscr{H})$$

and

$$\int_0^t db \eta \quad \text{for } \eta \in L^2_{\text{loc}}(\mathbf{R}^+,\ \rho(s) ds, \mathscr{H})$$

where $t \geqslant 0$. These spaces of integrands are obtained by completions of the simple $\mathscr{H}$-valued processes with respect to the norms determined by the right hand sides of the isometry relations. The isometry relations remain valid for such ξ and η.

Moreover, these integrals are centered $\mathscr{H}$-valued martingales. For further details see [4]. The converse is the martingale representation theorem [22].

THEOREM 4.1. *Every $\mathscr{H}$-valued martingale $\{\zeta_t : t \geqslant 0\}$ can be written uniquely as*

$$\zeta_t = \alpha\Omega + \int_0^t db^*\xi + \int_0^t db\eta$$

for $\alpha \in \mathbf{C}$, and processes $\xi \in L^2_{\text{loc}}(\mathbf{R}^+, (1 - \rho(s))ds, \mathscr{H})$ and $\eta \in L^2_{\text{loc}}(\mathbf{R}_+, \rho(s)ds, \mathscr{H})$.

We wish to discuss the formulation of a Doob-Meyer decomposition for the "square of ζ_t". We have seen, in Section 1, that for $\zeta \in \mathscr{H}$, $|\zeta|^2$ defines an element of $L^1(\mathfrak{A}'')$, in analogy whit the tracial theory of non-commutative integration. It is within this framework that the Doob-Meyer decomposition can be satisfactorily formulated.

THEOREM 4.2. *For any $\mathscr{H}$-valued martingale $\{\zeta_t : t \geqslant 0\}$ there is an L^1 martingale $\{Z_t : t \geqslant 0\}$ and a positive increasing L^1 process $\{A_t : t \geqslant 0\}$ null at $t = 0$ such that*

$$|\zeta_t|^2 = Z_t + A_t$$

for $t \geqslant 0$. Moreover this decomposition is unique.

Proof. Existence. By Theorem 4.1, we can write ζ_t uniquely as

$$\zeta_t = \alpha\Omega + \int_0^t db^*\xi + \int_0^t db\eta$$

for $t \geqslant 0$. Fix $t \geqslant 0$ and let (h_n) and (g_n) be sequences of simple $\mathfrak{A}$-valued processes such that $h_n\Omega \to \xi$ in $L^2_{\text{loc}}(\mathbf{R}^+, (1 - \rho(s))ds, \mathscr{H})$ and $g_n\Omega \to \eta$ in $L^2_{\text{loc}}(\mathbf{R}^+, \rho(s)ds, \mathscr{H})$. Then, it follows from the isometry relations, that

$$\zeta_t = \lim_n\left(\alpha\Omega + \int_0^t db^*h_n\Omega + \int_0^t dbg_n\Omega\right).$$

Write the right hand side as $\alpha\Omega + H_n\Omega + G_n\Omega$ and observe that $H_n, G_n \in \mathfrak{A}$. By Proposition 1.7,

$$\|\zeta_t\|^2 = L^1\text{-lim}\,\|\alpha\Omega + H_n\Omega + G_n\Omega\|^2 =$$

$$= L^1\text{-lim}(\alpha\mathbf{1} + H_n + G_n)^*(\alpha\mathbf{1} + H_n + \Omega_n)\Omega =$$

$$= L^1\text{-lim}(\alpha^*\alpha\mathbf{1} + \alpha(H_n^* + G_n^*)\Omega + \alpha^*(H_n + G_n)\Omega + H_n^*G_n\Omega + G_n^*H_n\Omega +$$

$$+ H_n^*H_n\Omega + G_n^*G_n\Omega).$$

By [15, 16] we can write

$$H_n^*H_n\Omega = F_n'\Omega + \int_0^t h_n^*h_n(1-\rho(s))\mathrm{d}s\Omega$$

and

$$G_n^*G_n\Omega = F_n''\Omega + \int_0^t g_n^*g_n\rho(s)\mathrm{d}s\Omega$$

where $F_n'\Omega$ and $F_n''\Omega$ are stochastic integrals. Thus, we can write $(\alpha\mathbf{1} + H_n + G_n)^*(\alpha\mathbf{1} + H_n + G_n)\Omega$ as $Z_n + A_n$ where Z_n is equal to $|\alpha|^2\Omega$ plus an $\mathscr{H}$-valued stochastic integral, and

$$A_n = \int_0^t h_n^*h_n\Omega(1-\rho(s))\mathrm{d}s + \int_0^t g_n^*g_n\Omega\rho(s)\mathrm{d}s.$$

Clearly $A_n \in \mathfrak{A}_t\Omega$.

Now as an element of $L^1(\mathfrak{A}'')$, A_n is given by

$$A_n(x) = \int_0^t (x\Omega, Jh_n^*h_n\Omega)(1-\rho)\mathrm{d}s + \int_0^t (x\Omega, Jg_n^*g_n\Omega)\rho\mathrm{d}s =$$

$$= \int_0^t (xJh_n\Omega, Jh_n\Omega)(1-\rho)\mathrm{d}s + \int_0^t (xJh_n\Omega, Jg_n\Omega)\rho\mathrm{d}s$$

for $x \in \mathfrak{A}''$.

But then it is easily verified, by repeated application of Schwarz, inequality, that

$$L^1\text{-}\lim A_n = \int_0^t (\cdot J\xi, J\xi)(1-\rho)\mathrm{d}s + \int_0^t (\cdot J\eta, J\eta)\rho \mathrm{d}s = A_t.$$

We note that $A_t \in L^1(\mathfrak{A}_t'')$.

It follows that Z_n converges in $L^1(\mathfrak{A}'')$ to $Z_t = |\zeta_t|^2 - A_t$. Now, for $0 \leqslant s \leqslant t$, $E_s Z_n$ is obtained from Z_n by replacing the approximating sequences (h_n) and (g_n) by $(\chi_{[0,s)}h_n)$ and $(\chi_{[0,s)}g_n)$, respectively. Thus, repeating the argument above, we obtain $L^1\text{-}\lim E_s Z_n = |\zeta_s|^2 - A_s$. Since E_s is $L^1(\mathfrak{A})$ continuous, we deduce that $E_s Z_t = |\zeta_s|^2 - A_s$; i.e. $\{Z_t : t \geqslant 0\}$ is an L^1 martingale. Clearly A_t is positive for each $t \geqslant 0$, $A_0 = 0$, and $A_t \geqslant A_s$ in $L^1(\mathfrak{A}'')$ for $0 \leqslant s \leqslant t$. This establishes the existence of a Doob-Meyer decomposition for $\{|\zeta_t|^2 : t \geqslant 0\}$.

We note immediately that, since

$$A_t = \int_0^t (\cdot J\xi, J\xi)(1-\rho)\mathrm{d}s + \int_0^t (\cdot J\eta, J\eta)\rho\, \mathrm{d}s$$

then, by Theorem 3.4, $\{A_t : t \geqslant 0\}$ is natural. Subject to being natural, the increasing process in the Doob-Meyer decomposition of $\{|\zeta_t|^2 : t \geqslant 0\}$ has to be unique (see the proof of uniqueness in Theorem 5.2 below). However something stronger holds in this context; subject to being null at 0, the increasing process in the Doob-Meyer decomposition has to be unique.

Uniqueness. The proof of the uniqueness of the decomposition proceeds as for that in [9]. We begin by noting that there is a σ-strongly dense set of elements $y \in \mathfrak{A}''$ such that $s \mapsto e_s(y)$ is continuous as a map from $\mathbf{R}^+$ into $L^\infty(\mathfrak{A}'')$. Indeed such a set is furnished by the polynomial algebra generated by $\{b^*(f), b(g) : f, g \in L^2(\mathbf{R}^+)\}$. We claim that for such y and any increasing L^1 process $\{A_t : t \geqslant 0\}$ null at $t = 0$ we have

$$\sum_{i=1}^{k(n)} (A_{s_{i+1}^n} - A_{s_i^n})(e_{s_i^n}(y)) \to A_t(y) \tag{4.1}$$

as $n \to \infty$, where $0 = s_1 < s_2 < \ldots < s_{k(n)+1} = t$ is a subdivision, σ_n, of $[0, t]$

with mesh $\sigma_n \to 0$ as $n \to \infty$. We have, for $\varepsilon > 0$ and sufficiently large $n \in \mathbf{N}$,

$$\left| A_t(y) - \sum_{i=1}^{k(n)} (A_{s_{i+1}^n} - A_{s_i^n})(e_{s_i^n}(y)) \right| =$$

$$= \left| \sum_{i=1}^{k(n)} (A_{s_{i+1}^n} - A_{s_i^n})(y) - \sum_{i=1}^{k(n)} (A_{s_{i+1}^n} - A_{s_i^n})(e_{s_i^n}(y)) \right| = \qquad \text{since } A_0 = 0$$

$$= \left| \sum_{i=1}^{k(n)} (A_{s_{i+1}^n} - A_{s_i^n})(e_{s_{i+1}^n}(y) - e_{s_i^n}(y)) \right| \leqslant \qquad \text{since } E_{s_{i+1}} A_{s_{i+1}} = A_{s_{i+1}} \text{ and } E_{s_{i+1}} A_{s_i} = A_{s_i},$$

$$\leqslant \sum_{i=1}^{k(n)} \| A_{s_{i+1}^n} - A_{s_i^n} \|_1 \, \| e_{s_{i+1}^n}(y) - e_{s_i^n}(y) \| <$$

$$< \sum_i \| A_{s_{i+1}^n} - A_{s_i^n} \|_1 \, \varepsilon = \qquad \text{by the uniform continuity of } s \mapsto e_s(y) \text{ on } [0, t],$$

$$= \varepsilon \sum_i \| A_{s_{i+1}^n} - A_{s_i^n} \|_1 =$$

$$= \varepsilon \sum_i (A_{s_{i+1}^n} - A_{s_i^n})(\mathbf{1}) = \qquad \text{since } A_{s_{i+1}^n} - A_{s_i^n} \text{ is positive.}$$

$$= \varepsilon A_t(\mathbf{1}) = \varepsilon \| A_t \|_1 ,$$

and we have proved the claim.

To prove the theorem, suppose that $|\zeta_t|^2 = Z_t' + A_t' = Z_t + A_t$, $t \geqslant 0$ are Doob-Meyer decompositions of $|\zeta_t|^2$. Then, by linearity, $A_t' - A_t = Z_t - Z_t'$ satisfies Equation 4.1. But for all i, $(Z_{s_{i+1}} - Z_{s_i})(e_{s_i}(y)) = 0$, for all $y \in \mathfrak{A}''$, for any L^1 martingale $\{Z_s : s \geqslant 0\}$. Hence $(Z_t' - Z_t)(y) = 0$, for $t \geqslant 0$. Since this holds for a σ-strongly dense set of $y \in \mathfrak{A}''$, we deduce that $Z_t = Z_t'$, $t \geqslant 0$. The result follows. ∎

In the terminology of [9], we would say that $(\mathfrak{A}'', \omega, (\mathfrak{A}_t''))$ is tempered, and the process $\{A_t : t \geqslant 0\}$ is nearly natural.

If $\rho \equiv 1/2$, then ω is a tracial state on $\mathfrak{A}''$ and $\mathfrak{A}''$ is the hyperfinite II_1 factor [11]. Let $\Psi(f) = b^*(f) + b(f)$, $f \in L^2(\mathbf{R}^+)$, f real, and let $\mathscr{C}$ and $\mathscr{C}_t$, $t \geqslant 0$, be the von Neumann algebras generated respectively, by the operators $\{\Psi(f) : : f \in L^2(\mathbf{R}^+), f \text{ real}\}$ and $\{\Psi(f) : f \in L^2(\mathbf{R}^+)\ f \text{ real, supp} f \subseteq [0, t]\}$ restricted to the cyclic subspace $\mathscr{K}$ generated by Ω. Then $\mathscr{K}$ is naturally isomorphic to the antisymmetric Fock space over $L^2(\mathbf{R}^+)$. Moreover, under this isomorphism, $\mathscr{C}$, $\mathscr{C}_t$, $t \geqslant 0$, are unitarily equivalent to the corresponding algebras of [3], and one can see that the Itô-Clifford theory of [3] is subsumed by the analysis given here.

5. DOOB-MEYER DECOMPOSITIONS OVER THE C.C.R.

We consider the representation of the C.C.R, over $L^2(\mathbf{R}^+)$ given by the gauge invariant quasi-free state ω with two point function

$$\omega(a^*(f)a(g)) = \int_0^\infty f(s)\overline{g(s)}\tau(s)\mathrm{d}s$$

where $\tau \in L^\infty_{\mathrm{loc}}(\mathbf{R}^+)$, $\tau > 0$ a.e. and f and g belong to the domain of $\tau^{1/2}$ as a multiplication operator on $L^2(\mathbf{R}^+)$.

The creation and annihilation operators are given concretely as unbounded operators on $\mathscr{H}$, the tensor product of two copies of the symmetric Fock space over $L^2(\mathbf{R}^+)$.

For $f \in D(\tau^{1/2})$, we define the Weyl operator $W(f) = \exp(a^*(f) - a(f))^-$. Then $W(f)$ is unitary and satisfies $W(f)W(g) = \mathrm{e}^{i\alpha}W(f+g)$ with

$$\alpha = \mathrm{Im} \int_0^\infty f(s)\overline{g(s)}\mathrm{d}s, \quad f, g \in D(\tau^{1/2}).$$

Denote by $\mathscr{W}$ the linear span of $\{W(f) : f \in L^2(\mathbf{R}^+),\ \mathrm{supp} f \text{ compact}\}$ and let $\mathscr{A}$ be the C^*-algebra generated by $\mathscr{W}$ while $\mathscr{A}_t$ is that C^*-subalgebra of $\mathscr{A}$ generated by $\{W(f) : f \in L^2(\mathbf{R}^+),\ \mathrm{supp} f \subseteq [0, t]\}$, $t \geqslant 0$. Then $\{\mathscr{A}_t'' : t \geqslant 0\}$ is a continuous filtration of von Neumann subalgebras of $\mathscr{A}''$. Moreover $\Omega = \Omega_0 \otimes \Omega_0$, where Ω_0 is the Fock vacuum vector, is cyclic and separating for $\mathscr{A}''$ on $\mathscr{H}$. We also note that because of the Weyl C.C.R., $\mathscr{W}$ is a $*$-subalgebra strongly dense in $\mathscr{A}''$. Furthermore, $\mathscr{W}\Omega$ is dense in $\mathscr{H}$.

Just as before, one can define stochastic integrals $\int_0^t \mathrm{d}a^*\xi$ and $\int_0^t \mathrm{d}a\eta$ for any processes $\xi \in L^2_{\mathrm{loc}}(\mathbf{R}^+, (1 + \tau(s))\mathrm{d}s, \mathscr{H})$ and $\eta \in L^2_{\mathrm{loc}}(\mathbf{R}^+, \tau(s)\mathrm{d}s, \mathscr{H})$. These are centred $\mathscr{H}$-valued martingales and satisfy the isometry relation

$$\left\|\int_0^t \mathrm{d}a^*\xi + \int_0^t \mathrm{d}a\eta\right\|^2 = \int_0^t \|\xi\|^2(1+\tau)\mathrm{d}s + \int_0^t \|\eta\|^2\tau\,\mathrm{d}s$$

for any $t \geq 0$. See [4] for further details. The following representation theorem is given in [23].

THEOREM 5.1. *For any $\mathscr{H}$-valued martingale $\{\zeta_t : t \geq 0\}$ there exist unique $\alpha \in \mathbf{C}$ and processes $\xi \in L^2_{\mathrm{loc}}(\mathbf{R}^+, (1+\tau(s))\mathrm{d}s, \mathscr{H})$ and $\eta \in L^2_{\mathrm{loc}}(\mathbf{R}^+, \tau(s)\mathrm{d}s, \mathscr{H})$ such that*

$$\zeta_t = \alpha\Omega + \int_0^t \mathrm{d}a^* \xi + \int_0^t \mathrm{d}a\eta$$

for all $t \geq 0$.

We can now state and prove the main result of this section.

THEOREM 5.2. *Let $\{\zeta_t : t \geq 0\}$ be an $\mathscr{H}$-valued martingale. Then there is an L^1 martingale $\{Z_t : t \geq 0\}$ and a natural increasing L^1 process $\{A_t : t \geq 0\}$ null at $t = 0$ such that $|\zeta_t|^2 = Z_t + A_t$, $t \geq 0$. Moreover, such a decomposition is unique.*

Proof. Existence. Write $\zeta_t = \alpha\Omega + \int_0^t \mathrm{d}a^*\xi + \int_0^t \mathrm{d}a\eta$ as given by Theorem 5.1. Then, as in [23], we approximate ξ and η by simple $\mathscr{W}\Omega$ valued processes $(h_n\Omega)$ and $(g_n\Omega)$, respectively, where $h_n\Omega \to \xi$ in $L^2_{\mathrm{loc}}(\mathbf{R}^+, (1+\tau(s))\mathrm{d}s, \mathscr{H})$ and $g_n\Omega \to \eta$ in $L^2_{\mathrm{loc}}(\mathbf{R}^+, \tau(s)\mathrm{d}s, \mathscr{H})$. Then

$$|\zeta_t|^2 = L^1\text{-}\lim \left| \alpha\Omega + \int_0^t \mathrm{d}a^* h_n\Omega + \int_0^t \mathrm{d}a\, g_n\Omega \right|^2$$

as in Section 4.

Write the right hand side as $|\alpha\Omega + H_n\Omega + G_n\Omega|^2$. This is the functional

$$\mathscr{A}'' \ni x \mapsto |\alpha\Omega + H_n\Omega + G_n\Omega|^2(x) =$$

$$= (xJ(\alpha\Omega + H_n\Omega + G_n\Omega),\ J(\alpha\Omega + H_n\Omega + G_n\Omega)) =$$

$$= ((\alpha\Omega + H_n\Omega + G_n\Omega),\ JxJ(\alpha\Omega + H_n\Omega + G_n\Omega)) =$$

$$= ((\alpha^*\mathbf{1} + H_n^* + G_n^*)(\alpha\mathbf{1} + H_n + G_n)\Omega,\ JxJ\Omega) =$$

(note that these manipulations with the unbounded operators H_n, G_n are justified)

$$= \left(\alpha^*\alpha\Omega + F_n\Omega + F_n'\Omega + \int_0^t h_n^* h_n\Omega(1+\tau)\mathrm{d}s + \int_0^t g_n^* g_n\Omega\tau\, \mathrm{d}s,\ JxJ\Omega\right) =$$

by [16], where F_n and F_n' are stochastic integrals,

$$= \left(x\Omega,\ J\left(\alpha^*\alpha\Omega + F_n\Omega + F_n'\Omega + \int_0^t h_n^* h_n \Omega(1+\tau)\,\mathrm{d}s + \int_0^t g_n^* g_n \Omega\tau\,\mathrm{d}s\right)\right).$$

Thus $|\alpha\Omega + H_n\Omega + G_n\Omega|^2 = Z_n + A_n$, where $Z_n = \alpha^*\alpha\Omega + F_n\Omega + F_n'\Omega$ and

$$A_n = \int_0^t h_n^* h_n \Omega(1+\tau)\,\mathrm{d}s + \int_0^t g_n^* g_n \Omega\tau\,\mathrm{d}s.$$

We see, as in Section 4, that

$$L^1\text{-}\lim A_n = \int_0^t (\cdot J\xi, J\xi)(1+\tau)\mathrm{d}s + \int_0^t (\cdot J\eta, J\eta)\tau\,\mathrm{d}s \equiv A_t,$$

and that $Z_t \equiv |\zeta_t|^2 - A_t$ determines an L^1 martingale.

Evidently, A_t is an L^1 process, increasing, and with $A_0 = 0$. Furthermore by Theorem 3.4, it follows that $\{A_t : t \geqslant 0\}$ is natural, and the existence part of the proof is complete.

Uniqueness. Suppose that $|\zeta_t|^2 = Z_t + A_t = Z_t' + A_t'$ are two possible Doob--Meyer decompositions with each increasing process natural. Then $Z_t - Z_t'$ being the difference of two natural processes will itself be natural. But it is also a martingale and as we remarked in Section 3 a natural L^1 martingale is identically zero. So $A_t = A_t'$ and the uniqueness of the decomposition follows. ▨

As an example of Theorem 5.2, consider the exponential martingales discussed in [13, 16, 23]. For $t \geqslant 0$, and $f \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$, put $M_t = \varkappa(t)W(f_t)$ where $f_t = f\chi_{[0,t]}$ and $\varkappa(t) = \exp\left(\frac{1}{2}\int_0^t (1 + 2\tau(s))|f(s)|^2\mathrm{d}s\right)$. Then $\{M_t : t \geqslant 0\}$ is a $\mathscr{W}$-valued martingale with $M_t^* M_t = \varkappa(t)^2 \mathbf{1}$ for all $t \geqslant 0$. Thus

$$M_t^* M_t = \mathbf{1} + (\varkappa(t)^2 - 1)\mathbf{1}$$

gives an L^∞-valued Doob-Meyer decomposition. To see that this agrees with that constructed in Theorem 5.2, we observe that M_t and $M_t\Omega$ agree as elements of L^2, and $M_t^* M_t$ and $|M_t\Omega|^2$ agree as elements of L^1.

Now, $M_t\Omega$ has the representation

$$M_t\Omega = \Omega + \int_0^t \mathrm{d}a_s^* f(s) M_s\Omega - \int_0^t \mathrm{d}a_s \overline{f(s)}\, M_s\Omega$$

for $t \geqslant 0$. Hence, according to Theorem 5.2,

$$|M_t\Omega|^2 = Z_t + A_t$$

where

$$A_t = \int_0^t (\cdot Jf(s)M_s\Omega, Jf(s)M_s\Omega)(1 + \tau(s))\mathrm{d}s + \int_0^t (\cdot J\overline{f(s)}M_s\Omega, J\overline{f(s)}M_s\Omega)\tau(s)\mathrm{d}s =$$

$$= \int_0^t (\cdot\Omega, \Omega)|f|^2\varkappa^2(1 + \tau)\mathrm{d}s + \int_0^t (\cdot\Omega, \Omega)|f|^2\varkappa^2\tau\, \mathrm{d}s =$$

$$= \int_0^t |f|^2\varkappa^2(1 + 2\tau)\, \mathrm{d}s\, |\Omega|^2 =$$

$$= (\varkappa(t)^2 - 1)\, |\Omega|^2, \quad t \geqslant 0.$$

But also $|M_t\Omega|^2 = \varkappa(t)^2|\Omega|^2$ and so $Z_t = |M_t\Omega|^2 - A_t = |\Omega|^2$.

Thus, the L^1-decomposition given by Theorem 5.2 is

$$|M_t\Omega|^2 = |\Omega|^2 + (\varkappa(t)^2 - 1)|\Omega|^2, \quad t \geqslant 0.$$

Since $|\Omega|^2 = \mathbf{1}$ in L^1, we recover the formula for $M_t^* M_t$ given above.

6. RANDOM INNER PRODUCTS

The existence of a unique increasing process in the Doob-Meyer decomposition of $\{|\zeta_t|^2 : t \geqslant 0\}$ allows us to define the random inner product between L^2 martingales, and leads to a characterisation of stochastic integrals.

The analysis here can be carried out for both the fermion (C.A.R.) and the boson (C.C.R.) theories, but we will only present the details for the former case and simply comment on the minor modifications required to give the analogous results for the boson case.

DEFINITION 6.1. Denote by $\mathfrak{M}_0^2$ the collection of centered L^2 martingales; i.e. those L^2 martingales $\{\zeta_t\}$ for which $(\zeta_t, \Omega) = 0$.

Thus, an L^2-martingale $\{\zeta_t\}$ belongs to $\mathfrak{M}_0^2$ if and only if $P_0\zeta_t = \zeta_0 = 0$. It will be convenient to sometimes denote a martingale $\{\zeta_t : t \geqslant 0\}$ simply by ζ.

DEFINITION 6.2. The *random inner product* of ζ' and ζ'' in $\mathfrak{M}_0^2$ is the L^1-process

$$\langle \zeta', \zeta'' \rangle_t(\cdot) = \int_0^t (\cdot J\xi', J\xi'')(1-\rho)\mathrm{d}s + \int_0^t (\cdot J\eta', J\eta'')\rho\, \mathrm{d}s \tag{6.1}$$

for $t \geqslant 0$, where

$$\zeta_t' = \int_0^t \mathrm{d}b^*\xi' + \mathrm{d}b\eta', \quad \text{for } t \geqslant 0,$$

and ζ_t'' is defined similarly.

Evidently $\langle \cdot , \cdot \rangle$ is sesquilinear (linear in the second position). By Schwarz' inequality and the isometry relations, it is easy to see that if (ζ_n') and (ζ_m'') are sequences in $\mathfrak{M}_0^2$ such that $(\zeta_n')_t \to \zeta_t'$ and $(\zeta_m'')_t \to \zeta_t''$ in L^2 for a given $t \geqslant 0$, then $\langle \zeta_n', \zeta_m'' \rangle_t \to \langle \zeta', \zeta'' \rangle_t$ in L^1 as $n, m \to \infty$.

If we denote by $\langle \zeta \rangle$ the unique (natural) increasing process in the Doob-Meyer decomposition of $|\zeta_t|^2$, $\{\zeta_t\} \in \mathfrak{M}_0^2$, then clearly $\langle \zeta \rangle = \langle \zeta, \zeta \rangle$. The random inner product $\langle \zeta', \zeta'' \rangle$ can thus be expressed in terms of the quadratic variation process $\langle \cdot \rangle$ by polarisation. In [6, 9], the random inner product was called the bracket process.

Given $\zeta', \zeta'' \in \mathfrak{M}_0^2$, $\langle \zeta', \zeta'' \rangle$ defines a σ-additive L^1-valued Borel measure on any bounded interval of $\mathbf{R}^+$ by

$$E \mapsto \langle \zeta', \zeta'' \rangle_E(\cdot) = \int_E (\cdot J\xi', J\xi'')(1-\rho)\mathrm{d}s + \int_E (\cdot J\eta', J\eta'')\rho\, \mathrm{d}s \tag{6.2}$$

for any Borel set $E \subseteq [0, t]$. To see that this is σ-additive note that it is clearly additive because the right hand side of Equation 6.2 is additive. Moreover it is clearly weakly countably additive. The result follows from a theorem of Pettis.

It is clear from Equation 6.2 that this measure is absolutely continuous with respect to Lebesgue measure. We also observe that this measure has finite variation on any bounded interval $[0, t]$. Indeed, if $|\langle \zeta', \zeta'' \rangle|(E)$ denotes the variation of a Borel set $E \subseteq [0, t]$ then, by definition, we have, for $\zeta \in \mathfrak{M}_0^2$

$$|\langle \zeta, \zeta \rangle|(E) = \sup \sum_i \|\langle \zeta, \zeta \rangle_{E_i}\|_1$$

where the supremum is taken over all finite partitions $\{E_i\}$ of E, each E_i a Borel set. Since $\langle \zeta, \zeta \rangle_E$ is $L^1(\mathscr{M})^+$-valued and the norm is additive on $L^1(\mathscr{M})^+$ then,

$$|\langle \zeta, \zeta \rangle|(E) = \|\langle \zeta, \zeta \rangle_E\|_1$$

which is finite. The result for $|\langle \zeta', \zeta'' \rangle|(E)$ follows by polarisation.

One can now construct L^1-valued integrals for suitable L^∞-valued integrands via the bilinear mapping $L^\infty \times L^1 \to L^1$ given by $(x, y) \mapsto xy$, where the juxtapositioning denotes the module action of $\mathscr{M}$ on $L^1(\mathscr{M})$ discussed in Section 1. The admissible integrands are given as suitable limits of L^∞-valued step functions. We are only interested here in adapted integrands and because of the particular structure we have, we will be able to define the integral for certain strong limits of step-functions.

Fix $t \geqslant 0$, and let $y(s)$ be an elementary L^∞-valued process on $[0, t]$; thus y has the form $y(s) = z\chi_{[c, d)}(s)$, for $0 \leqslant c \leqslant d \leqslant t$ and $z \in \mathfrak{A}_c''$. Then for $\zeta', \zeta'' \in \mathfrak{M}_0^2$, we define

$$\int_0^t y(s)\,\mathrm{d}\langle \zeta', \zeta'' \rangle_s(\cdot) = z\langle \zeta', \zeta'' \rangle_{[c, d)}(\cdot) =$$

$$= \langle \zeta', \zeta'' \rangle(\cdot z) = \int_c^d (\cdot zJ\xi', J\xi'')(1 - \rho)\mathrm{d}s + \int_c^d (\cdot zJ\eta', J\eta'')\rho\,\mathrm{d}s =$$

$$= \int_0^t (\cdot yJ\xi', J\xi'')(1 - \rho)\mathrm{d}s + \int_0^t (\cdot yJ\eta', J\eta'')\rho\,\mathrm{d}s$$

where

$$\zeta_t' = \int_0^t \mathrm{d}b^*\xi' + \mathrm{d}b\eta', \quad \text{for } t \geqslant 0,$$

and ζ_t'' is defined similarly. By linearity, $\int_0^t y(s)\,\mathrm{d}\langle \zeta', \zeta'' \rangle_s$ is similarly defined for any simple L^∞-valued process y on $[0, t]$; i.e. any finite sum of elementary processes. The class of integrands to which we wish to extend the integral are given by the following definition.

Definition 6.3. Let $\mathscr{P}[0, t]$ denote the linear space of processes $y : [0, t] \to L^\infty$ such that there is a sequence of simple L^∞-valued processes (y_n) satisfying

$\|y_n(\cdot)\| \leqslant M$ (Lebesgue) a.e. on $[0, t]$, for all n, for some $M > 0$, and $y_n(\cdot)$ converges strongly to $y(\cdot)$ (Lebesgue) a.e. on $[0, t]$ as $n \to \infty$. We say that (y_n) *defines* y.

Evidently, if $y \in \mathscr{P}[0, t]$, then there is $M > 0$ such that $\|y(\cdot)\| \leqslant M$ Lebesgue a.e. on $[0, t]$.

LEMMA 6.4. *Let* $\xi \in L^2([0, t], (1-\rho)\mathrm{d}s, \mathscr{H})$ *and* $\eta \in L^2([0, t], \rho\mathrm{d}s, \mathscr{H})$. *Then for any* $y \in \mathscr{P}[0, t]$ *we have* $y\xi \in L^2([0, t], (1-\rho)\mathrm{d}s, \mathscr{H})$ *and* $y\eta \in L^2([0, t], \rho\mathrm{d}s, \mathscr{H})$.

Proof. Let (y_n) be a defining sequence for y in $\mathscr{P}[0, t]$. Then $y\xi = \lim y_n\xi$ (Lebesgue) almost everywhere in $\mathscr{H}$ and so $y\xi$ is a measurable process. Furthermore, for some $M > 0$, $\|y_n\xi\| \leqslant M\|\xi\|$ and $M\xi \in L^2([0, t], (1-\rho)\mathrm{d}s, \mathscr{H})$. By the dominated convergence theorem we deduce that $y\xi \in L^2([0, t], (1-\rho)\mathrm{d}s, \mathscr{H})$ and $y_n\xi \to y\xi$ in $L^2([0, t], (1-\rho)\mathrm{d}s, \mathscr{H})$. The argument for $y\eta$ is virtually identical. ▨

PROPOSITION 6.5. *Let* $y \in \mathscr{P}[0, t]$ *and let* (y_n) *be a defining sequence for* y *Then for any* $\zeta', \zeta'' \in \mathfrak{M}_0^2$:

$$L^1\text{-}\lim_n \int_0^t y_n(s)\,\mathrm{d}\langle\zeta', \zeta''\rangle_s(\cdot) = \int_0^t (\cdot\, yJ\xi', J\xi'')(1-\rho)\mathrm{d}s + \int_0^t (\cdot\, yJ\eta', J\eta'')\rho\mathrm{d}s.$$

Proof. First we note that, by Lemma 6.4, the integrals on the right hand side are well defined for any $y \in \mathscr{P}[0, t]$.

Now for any $x \in L^\infty$ with $\|x\| \leqslant 1$, we have

$$\left|\int_0^t y_n(s)\,\mathrm{d}\langle\zeta', \zeta''\rangle_s(x) - \int_0^t (xyJ\xi', J\xi'')(1-\rho)\mathrm{d}s - \int_0^t (xyJ\eta', J\eta'')\rho\mathrm{d}s\right| =$$

$$= \left|\int_0^t (x(y_n - y)J\xi', J\xi'')(1-\rho)\mathrm{d}s + \int_0^t (x(y_n - y)J\eta', J\eta'')\rho\mathrm{d}s\right| \leqslant$$

$$\leqslant \int_0^t \|(y_n - y)J\xi'\|\,\|J\xi''\|(1-\rho)\mathrm{d}s + \int_0^t \|(y_n - y)J\eta'\|\,\|J\eta''\|\rho\mathrm{d}s.$$

But, by the remark made in the proof of Lemma 6.4, $y_nJ\xi' \to yJ\xi'$ in $L^2([0, t], (1-\rho)\mathrm{d}s, \mathscr{H})$ and $y_nJ\eta' \to yJ\eta'$ in $L^2([0, t], \rho\mathrm{d}s, \mathscr{H})$ and so by an application of Schwarz' inequality we see that the expression above is arbitrarily small for all sufficiently large n. ▨

DEFINITION 6.6. Let $y \in \mathscr{P}[0, t]$ with defining sequence (y_n) and let $\zeta', \zeta'' \in \mathfrak{M}_0^2$. We define the integral

$$\int_0^t y(s)\, \mathrm{d}\langle \zeta', \zeta'' \rangle_s$$

to be $L^1\text{-}\lim_n \int_0^t y_n(s)\, \mathrm{d}\langle \zeta', \zeta'' \rangle_s$. Thus by Proposition 6.5, we have

$$\int_0^t y(s)\, \mathrm{d}\langle \zeta', \zeta'' \rangle_s = \int_0^t (\cdot\, yJ\xi', J\xi'')(1-\rho)\mathrm{d}s + \int_0^t (\cdot\, yJ\eta', J\eta'')\rho\mathrm{d}s.$$

We now wish to construct stochastic integrals with respect to any martingale in $\mathfrak{M}_0^2$.

DEFINITION 6.7. For any elementary process on $[0, t]$ say $y = z\chi_{[c,d)}$, and $\zeta \in \mathfrak{M}_0^2$ we define the *stochastic integral* $\int_0^t y\mathrm{d}\zeta$ by

$$\int_0 y\mathrm{d}\zeta = z(\zeta_d - \zeta_c).$$

If $\zeta_s = \int_0^s (\mathrm{d}b^*\xi + \mathrm{d}b\eta)$, then by the construction of these integrals, we have

$$\int_0^t y\mathrm{d}\zeta = z\int_c^d (\mathrm{d}b^*\xi + \mathrm{d}b\eta) = \int_c^d (\mathrm{d}b^*\beta(z)\xi + \mathrm{d}b\beta(z)\eta) =$$

since $z \in \mathfrak{A}_c''$, and where β is the parity automorphism,

$$= \int_0^t (\mathrm{d}b^*\beta(y)\xi + \mathrm{d}b\beta(y)\eta).$$

The stochastic integral $\int_0^t y \mathrm{d}\zeta$ for any simple process y on $[0, t]$ is defined by linearity and, as above, obeys

$$\int_0^t y \mathrm{d}\zeta = \int_0^t (\mathrm{d}b^* \beta(y)\xi + \mathrm{d}b\beta(y)\eta).$$

PROPOSITION 6.8. *Let $y \in \mathscr{P}[0, t]$ with defining sequence (y_n) and let $\zeta \in \mathfrak{M}_0^2$ with*

$$\zeta_s = \int_0^s (\mathrm{d}b^* \xi + \mathrm{d}b\eta).$$

Then

$$L^2\text{-}\lim_n \int_0^t y_n \mathrm{d}\zeta = \int_0^t (\mathrm{d}b^* \beta(y)\xi + \mathrm{d}b\beta(y)\eta).$$

Proof. Since β is spatial and commutes with the conditional expectations e_s, $s \geqslant 0$ (see [3]), it is clear that $\beta(y) \in \mathscr{P}[0, t]$ with defining sequence $\beta(y_n)$. Thus $\beta(y)\xi$ and $\beta(y)\eta$ are processes and by Lemma 6.4, $\beta(y_n)\xi \to \beta(y)\xi$ in $L^2([0, t], (1 - \rho)\mathrm{d}s, \mathscr{H})$ and $\beta(y_n)\eta \to \beta(y)\eta$ in $L^2([0, t], \rho \mathrm{d}s, \mathscr{H})$. Hence by the isometry relations we have

$$\int_0^t y_n \mathrm{d}\zeta = \int_0^t (\mathrm{d}b^* \beta(y_n)\xi + \mathrm{d}b\beta(y_n)\eta) \to$$

$$\to \int_0^t (\mathrm{d}b^* \beta(y)\xi + \mathrm{d}b\beta(y)\eta)$$

in $\mathscr{H}$ as $n \to \infty$. ▨

This result allows us to make the following definition.

DEFINITION 6.9. For $\zeta \in \mathfrak{M}_0^2$ and $y \in \mathscr{P}[0, t]$, the *stochastic integral* $\int_0^t y \mathrm{d}\zeta$ is defined to be

$$\int_0^t y \mathrm{d}\zeta = L^2\text{-}\lim \int y_n \mathrm{d}\zeta$$

where (y_n) is any defining sequence for y. Proposition 6.8 gives us an explicit expression for this limit.

Recall that for any vector $\zeta \in \mathscr{H}$, there is $\alpha \in \mathbf{C}$ and processes ξ in $L^2(\mathbf{R}^+, (1-\rho)\mathrm{d}s, \mathscr{H})$ and $\eta \in L^2(\mathbf{R}^+, \rho\mathrm{d}s, \mathscr{H})$ such that

$$\zeta = \alpha\Omega + \int_0^\infty \mathrm{d}b^*\xi + \int_0^\infty \mathrm{d}b\eta$$

moreover α, ξ, and η are unique [22].

Definition 6.10. With the notation as above, define the map $\tilde{\ }: \mathscr{H} \to \mathscr{H}$ by

$$\zeta \mapsto \tilde{\zeta} = \bar{\alpha} + \int_0^\infty (\mathrm{d}b^* J\xi + \mathrm{d}bJ\eta).$$

We note that since J commutes with the projections P_s, $s \geqslant 0$, both $J\xi$ and $J\eta$ are processes, and since J is a bounded (conjugate-linear) operator $J\xi \in L^2(\mathbf{R}^+, (1-\rho)\mathrm{d}s, \mathscr{H})$ and $J\eta \in L^2(\mathbf{R}^+, \rho\mathrm{d}s, \mathscr{H})$ and hence $\tilde{\zeta}$ is well-defined. Using the isometry relations, it is easy to see that $\tilde{\ }$ is a conjugate-linear isometry on $\mathscr{H}$. Obviously $(\tilde{\zeta})^\sim = \zeta$ for all $\zeta \in \mathscr{H}$. Thus $\tilde{\ }$ is a conjugation on $\mathscr{H}$.

If $\zeta_t = \alpha\Omega + \int_0^t \mathrm{d}b^*\xi + \int_0^t \mathrm{d}b\eta$, $t \geqslant 0$, is an L^2-martingale then $\{\tilde{\zeta}_t : t \geqslant 0\}$ is an L^2 martingale given by $\bar{\alpha}\Omega + \int_0^t (\mathrm{d}b^* J\xi + \mathrm{d}bJ\eta)$, $t \geqslant 0$. Since $\{\zeta_t\} \in \mathfrak{M}_0^2 \Leftrightarrow \alpha = 0$ it is clear that $\zeta \in \mathfrak{M}_0^2 \Leftrightarrow \tilde{\zeta} \in \mathfrak{M}_0^2$.

Definition 6.11. Let $\mathscr{P}$ denote the linear space of processes $y : \mathbf{R}^+ \to L^\infty$ such that $y \restriction [0, t] \in \mathscr{P}[0, t]$ for all $t \geqslant 0$.

Theorem 6.12. *For any $y \in \mathscr{P}$ and martingales $\zeta', \zeta'' \in \mathfrak{M}_0^2$ and $t \geqslant 0$, we have*

$$\beta(y)(s)\,\mathrm{d}\langle \tilde{\zeta}', \zeta'' \quad = \left\langle \left(\int y\mathrm{d}\zeta' \right)^\sim, \zeta'' \right\rangle_t$$

where $\int y\mathrm{d}\zeta'$ *denotes the L^2-martingale* $\left\{ \int_0^t y\,\mathrm{d}\zeta' : t \geqslant 0 \right\}$.

Proof. Since $y \in \mathscr{P}$, it follows from Proposition 6.8 that for any $t \geqslant 0$,

$$\int_0^t y\,d\zeta' = \int_0^t (db^*\beta(y)\xi' + db\beta(y)\eta')$$

which defines an L^2 martingale in $\mathfrak{M}_0^2$. Thus by definition of the random inne product, Equation 6.1,

$$\left\langle \left(\int y\,d\zeta' \right)^{\sim}, \zeta'' \right\rangle_t (\cdot) = \int_0^t (\cdot\,\beta(y)\xi', J\xi'')(1-\rho)\,ds + \int_0^t (\cdot\,\beta(y)\eta', J\eta'')\rho\,ds.$$

On the other hand, by Proposition 6.5, we have

$$\int_0^t \beta(y)(s)\,d\langle \tilde{\zeta}', \zeta''\rangle_s(\cdot) = \int_0^t (\cdot\,\beta(y)\xi', J\xi'')(1-\rho)\,ds + \int_0^t (\cdot\,\beta(y)\eta', J\eta'')\rho\,ds.$$ ▨

The following result gives a characterisation of the process $\left\{ \int_0^t y\,d\zeta : t \geqslant 0 \right\}$.

THEOREM 6.13. *Let $y \in \mathscr{P}$ and $\zeta' \in \mathfrak{M}_0^2$. Then $\left\{ \int_0^t y\,d\zeta' : t \geqslant 0 \right\}$ is the unique element ζ, say, of $\mathfrak{M}_0^2$ which satisfies*

$$\int_0^t \beta(y)(s)\,d\langle \tilde{\zeta}, \zeta''\rangle_s = \langle \tilde{\zeta}, \zeta''\rangle_t \tag{6.3}$$

for all $t \geqslant 0$ and for every $\zeta'' \in \mathfrak{M}_0^2$.

Proof. By Theorem 6.12, ζ satisfies the equation. Conversely, suppose that γ in $\mathfrak{M}_0^2$ also satisfies Equation 6.3 for all $t \geqslant 0$ and $\zeta'' \in \mathfrak{M}_0^2$. Then we get

$$\langle \tilde{\zeta}, \zeta''\rangle_t = \langle \tilde{\gamma}, \zeta''\rangle_t$$

for all $t \geqslant 0$ and all $\zeta'' \in \mathfrak{M}_0^2$. In particular, we may take $\zeta'' = (\zeta - \gamma)^{\sim}$ to obtain $\langle (\zeta - \gamma)^{\sim}, (\zeta - \gamma)^{\sim}\rangle_t = 0$ for all $t \geqslant 0$. Thus $\langle (\zeta - \gamma)^{\sim}, (\zeta - \gamma)^{\sim}\rangle_t(1) = 0$ for all $t \geqslant 0$, and it follows that this can only be true if $(\zeta - \gamma)^{\sim}$ is the zero martingale. Hence $\zeta = \gamma$ and the proof is complete. ▨

THEOREM 6.14. *For given* $\zeta' \in \mathfrak{M}_0^2$ *and* $y \in \mathscr{P}$, $\left\{\int_0^t y(s)\,d\zeta'_s : t \geq 0\right\}$ *is the unique element* ζ, *say, of* $\mathfrak{M}_0^2$ *satisfying*

$$(6.4)\qquad (\zeta_t, \zeta''_t) = \int_0^t \beta(y)(s)\, d\langle \tilde{\zeta}', \tilde{\zeta}''\rangle_s(\mathbf{1})$$

for all $t \geq 0$ *and for any* $\zeta'' \in \mathfrak{M}_0^2$.

Proof. For any $\zeta'' \in \mathfrak{M}_0^2$ and $t \geq 0$,

$$(\zeta_t, \zeta''_t) = (\zeta''_t{}^*\zeta_t)(\mathbf{1}) = \qquad \text{by Definition 1.6}$$

$$= Z_t(\mathbf{1}) + \langle \zeta'', \zeta\rangle_t(\mathbf{1}) =$$

where $\{Z_t : t \geq 0\}$ is an L^1 martingale, by (polarising) Theorem 4.2,

$$= Z_0(\mathbf{1}) + \langle \zeta'', \zeta\rangle_t(\mathbf{1}) = \langle \zeta'', \zeta\rangle_t(\mathbf{1}) = \qquad \text{since } Z_0 = 0$$

$$= \int_0^t (J\zeta'', J\beta(y)\xi')(1-\rho)\, ds + \int_0^t (J\eta'', J\beta(y)\eta')\rho\, ds$$

by definition of the random inner product, since $\zeta_t = \int_0^t (db^*\beta(y)\xi' + db\beta(y)\eta')$ for $t \geq 0$. Hence

$$(\zeta_t, \zeta''_t) = \int_0^t (\beta(y)\xi', \xi'')(1-\rho)ds + \int_0^t (\beta(y)\eta', \eta'')\rho ds =$$

$$= \int_0^t \beta(y)(s)\, d\langle \tilde{\zeta}', \tilde{\zeta}''\rangle_s(\mathbf{1})$$

for all $t \geq 0$.

If $\gamma \in \mathfrak{M}_0^2$ also satisfies Equation 6.4, then $(\zeta_t - \gamma_t, \zeta''_t) = 0$ for every $\zeta'' \in \mathfrak{M}_0^2$ and every $t \geq 0$. Taking $\zeta'' = \zeta - \gamma$ gives $\zeta = \gamma$ and the uniqueness is proved. ∎

Suppose that $\zeta', \zeta'' \in \mathfrak{M}_0^2$. Then trivial modifications to the proof of Proposition 6.5 yield the existence of

$$\int_0^t x(s)\, d\langle \zeta', \zeta''\rangle_s y(s)^*$$

for $t \geqslant 0$, for any $x, y \in \mathscr{P}$ (as the L^1 limit of $\int_0^t x_n(s)\,\mathrm{d}\langle \zeta', \zeta''\rangle_s y_k(s)^*$ as $n \to \infty$, $k \to \infty$, where (x_n) and (y_k) are defining sequences for x and y respectively). Moreover, we have

$$\int_0^t x(s)\,\mathrm{d}\langle \zeta', \zeta''\rangle_s y(s)^* = \int_0^t (y(s)^* \cdot x(s)J\xi'(s), J\xi''(s))(1-\rho(s))\mathrm{d}s +$$

$$+ \int_0^t (y(s)^* \cdot x(s)J\eta'(s), J\eta''(s))\rho(s)\mathrm{d}s =$$

$$= \int_0^t (\cdot\, xJ\xi', yJ\xi'')(1-\rho)\,\mathrm{d}s + \int_0^t (\cdot\, xJ\eta', yJ\eta'')\rho\,\mathrm{d}s$$

for all $t \geqslant 0$.

We use this in proving a version of the Kunita-Watanabe Inequality.

THEOREM 6.15. *Let $\zeta', \zeta'' \in \mathfrak{M}_0^2$, and let $x, y \in \mathscr{P}$. Then, for any $u, v \in L^\infty$ and $t \geqslant 0$,*

$$\left|\int_0^t x(s)\,\mathrm{d}\langle \zeta', \zeta''\rangle_s y(s)^*(v^*u)\right|^2 \leqslant$$

$$\leqslant \left(\int_0^t x(s)\,\mathrm{d}\langle \zeta', \zeta'\rangle_s x(s)^*(u^*u)\right)\left(\int_0^t y(s)\,\mathrm{d}\langle \zeta'', \zeta''\rangle_s y(s)^*(v^*v)\right).$$

Proof. By the remarks above

$$\left|\int_0^t x(s)\,\mathrm{d}\langle \zeta', \zeta''\rangle_s y(s)^*(v^*u)\right|^2 =$$

$$= \int_0^t (uxJ\xi', vyJ\xi'')(1-\rho)\mathrm{d}s + \int_0^t (uxJ\eta', vyJ\eta'')\rho\,\mathrm{d}s\,\Big|^2 \leqslant$$

$$\leqslant \left(\int_0^t \|uxJ\xi'\|\,\|vyJ\xi''\|(1-\rho)\mathrm{d}s + \int_0^t \|uxJ\eta'\|\,\|vyJ\eta''\|\rho\mathrm{d}s\right)^2 \leqslant$$

$$\leqslant \left\{\left(\int_0^t \|uxJ\zeta'\|^2(1-\rho)\,ds\right)^{1/2}\left(\int_0^t \|vyJ\zeta''\|^2(1-\rho)\,ds\right)^{1/2} + \right.$$

$$\left. + \left(\int_0^t \|uxJ\eta'\|^2\rho\,ds\right)^{1/2}\left(\int_0^t \|vyJ\eta''\|^2\rho\,ds\right)^{1/2}\right\}^2 \leqslant$$

$$\leqslant \left(\int_0^t \|uxJ\zeta'\|^2(1-\rho)\,ds + \int_0^t \|uxJ\eta'\|^2\rho\,ds\right) \times$$

$$\times \left(\int_0^t \|vyJ\zeta''\|^2(1-\rho)\,ds + \int_0^t \|vyJ\eta''\|^2\rho\,ds\right) =$$

using $(ab+cd)^2 \leqslant (a^2+c^2)(b^2+d^2)$ for $a, b, c, d \in \mathbf{R}$

$$= \left(\int_0^t x(s)\,d\langle\zeta',\zeta'\rangle_s x(s)^*(u^*u)\right)\left(\int_0^t y(s)\,d\langle\zeta'',\zeta''\rangle_s y(s)^*(v^*v)\right). \qquad \blacksquare$$

In analogy with classical probability theory, we define the mean or expectation of $x \in L^\infty$ by $\omega(x)$. This can be written as $\omega(x) = (x\Omega, \Omega) = \omega_x(\mathbf{1})$. Similarly, the expectation of $\zeta \in L^2$ is defined as (ζ, Ω). This can be written as $(\zeta, \Omega) = (\Omega, J\zeta) = \omega_\zeta(\mathbf{1})$. Furthermore, $\omega_\zeta(\mathbf{1}) = \omega_x(\mathbf{1})$ if $\zeta = x\Omega$. The expectation is thus simply evaluation at $\mathbf{1}$ via the embeddings of L^∞ and L^2 into $L^1 = L^\infty_*$. It is therefore natural to define the expectation of an element $x \in L^1$ to be $\omega_x(\mathbf{1})$. As we saw at the begining of the proof of Theorem 6.14, for $\zeta', \zeta'' \in L^2$, the expectation of the product $(\zeta'\zeta'')$ is given by $(\zeta'\zeta'')(\mathbf{1}) = (J\zeta', J\zeta'') = (\zeta'', \zeta')$. Thus Theorem 6.14 can be considered to be an equality of expectations, and, by setting $u = v = \mathbf{1}$ in Theorem 6.15 we obtain an inequality involving expectations.

The next result is along similar lines.

COROLLARY 6.16. *For any $\zeta', \zeta'' \in \mathfrak{M}_0^2$ and $x, y \in \mathscr{P}$,*

$$\left|\left(\int_0^t x(s)\,d\zeta'_s \int_0^t y(s)\,d\zeta''_s\right)(\mathbf{1})\right|^2 \leqslant$$

$$\leqslant \left(\int_0^t \beta(x)(s)\,d\langle\tilde\zeta',\tilde\zeta'\rangle_s \beta(x)(s)^*(\mathbf{1})\right)\left(\int_0^t \beta(y)(s)\,d\langle\tilde\zeta'',\tilde\zeta''\rangle_s \beta(y)(s)^*(\mathbf{1})\right)$$

for all $t \geqslant 0$.

Proof. Exactly as in the proof of Theorem 6.14, we see that

$$\left(\int_0^t x(s)\mathrm{d}\zeta'_s \int_0^t y(s)\mathrm{d}\zeta''_s\right)(\mathbf{1}) = \int_0^t \beta(x)(s)\,\mathrm{d}\langle\tilde{\zeta}', \tilde{\zeta}''\rangle_s \beta(y)(s)^*(\mathbf{1}).$$

The result now follows from Theorem 6.15. ▨

The changes to be made for the boson case are quite trivial; b^*, b are replaced by a^* and a, while $1-\rho$, ρ are replaced by $1+\tau$ and τ, $\mathfrak{A}$ by $\mathscr{A}$ and β, the parity automorphism, is simply left out, i.e. is replaced by the identity automorphism. Apart from the obvious changes the definitions and results are identical to those presented above.

7. STOPPING

We will use the results of the last section to obtain a characterisation of stopped martingales. First we require the following observation.

LEMMA 7.1. *$\mathscr{P}$ contains the set of bounded L^∞-valued increasing processes.*

Proof. Let $y:\mathbf{R}^+\to L^\infty$ be a bounded increasing process. Clearly we may assume that $y(0)=0$. For $t\geqslant 0$, let $z_t=\inf_{s>t} y(s)-\sup_{s<t} y(s)$. Then

$$z_t\neq 0 \Leftrightarrow z_t^{1/2}\neq 0 \Leftrightarrow z_t^{1/2}\Omega\neq 0 \Leftrightarrow$$

since Ω is separating

$$\Leftrightarrow (z_t^{1/2}\Omega, z_t^{1/2}\Omega) = (z_t\Omega, \Omega) = \inf_{s>t}\gamma(s) - \sup_{s<t}\gamma(s)\neq 0$$

where $\gamma(s)=(y(s)\Omega,\Omega)$ for $s\in\mathbf{R}^+$. But γ is a bounded increasing function and so it follows that $z_t=0$ except possibly for at most a countable number of values of $s\in\mathbf{R}^+$. Hence $s\mapsto y(s)$ is Lebesgue a.e. strongly continuous on $\mathbf{R}^+$.

Now for any $t>0$, put $y_n=\sum_{i=0}^{n-1} y(s_i)\chi_{[s_i,s_{i+1})}$ where $s_i=it/n$. Then (y_n) is a defining sequence for $y\upharpoonright[0,t]$ and so $y\in\mathscr{P}$. ▨

DEFINITION 7.2. A *stopping time* τ is a projection-valued increasing process $\{p_t : t\geqslant 0\}$ with $p_0=0$ and $\sup_t p_t=\mathbf{1}$.

For any martingale ζ in $\mathfrak{M}_0^2$, the corresponding stopped martingale is the process $\left\{\int_0^t q_s\mathrm{d}\zeta_s : t\geqslant 0\right\}$ which we denote by $\zeta_{\tau\wedge\cdot}$, where $q_s=\mathbf{1}-p_s$, $s\in\mathbf{R}^+$.

For $\zeta', \zeta'' \in \mathfrak{M}_0^2$, the stopped L^1-process $\langle \zeta', \zeta'' \rangle_{\tau\wedge\cdot}$ is the process $\left\{\int_0^t q_s \, d\langle \zeta', \zeta'' \rangle_s : t \geq 0\right\}$.

The motivation for these definitions is given in [2, 8]. By Lemma 7.1, $s \mapsto q_s$ belongs to $\mathscr{P}$ and so the integrals above are well defined according to the discussion of Section 6. Furthermore, we can write these explicitly: (with the obvious notation)

$$\zeta_{\tau\wedge t} = \int_0^t db_s^* \beta(q_s)\xi(s) + \int_0^t db_s \beta(q_s)\eta(s),$$

$t \geq 0$, and

$$\langle \zeta', \zeta'' \rangle_{\tau\wedge t}(\cdot) = \int_0^t (\cdot\, q_s J\xi'(s), J\xi''(s))(1 - \rho(s))ds + $$

$$+ \int_0^t (\cdot\, q_s J\eta'(s), J\eta''(s))\rho(s)ds.$$

Evidently $\zeta_{\tau\wedge\cdot} \in \mathfrak{M}_0^2$.

As in Section 6 we will only write out the fermion theory. The usual obvious changes immediately give the corresponding boson formulae.

The following symmetry follows directly from the definition of the random inner product as an integral.

PROPOSITION 7.3. *Let $\zeta', \zeta'' \in \mathfrak{M}_0^2$. Then for any stopping time τ, we have*

$$\langle \zeta'_{\tau\wedge}, \zeta'' \rangle = \langle \zeta'_{\tau\wedge}, \zeta''_{\tau\wedge} \rangle = \langle \zeta', \zeta''_{\tau\wedge} \rangle$$

Proof. Note that

$$(\cdot\, J\beta(q_s)\xi'(s), J\xi''(s)) = (\cdot\, J\beta(q_s)JJ\beta(q_s)\xi'(s), J\xi''(s)) =$$

$$= (\cdot\, J\beta(q_s)\xi'(s), J\beta(q_s)\xi''(s)) = (\cdot\, J\xi'(s), J\beta(q_s)\xi''(s))$$

with a similar equation for η replacing ξ. The result follows. ▨

A characterisation of the stopped martingale $\zeta_{\tau\wedge\cdot}$ is given in the next theorem.

THEOREM 7.4. *Let $\zeta' \in \mathfrak{M}_0^2$ and let $\tau = \{p_t : t \geq 0\}$ be a stopping time. Then $\zeta'_{\tau\wedge\cdot}$ is the unique element of $\mathfrak{M}_0^2$ satisfying*

$$\langle \tilde{\zeta}', \zeta'' \rangle_{\beta(\tau)\wedge\cdot} = \langle (\zeta'_{\tau\wedge\cdot})^{\sim}, \zeta'' \rangle \tag{7.1}$$

for all $\zeta'' \in \mathfrak{M}_0^2$, where $\beta(\tau)$ is the stopping time $\{\beta(p_t) : t \geq 0\}$.

Proof. The left hand side of 7.1 is given by

$$\langle \tilde{\zeta}', \zeta'' \rangle_{\beta(\tau)\wedge t}(\cdot) = \int_0^t (\cdot\, \beta(q_s)\xi'(s), J\xi''(s))(1 - \rho(s))\mathrm{d}s +$$

$$+ \int_0^t (\cdot\, \beta(q_s)\eta'(s), J\eta''(s))\rho(s)\mathrm{d}s$$

for $t \geqslant 0$. But the right hand side of this equation is precisely that of Equation 7.1 at t. Thus $\zeta'_{\tau\wedge\cdot}$ does indeed satisfy Equation 7.1.

If $\gamma \in \mathfrak{M}_0^2$ also satisfies Equation 7.1 for all $\zeta'' \in \mathfrak{M}_0^2$, then $\langle (\zeta'_{\tau\wedge\cdot} - \gamma)^\sim, \zeta'' \rangle = 0$ for all $\zeta'' \in \mathfrak{M}_0^2$ and as in the proof of Theorem 6.13, we deduce that $\gamma = \zeta'_{\tau\wedge\cdot}$. ▨

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